A QUALITATIVE STUDY COMPARING THE INSTRUCTION ON VECTORS BETWEEN A PHYSICS COURSE AND A TRIGONOMETRY COURSE

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In the beginning was the Word, and the Word was with God, and the Word was God. He was with God in the beginning. Through him all things were made; without him nothing was made that has been made. … The Word became flesh and made his dwelling among us. (John 1:1-3, 14a)

If I believe that language creates power, then I am filled with hope from His name being *the Word.* If reality is created by words, then it makes sense that *through Him all things were made*. If words are signifiers for signifieds, then it makes sense that *the Word was God* and *the Word became flesh and made His dwelling among us*. If reality is beyond our human comprehension, then Jeremiah 17:9 statement *The heart is deceitful and desperately wicked. Who can know it?* has greater implications than I first recognized. I cannot know myself, I cannot know others, I cannot know the world. Great darkness comes from recognizing that there is no truth on earth amongst humans.

So, when I say that this dissertation would not be possible without the gospel of Jesus Christ, I really mean that. I wake in the morning because He *is* the Word. There may be no truth in me or on earth, but Jesus said, "*I am …the truth.*" I'm thrilled He doesn't just speak truth—He *is* truth. It's a seemingly-crazy grammatical statement, but the result of the collapse between signifier and signified means He *does not change like shifting shadows*—as words can do in separate communities of practice. He *is* the Word, *is* Truth, *is* Light, *is* the Way…all these metaphors because one word is not enough. I am filled with hope and eager expectation of knowing Him better as I study words in general.

To my committee, thank you for your time and investing a piece of your life in me. Often after I left your office a rush of deep-seated emotions would fill my heart because of my thankfulness for your willingness to sit and talk with me. I recognize how busy you are, and I never took it for granted that you paused to spend your time, which is a piece of your life, with me.

To my parents, I love you. Thank you Mom for the times you sat with me in encouragement of my forward progress. Your service of listening and encouragement is unmatched. Thank you Dad for pushing me and for believing in me. Your gifts of love have made my work easier.

To Kansas Conrady, thank you for your friendship, and thank you for being a sounding board to help me get my thoughts objectified. For countless hours, I've enjoyed discussing with you everything from Sfard to good food! You are the best traveling buddy ever, and I hope academia continues our adventures.

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countless others who's interest in my progress and encouragement have been instrumental in keeping me encouraged, my sincerest thanks. I thank God for you. Sometimes your questions and encouragement were so perfectly timed to keep me encouraged—I know there is no way for you to ever know how much they meant and how perfectly timed they were.

This project was possible because of the conversations I had with numerous science and engineering teachers, faculty, and students over the years. As the opening vignette explains, my curiosity about applying mathematics in science courses began with conversations with Dan Allen and Jimmy Parker, physics teachers at Abilene High. At the end of my masters at Oklahoma State, I got the opportunity to begin to explore "the math problem" in observing a physics course and discussing my observations with Dr. Bruce Ackerson, an OSU Physics faculty member. He then introduced me to Dr. Alan Cheville in Electrical Engineering, and I spent the beginning of my doctoral work having the fantastic opportunity to serve on the NSF grant *Engineering Students for the 21st Century*.

During my time on the grant, I was invited into classrooms for direct observations of instruction, to innumerable meetings discussing visions for and problems of engineering students' learning, and to attend several engineering-education national conferences. Through these valuable experiences, I came to sense the deep importance of educational researchers sharing expertise on how educational assumptions directly impact the product of our research and teaming with science and engineering instructors who desire to do educational research. I observed the impact of the engineering research in journals and at conferences was limited by some antiquated assumptions. Hear me clearly, I mean this as no means critical of my engineering colleagues; I myself worked within these assumptions until my doctoral work. So, how do engineering and science faculty who teach, do content research, and do service for the university come to understand choices in *assumptions* within educational research when they want to add educational research to their triumphs?

As a result, as I neared my time to begin my dissertation, I felt a sense of duty to return to the project early in my own travels that helped me see how our assumptions might change. I hope my work helps you in your own travels. Thank you especially to Drs. Alan Cheville, Karen High, Chuck Bunting, and Jim West for the opportunity to enter your world to observe mathematics and its use in your field.

Funny fact? As I stood in Hawaii overlooking its beauty, I nostalgically contemplated the path that brought me there. Random conversations in Texas led to a fellowship in Oklahoma that paid my way to an engineering conference in Hawaii. At that very moment and at that very place in Hawaii while I considered it all, Dan Allen and Jimmy Parker walked up.

We've come full circle.

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Title of Study: A QUALITATIVE STUDY COMPARING THE INSTRUCTION ON VECTORS BETWEEN A PHYSICS COURSE AND A TRIGONOMETRY COURSE

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Abstract: Science and engineering instructors often observe that students have difficulty using or applying prerequisite mathematics knowledge in their courses. This qualitative project uses a case-study method to investigate the instruction in a trigonometry course and a physics course based on a different methodology and set of assumptions about student learning and the nature of mathematics than traditionally used when investigating students' difficulty using or applying prerequisite mathematics knowledge. Transfer theory examined within a positivist or post-positivist paradigm is often used to investigate students' issue applying their knowledge; in contrast, this qualitative case-study is positioned using constructionism as an epistemology to understand and describe mathematical practices concerning vectors in a trigonometry and a physics course. Instructor interviews, observations of course lectures, and textbooks served as the qualitative data for in-depth study and comparison, and Saussure's (1959) concept of signifier and signified provided a lens for examining the data during analysis. Multiple recursions of within-case comparisons and across-case comparison were analyzed for differences in what the instructors and textbooks explicitly stated and later performed as their practices. While the trigonometry and physics instruction differed slightly, the two main differences occurred in the nature and use of vectors in the physics course. First, the "what" that is signified in notation and diagrams differs between contextualized and context-free situations, and second, physics instruction taught vectors very similar to trigonometry instruction when teaching the mathematics for doing physics, but once instruction focused on physics, the manner in which vector notation and diagrams are used differed different from what is explicitly stated during mathematics instruction.

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CHAPTER I

INTRODUCTION

> *Slightly exasperated from the day of teaching, a physics teacher joins me in the workroom lamenting, "The students don't know the math so they cannot learn the physics. I have to teach the math alongside the physics. I don't have time for that because then I can't cover all the physics material!"*
>
> *Concerned about my colleague's frustrations, I grab a sheet of paper so he can describe the physics problems he is teaching and what math is required for them. It turns out the students do not know how to add vectors geometrically and are too weak algebraically to solve the equations properly. Solving equations is taught in any algebra course; vectors are taught only in a trigonometry course, and I teach most of the trigonometry course sections at this high school. I happen to know my students completed the vector unit just a few weeks prior, and they did very well. However, as we continue to talk, he mentions specific students' names who are struggling, and I am shocked to realize they are my students.*
>
> *Thus, the next day in class, I decide to perform a little experiment. I asked my students to solve a problem requiring vector addition—requiring the very process the physics teacher claimed the students couldn't do. To my surprise, again, the students successfully displayed their skill—even the ones he had mentioned by name.*
>
> *So my mind swims with several questions: what is the problem?, why if students learn the mathematics in my class are they not able to display their knowledge when applying it in their physics class?, and what is causing this gap in transferring their skills from one setting to another?*

Typically, science and engineering departments require students to have learned Algebra, Trigonometry, and/or Calculus I from either high school or college mathematics courses prior to enrolling in their instruction. The advantage for science and engineering departments to employ this type of curriculum alignment and division of labor with the mathematics department is so that science and engineering concepts can be taught fluidly without simultaneously teaching the

the underlying mathematical concepts. This multiple-course-curriculum alignment is designed for students first to acquire mathematical fluency in mathematics courses and then successfully employ their mathematical knowledge during the science and engineering courses; however, science and engineering instructors often complain that the prerequisite math courses do not prepare the students for their courses, and as a result, the instructors feel they still have to teach the mathematics along with the science and engineering.

An example of this problem is students' limited knowledge of vector concepts during their physics courses. In 1995, Knight offered in *The Physics Teacher,* a practitioner's journal, a study in which 86% of the students in his study reported remembering that they had studied vectors prior to the physics course, but when their knowledge was evaluated, only a third of the students came with sufficient knowledge, and "a full 50% entered with no useful knowledge of vectors at all" (p. 77). Similarly, Nguyen and Meltzer (2003) found that even after a full semester of physics "more than one quarter of students beginning their second semester of study in the calculus-based physics course, and more than half of those beginning the second semester algebra-based sequence, were unable to carry out two-dimensional vector addition," and despite having received credit for a full semester of physics, "many students retained significant conceptual difficulties regarding vector methods that are heavily employed through the physics curriculum" (p. 630). While these studies contribute evidence that students lack requisite mathematical vector knowledge, they do not explain why the phenomenon occurs. Further research is necessary to seek an explanation.

Researcher Statement

Before building my argument and explanation for the background of the problem, I need to state my position as researcher (Patton, 2002). The introductory story that opens this chapter happened when I taught high school algebra and trigonometry. This question of why students were able to do the vector addition in my course and not in the physics course continued to haunt me when I returned to the university for my master's and doctoral degrees. The requirements for all three of my post-secondary degrees have been situated within mathematics education. While this study bridges two content fields, specifically mathematics and physics, I investigated this question situated from within and coming out of mathematics education. This view impacts this project in several ways. First, when I use the word *vector,* I'm referencing a mathematical, physical object. As I worked through this study, I came to recognize that people outside of the field of mathematics sometimes reference vector quantities as vectors. I am clarifying that I will not use *vector* to reference *vector quantities.*

Second, the background of the problem is built from my perspective as a mathematics educator and researcher who is attempting to understand the factors that contribute to students' misunderstanding and use of mathematics in general and vectors in particular. Instruction on two-dimensional vectors is a small unit in trigonometry, and I know trigonometry instructors who skip teaching vectors entirely; however, the large number of vectors illustrated throughout physics textbooks suggests vectors have a strong presence in physics instruction. Other than solving equations for a particular unknown, working with vectors is the most common mathematical practice used in physics. Therefore, I have come to value the investigation of vectors because of its importance for students' success in physics, and being successful in physics is required for most science and engineering degrees.

Background to the Problem

Historically transfer theory was used to explore why students have a difficult time applying their prerequisite knowledge to a new context. The problem with traditional transfer theory is that it assumes that students learned what they were supposed to learn with varying degrees of acquisition, but over the past decades, researchers have gradually adjusted their assumption about student learning from students *acquiring* knowledge to students *constructing* knowledge (Kieran, Forman, & Sfard, 2001). This change in assumption calls for the focus of research to shift from just measuring the degree of acquisition to describing "both the 'how' and the 'what' of learning" (Sfard, 2007, p. 566).

The change in assumptions about how students learn is accompanied by changes in assumptions about the nature of mathematics itself. Historically, mathematical symbols have been viewed as having fixed referents with the capability of embodying those fixed referents, but more recently math researchers recognize the multiple, nuanced meanings symbols hold depending on the context in which they are used (e.g. Sfard, 2000). Students construct their understanding of symbols and their meanings from the instructional contexts in which they learn and use them, and these meanings may not match across communities of practice. Sfard (2003) writes, “The act of naming and symbolizing is, in a sense, the act of inception, and using the words and symbols is the activity of constructing meaning” (p. 374).

Evans (1999) called for using Saussure’s linguistic theory to analyze the similarities and differences between the practices of school mathematics and other contexts. After noting the complications with using traditional transfer theories as a means of investigation, he argued that the use of Saussure’s linguistic theory might produce knowledge for instructors to help students bridge the communities. While his interest was in the relationship between school mathematics and out-of-school mathematics in work and everyday activities, this project’s interest is in the relationship of school mathematics to another discipline, specifically physics.

Applied Purpose

Using a different set of assumptions than traditionally used, this project investigated the instruction in a trigonometry and physics course to identify any differences in instruction that might cause students to struggle applying their vector knowledge between the two courses. With the former set of assumptions of how students learn, transfer theory was used as a lens to describe and investigate the problem of students applying prerequisite mathematics knowledge, but with the current set of assumptions of how students learn based on constructivist learning theories and an understanding of mathematics as subjective, concepts from the field of semiotics is used as the lens. A description of the instruction in which students learn is emphasized in the analysis because the instruction provides the tools and materials students generally use to construct their knowledge and become literate members of the communities of practice. By framing the learning and using of mathematics as a type of mathematical literacy, the study of the nature of mathematics in separate communities of practice is possible.

Based on the assumption that the math and physics communities are separate communities of practice and based on the assumption that mathematics itself is “a configuration of evolving, historically contingent literacies” (Cobb, 2004, p. 334), the purpose of this project is to begin to characterize the various practices of the two disciplines with respect to vectors and describe the possible differences so math and physics instructors, curriculum designers, and policy makers begin to recognize any instructional differences between a trigonometry course and a physics course and these vested interest groups can intervene in curricular and classroom practices. If the “act of naming and symbolizing is, in a sense, the act of inception,” then the following research questions guide the description of inception:

1. In the act of defining vectors, what are the similarities and differences between trigonometry and physics instruction?
2. In the act of symbolizing vectors, what are the similarities and differences between trigonometry and physics instruction?

If “using the words and symbols is the act of constructing meaning,” then the following research question guides the description of the meaning given to vectors by the two communities of practice:

3. In the activities of using vectors, what are the similarities and differences between trigonometry and physics instruction?

Significance of the Study

Some researchers would argue that testing what students understand about vectors can be investigated by means of a testing instrument comprised of validated, standardized questions requiring the use of vectors, which is what Knight (1995) offered readers. However, based on the assumptions made in this project, such an instrument would be laden with the researcher's choices concerning register and literacy practices. Register is a term used in literacy to describe the word and grammar choices people make for particular purposes in particular social settings. When an instrument is designed, the author selects the register used in the instrument; as a result, the instrument can only measure students' understanding of the particular literacy practices used by the instrument. Thus, such an instrument would not examine *what* students know about vectors; rather, it would seek to answer the binary question of whether the students know *the* particular register and literacy pattern used by the instrument. The use of an instrument manipulates the phenomenon of interest; therefore, this project seeks to contribute a description of the course discourses on vectors from both a trigonometry and physics course as a means of reflecting on what students might be experiencing during instruction as they develop their understanding. The significance of this study rests on whether the descriptions presented can offer insight to teachers and curriculum designers of the phenomena of interest to increase awareness of vectors' multiple meanings and to modify teaching practices that might shape student experiences.

Methodology

This case study project extends previous research by making a methodological and theoretical shift from the existing body of scholarship described in Chapter II. This project accepts Cobb's (2004) charge for mathematics educators and language and literacy educators to collaborate to *describe* the "development of a particular mathematical literacy and the means by which that process of development was supported and organized" by the classroom activities (p. 333). Consequently, the purpose of this project was to begin to characterize the various practices of two academic disciplines, specifically trigonometry and physics, with respect to the concept of vectors and to describe any differences in their practices. These practices are modeled and described in course instruction; therefore, the research design for this project required accessing and objectifying the instruction for analysis while not stripping the instruction of its complexity.

Lemke (1998) writes, “The essential context-sensitivity of meaning-based phenomena strongly suggests that if we are interested in, say a classroom phenomenon, that we study it in situ” (p. 8). Therefore, qualitative research is the most appropriate form of inquiry for this project because qualitative research supports what is termed “naturalistic” inquiry of naturally-occurring events and processes as they unfold and encourages the use of unobtrusive measures for collecting and analyzing data (Patton, 2002).

One trigonometry course and one physics course were selected for analysis, and the two courses were designed as separate case studies for in-depth study and comparison (Patton, 2002). Instructor interviews, observations of course lectures, and textbooks served as the qualitative data, and Saussure’s (1959) concept of the duality of a sign in having both a signifier and signified as a way of examining the data during analysis. Multiple recursions of within-case comparisons and across-case comparison were analyzed for differences in what the instructors and textbooks explicitly stated and later performed as their practices. Further details concerning the methodology are included in Chapter III.

Assumptions and Limitations

An assumption of this study is that my videotaping the course lectures did not affect the instructors’ content and direction of instruction. Because only two courses are used for the study, the findings are not intended to be generalizeable for all mathematics and physics teachers in the general population and nor can they be generalizeable for courses which do not focus on the traditional lecture as a primary means for providing information to the students. The inclusion of the course textbooks in the analysis serves to regulate this assumption and these limitations.

Organization of the Study

This project is presented through a six chapter organizational format. The first chapter provides an introduction, the background and statement of the problem, the purpose of the study, assumptions and limitations, and definitions of terms that will be used throughout the study. A review of relevant literature framing this project is presented in Chapter II. Chapter III presents the methodology of the study and the specific information relating to the research design, the participants, the types of data and their procedures for collecting, and the procedures for analysis of the data. Chapter IV and V provide the description of and comparison across the courses' instruction to answer the research questions. Chapter VI presents the overall findings of the project, the conclusions, the implications of the study, and the call for additional research.

CHAPTER II

REVIEW OF LITERATURE

The purpose of this chapter is to review the research that is relevant to identifying reasons why students struggle with applying their mathematics knowledge in science and engineering courses. This chapter begins describing the concept of transfer, the assumptions underlying its use in research, and its limitations as a foundation for refining and developing curriculum. The chapter then describes studies that have identified students' problems with mathematics in science and engineering in general and with vectors in particular. The chapter closes with a description of the gradual shift that has gained momentum in mathematics education over the past two decades from its objectivist assumptions toward more relative assumptions concerning meaning production and the effects of the shift on research. That shift has directly resulted in viewing mathematics not as a static, fixed body of knowledge but as a fluid practice developed and maintained by the communities of people who use it. By framing the learning and using of mathematics as a mathematical literacy, the study of the nature of mathematics in separate communities of practice is possible.

Transfer

Marini and Genereux (1995) state, "Broadly defined, transfer involves prior learning affecting new learning or performance" (p. 2). According to a review of literature by Macaulay and Cree (1999), there is much controversy about an exact definition for transfer of learning;

however, they suggest that Marini and Genereux's (1995) definition provides a universal description on which most experts would agree. In addition, debates flourish concerning the nature of transfer, the extent to which it occurs, and the nature of its underlying mechanisms (Barnett & Ceci, 2002, p. 612).

According to Lobato (1996, 2006), the historical development of the research on transfer of learning can be categorized by two trends in their theoretical perspectives: those situated within a classical model of transfer and those situated within a contemporary model of transfer. The classical model is dominated by experimental designs in which a control group does not receive the initial instruction that an experimental group receives, and the research seeks to find whether the initial instruction assists the success of the experimental group in a statistically significant manner greater than the control group's success. Any study considered to follow the classical model of transfer "typically involves pre-defining the underlying concept that should transfer and then seeking evidence for transfer. Studies based on these traditional views of transfer often show little support for the occurrence of transfer" (Robello, 2005, p.4). Showing little support for the occurrence of transfer is problematic because it is a goal of education, particularly math education, for students to be able to use prior learning to successfully inform new learning, life, or work activities.

Debates and criticism for the classical model of transfer began around the turn of the 20th century and gained momentum in the 1980s and 1990s (Barnett & Ceci, 2002; Lobato, 2006). Many researchers have critiqued the classical study of transfer (e.g., Beach, 1999; Evans, 1999; Greeno, 1997; Packer 2001). In particular, Lave's work, *Cognition in Practice: Mind, Mathematics, and Culture in Everyday Life* (1988) has opened the pressing question of whether transfer can ever occur if all learning is situated in context. Her research focused on grocery shoppers and compared their in-school and out-of-school mathematical abilities. She noticed the adults were not recognizing the sameness of the math activities across the separate communities of practice (the formal, academic activities of school and the everyday activities required for smart shopping). The adults demonstrated powerful mathematical reasoning skills when shopping but were unsuccessful when they addressed the same mathematical concepts on paper. Lave questioned the theorists' and researchers' conceptualization of transfer, but "she also pointed to researchers' rethinking of transfer along many dimensions. She did this by bringing to bear the assumptions about knowledge, learners, and context from a situated cognition perspective" (Lobato, 2006, p. 438). Lave's work has prompted researchers to reconsider the assumptions underlying transfer.

As a result of the debates, Lobato (2006) states some researchers have therefore abandoned transfer as a research construct (e.g., Carraher & Schliemann, 2002), others have side-stepped the debate by blurring the term as synonymous with *learning* (e.g., Campione, Sharpiro, & Brown, 1995), others have made methodological adjustments (e.g., Mayer, 1999; Novick, 1988), and others have justified results by creating categories and taxonomies of transfer (e.g., Butterfield & Nelson, 1991; Barnett & Ceci, 2002). Concerning the researchers' reactions to the debates, Lobato remarks,

> Although these organizational and methodological changes have provided important insights into the occurrence of transfer (or, more accurately, the lack thereof), it is also important to note that these adjustments can be adopted without addressing the concerns raised by Lave (1988) and many others regarding the conceptual foundations of transfer. (p. 435)

Lobato highlighted the concerns raised by Lave and others by providing five theoretical problems at the conceptual foundation of the classical transfer approach. For the purposes of this study, I highlight two.

First, classical transfer studies predefine "what" will be transferred and seek evidence of its use by the subjects. The problem is that the "what" in classic transfer studies is patterned after an expectation based on typical methods used by experts, and the research design accepts "as evidence of transfer only specific correspondences defined a priori as being the 'right' mappings"

(Lobato, 2006, p. 434). As a result, this research lens overlooks *what was* transferred by the novice learners because it does not take into consideration the complexity of a mathematical concept; instead, it assumes students learn in varying degrees specific mathematics concepts as having fixed meanings. The assumption is contradicted by research described later in this chapter.

A second theoretical problem provided by Lobato (2006) is

> the 'applying knowledge' metaphor of transfer suggests that knowledge is theoretically separable from the situation in which it is developed or used, rather than a function of activity, social interactions, culture, history, and context. As a result, this view of transfer is severely limited by ignoring the contribution of the environment, artifacts, and other people to the organization and support of the generalization of learning. (p. 434)

The underlying assumption is that students who have been taught properly will find transfer unproblematic, but the contextualization of mathematical concepts is often highly influential in understanding and using the concepts themselves (Evans, 1999). These realizations concerning the conceptual foundations of transfer have sparked strong critiques of the traditional view of transfer, and contemporary models have emerged to negotiate the reconciliation of transfer with changes in metaphors on learning. These shifts are a result of new conceptions about the nature of mathematics as being less concrete than traditional conceptions of mathematics, which is further explained later in this chapter.

In contrast to the classical model of transfer, the contemporary models (e.g., Bransford & Schwartz, 1999; Greeno, Smith & Moore, 1993; Greeno, 1997, 2006; Lobato 1996, 2003, 2006; van Oers, 2004) avoid pre-defining and measuring the concepts that should transfer to understand the nature of transfer, its processes, and its influences from the learner's perspective. These models include socio-cultural aspects in their discussion of transfer and "describe transfer as the dynamic construction of knowledge in the target scenario, rather than applying what they have learned previously" (Robello, 2005, p.6). Lobato (2006) clarifies, "Some of the alternative transfer perspectives have emerged not in order to offer an improved approach to the same phenomenon captured by classical measures, but to explore a different (but related) underlying phenomenon" (p. 436). For example, Lobato's alternative approach uses ethnographic methods to note similarities students create—whether "right" or "wrong." Her method allows for a broader inclusion of students' transfer than the narrow scope researchers usually consider, which requires a complete overhaul of the very definition of transfer and its surrounding metaphors.

Studies Conducted Concerning the Transfer of Mathematics to Science and Engineering

Mathematics has often been a topic researchers use to investigate transfer because the very nature of its content seems completely separate from cultural factors; as a result, a full review of the literature investigating transfer of mathematics is beyond the purpose of this project. This section reviews studies investigating transfer of mathematics to science and engineering courses using the traditional and contemporary models with brief discussions about their assumptions concerning the nature of mathematics.

Bassok and Hoyoak (1989) evaluated students' ability either to transfer isomorphic algebra questions to physics or vice-versa. The researchers selected arithmetic-progression word problems (algebra) and problems involving motion in a straight line with constant acceleration (physics). The study found that students who learned the algebra were able to successfully complete the physics problems, but if the students learned the physics problems first, they did not recognize how to complete the isomorphic algebra problems.

Bassok returned to this concept in 1990 to investigate whether the lack of transfer from physics to mathematics would change if students were given information concerning mapping of the relationship between the two topics. The study changed the mathematical content to be geometric progressions (algebra) and banking (finance) rather than algebra and physics. The study found "abstraction and transfer can be obtained following training in content-rich quantitative domains and are not limited to content-free algebraic training" (p. 531), but analysis showed transfer was somewhat lower.

Potgieter, Harding, and Engelbrecht (2008) sought to determine whether mathematically related difficulties that students experience in chemistry are due to deficiencies in their mathematics foundation or due to the complexity introduced by transfer of mathematics to a new scientific domain. The study compared students' ability either to solve questions written within the chemistry context referencing the Nernst equation or questions written stripped of all the chemistry context referencing an equivalent logarithmic equation to the Nernst equation. The findings prompted the authors to state, "The answer seems to be clear; the problem lies at the mathematics side and is not due to the transfer of mathematics to an application" (p. 197).

The Bassok studies and the Potgieter et al. study differ in the manner in which they studied transfer, but all three studies are based on the traditional views of transfer because they pre-defined the underlying concept that should transfer and sought evidence for transfer by quantifying the students' degree of knowledge. Just as Lobato described, the authors predefined the math and science concepts that would be transferred and sought evidence for it. Note that the researchers for these studies selected the mathematics and science content as being representative of the entire discipline. In other words, arithmetic progression represented the entire field of algebra, straight-line motion represented the field of physics, and the Nernst equation represented the field of chemistry. By assuming these specific topics adequately represent the entire discipline, the complexities for teaching and learning of the particular topic and the effects of the contexts in which the students learned the topics are not considered in the design. The researchers' selection of the mathematics and science content was assumed to be fixed and a neutral factor in the studies' results, which is not an assumption supported by the present study.

Ozimek (2004) examined students' retention and transfer from trigonometry to physics. From the traditional view of transfer, he found no evidence of transfer; however, from the contemporary perspective, he found that students do transfer what they learned in trigonometry to physics. Using just the contemporary models, his advisor and others at Kansas State extended his research and found similar results when investigating students' transfer from calculus to physics (Cui, Rebello, Fletcher, & Bennett, 2006). Using the contemporary models, both studies found that students needed specific scaffolding to connect the mathematics knowledge with the physics problem during the transfer process. In addition to these findings, they also observed that when students were asked to solve novel physics problems requiring calculus in which they had no prior connection to a similar physics problem, "Students often tended to use oversimplified algebraic relationships to avoid using calculus because they do not understand the underlying assumptions of the relationships" (Rebello, Cui, Bennett, Zollman, & Ozimek, 2007, p.20).

These studies suggest students do transfer their mathematics skills to physics. If this is the case, then the question of "what's the problem?" seems to go unanswered. These studies do not identify the problem and do not offer recommendations for curricular adjustments; therefore, transfer may not be a sufficient theory to identify the varied factors interfering with students' learning. The following section reviews the literature from any theoretical background identifying students' problems with mathematics in science and engineering.

Identifying the Problems with Mathematics in Science and Engineering

Mathematics is seen as the foundation and life-blood of science and engineering, and while mathematics is viewed as one of the essential tools for doing science and engineering, it is also one of the confounding variables interfering with students' learning (Varsavsky 1995). Yet, James (2008) search of the literature found few relevant articles focused on the topics of algebra, trigonometry, and calculus in science and engineering. As a result, James analyzed conference papers from the 2006 American Society for Engineering Education (ASEE) conference proceedings archives to list any problematic areas the authors mentioned about algebra, trigonometry, or calculus and to report the authors' assumptions, actions, and future vision for other researchers interested in building from these initial works. For the articles analyzed, engineering faculty did not report the problems with students' understanding of freshman-level mathematics in their courses, and they did not provide suggestions for future research springing from their current work—except to repeat their work with larger sample sizes. Engineering faculty seemed to participate in programs and interventions interrelated with mathematical learning or adjustments without identifying a problem with students' learning of the mathematics in their courses (James, 2008).

If engineering and science faculty perceive specific problems with students' use of algebra, trigonometry and calculus in their students' coursework, they are generally not investigating the cause of the problems in their research or explicitly reporting the problems in their literature. Artigue, Batenero, and Kent (2007) remark that in engineering research,

> Educational research papers, if written at all, tend to take the form of descriptive reports, not much connected with the research literature of the world of mathematics education. Where a distinct research methodology is followed, the pre-test/intervention/posttest approach is still quite common, which is nowadays out of favor amongst socioculturally influenced mathematics educators. (p. 1031)

In following a research methodology that is out of favor with mathematics educators, engineering and science educators miss an opportunity to influence and make connections with mathematics educators.

Articles describing students' difficulties with mathematics in science and engineering are not completely absent from the literature. Varsavsky (1995) surveyed engineering faculty concerning what mathematical skills engineering students most needed and at what point in their coursework. Of her ten findings, four are worth mentioning here because of their relevance to the current study. First, Varsavsky reports engineering instructors are not fully aware of the mathematical background of their students. For example, "Differentiation and integration are assumed knowledge from the day students enter university, even though students have not necessarily done calculus" yet (p. 343). Second, she noted various branches of engineering have different patterns concerning how much mathematics and what types of mathematics are utilized; polynomials, exponential, logarithmic and trigonometric functions are reported as the most used functions. Third, she noted engineering instructors differ in the techniques they use from those techniques taught by the mathematics department. She provides an example of a difference in technique: the engineers' use of Cramer's rule for solving linear equations in contrast to the math department's use of Gaussian elimination or matrix inversion. Fourth, the faculty surveyed reported students having "serious difficulties with their algebraic skills, abstract concepts and modeling" (p. 344). Varsavsky recommends that in future research "special attention must be paid to the techniques and notation used across the engineering and mathematics subjects" (p. 345), and she asserts that the mathematical needs of the different branches of engineering must be identified and addressed. This current study contributes a case report of some of the mathematical needs, techniques, and notation used in physics.

Scholarship using quantitative methods to investigate variables interfering with students learning is particularly well-represented in the literature. For example, Meltzer (2002) investigated variables that may contribute to learning gains from conceptual physics instruction using physics pre- and post-testing and students' college entrance exam scores. He found that students' initial physics conceptual knowledge was not correlated to their learning gains while students' mathematics skills or a factor related to mathematics skills were associated with the learning gains. By using college entrance exams that quantify students overall understanding of mathematics, the project did not identify *what* particular mathematics skills might have affected the results, which does not provide advice for curricular management. The author concludes by stating future research needed to identify and measure factors that help with "understanding and addressing students learning difficulties in physics" (p.1267). These studies point to the need for research, such as the current study, that directly observes and investigates the use of mathematics in other disciplines with depth and detail.

Using informal mixed-method approaches, Rebmann and Viennot (1994) and Clement, Lochhead, and Monk (1981) found that students struggle with translating physical situations into and out of algebraic notation. Both studies presented students with specific questions, briefly reported their scores, and focused on describing students' difficulties. Clement, Lochhead, and Monk (1981) claim from reflecting on their own teaching that students are generally given algebraic formulas in a mathematics coursework and do not have to create them; as a result, when students are asked to write algebraic expressions expressing connections to the physical world, their limited understanding of the meaning of variables and equations becomes apparent. The authors conclude

> What makes teaching (and learning) of these translation skills so difficult is that behind them there are many unarticulated mental processes that guide one in constructing a new equation on paper. These processes are not identical with the symbols; in fact, the symbols themselves, as they appear on the blackboard or in a book, communicate to the student very little about the processes used to produce them.

They call for teachers to explicitly adjust curriculum to help students acquire and develop these translation skills. This study provides some information to teachers to support their adjustment of their curriculum.

Rebmann and Viennot (1994) also identified two related skills with which students struggled concerning translating physics into algebraic expressions. The article described when physical quantities are translated into numerical values, the sign of a physical quantity is a combination of the parameters of the physical situation and the student's choice of coding (e.g. direction of the axes). This reasoning for selecting the sign of a physical quantity is different than the reasoning necessary to select the sign of the physical quantity when it is being related to other variables in an algebraic relationship (e.g. an equation defining a relationship between currents and voltages).

First, Rebmann and Viennot suggest that "students made no clear distinction between the 'sign of' a physical quantity and the 'sign in' a given relationship" (p. 724); as a result, students often just "manipulate algebraic expressions in a quasiautomatic way, and if they do not find the expected result, they just change as many signs as necessary" (p. 726). Second, they identify that this struggle with sign conventions occurs when working with common diagrams (e.g., diagram of the first law for a heat engine) because they have implicit meanings that students miss. The authors call for additional arrows to be added making the implicit meanings more explicit. Similar to Clement, Lochhead, and Monk (1981), the authors conclude "students should have definite opportunities to work on a precision translation from one language—verbal, algebraic, or diagrammatic—to another, and vice versa" (Rebmann and Viennot, 1994, p. 726).

In non-research related articles, Breitenberger (1992) and Vondracek (1999) identified factors contributing to student difficulties and shared them with their colleagues. Breitenberger states he made a systematic observation for 11 years of his first-year graduate students and noticed a decline in the students' mathematical abilities. His observations led him to believe that content is being eliminated from the mathematics curriculum and that students "regarded mathematics as mechanical method, not as constructive thinking" (p. 318). Based on the broad international pool of his students, he concludes that the problem must be more complex than indicting mathematics teachers for flawed approaches.

Vondracek, on the other hand, offered "a relatively quick and simple method that worked fairly well" (p. 32) for students' success, retention, and later recruitment of students in physics. He noticed "most of the equations used in an introductory, noncalculus physics class can be broken down into the general form $a = b/c$" (p. 32). As a result, he began early in the course "drilling" students how to write the equation $a = b/c$ for any of the three variables so that throughout the course when new topics introduced equations of the same form, students could easily use them. Vondracek expresses that students' success with the algebra results in their increased ability to understand the difference between direct and indirect relationships, and understanding the physical world increased their ability to understand context-free algebraic relationships and algebraic expressions.

The Teaching and Learning of Vectors

Searching the literature for articles related to the difficulties with teaching and learning vectors in mathematics or physics produced two categories of articles: those that focused on vectors as mathematics and those that focused on vectors as vector quantities. Many articles used *vector* in the title, but they did not seem to address mathematics—rather, they discussed vector quantities. Usually the articles discussed students' misconceptions with motion, forces, or other vector quantities. For example, Roche's article *Introducing Vectors* (1997) described approaches to teaching vectors that offer to minimize students' misunderstandings, but despite the fact that the article's subheadings are "vector graphics," "vector algebra," and "unit vectors," the article addresses how to teach physical *vector quantities* that are being graphed and expressed algebraically. Throughout these articles, the word *vector* referenced physical quantities—not a mathematical object. Other articles (e.g. Aguirre & Rankin, 1989 or Aguirre, 1988) reference physical quantities in their titles, which signals the articles focus on physical quantities, but use the words *vector characteristics* to reference characteristics of vector quantities rather than characteristics of mathematical vectors.

Knight (1995) and Nguyen and Meltzer (2003) offer studies using instruments with 7-9 questions and some free-response problems evaluating students' knowledge of mathematical vectors for physics. Both articles mention the importance of vector concepts to the physics curriculum and "the surprising lack of published research regarding student learning of vector concepts" (Nguyen, 2003, p.630). Knight found 86% of the students in his study reported remembering that they had studied vectors prior to a first-semester physics course, but when their knowledge was evaluated, only a third of the students came with sufficient knowledge, and "a full 50% entered with no useful knowledge of vectors at all" (p. 77).

Responding to Knight's study, Nguyen and Meltzer (2003) organized their project to question both first and second year physics students. Similar to Knight, they found 75-90% of their first-semester students reported having studied vectors before. Despite the exposure, their results found the first-semester students scored low, similar to Knight's findings. The researchers also found that even after a full semester of physics students' scores only showed a small performance improvement. Surprisingly, more than one quarter of students beginning their second semester of study in the calculus-based physics course, and more than half of those beginning the second semester algebra-based sequence, were unable to carry out two-dimensional vector addition. These scores show that despite having received credit for a full semester of physics, "many students retained significant conceptual difficulties regarding vector methods that are heavily employed through the physics curriculum" (p. 630).

While discussing students' minimal improvement in scores after taking a full semester of physics, Nguyen and Meltzer (2003) state,

> It seems that the bulk of students' basic geometrical understanding of vectors was brought with them to the beginning of their university physics course and was little changed by their experiences in that course, at least during the first semester." (p. 635)

The authors note that first-semester courses generally provide instruction for these concepts in less than one lecture; as a result, the authors advocate for significant additional instruction.

In Nguyen and Meltzer's discussion they also noted students' frequent imprecision in accurately copying the magnitude and/or the direction of the vectors as they solved the problems. The researchers write,

> many of the students' errors could perhaps be traced to a single general misunderstanding, that is, of the concept that vectors may be moved in space in order to combine them as long as their magnitudes and directions are exactly preserved. We suspect that, to some extent, this misunderstanding results in part from lack of a clear

> concept of how to determine operationally a vector's direction (through slope, angle, etc.). (2003, p. 635)

The authors support future projects investigating whether introducing vectors on and off grids would help develop the concepts.

These two studies are helpful in contributing evidence that some students lack sufficient mathematical vector knowledge prior to participating in a physics course and after taking a full-semester of a physics course. Further research is needed to offer an explanation as to why the phenomenon occurs. This project recommends investigating instruction practices with depth and detail as a way to identify some of the mathematical needs, techniques, and notation used in physics regarding vectors and any differences between the instruction between physics and trigonometry.

Shifts in Mathematics Education: Effects in Research

Over the past decades, researchers have gradually adjusted their stance to a more relative perspective concerning students' learning (Schoenfeld, 1992, p. 334). Part of the new stance posits that students do not *acquire* knowledge, they *construct* their knowledge. This stance is called constructivism, which is a theory about knowledge and learning that often shapes decisions about methods for teaching. Although constructivism is sometimes considered an epistemological position (Noddings, 1990) (different from *constructionism*), the field of mathematics education often references constructivism as a learning theory. There are various forms of constructivism (Phillips, 1995; Crotty, 2003), but in general, constructivism claims each student actively constructs his own knowledge, and for learning to occur, the student is required to gather and synthesize the meaning of the information. Because students construct their own knowledge, no assumption is made that their constructions are identical to the instruction. Instead, each student uses the instruction and countless factors from the social setting to make sense of the material, to connect it to their prior learning, and to construct new understanding. Thus, each student's constructions are rather unique.

Cobb (1994) suggests that this theory of learning has led educational researchers to increase their emphasis on the role of context and culture in shaping student learning. Rather than focusing on quantifying the degree of acquisition, some research designs focus on *describing* students' learning. Thus, the question of "*What* did the students learn?" has extreme importance because learners are constructing their knowledge within an array of cultural contexts, which a variety of research designs overlook, including using traditional transfer theories. In describing students' learning, many educational researchers have turned from using and producing a classical background-method-sample-findings-discussion structure in their research design and write-ups to using and producing highly-variable research projects from the use of qualitative methods that often rely on extensive, detailed transcripts as essential data for insights and expression (Kieran, Forman, & Sfard, 2001, p. 1).

Historically, mathematical symbols have been viewed as having fixed referents and embodying those fixed referents, but more recently researchers recognize the multiple, nuanced meanings symbols hold depending on the context in which they are found (Kieran, 2007, p. 707). In settings like an algebra course, students must construct their understanding of mathematical concepts and process along with the symbols and symbol systems used to describe, manipulate, and work with them (Sfard, 2000). Students are required to learn the meaning of the mathematical symbols, notations, and symbol systems, and they are required to use them to express their own thinking. Students must come to understand the nuanced meanings of symbols that morph from context to context without realizing, necessarily, that they are not fixed concepts.

The meaning of mathematical symbols is not fully transmitted to students with explicit words; instead, students develop their understanding while participating in course activities. For example, beginning algebra students often struggle with the different symbolic meanings an addition sign can have in various contexts. Most students develop an understanding of the symbol for addition in elementary school as an operational signifier. From their experiences when they see $5 + 2$, they know the plus sign is a symbol calling on them to perform the operation of addition. However, once the students enter algebra, they must read the symbol with respect to its context to recognize the addition sign's meaning as operational or as structural. For example, if an expression where the plus sign is nested between unlike terms, such as $x + 2$, the symbol is a structural signifier because $x + 2$ is an object representing a particular number, but if the expression is part of an equation, such as $x + 2 = 7$, then the signifier will be read as operational as we "un-do" the operation to solve the equation. If the expression is part of the equation of a line, such as $y = x + 2$, the addition sign is viewed structurally, but if working with graphs and their transformations, such as $y = f(x) + 2$, the same addition sign will switch back and forth between being operational and structural depending on the moment of the reader's reference.

In Sfard and Linchevski's (1994) study, the authors note several instances in which the context of a symbol marks the intended meaning for its reader. Sfard and Linchevski open their article by using 3(x+5) + 1 as an example of how this one expression can be viewed as

- directions for a computational process,
- an object representing a particular number,
- a function,
- a family of functions, or
- simply a string of random symbols based on the context in which it is found. (p. 191)

Thus, the context builds meaning for the expression that is not inherent within the symbols themselves, and the question of "*What* did the students learn?" becomes even more important because students *construct* their own knowledge of the symbols, their referents, and their separate uses; therefore, students may or may not *acquire* a complete understanding of all the nuanced and intended meanings of the symbols situated within all the contexts in which the symbol is used.

Not only is the meaning of mathematical symbols not always explicitly stated with words, the meaning of symbols sometimes develops unintentional meanings during instructional activities. Several studies have shown perceptions of the learners' meaning of the equal sign do not always match the meanings intended by mathematics teachers and the mathematics community in general (Prediger 2009; Molina, Castro, & Castro 2009; Kieran 1981; Jones & Pratt, 2005; Falkner, Levi, & Carpenter 1999; Saenz-Ludlow & Walgamuth, 1998). "The symbol which is used to show equivalence, the equal sign, is not always interpreted in terms of equivalence by the learner" (Kieran 1981, p. 317). Because students repeatedly see the equal sign separating the problem from the answer and representing the operating button on a calculator, they come to believe the meaning of an equal sign is operational—a "do something" signal. Jones and Pratt (2005), Falkner, Levi, and Carpenter (1999), Saenz-Ludlow and Walgamuth (1998) found that in adjusting the student-learning activities, students in their studies seemed to adopt equivalence as the meaning of an equal sign. Saenz-Ludlow writes, "Hence, it was clear that the children needed to experience a variety of numerical equalities to continue their progressive understanding of the meanings of the equal sign" (p. 182). The mathematics class activities were unintentionally causing students to interpret the meaning of the equal sign differently than would have been explicitly stated by the instructor.

The current project deviates from the scholarship described earlier in the chapter to make a theoretical shift to assume mathematical symbols can change meanings depending on context in which they are used and the contexts in which they are developing. While transfer studies and the Meltzer (2002) study seemed to assume that the specific mathematical content used in the investigation would not affect the results because the content is often conceptualized as static and fixed in meaning, this project investigated what happens if such an assumption was not followed.

Mathematical Symbols: How Do They Get Their Meaning?

In the fall of 1995, a small international group met at Vanderbilt University for a symposium to discuss learning in reference to symbolizing, communicating, and mathematizing.

The result of the symposium was the book *Symbolizing and Communicating in Mathematics Classrooms: Perspectives on Discourse, Tools, and Instructional Design* (Cobb, Yackel, & McClain, 2000). The book is organized into two parts: the first relating to theoretical issues and theory development, and the second relating to instructional design. The authors of the theoretical chapters offer "distinct but complementary views" on the processes of symbolizing and meaning making (p. 8). The authors focus on the activity of symbolizing because of their assumption concerning the reflexive relationship between symbol use and mathematical meaning, arguing, "the ways that symbols are used and the meaning they come to have are mutually constitutive and emerge together" (Cobb, 2000, p. 18). Within the book, Sfard (2000) writes, "today's student is usually thrown straight into a predetermined mathematical conversation, governed by a set of ready-made rules" (p. 55). Learners, therefore, construct their knowledge of ready-made mathematical symbols and their referents, and they do so from the process of observing communities of practice use them and then participating in the practices by using the symbols themselves.

As early as in 1999, Evans called for using Saussure's linguistic theory to analyze the similarities and differences between the practices of school mathematics and other contexts. After noting the complications with using traditional transfer theories as a means of investigation, he argued that the use of Saussure's linguistic theory might produce knowledge for instructors to help students bridge the communities. While his interest was in the relationship between school mathematics and out-of-school mathematics in work and everyday activities, this project's interest is in school mathematics to another discipline, specifically physics.

When Saussure (1959) introduces the words *sign, signifier,* and *signified*, he maps *signifier,* and *signified* to the words *concept* and *sound-image.* He argues, "Some people regard language…as a naming-process only—a list of words, each corresponding to the thing it names. … (but) the linguistic sign unites, not a thing and a name, but a concept and a sound-image" (p. 65). He criticizes the simple belief that words directly correspond to the thing it names because the belief assumes that "ready-made ideas exist before words" (p. 65), and he uses the concept of a tree as an example. The word *tree* is a signifier for the concept/idea that distinguishes it from, say, a bush—or even a pencil, mountain, or school. The concept of what a tree *is* is not the tree itself; the concept of what a tree *is* is an idea that links the tree with other similar objects characterized by the same idea. Saussure names the concept/idea as the *signified* and the sound-image/word as the *signifier*. Together, the signifier and signified are part of the linguistic sign. Figure 2.1 is his manner of representing the separation and unity of the signified and signifier as parts of the sign.

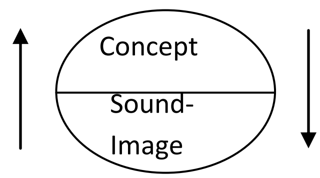


Figure 2.1. Saussure's depiction of the two parts of the linguistic sign

Borrowing this language and ideas from Saussure (1959), mathematical symbols, objects, and vocabulary could all be considered to be signs. Each sign has two parts: its signifier and its signified. The signifier is the visually accessible form of the sign, and the signified is the concept and/or meaning that is being represented by the signifier. For example, "½" or "half" are both signifiers. The use of the numbers 1 and 2, where 1 is above the 2 and has a line between them, signifies the same thing as the word using the letters *h,a,l,f*, and in both cases, what they signify is the quantity of half of an object or half of a set of objects. Children are not born with an understanding of the quantity of half—they must construct it, and they also must construct an understanding of the various signifiers that represent the signified, which occurs in a particular, richly-variable context.

Learners construct the meaning of a symbol much like the meaning of any word: through context and use (Sfard, 2003). For example, if a person is at a meeting where everyone has decided to "table" a discussion, the context in which the meeting occurs decides the use of the word "table." People in Britain use "table" to express the desire to place a discussion on the agenda, but people in the U.S. use "table" to express the desire to remove it from consideration from the meeting. Thus, the same word expresses two exactly opposite meanings. Context and culture provides people an understanding of what the word is to mean. The two communities have developed separate practices in using the word "table" to convey meaning, and, as a result, the meaning of the word is not universal.

Not only do learners come to understand symbols by observing their use in culture, they come to understand symbols as learners participate in using them. Students use the symbols and symbol systems even before they know exactly what the symbols mean and signify, and as a result, the student constructs the meaning of the symbols from the process of using them (Sfard, 2000, 2003). Continuing with the concept of *half* as an example, Van de Walle (2010) recommends that learners construct a meaning of half and other rational numbers from sharing, partitioning, and iteration tasks before learning the fraction notation, and once the notation is learned, he recommends students use the fraction notation. While using the notation, students continue to develop their understanding of the signifier and what it signifies. Studies show if course activities use multiple styles of area, length, and set models to illustrate fractional quantities, rather than just circle-style area models, students' understanding of fractional concepts, quantities, and symbols are broadened and deepened (Behr, Harel, Post, & Lesh, 1992). Activities that develop the multiple meanings of fractions also broaden and deepen student success (Thompson & Saldanha, 2003). The context and use of the fractional notation provides the meanings students develop.

Often, a mathematical object is introduced by providing a definition. The definition serves to connect the signifier to the signified, but for the learner, understanding occurs not from being introduced to a definition, but from observing and participating in the symbolizing within the context of learning. Sfard (2003) summarizes by saying, "The act of naming and symbolizing is, in a sense, the act of inception, and using the words and symbols is the activity of constructing meaning" (p. 374).

Mathematics as a Configuration of Literacies

Lave (1988) has been seminal in questioning whether mathematical reasoning is naturally universal across communities of practice. As a result of her study, "there has been a gradual, but by no means universal, shift to the view of mathematics as a configuration of evolving, historically contingent literacies" (Cobb, 2004, p.334); therefore, Cobb charged mathematics educators and language and literacy educators to collaborate to describe "students' development of a particular mathematical literacy and the means by which that process of development was supported and organized" by the classroom activities (p. 333).

The word *literacy* has multiple formal and colloquial meanings. In its general sense, *literacy* is defined as reading, writing, and thinking with understanding as a result of being competent and knowledgeable in specialized areas. As a result, literacy is characterized as not just the ability to read and write but is now characterized as a metaphor for describing a person's understanding of the content, processes, and manners of communication (verbal, written, and symbolic) of a specialized area.

Specialized areas develop, maintain, and adjust vocabulary and symbols to express their content, processes, and ways of thinking. Thus, a particular literacy is socially constructed and maintained by its community of practice. Wenger (2006) writes, "Communities of practice are groups of people who share a concern or a passion for something they do and learn how to do it better as they interact regularly." Because separate communities of practice—even within the same specialized area—exist, separate literacies form, and over time, these communities of practice adjust their content, processes, and ways of thinking resulting in the literacies being historically-contingent and evolving. Heller and Greenleaf (2007) explain,

> researchers have challenged the assumption that literacy learning is basically a solitary activity. Rather, people learn by interacting with others (especially with people who are

> more knowledgeable in the area than they are), gradually becoming familiar with and internalizing their ways of doing things (their practices). Every academic discipline, or content area, has its own set of characteristic literacy practices….To enter any academic discipline is to become comfortable with its ways of looking at and communicating about the world. (p. 7)

Instructors have developed an understanding of the content, processes, and manners of communication (verbal, written, and symbolic) of their specialized area, and students become familiar and internalize the instructors' ways of doing (their practices) as they participate in the course.

In a mathematics class, instructors are seasoned members of the mathematical community of practice, and students are required to learn both the mathematical content and processes, and the accompanying symbols and manners of symbolizing required by the content and processes. Likewise, in a physics class, instructors are seasoned members of the physics community of practice, and students are required to learn both the mathematical content and processes and the physics content and processes and the accompanying symbols and manners of symbolizing. The aspects of what instructors know and students must learn are the very elements necessary to be considered mathematically literate in the particular subject matter being taught.

Summary

Historically transfer theory was used to explore why students have a difficult time applying their prerequisite knowledge to a new context. A problem with traditional transfer theory is that it provides few suggestions for curricular changes or for determining what students did learn. Over the past decades, researchers have gradually adjusted their assumptions about students learning, the goal of research, and the nature of mathematics. Recently researchers recognize the multiple, nuanced meanings symbols hold depending on the context in which they are used (Sfard, 2000). Students construct their own understanding of symbols and their meanings from the instructional contexts in which the students learn and use the symbols, and these interpretations vary across different communities of practice.

This project extends previous research by making a methodological and theoretical shift from the existing body of scholarship described in this chapter. This project accepts Cobb's (2004) charge for mathematics educators and language and literacy educators to collaborate in order to *describe* "students' development of a particular mathematical literacy and the means by which that process of development was supported and organized" by the classroom activities (p. 333). The methodological shift is a result of wanting to describe the actual mathematics instructors model in classroom contexts that students must internalize to be considered mathematically literate. The theoretical shift builds from Evans (1999) contention that semiotics is a useful theory to approach the application of mathematics across communities of practice to avoid the pitfalls of transfer theory. In addition, the particular focus of the analysis will be on the act of naming, symbolizing, and using vectors resulting from Sfard's (2003) assertion that "The act of naming and symbolizing is, in a sense, the act of inception, and using the words and symbols is the activity of constructing meaning" (p. 374).

CHAPTER III

METHODOLOGY

The purpose of this project was to begin to characterize the various practices of two academic disciplines, specifically trigonometry and physics, with respect to the concept of vectors and to describe any differences in their practices. A qualitative case study of the two courses' instruction was conducted and analyzed to describe the meanings given to vectors. The strength of this qualitative case study approach is in its unobtrusive methods of data collection and analysis within the complex system of instruction and in its production of a rich, thick description as documentation of the mathematical practices. This chapter addresses methodology for the study including the theoretical perspective, research design, data selection, and analysis methods used to develop an understanding of the following research questions:

1. In the act of defining vectors, what are the similarities and differences between trigonometry and physics instruction?
2. In the act of symbolizing vectors, what are the similarities and differences between trigonometry and physics instruction?
3. In the activities of using vectors, what are the similarities and differences between trigonometry and physics instruction?

Theoretical Perspective

The majority of the research in mathematics, science, and engineering is conducted under a positivist or post-positivist paradigm in which the purpose of research is to contribute to a collection of hypothesized-and-tested, universally-verifiable statements to explain a reality. In contrast, this particular educational study is positioned using constructionism as an epistemology and interpretivism as a theoretical perspective in which the purpose of research is to understand and reconstruct a description of a phenomena from a novel perspective for the overall objective of accumulating a more informed and sophisticated description of the possible multiple realities under study (Crotty, 2003). Briefly stated, these differences concern beliefs about the construction and acquisition of research knowledge. Studies conducted under positivism demand measuring; studies conducted under constructionism often involve description. This positioning directs the aim of this inquiry toward understanding the mathematical practices concerning vectors in the two courses and reconstructing descriptions of these practices for the reader. The researcher assumes vectors and vectors' uses may have multiple meanings, and these meanings result from being constructed and reconstructed over the course of history and within cultures both by participants and for participants within their communities of practice.

As Chapter II describes, changes in the educational research community have led to new directions in research which include qualitative studies from diverse theoretical perspectives using varied methodologies. This project situates itself on the following assumptions: students and teachers construct their own knowledge, the meaning and use of mathematical symbols emerges from contextual use, and literacy theory provides an avenue for exploring the problem based on the other assumptions. These assumptions are briefly expressed in the following three paragraphs.

First, this project situates itself within a theory of learning referred to as constructivism. Although constructivism is sometimes considered an epistemological position (Noddings, 1990) (different from the term *constructionism* used above), the field of mathematics education often references constructivism as a learning theory. Using this learning theory, this project's research design is situated within the entire change in metaphor necessary to support the belief that students construct their understanding rather than acquire a pre-packaged, fixed set of knowledge

(Phillips, 1995; Sfard, 1998). From this perspective, a shift in metaphor for learning theories and research means that no longer is *knowledge acquired*—instead, *understanding is constructed*. Where *knowledge* is considered an object that a person either has or does not have, *understanding* is considered a measure of the quality or quantity of connections that a person has made with a particular idea and his existing ideas. Acquiring knowledge suggests knowledge is pre-packaged and complete—something a person either has or does not have. In contrast, constructing understanding suggests the process of understanding has many facets and parts that have to be assembled together—something a person does along a continuum of completion (Van de Walle, 2010). If knowledge is pre-packaged, then teachers can give it to their students, but if understanding must be assembled, then teachers can only support an individual and facilitate the student's own learning process. To construct understanding, tools and materials are necessary. The *tools* learners use to develop their understanding are their existing ideas, reflective thoughts, and learned experiences; the *materials* the learners use are situated in the context of learning—things that can be seen, heard, or touched.

Secondly, this project assumes symbols do not hold their fixed referents, and students construct their own understanding of symbols and symbol systems as a result of their experiences observing and participating in the activities of their coursework (Sfard, 2000). The larger body of ideas from which Sfard (2000) draws these ideas is from Saussure (1959) in the field of semiotics. Students may use symbols and symbol systems prior to fully understanding their meanings, and by using them, the students dynamically develop further understanding (Cobb, 2000). The relationship between understanding and use is recursive.

Third, this project supports the view that mathematics is a collection of multiple, evolving, historically-contingent literacies, and as such, it is constructed and reconstructed by communities of people who use mathematics in their practices (Cobb, 2004; Wenger, 2006). Among the forces that develop and maintain these practices are instructors and textbook authors, who reflect and develop values and goals for students to internalize to be literate members of their particular disciplines. Instructors and textbooks have intended and unintended classroom norms and practices that serve as materials for students constructing their understanding of the discipline's concepts, processes, and manners of communication (verbal, written, and symbolic). Practices include, but are not limited to descriptions, drawings, linguistic expectations in language, processes modeled, and processes assigned to exercise. Students construct their understanding of mathematical processes, concepts, and manners of communication amid the cultural settings where they both learn and use their understanding.

Thus, this project is grounded in the view that any individual student's internal cognition and learning are situated within the practices of the particular community in which they learn. Exploring the practices of a community provides a description of some of the *materials*, in the very broadest sense, students used to construct their understanding. Because mathematics is not a fixed concept and, as such, can develop multiple meanings and uses in separate communities of practice, this project assumes the possibility that mathematics used by any two disciplines may have differences that effect student learning success.

Research Design

To investigate a viable reason why some students struggle with applying their vector knowledge from a trigonometry course to a physics course (Knight, 1995; Nguyen & Meltzer, 2003), student understanding could be the focus of this research design; however, since students' understanding may still be developing, studying their practices would not be as strong indicator of the possible differences across the communities of practice. Because vectors may have multiple meanings and uses across communities of practice, the focus of this study is to describe the mathematical practices concerning vectors used by seasoned members of the mathematics and physics communities. These practices are modeled and described in course instruction; therefore, the research design for this project required accessing and objectifying the instruction for analysis while not stripping the instruction of its complexity. Lemke (1998) writes, "The essential context-sensitivity of meaning-based phenomena strongly suggests that if we are interested in, say a classroom phenomenon, that we study it in situ" (p. 8). Therefore, qualitative research is the most appropriate form of inquiry for this project because qualitative research supports what is termed "naturalistic" inquiry of naturally-occurring events and processes as they unfold and encourages the use of unobtrusive measures for collecting and analyzing data (Patton, 2002). The data sources were course textbooks, instructor lectures, and instructor interviews.

There are multiple qualitative approaches (Patton, 2002; Merriam, 1998). A case study approach was selected based on four criteria provided by Merriam (1998). First, a case study approach requires that data come from a bounded unit. In this case, the instruction in a particular area of mathematics, such as vectors, in two specific courses is a bounded unit that a researcher can examine in depth and detail. Second, a case study approach allows for multiple methods of data collection and data analysis. In this study, interviews, lectures, and textbooks are needed to weave a holistic description of each course's instruction. Third, a case study "can be characterized as being particularistic, descriptive, and heuristic" (Merriam, 1998). This study is particularistic because it focuses on the particular topic of describing how the two communities teach and use vectors in their instruction. Such a description may provide insight to the more general problem of why some students' struggle to apply their vector knowledge between a trigonometry course and physics course. This study is descriptive because the product of the study produced a rich, thick description as documentation of the mathematical practices. This study is heuristic because it provides insights into the two communities' practices to illuminate potential problems for students in their learning.

The intent of this case study is to be descriptive to provide a detailed account of the act of naming, symbolizing, and using vectors in both courses (Merriam, 1998). The research questions provide detailed information about the act of naming, symbolizing, and using vectors in each classroom context (described in Chapters IV and V) before any type of theorizing about the patterns occurs (described in Chapter VI). The researcher organized the data from each discipline as separate cases. Patton explains, "Case analysis involves organizing the data by specific cases for in-depth study and comparison" (2002, p. 447). Describing the practices of each discipline from individual interviews with instructors, observing and videotaping course lectures, and analyzing textbooks occurred first prior to comparing across the cases. Comparing across the disciplines usually resulted in returning to expand the descriptions of the individual cases.

Data Selection & Method of Collection

The course instruction was sampled from three qualitative sources: course lectures, course textbooks, and instructor interviews. Lectures and textbooks were selected because they were chosen by the instructor as the main vehicles in supplying students with the course instruction. Instructor interviews were selected as a means to compare the intended instruction to the enacted instruction (Marsh & Willis, 2003). This triangulation of data sources served as a means of supporting a rich description of the course instruction and to test for consistency and minimize the limitations of basing analysis on any one source. The three data sources were collected for both the physics course and the trigonometry course to construct the individual, in-depth studies on the two separate cases and later develop the cross-comparison.

Course Lectures

Instructors' lectures include reflective talk on the content material and student questions along with any modeling of the activities in which the students themselves should internalize. Trigonometry and physics lectures include vectors as part of their instruction; as a result, course lectures on vectors and using vectors served as the ideal focus for data collection. The method for collecting data from the lectures was by attending each lecture and videotaping the instructor and all visual devices used during instruction without capturing student behavior. This allowed the multiple meaning-making systems of verbal activity, written activity, and gestures to be later objectified into separate but interlaced transcriptions. The video sometimes captured student voices, but their voices were not included in the analysis.

Videos included lessons which develop an understanding of vectors and their practices. For the trigonometry course, portions of two class sessions were dedicated to vector instruction.

Videoing occurred during week 13 of a spring semester. In physics, vectors span most of the entire semester. This project limited its investigation to the instruction prior to the first exam. The physics lectures were videoed during weeks 2 and 3 of a fall semester.

Course Textbooks

The second selection of data is students' course textbooks because they serve as another primary source for instruction and support in student learning. Although students may use other written resources beside the textbook as learning resources, such as the internet, the textbook still serves as the primary written resource provided for students. Course textbooks serve as a source for analyzing the communities' practices because they are developed for the students with intention, by multiple experts collaborating as a unified voice as the instructional narrative, and after multiple drafts.

Textbooks offer the instructional narrative with supporting example problems and sets of practice problems for homework. This project excluded analysis of the homework practice problems and focused on the instructional narrative because it provided the course content, reflections on the content, and modeling of example problems.

Instructor Interviews

The third selection of data was an interview focused on the instructors' intentions for the course instruction of vectors prior to my videoing their course lecture. The interview questions were designed for the instructors to describe what they planned to teach, describe what they believed students should know about vectors after the lesson and homework, and describe what they assumed students already knew about vectors prior to their instruction (See Appendix 1).

Patton (2002) states that "Regardless of how sensitively observations are made, the possibility always exists that people will behave differently under conditions where an observation or evaluation is taking place than they would if the observer were not present" (p. 291). If people act differently when being observed, the purpose for including interviews was to compare the instructors' enacted course instruction when being videoed to their description of their intended course instruction when not being videoed. The interviews provided data concerning the intended course instruction—as possibly distinct from the enacted instruction during the lectures. They also provided information about the instructors' thoughts that contextualize elements of the course, characteristics of the students, and aspects of their teaching philosophies that are beyond the specific course instruction provided by observing the lectures and reading the textbooks.

Even during the interview, the instructors were still within a condition in which they were being observed; as a result, I elected to minimize my footprint in making the instructors feel "under the microscope" by audio recording rather than videoing the interviews and minimized the length of the interview by inquiring about demographic information through an email prior to the first interview (See Appendix 2).

Participants and Setting

University courses in trigonometry and physics were chosen for analysis for this project based on their convenience to the researcher. Courses chosen to be videotaped were at a large mid-western university. The courses were a freshman-level Trigonometry course and a sophomore-level, non-calculus-based General Physics course. The non-calculus-based physics course was chosen over a calculus-based physics course for two reasons: convenience to the researcher and its similarity to physics instruction at the high school level.

Patton (2002) writes, "Purposeful sampling focuses on selecting information-rich cases whose study will illuminate the questions under study" (p. 230). Instructors were purposefully sampled as established instructors in their field who use the typical, traditional method of lecture, which offers course content, reflection on content, and modeling of practice problems. Because this project assumes people's knowledge is based on their learning experiences and seeks to describe the practices within an American context, instructors were selected as learners of mathematics and physics from their own elementary, secondary, and post-secondary educations being entirely from within the US.

The trigonometry instructor's undergraduate degree was in chemical engineering. As a result of needing a job, she stumbled into teaching mathematics. At the time of this project, she had happily taught mathematics for over twenty years and returned to school to acquire both her master's and doctoral degrees in mathematics education. She has taught Trigonometry at the collegiate level about 20 times.

The physics instructor's undergraduate, masters, and doctoral degrees are in physics, and teaching at the university-level is part of his permanent occupational goals. He has taught algebra-based, calculus-based, and inquiry-based physics numerous times over 15 years. While he was instructor at a smaller university earlier in his career, he taught the entire range of physics courses and a variety of mathematics courses, including Trigonometry. The instructor also did extensive amounts of tutoring of individual and groups of students in physics and mathematics to earn income.

Note the trigonometry instructor has an engineering background, and the physics instructor has taught and tutored trigonometry. This similarity in the background and experiences of the two instructors, considered a homogenous sampling because it reduces the variation in instruction (Patton, 2002), strengthens the richness of the comparisons between the two courses.

Analysis Methods

Verbatim transcripts were created for each video session and interview by the researcher for the purpose objectifying and re-presenting the fluid activity during the course in a form to be further analyzed by the researcher. Decisions about what to include and how to present verbal words on paper have practical and theoretical consequences (Ochs, 1979; Johnstone, 2002; Cameron, 2001). Transcripts focused on the instructor's verbatim verbal statements and included a few paralinguistic features to maintain an oral element which supports meaning development. The leading elements of intonation units and transitional continuity, which work to support listener meaning making, were designed in the transcriptions using the conventions presented by Du Bois, Scheutze-Coburn, Cumming, and Paolino (1993). Separation of intonation units were generally marked by carriage returns, unless the flow of speech seemed better represented by not forcing the vertical break. Transitional continuity between intonation units were noted by periods (for finality) and commas (for continuing). These leading elements were sufficient in making aspects of the speech act accessible for this study; therefore, other finely detailed qualities of the speech act were not included.

Despite having these written transcripts, the videos were generally the primary source of data during analysis because the videos allowed the instruction to be repeatedly analyzed in situ with the weaving of verbal, written, gestural communication still intact. Verbal, written, and gestural communication each have unique characteristics, but they are also highly dependent on one another and influence meaning making for the listener/viewer (Singer, Radinsky, & Goldman, 2008).

Table 4.1

Trigonometry Instructor Transcript, Day 1 – Defining Vectors

Written	Verbal	Gesture
vectors ⇒		
	22. Vectors are directed line segments.	
23. directed line segments	23. ...	
	24. So, if you think about in geometry what you learned about a line segment,	
	25. right?	
26. (black dot, line coming out of black dot, and then another black dot at other end)	26. We know a line segment has end points,	
	27. and then it's a fixed length.	27. slides pen back and forth across length of segment
	28. So, with vectors,	
	29. the only difference there is,	
	30. it has endpoints, it's a fixed length,	
	31. But, we have to show direction.	
32. > (adds a > at one end of segment to make arrow)	32. So, we might say well, the vector direction is this way so we're going to put a little arrow here,	

In this paper, when necessary to communicate these elements intact for the reader, a three-part transcript is provided. The interlacing of a verbal, written and gestural transcript occurs by using a table with three columns to relate the transcripts together as they were in real time. The researcher's preference for verbal communication as a primary data source for meaning making is reflected in placing the verbal transcript between the other two transcripts. In this way, the verbal transcript "holds" the three together. Table 4.1 has been provided as an example.

In the written column, words that were specifically written have been typed directly into the transcript. When the researcher is required to describe what is written without being able to just explicitly state it, the descriptive words are set apart within parentheses. The gesture column is a description by the researcher of the gestures, and the verbal column is the verbatim transcript of what the instructor said with the above-mentioned paralinguistic features.

The researcher used Saussure's (1959) concept of the duality of a sign in having both a signifier and signified as a way of examining the data during analysis. The researcher considered mathematical symbols, objects, and vocabulary as having two parts: a signifier and a signified. My orientation as I analyzed the data was to focus on collecting and describing the introduction and use of the visually accessible form of the signs, which are referenced as *signifiers*, as flat objects without any meanings. As the descriptions of the signifiers' introductions and uses accumulated, the overall meanings for the vector signifiers, which are referenced as *signifieds*, could be weaved and constructed.

Figure 3.1 provides a visual representation of my method of analysis. The darkly shaded boxes provide the titles of the two cases: a trigonometry course and a physics course. Within each case, the three data sources (interviews, lectures, and textbooks) have been linked and represented by boxes. Overall, my analysis was primarily situated within the instruction provided by the textbooks and the lectures; therefore, the boxes representing textbooks and lectures are larger than the boxes representing interviews. Interviews served as a smaller voice in the overall analysis. The arrows communicate the *within-case* comparisons and the *across-case* comparison. The dotted, lighter-shaded box represents the innumerable recursions of doing the *within-case* and the *across-case* comparisons between the lectures and textbooks for each separate mathematical practice. In addition, these *within-case* and the *across-case* comparisons each required three separate recursions in order to compare what the instruction explicitly said to what was actually done during the instruction. The three layers are listed to the left of the box with the titles “SAY vs. SAY,” “DO vs. DO,” and “SAY vs. DO.”

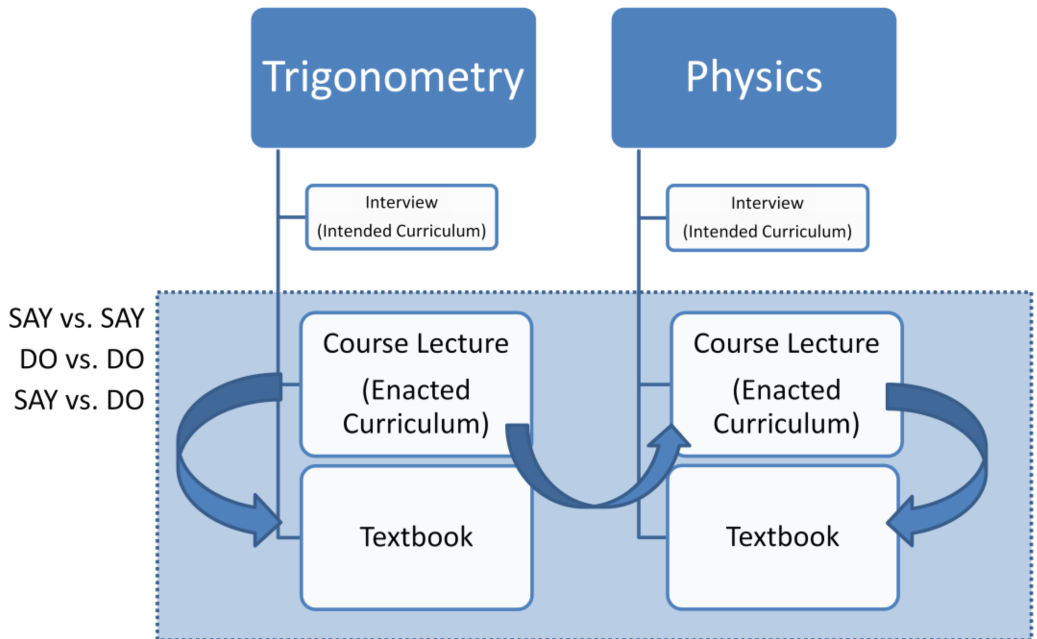


Figure 3.1. Visual representation of the method of analysis

For example, an instructor said the notation for vectors is $\vec{v}$. During the “SAY vs.SAY” layer, the method of analyzing this statement was to collect in an exhaustive manner all other explicit statements by the instructor toward this notation, and then compare the exhaustive list of explicit statements appearing in the textbook. Contrasting instruction in the lectures and textbook is considered a within-case comparison and was repeated for the other instructor and textbook as a second within-case comparison. Finally, a cross-case comparison was made of the explicit statements toward this notation. During the “DO vs DO” layer, each time the notation was used during the instruction, the notation was exhaustively analyzed for patterns in how it was used. This process was repeated for the textbook’s use of the notation, and then a within-case comparison was made of how the instructor and textbook used the notation. After completing the within-case comparison of the other instructor and textbook's use of the notation, a cross-case analysis was completed concerning how the four data sources used the notation. During the layer of "SAY vs. DO," the final set of within-case and across-case comparisons comprehensively analyzed differences in what the instructors and textbooks said about the notation verses the patterns of how they actually used the notation.

These *within-case* and *across-case* comparisons for each of the "SAY vs. SAY," "DO vs. DO," and "SAY vs. DO" layers occurred for each notation, each vocabulary word, and any other object of interest described for the study. Further details concerning the analysis methods are included in Chapters IV and V as the data is introduced. The writing and rewriting of Chapters IV and V was part of the method of analysis because they caused layers of synthesis and condensing. Returning to the data repeatedly was necessary to write a rich description and cross-analysis of each instructor's and textbook's explicit statements and use of vectors through their verbalization, gestures, and written representations.

Role of the Researcher

Having taught trigonometry at the high school and college level multiple times, I have *an insider's view* to the language and notation often used in a trigonometry course in teaching vectors (Patton, 2002, p. 268). In contrast, I'm an outsider to the language and notation used in a physics course. My position offers the advantage to analyze the instruction with a strong mathematical background and with fresh eyes to the physics instruction. With this position during analysis, I am able to observe my own struggles in transferring my mathematics knowledge to physics.

At first, a disadvantage of my position was over-looking what seemed to me routine behaviors, but because my analysis was sensitized by semiotics, strictly enforced creating the connections between signifier verses signified, and contrasted what was said verses what was done, instances of my over-looking these routine behaviors surfaced. These clarifications were strengthened further by the multiple iterations of my watching the videos and continuous comparisons across cases.

While videotaping the course lectures, I attempted to be as unobtrusive as possible. I attempted to have my video equipment set up before students entered the room and to keep my movements to a minimum. During videotaping, my role was strictly an observer. My purpose for attending was announced during the first class session I attended, and an explanation was offered to the students that my video was to capture instruction and my analysis would not include student questions and comments.

My role as a researcher is to conduct my project ethically. I have protected the identity of my participants. I have reported what I have observed and spent extensive time analyzing the data in depth and detail (Bailey, 2012). While writing every sentence, I have questioned myself asking “how do you know that’s true?” to interrogate my analytic processes and claims, following the guidelines for qualitative inquiry in which the researcher is the “primary instrument” of data collection, analysis, and interpretation (Patton, 2002). I have also sought peer review of my research and findings (Patton, 2002).

Trustworthiness Criteria

The value of any research project is dependent on its quality and trustworthiness. Lincoln and Guba (1985) offer four criteria for judging the quality of naturalistic inquiries. First, the credibility of this project is first addressed through the triangulation of the data collected, the case-study design, and the length of time spent in analysis. By collecting descriptions of the instruction through interviews, course lectures, and textbooks, inconsistencies and contradictory statements were better understood from the holistic understanding offered by comparing the data of all three sources. Credibility for the findings can also be argued because the qualitative, case-study design allowed me direct access to the phenomena of interest to gain an in-depth understanding of the meanings for vectors without having a data-collection instrument reshaping the data between the participants and me (Merriam, 1998, p. 203). In addition, the credibility of the findings can be argued based on my multiple, reoccurring observations of the instructional videos and numerous, repeated readings of the textbooks. Analysis occurred over several years with times of removal and times of intense immersion. Collectively, the time spent immersed and removed allowed for fresh thoughts, objectification of previous thoughts, retracing of previous analysis, and stimulation from outside conversations with other educators and researchers in various fields. Having triangulation of data, a case-study design, and a lengthy time of immersion with the data provides the reader assurances of the convergence of evidence.

Second, the transferability of this project is dealt with by providing readers with rich, detailed descriptions of the contexts, data, and participants. By providing the rich descriptions, the reader may determine how well another context or participant matches this research situation.

Third, the dependability of this project is substantiated by offering readers a detailed description of the analysis methods, the positionality of the researcher, and of the theoretical perspective used to analyze the data. These descriptions offer the reader some assurances that the findings are consistent with the data collected.

Finally, the confirmability of the project is addressed by linking any claims, findings, and interpretations directly to the supporting data.

Organization of the Study

Chapters IV and V will provide the analysis and description of the course instruction in the trigonometry course and the physics course. Chapter VI will present the overall findings of the project, the conclusions, the implications of the study, and the call for additional research.

CHAPTER IV

ANALYSIS OF THE DATA: INCEPTION OF *VECTOR*

Sfard (2003) writes, "The act of naming and symbolizing is, in a sense, the act of inception, and using the words and symbols is the activity of constructing meaning" (p. 374). If the "act of naming and symbolizing is, in a sense, the act of inception," then the following research questions guide the description of inception:

1. In the act of defining vectors, what are the similarities and differences between trigonometry and physics instruction?
2. In the act of symbolizing vectors, what are the similarities and differences between trigonometry and physics instruction?

If "using the words and symbols is the act of constructing meaning," then the following research question guides the description of the meaning given to vectors by the two communities of practice:

3. In the activities of using vectors, what are the similarities and differences between trigonometry and physics instruction?

While Chapter V addresses the third research question, Chapter IV addresses the first two research questions in the order in which they are presented. The chapter begins with a description of how the instructors and textbooks introduce and define vectors and is followed by how the instructors and textbooks initially symbolized vectors.

The chapter presents in detail the data sources and analytic decisions that led to the findings, which are further presented in chapter 6. Analytic decisions were organized as within-case comparisons and across-case comparisons (Merriam, 1998), and this chapter is the product of the overall case study. Patton (2002) explains that a *case study* is both an analytic process and an analytic product. He writes, “The case study approach to qualitative data analysis constitutes a specific way of collecting, organizing, and analyzing data; in that sense it represents an analysis *process*….The analysis process results in a *product*; a case study” (p. 447, emphasis in the original). This project collected data for an in-depth description and analysis of each discipline’s instruction as a case before comparing across the two individual cases. After comparing instructors’ instruction across cases, the instructors’ instruction was compared within the case to their respective textbooks.

Generally, this method of comparing across the cases and within the cases resulted in returning to further analyze and in developing the cases with more specificity than first used. The continuous weaving within and across the cases produced the descriptions provided in this chapter. The representation of that analysis compares what the instructor says about vectors against what they do as they work with vectors, it compares the instructors’ instruction against their textbooks’ instruction, and it compares the trigonometry instruction to the physics instruction.

Explicitly defining *Vector*

This portion of the chapter describes the explicit definitions provided by the instructors and textbooks, and it has six sections: an analysis of the instruction by the trigonometry instructor, an analysis of the instruction by the physics instructor, a cross-analysis of the similarities and differences between the instruction provided by the instructors, an analysis of the instruction by the trigonometry textbook, an analysis of the instruction by the physics textbook, and an overall cross-analysis.

These six sections are a result of analyzing what each instructor and textbook *says* a vector *is*. The researcher looked for the first instances of where the word *vector* occurred and anywhere where the instruction made statements of "a vector is…." These moments of instruction were then compared across the instructors and textbooks.

The results of analysis reveal key components of the trigonometry textbook's definition differ significantly from the trigonometry instructor's and physics textbook's and instructor's definitions. Results of analysis also reveal strong similarities among the trigonometry instructor's, the physics textbook's, and physics instructor's definitions.

Trigonometry Instructor's Instruction

The trigonometry course chosen for this study was at a mid-sized, mid-western, public university. The class met in the mid-morning of Mondays, Wednesdays, and Fridays for 50 minutes. This vector unit was taught a week and a half prior to the final exam. She remarked during the interview that she always feels during this time of the semester that teaching vectors is all "for naught" because it's the end of the semester and the students have tunnel vision toward finals and the summer. At the beginning of the second lecture when students were invited to ask questions about their vector homework from the previous class session, the students responded instead with inquiries about the upcoming final exam, and when she brought them back on topic concerning questions about vectors, the students again responded with more inquiries about the final exam. No questions concerning the vector homework were given.

Despite the students' focus on finals, the instructor expressed her hope that students would get into a later course feeling that they know how to do "basic operations with vectors." She expressed her wish that the course had more time to "work a bunch of application problems" with vectors, but the time constraints force her to focus on teaching how to add vectors geometrically and algebraically, how to multiply a vector by a scalar, how to find dot products, how to find the angle between two vectors, and, if she had time, how to do cross products. Her instruction did include each of these elements except cross products. In addition to basic operations, she commented that much of the instructional time would be spent introducing new vocabulary and terminology, which it was. She did not assume her students had ever learned vectors before "because unless they took trig or physics in high school, I don't know where they would have gotten it."

When one enters the room, a projection screen and series of white boards ran along the right-hand wall, and about 40 plastic chairs with their accompanying, fixed desktops were arranged in narrowly-spaced columns to face the whiteboard and projection screen. Over 30 students were in the class. During her lectures, she predominately relied on the white board to convey her written work. The projection screen switched from displaying her use of a computer-generated application of a graphing calculator, her written work on grid paper or diagrams from the book using an Elmo device, and her PowerPoint listing the unit's topics and vocabulary. During the interview, she remarked that she uses the PowerPoint to remind herself of the topics she is supposed to teach that day.

The trigonometry lectures on vectors spanned a day and a half. The unit began with a brief narrative of what vectors were and followed with 14 example problems where the instructional narration was interwoven. Of the 14 example problems, 13 were taken as specific problems from the book, and one example was created by the instructor during the lecture. She remarked during the interview that the problems she selects are the even-numbered problems that are similar to the odd-numbered problems she assigns for homework. The example problems partitioned the trigonometry instruction into *segments*. Thus, the vector unit in the trigonometry course spanned 15 segments: the introductory narrative and the following 14 example problems. The next section offers the data concerning how the trigonometry instructor defines vectors.

Explicit definitions. After spending the first 16 minutes of class answering questions about the previous homework assignment using complex numbers, the trigonometry instructor initiated the new instruction by announcing to her class that "Today we are going to do vectors." After writing "vectors" on the board, she introduced the word *vector* with its similarity to the words *vehicle* and *convection* resulting from their similar meaning in Latin. Moving to stand formally in front of the white board, she then began her formal instruction by defining vectors. The transcript in which she defines vectors is Table 4.1.

Table 4.1

Trigonometry Instructor Transcript, Day 1 – Defining Vectors

Written	Verbal	Gesture
vectors ⇒		
	22. Vectors are directed line segments.	
23. directed line segments	23. ... 24. So, if you think about in geometry what you learned about a line segment, 25. right?	
26. (black dot, line coming out of black dot, and then another black dot at other end)	26. We know a line segment has end points, 27. and then it's a fixed length. 28. So, with vectors, 29. the only difference there is, 30. it has endpoints, it's a fixed length, 31. But, we have to show direction.	27. slides pen back and forth across length of segment
32. > (adds a > at one end of segment to make arrow)	32. So, we might say well, the vector direction is this way so we're going to put a little arrow here,	

Notice in line 22 that the instructor defined vectors to be "directed line segments." She went on in lines 26-27 to characterize line segments as having end points and a fixed length, and her gesture during line 27 conveys that the length is bound between the two endpoints. In lines 28-31, she equated vectors with line segments except for having the additional characteristic of direction, which is depicted with "a little arrow" (line 32). This statement matches the first statement on her PowerPoint outlining the goals and order of her instruction for the day, which said, "line segment vs. directed line segment." Thus, a vector is a line segment (a bounded length between the two endpoints) with direction depicted with a little arrow.

Later in the same lesson, she provides a second and third definition of a vector, but her body language and tone do not give the same authority to these definitions as she did for the first. As she builds up to start her third instructional segment, she reminds her students in one sentence that "vectors in space are defined by an initial point and a terminal point," (T1-427). This statement emphasizes that the nature of a vector is determined by its initial and terminal point and provides a second, less-formal definition concerning the defining nature of a vector. A third definition occurs while she provides instruction on how to find the magnitude of a vector. As she introduces the formula $\|\overrightarrow{PQ}\| = \sqrt{a^2 + b^2}$ and after writing the formula on the board, she asks the class, "So what does that mean a squared plus b squared?" Without waiting for an answer, she moves to write the equation $\vec{\mathrm{v}} = \mathrm{ai} + \mathrm{bj}$ and says "That's my definition of a vector. Generically speaking." The transcript is in Table 4.2.

Table 4.2

Trigonometry Instructor Transcript, Day 1 – An Algebraic Definition

Written	**Verbal**	**Gesture**
$\|\overrightarrow{PQ}\| = \sqrt{a^2 + b^2}$	616. Ok. 617. So what does that mean a squared plus b squared?	617. gestures at what she has written
$\vec{v} = ai + bj$	618. For any vector is a i plus b j. 619. That's my definition of a vector. 620. Generically speaking. 621. a i plus b j.	618. moves to a different place on the board to write
	622. So it's the coefficient of the i term, 623. which is the x, 624. the coefficient of the b term, 625. which is the y, 626. and so that's my Pythagorean Theorem right there.	622. moves to stand out of the way of the board

This conversation about a second definition of vectors opens in line 616 immediately after she has written the magnitude formula $\|\overrightarrow{PQ}\| = \sqrt{a^2 + b^2}$, which has been introduced as a result of being asked to find the magnitude of a vector given only that the vector equals $6i + 3j$. Notice she comments in line 619 that $\vec{v} = ai + bj$ is "my definition of vector." Notice she clarifies in line 620 that the definition is "generically speaking," and restates to clarify in line 621 that the object in which she speaks is "a i plus b j." Thus, the definition of a vector, generically speaking, is the equation $\vec{v} = ai + bj$. Notice she does not clarify or expand on this definition, nor does she comment on what she means by saying the definition is "generically speaking." She simply continues her flow of solving the overall problem of finding the magnitude of the given vector. While this definition seems to be of little importance now, later analysis reveals this style of algebraic notation is the primary way vectors are symbolized algebraically in the remaining instruction.

Summary. Thus, the instructor provides three definitions for vectors during her direct, observable instruction. In her opening instruction, she states, "vectors are directed line segments" (line 22) and writes her statement on the board. In the third segment she states, "vectors in space are defined by an initial point and a terminal point" (T1-427), and in the fifth segment while providing instruction on how to calculate the magnitude of a vector, she states that $\vec{v} = ai + bj$ is her "definition of a vector, generically speaking" (lines 619-620). The second and third definitions are given very little emphasis as they were introduced as passing comments with no supporting clarification or elaboration, but, in contrast, direct emphasis is given to the first definition, and the lecture seems to unfold from this definition.

Physics Instructor's Instruction

The physics course chosen for this study was at a large, mid-western, public university. The class met in the mid-morning of Mondays, Wednesdays, and Fridays for 50 minutes. The first three days of instruction were spent introducing the course, answering the questions of

“What is Science?” and “What is Physics?” along with reasons why to study the two topics, and reviewing how to use conversion factors and significant digits. Vectors were introduced on the fourth day of class after addressing some course house-keeping issues and finishing the review on significant digits.

Two doors provide access to the large lecture hall from the rear. Each door opens to a descending staircase providing access to the rows of fixed, wooden seats with their stored tabletops, and each staircase ends at the sunken stage where two work stations are fixed for the lecturer. One work station is a large table in the center of the room with nothing on it, and the other station is at stage right loaded with technological devices, which include a document camera and a computer screen. Three enormous projector screens hide the what-would-have-been-considered-enormous-at-one-time granite chalkboards paneling the front of the classroom. Generally, the three screens all display his PowerPoint presentation, which structures the instruction given. When working an example problem, the middle screen displayed the PowerPoint slide providing the original information while the two outer screens would rely on the document camera to display the instructor’s written work as he solved the problem on notebook paper. About 100 students were generally scattered in clumps throughout the room.

Five consecutive lectures were analyzed, beginning with the lesson that introduced vectors. The fifth lecture marked the end of instruction for the first exam. Instruction during the lectures was segmented by changes in PowerPoint slides. Some slides provided instruction, and others provided house-keeping issues for the course. As students entered class each day, the screens hosted a Demotivator® slide from www.despair.com. For example, on the day projectile motion was introduced, the introduction slide hosted a photograph of a high jumper midair and colliding with the high bar. Under the photograph was the quote, “RISKS: If you never try anything new, you’ll miss out on many of life’s great disappointments.” The slide was selected because of its humorous sarcasm in motivating students to take risks in their learning and because the photograph hosted the physics concept of the day, projectile motion.

In this analysis, all introductory and house-keeping slides were not used. The physics lectures can be segmented into 18 segments based on the PowerPoint's demarcations: four provided specific example problems, 2 provided problem-solving strategies, and the remaining 12 were narrative in nature providing the lesson content. The two problem-solving strategy slides were also not included in this analysis.

During the interview, the physics instructor explained he develops his lectures with the assumption that students know nothing about vectors even though trigonometry is a formal prerequisite for his course. He recently asked his class to state by a show of hands whether they had a trigonometry course before, and he reports a quarter of them claimed they had not. As a result, the instructor states that he believes that half his students "have conceptual difficulty with (vectors), or they've never seen it."

Knowing his instructional time with the students was limited, the instructor used the textbook as a topical guide, selected only "3-5 underlying ideas," and focused his instruction during class "on what I see as the big ideas, especially the ideas that people have trouble." He charges students to read the textbook before class because his lectures do not incorporate all needed topics so that he can incorporate discussions about common student troubles he has observed during his extensive years of tutoring. As a result of his experience tutoring, the instructor states, "I think students struggle with vectors a lot.…(But) if they internalize vectors, they have very few problems" with the physics. He elaborates on the meaning on the meaning of the phrase, "internalizing vectors," by explaining students need to "be able to talk about physical quantities in terms of vectors," describe vectors in terms of components, and be able to add them, subtract them, and multiply them by a scalar.

The first thing he wanted his students to recognize was "that there are two major kinds of structures" that are used in physics: scalars and vectors. He explained how most students do not see vectors and scalars as different from one another; therefore, his instruction began by describing scalars as quantities and vectors as quantities requiring the additional description of direction within a referenced system. During his initial instruction, he also planned to describe components, how to deal with components, and component transformations between rectangular and polar coordinates. His instruction did include each aspect of these topics; the next section offers the data of how the instructor began his instruction and defined vectors.

Explicit definition. Analysis of the first physics lecture provides the instructor's definition of vector. The definition comes during the second slide, which is a further elaboration of the topics began on the first slide concerning vector quantities.

On the first slide, the instructor states that all physical quantities in physics are chopped "up into two categories based on whether we are talking about things that only have size, or whether they have size and another parameter, which we would call—just, uh, direction, in their description" (lines 1-8). To explain the differences, he poses two questions: "How many people are in the room?" and "Where are you?" He explains that the first question "is answerable using numerical values that we call scalars" (line 16), and the second question requires "some sort of an external reference set" (74) to describe the physical quantity, which is "one of the features of vectors" (85). While transitioning to the next slide, he states, "So, in the next few slides, we are going to try to examine the question 'what is a vector?' because vectors form the foundational concept of the remainder of physics, that we use all of this semester, as well as next semester" (116-118). Notice he is using the word "vector" even though he has not yet defined it.

After changing slides, he says, "Well, let's see if we can talk a little more about vector quantities, and how you might understand it" (142-144) by taking "a closer look at" (145) the question of "where are you?" After reiterating the need for a coordinate system and the qualities of the standard coordinate plane, he defines vectors. Table 4.3 is an excerpt of the transcript where he defines what vectors are for the class.

Table 4.3

Physics Instructor Transcript, Day 1 – Defining Vectors

Written	Verbal	Gesture
177. (A bullet is added to the PowerPoint reading,) Vectors extend from one point in the CS plane to another.	177. Ok. 178. If you would like to describe the position of this purple dot in the context of this coordinate plane, 179. you might be able to use a vector. 180. a vector is just some sort of construction that goes from one place to another in this coordinate plane.	178. uses cursor to circle around dot and then throughout the coordinate plane 179. points cursor toward written words in bullet 180. waves hand from one place to another. then gestures back toward the coordinate plane on screen behind him
181. (A bullet is added to the PowerPoint reading,) Vectors are not required to originate at the CS origin.	181. It is not true, 182. that all vectors have to begin at the origin. 183. This is another thing that can-- 184. that can-- 185. that can give you conceptual difficulty.	
186. (A red vector appears starting at origin and ending at purple dot.)	186. So if I'd like to describe where this position is, 187. I can simply create a vector, 188. which is a line pointing from one place to the other in this coordinate system.	
	189. Notice that, 190. the vector I drew to describe the position of this purple dot, starts at the origin for my convenience. 191. It didn't have to, 192. and it also ends at the location that we're interested in.	190. points cursor at origin 192. points cursor at the purple dot

Notice the instructor offers three separate statements to define a vector. First, in line 180 he says that "a vector is just some sort of construction that goes from one place to another in this coordinate plane." Meanwhile the slide behind him provides the second statement and reads "Vectors extend from one point in the CS plane to another." The initials "CS" are the personal abbreviation for "coordinate system" the instructor uses for brevity on the slides. Third, in lines 187-188 he restates "a vector, which is a line pointing from one place to the other in this coordinate system." Thus, there are three separate statements working to define what a vector is, and they have similarities and differences.

In comparing his two verbal statements, notice the difference between the first verbal statement that states a vector is "just some sort of construction" and the second verbal statement that states a vector is "a line." The instructor's initial definition that a vector is "*just some sort of* construction" (emphasis added) seems unscripted and informal based on his tone and elongated delivery of the statement; thus, when he redefines a vector as "a line" just seconds later, analysis must consider whether the instructor used the word *line* in an informal, colloquial sense (as a straight mark) or in the formal, mathematical sense (as an infinite set points capable of being distinguished by a particular linear equation).

Notice the differences between the verbal statements of whether vectors are "lines pointing" or "constructions going" is not further distinguished by the written text of the slide. Instead, the sentence on the slide states "vectors extend," which does not seem to state what a vector *is*, just what it *does*. This distinction is important because if the sentence on the slide is read as a definition, then any object that "extends from one point in the CS plane to another" would be considered a vector. For example, a curved path and line segment, which both extend from one place to another in a coordinate plane, would also be considered vectors. Because the written statement appears on the PowerPoint simultaneously with his verbal statement defining vectors, arguably, the intent of the written statement was to serve as a definition, but it has not been worded as such. As a result, the three statements do not seem to provide a formal definition of what a vector *is*.

While the statements do not seem to converge on a formal definition, they do seem very similar in the manner in which they personify and describe vectors. Notice the similarities between how all three statements unfold. All three statements personify vectors as active (going/pointing/extending); all three statements have vectors going/pointing/extending "from one point/place to another/other;" and all three statements have the vectors going/pointing "in the coordinate plane/system."

Summary. All three statements seemingly parallel each other in their latter portions as objects that go *from one point/place to another/other in the coordinate system/plane*, but the beginnings of each seem too informal and different for the analysis to converge in order to state how the physics instructor formally defines vectors to the class.

Cross-Analysis of Class Instruction

Both instructors provide three definitions for vectors during their instruction. In her opening instruction, the trigonometry instructor states that a vector is a "directed line segment" and emphasizes her statement by writing it on the board. Direct emphasis is given to the first definition, and the lecture seems to unfold from this definition. Little emphasis is given to the other two definitions because they are introduced as passing comments with no supporting clarification or elaboration. All three of the physics instructor's statements are informal definitions and seemingly parallel each other in their latter halves stating vectors go/point/extend *from one point/place to another/other in the coordinate system/plane*.

Thus, in both of these cases of physics and trigonometry instruction, a vector is some kind of geometric object. Both instructors draw arrow-like objects while defining them, and the definitions seem fairly equivalent for another two reasons. First, both agree that they begin and end at endpoints. In the trigonometry instruction, vectors are defined as "directed *line segments*" (emphasis by researcher), and the instructor clarifies that *line segments* have end points (line 26), which is where the vectors begin and end. Similarly, in the physics instruction, vectors begin and end at endpoints because they go/point/extend from *one point/place* to *another/other* in the coordinate system/plane (emphasis by researcher). Secondly, both definitions seem to be equivalent because they agree that the line segment will show direction. In the trigonometry instruction, vectors explicitly are stated to be "*directed* line segments" (emphasis by researcher), and the instructor clarifies "we have to show direction. So, we might say well, the vector direction is this way so we're going to put a little arrow here" (line 31-32). In the physics instruction, vectors are "lines pointing" and "constructions going," which are verbs requiring direction, and they go/point *from* one point/place *to* another/other (emphasis added).

A marked difference between the definitions is that all three of the physics instructor's statements have vectors going/pointing/extending from one point/place to another/other *in the coordinate system/plane* (emphasis by researcher). The trigonometry instructor does not include a statement in her definition requiring vectors to be *in a coordinate system/plane*.

In conclusion, both instructors seem to define a vector as a geometric object that looks like an arrow, has endpoints, and shows direction. No obvious differences emerge in the analysis of their definitions other than the formality in which they are provided and the trigonometry instructor's omission in her definition that vectors are required to be *in a coordinate system/plane*.

Trigonometry Textbook's Instruction

The textbook used in the University's trigonometry course is *Trigonometry: Enhanced with Graphing Utilities, A Right Triangle Approach* (4^{th} ed.) by Sullivan and Sullivan (2006). The textbook spans 771 pages: 18 introductory pages, 660 instructional pages with accompanying practice problems, and 93 solution and index pages. The instructional pages are segmented into 7 chapters. Chapter 5 is labeled *Polar Coordinates; Vectors* and is segmented into seven sections. The first three sections address polar coordinates, and the last four address vectors. The four vector sections are titled *Vectors, The Dot Product, Vectors in Space*, and *The Cross Product*. The trigonometry instructor includes only the first two of the four vector sections in her course.

These two sections span 20 pages, 16.5 of which are used for instruction. The remaining pages contribute practice problems to be used for homework. The first section, called *Vectors*, has 10 instructional pages and is broken into 12 segments: an introduction and 11 instructional subheadings, 7 of which have example problems. The second section, called *The Dot Product*, has 6.5 instructional pages and is broken into 7 instructional subheadings.

Explicit Definition. An analysis of the textbook has the definition of vector as the opening sentence of section 5.4, which is the first of two vector sections in the book. Figure 4.1 is a snapshot of the opening three paragraphs in the first section and their related diagrams.

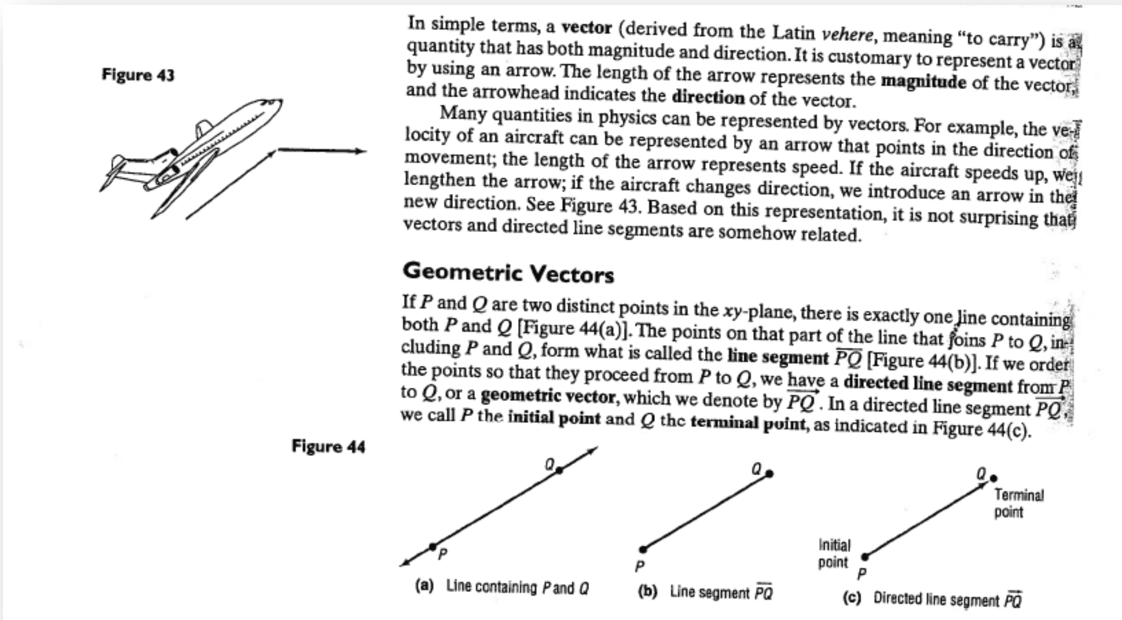


Figure 43

In simple terms, a **vector** (derived from the Latin *vehere*, meaning "to carry") is a quantity that has both magnitude and direction. It is customary to represent a vector by using an arrow. The length of the arrow represents the **magnitude** of the vector, and the arrowhead indicates the **direction** of the vector.

Many quantities in physics can be represented by vectors. For example, the velocity of an aircraft can be represented by an arrow that points in the direction of movement; the length of the arrow represents speed. If the aircraft speeds up, we lengthen the arrow; if the aircraft changes direction, we introduce an arrow in the new direction. See Figure 43. Based on this representation, it is not surprising that vectors and directed line segments are somehow related.

Geometric Vectors

If P and Q are two distinct points in the xy-plane, there is exactly one line containing both P and Q [Figure 44(a)]. The points on that part of the line that joins P to Q, including P and Q, form what is called the **line segment** $\overline{PQ}$ [Figure 44(b)]. If we order the points so that they proceed from P to Q, we have a **directed line segment** from P to Q, or a **geometric vector**, which we denote by $\overrightarrow{PQ}$. In a directed line segment $\overrightarrow{PQ}$, we call P the **initial point** and Q the **terminal point**, as indicated in Figure 44(c).

Figure 44

(a) Line containing P and Q (b) Line segment $\overline{PQ}$ (c) Directed line segment $\overrightarrow{PQ}$

Figure 4.1. Excerpt from p. 372 of the trigonometry textbook related to defining vectors.

Notice the first paragraph reads, "In simple terms, a **vector** (derived from the Latin *vehere*, meaning "to carry") is a quantity that has both magnitude and direction. It is customary to represent a vector by using an arrow" (p.372, emphasis in the original). Note that a vector is a *quantity* and that it is *represented* by an arrow. The paragraph continues by stating, "The length of the arrow represents the **magnitude** of the vector, and the arrowhead indicates the **direction** of the vector" (p.372, emphasis in the original). Note that the length and arrowhead of the arrow specifically *represent* the characteristics of the vector; they are not parts of the vector itself. In summary, all three sentences in the first paragraph indicate that a vector is a quantity represented by an arrow.

The third paragraph has been separated from the first two by a new section header stating, “Geometric Vectors.” Notice the third sentence reads, “If we order the points so that they proceed from *P* to *Q*, we have a **directed line segment** from *P* to *Q*, or a **geometric vector**, which we denote by $\overrightarrow{PQ}$” (emphasis in the original). Note the grammatical use of commas to indicate an appositive, creating the meaning that a geometric vector is another name for a directed line segment.

Summary. The textbook distinguishes between a vector and a geometric vector, and their definitions differ distinctly. The textbook explicitly states that a vector is “a *quantity* that has both magnitude and direction” (emphasis added by researcher), and that a directed line segment is called a geometric vector.

Cross-analysis of trigonometry definitions. In comparing the textbook’s discussion with the instructor’s instruction in the data collected, the reader should notice two dramatic differences. First, the reader should notice that the textbook’s distinction between vectors and geometric vectors is not a distinction made by the instructor. While the textbook introduces both words and defines each, the instructor only introduces the vocabulary word *vector*. Secondly, the reader should notice the mix-match of the instructor’s signifier vector with the textbook’s definition for a geometric vector. The instructor defines vectors as directed line segments, but the textbook states that directed line segments are called geometric vectors. The same definition is being linked with two different signifiers. In contrast to the instructor’s definition, the textbook states vectors are quantities with magnitude and direction. The instructor does not refer to vectors as quantities.

Physics Textbook’s Instruction

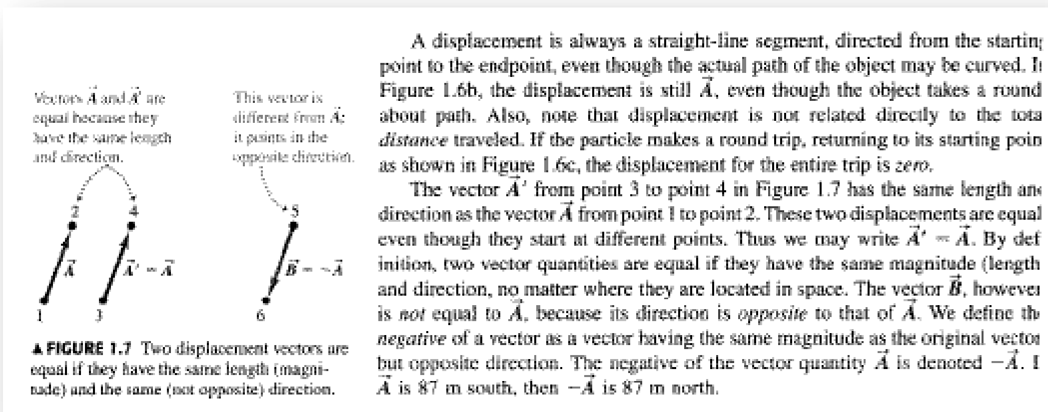
The textbook used by the large, mid-western, public university for its algebra-based physics courses is *Sears and Zemansky’s College Physics* (8^{th} ed.) by Young and Geller (2007). The textbook spans 1134 pages: 26 introductory pages, 1049 instructional pages with accompanying practice problems, 59 pages for the appendix, credits, and index. The instructional pages are segmented into 30 chapters. As one flips through the text, diagrams with vector arrows seem to span the textbook. The first unit concerned with mechanics, which is the first 10 chapters, hosts the bulk of the diagrams using vectors/arrows. The five lessons used for analysis in this project relate to the first 3 chapters of the book.

This analysis uses 54 instructional pages from the textbook spanning 14 sections across 3 chapters. The first chapter is segmented into eight sections and titled *Models, Measurements, and Vectors*. The first six sections address measurement, significant digits, and other related topics, and the last two introduce vectors. The first six sections are not included in the analysis because they do not correspond to the five videoed lessons; videoing began when instruction including vectors began. The two vector sections from Chapter 1 are titled *Vectors and Vector Addition* and *Components of Vectors*. These two sections provide 9.5 instructional pages. The second chapter is segmented into eight sections and provides 27 instructional pages and titled *Motion along a Straight Line*. The third chapter is segmented into 6 sections and titled *Motion in a Plane*, but only the first 4 are included in the analysis because they relate to the topics given during class. They provide 17.5 instructional pages.

The dialogue in the instructional pages is often interrupted visually by large blue boxes providing supportive problems and additional discussion. These blue boxes provide problems with their accompanying solutions and are labeled with one of three titles: Example, Conceptual Analysis, and Quantitative Analysis. Because these blue boxes also provide examples of the textbook's narration doing physics or mathematics, they have been included in the analysis. There are 35 of these boxes in the analyzed pages.

Explicit definition. Vectors are first introduced in the textbook in the section titled "Vectors and Vector Addition," but despite its title, the section does not provide a definition of a vector. The textbook begins its narrative in the first paragraph by comparing different types of physical quantities, which culminates into the second paragraph. The second paragraph begins, "When a physical quantity is described by a single number, we call it a **scalar quantity**. In contrast, a **vector quantity** has both a **magnitude** (the "how much" or "how big" part) and a **direction** in space" (p.12, emphasis in original). The paragraph continues by explaining that calculations with scalar quantities use regular arithmetic, but vector quantities are different. The third and fourth paragraphs use displacement as an example of a vector quantity to further elaborate how displacement requires both magnitude and direction to be quantified.

Finally in the sixth paragraph, the word *vector* is used. The word *vector* is used without introduction or definitions. The fifth and sixth paragraphs are provided in Figure 4.2, along with the accompanying diagram for the sixth paragraph.



▲**FIGURE 1.7** Two displacement vectors are equal if they have the same length (magnitude) and the same (not opposite) direction.

A displacement is always a straight-line segment, directed from the startin point to the endpoint, even though the actual path of the object may be curved. I Figure 1.6b, the displacement is still $\vec{A}$, even though the object takes a round about path. Also, note that displacement is not related directly to the tota *distance* traveled. If the particle makes a round trip, returning to its starting poin as shown in Figure 1.6c, the displacement for the entire trip is *zero*.

The vector $\vec{A}'$ from point 3 to point 4 in Figure 1.7 has the same length an direction as the vector $\vec{A}$ from point 1 to point 2. These two displacements are equal even though they start at different points. Thus we may write $\vec{A}' = \vec{A}$. By def inition, two vector quantities are equal if they have the same magnitude (length and direction, no matter where they are located in space. The vector $\vec{B}$, howeve is *not* equal to $\vec{A}$, because its direction is *opposite* to that of $\vec{A}$. We define th *negative* of a vector as a vector having the same magnitude as the original vecto but opposite direction. The negative of the vector quantity $\vec{A}$ is denoted $-\vec{A}$. I $\vec{A}$ is 87 m south, then $-\vec{A}$ is 87 m north.

Figure 4.2. Excerpt from p. 12 of the physics textbook introducing *vector.*

Notice the sixth paragraph begins, "The vector $\vec{A}'$ from point 3 to point 4 in Figure 1.7 has the same length and direction as the vector $\vec{A}$ from point 1 to point 2." Notice beside the paragraph is two blue arrows labeled $\vec{A}$ and $\vec{A}'$ and with large black dots at either end labeled 1, 2, 3, and 4. Notice above the arrows is a little comment reading, "Vectors $\vec{A}$ and $\vec{A}'$ are equal because they have the same length and direction." Notice below the diagram is a second comment reading, "Two displacement vectors are equal if they have the same length (magnitude) and the same (not opposite) direction." The reader should note that the use of the word *vector* is without introduction and without a definition in both the instructional dialogue and the comments accompanying the diagram. The use of the word *vector* in each case as a term to name and discuss the arrows in the accompanying diagram strongly suggests to the reader that the diagrammed arrows are vectors; thus, the context allows the reader to surmise that vectors must be arrows.

Cross-analysis of physics instruction. In comparing the introductions of the word *vector* provided by the physics textbook and the physics instructor, two similarities emerge. First, neither the textbook nor the in-class instruction provide a formal definition for vectors. While the textbook never provides a definition, the in-class instruction provides three informal ones. As a result, a formal definition has not been provided by the textbook or in-class instruction in the data gathered, which means neither source explicitly states what vectors *are*.

Second, while no formal definition has been provided, explicit statements made within the textbook and the in-class instruction uniformly seem to define vectors as arrow-like objects that go/extend/point *from one point to another* (emphasis by researcher). The textbook's first use of the word "vector" reads, "The vector $\vec{\boldsymbol{A}}'$ *from point 3 to point 4* in Figure 1.7 has the same length and direction as the vector $\vec{\boldsymbol{A}}$ *from point 1 to point 2*" (italics added by researcher). Notice the textbook's sentence states both vectors in the diagram extend from a particular point to another particular point. Similarly, in the instructor's verbal statements, he says that "a vector is just some sort of construction that goes *from one place to another* in this coordinate plane" and "a vector, which is a line pointing *from one place to the other* in this coordinate system" (emphasis by researcher). Other than the exchange of the words *other* and *another*, the latter portions of these verbal definitions are equivalent. In the instructor's written statement, the PowerPoint reads, "Vectors extend *from one point* in the CS plane *to another*" (emphasis by researcher). Notice the only difference is the placement of the prepositional phrase "in the coordinate system," which provides no change in the meaning the sentence provides. As a result, all three of the instructor's statements state that vectors go/point/extend *from one point/place to another/other* in the coordinate system/plane. In conclusion, a reader should notice the similarity between the three instructor statements and the textbook statement that vectors are arrow-like objects that go/extend/point *from one point to another* (emphasis by researcher).

A difference between the textbook and in-class instruction is that nothing in the sixth paragraph or in the accompanying diagram seems to reference explicitly that the points/places are required to be *in a coordinate system/plane*. While the diagram used during class situated the vector on a rectangular coordinate plane using both a grid and axes, the textbook has the vectors seeming to be free floating beside each other in white space. While the textbook diagram does not display axes and a grid, the vectors in the diagram seem to follow an informal reference system designed by the directions of left/right/up/down. Notice the fifth sentence of the sixth paragraph and the specific inclusion of the third vector in the diagram clarifies that the third vector with its same tilt and length as the original ones is not equivalent to the original ones because the direction of the arrow matters. The orientation of the third vector within the left/right/up/down system establishes its uniqueness. As a result, though the book does not explicitly state that a coordinate system is required, the textbook's diagram and narration argue the need for the existence and use of a fixed system/plane.

Cross-Analysis of the Explicit Defining of Vectors

Now that we have compared both the trigonometry and physics instructors' instruction and their instruction to their textbooks, this section will compare the instruction across all four sources. Analysis reveals key consistencies across five of the six definition-like statements: the trigonometry instructor's primary definition, the physics instructor's three definition-like statements, and the physics textbook statements. These five definitions of the word *vector* seem to lend themselves as vectors being geometric objects. Specifically, the trigonometry instructor says that vectors are "directed line segments," and the physics instructor states that a vector both is "just some sort of construction that goes from one place to another in this coordinate plane" and "a line pointing from one place to the other in this coordinate plane." Similarly, his PowerPoint slide states, "vectors extend from one point in the CS plane to another." The physics textbook does not formally define vectors, but the manner in which the word is used allows the reader to surmise that the arrows in the accompanying diagrams are vectors. As a result, all three sources are referencing a type of geometric object.

The key consistencies of these five definitions seems to conflict with the trigonometry textbook, which defines a vector as being "a quantity that has both magnitude and direction." The trigonometry textbook states that a vector is a quantity, an amount of something. Interestingly, the definition of *vector* from the trigonometry textbook seems equivalent to the physics textbook's definition and the physics instructor's definition of a *vector quantity*. The physics textbook states, "a **vector quantity** has both a magnitude and a direction in space" (emphasis in the original). These definitions seem incredibly similar, and they seem to say that quantities that have magnitude and direction have different labels (signifiers) depending on the particular community of practice. Specifically, the trigonometry textbook seems to label quantities with magnitude and direction as being *vectors*; whereas, the physics instructor and textbook seem to label quantities with magnitude and direction as *vector quantities*. The physics textbook's definition of *vector quantities* is similar to the physics instructor's, which states on his PowerPoint slide that "**vector** quantities have <u>both</u> size and direction" (emphasis in the original).

From the three sources in which their definitions are similar in referencing vectors as geometric objects, there are three similarities: they seem to be geometric arrows, they seem to begin and end at endpoints, and they seem to show direction. First, the three sources all introduce vectors and first visually mediate them as geometric objects that look like arrows. Second, the three sources seem consistent in conveying that vectors begin and end at particular places. The trigonometry instructor states vectors are defined as "directed *line segments*" (emphasis by researcher), and she clarifies that *line segments* have end points (line 26), which is where the vectors begin and end. The physics instructor states in his two verbal and one written statements that vectors go/point/extend from *one point/place* to *another/other* in the coordinate system/plane (emphasis by researcher). Similarly, the physics textbook states "The vector $\vec{A}'$ *from point 3 to point 4* in Figure 1.7 has the same length and direction as the vector $\vec{\mathrm{A}}$ *from point 1 to point 2*" (italics added by researcher). In all three cases, the vectors are beginning and ending at particular points/places, which are endpoints.

Third, all three cases seem consistent that vectors will show direction. The trigonometry instructor explicitly states "vectors are *directed* line segments" (emphasis by researcher), and the instructor clarifies "we have to show direction. So, we might say well, the vector direction is this way so we're going to put a little arrow here" (line 31-32). The physics instructor states vectors are "lines pointing" and "constructions going," which are verbs requiring direction, and they go/point *from* one point/place *to* another/other (emphasis added). Similarly, the physics textbook states "The vector $\vec{A}'$ *from* point 3 *to* point 4 in Figure 1.7 has the same length and direction as the vector $\vec{\mathrm{A}}$ *from* point 1 *to* point 2" (italics added by researcher). In all three cases, the vectors possess direction as described by going from one endpoint to the other endpoint.

A marked difference between the cases is the physics instructor's consistent inclusion of vectors' need to be in a coordinate system that the other two cases, the trigonometry instructor and physics textbook, do not mention. All three of the physics instructor's statements have vectors going/pointing/extending from one point/place to another/other *in the coordinate system/plane* (emphasis by researcher), but the trigonometry instructor and physics textbook do not include an explicit statement requiring vectors to be *in a coordinate system/plane* and both have the vectors in the initial diagrams free-floating in white space.

The Initial Act of Symbolizing Vectors

When defining or introducing vectors, both instructors and both textbooks provide diagrams to accompany their statements. This portion of the chapter describes the visual appearance of these initial diagrams and provides a description of the verbal statements characterizing the illustration. Because the initial visualization of vectors usually occurred concurrently with the explicit definitions, some instructional moments that were described earlier are revisited and further elaborated in order to describe how each source of instruction initially symbolized vectors.

This portion of the chapter has the same six sections: an analysis of the instruction by the trigonometry instructor, an analysis of the instruction by the physics instructor, a cross-analysis of the similarities and differences between the instruction provided by the instructors, an analysis of the instruction by the trigonometry textbook, an analysis of the instruction by the physics textbook, and an overall cross-analysis.

These six sections are a result of analyzing what each instructor and textbook *says* as they initially symbolize a vector and the result of what they drew as they initially symbolized the vectors. These moments of instruction were then compared across the instructors and textbooks for meaningful similarities and differences. Later sections of the chapter will return to compare the remaining diagrams to these initial depictions and whether the depictions match the explicit definitions described earlier.

The outcome of the analysis in this section of the chapter reveals consistency across the instructors and textbooks symbolizing vectors as arrows beginning and ending at endpoints, having a fixed length, showing direction by beginning at the initial point and ending at the terminal point, and following the direction of a fixed plane. The physics instructor's initial vector differed from the others' diagrams in two ways: it lacked large dots at either end of the arrow and it was situated with coordinate axes. The physics instructor insisted the axes were essential, and the trigonometry instructor seemed to indicate the necessity of the dots.

Trigonometry Instructor's Inception of the Visual Vector

As mentioned earlier, after spending the first 16 minutes of class answering questions about the previous homework assignment using complex numbers, the trigonometry instructor initiated the new instruction by announcing to her class that "Today we are going to do vectors." She then begins her instruction by writing the word *vectors* on the board with a double arrow after it, creating the impression that an elaboration will be written to follow. She then drifts across the classroom to close the classroom door as she familiarizes her students to the word by stating, "the word vector is Latin, and it means to carry" (lines 10-11), and she provides the related words *vehicle* and *convection* because they carry people and carry heat, respectively. Upon returning to the board, she provides the definition and draws two examples. The following paragraphs describe her introduction of the visual form of vectors.

After pronouncing the definition that "vectors are directed line segments" (22) and writing it on the board after the double arrow, she states, "So, if you think about in geometry what you learned about a line segment" (24), and she sketches a line segment: a black dot, an approximately 5-inch line coming out of the black dot, and another black dot at the following end. Addressing the line segment, she states, "We know a line segment has end points, and then it's a fixed length" (26-27), and then she segues to introduce vectors as she says, "So, with vectors, the only difference there is, it has endpoints, it's a fixed length, But we have to show direction." She then adds a ">" between the black dots at one end of the line segment in such a way that the tip of the point barely touches the black dot as she says, "So, we might say well, the vector direction is this way so we're going to put a little arrow here" (32). She then draws a second line segment with a different slant than the first and then immediately adds an arrow in a similar manner to the first but facing the other end of the line segment. Figure 4.3 has a snapshot from the video displaying the two example vectors. As she is drawing them, a student interrupts her, and after addressing his questions, she returns to face her two drawings to state "So, that's typically what our vectors look like" (54-55). Thus, vectors are symbolized in this example as line segments with an arrow at one end to depict direction.

Figure 4.3. The initial vectors from a video snapshot of the trigonometry instructor's lecture.

The student's interruption prompts an interesting interpretation of the sketch. Though the vector has been introduced as a line segment with large dots at either end and an arrow's tip at one end, colloquially, vectors look a lot like arrows or rays, which instigates the student to interrupt the instructor to ask whether it is the same as a ray. The instructor responds that vectors and rays are different because rays go on forever and vectors do not. The student interrupts her again with an adjustment to his thinking that still includes seeing vectors as a type of ray; however, after pausing to consider the student's question, the instructor states her resolve that the student should not consider the connection to ray, and she firmly states that vectors should just be considered directed line segments. The following lines are excerpted from their conversation. Only the teachers' words are included; the student's words are not included (as a result of being excluded by the standards set by the IRB).

22. Vectors are directed line segments.
23. …
24. So, if you think about in geometry what you learned about a line segment,
25. right?
26. We know a line segment has end points,
27. and then its a fixed length.
28. So, with vectors,
29. the only difference there is,
30. it has endpoints, it's a fixed length,
31. But, we have to show direction.
32. So, we might say well, the vector direction is this way so we're going to put a little arrow here,
33. Or we could say,

34. [student interrupts & questions] …
35. So, it's just what?
36. [student talking] …
37. No, it's not.
38. And that's where the confusion sets in.
39. It's not a ray.
40. Because remember in geometry a ray goes on infinitely. [draws a ray as she speaks and gestures at it. The sketch has an initial point with an arrow at the end.]
41. Yo- You couldn't really measure it.
42. [student talking] …
43. [repeating student while considering what he has said:] a ray inside a line segment?
44. Let's just stick with directed line segment.
45. Let's not use the word ray.
46. Ok?
47. cause a ray-
48. A ray means it goes on infinitely.
49. And we don't want that.
50. So, it's kinda like you're thinking that it's a ray and it hits a brick wall,
51. and it stops there?
52. Um,
53. Let's just say it's a directed line segment.
    [regains her line of thinking]
54. So,
55. that's typically what our vectors look like.

Notice in lines 37-39 that the instructor says that a vector is not a ray and recognizes that sometimes there is confusion. While speaking of a ray in line 40, she draws a ray on the board and gestures at it. The ray is drawn similarly to the vectors with the absence of the large dot at the tip of its arrow. In line 50 she is thinking out loud as she considers whether a vector can be "a ray and it hits a brick wall, and it stops there," meaning it hits the terminal point and stops. In line 53, she decides firmly that a vector should just be considered "a directed line segment." Figure 4.4 has a snapshot of the video displaying the two example vectors, which are drawn near each other on the right, and the example of the ray, which is drawn on the left.

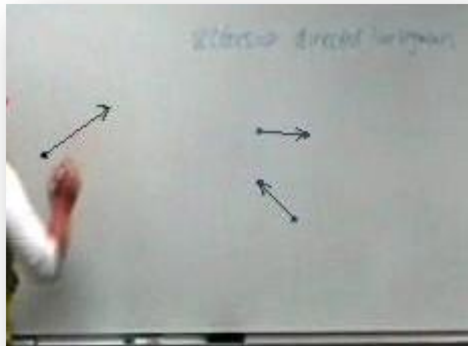


Figure 4.4. Illustrating a ray from a video snapshot of the trigonometry instructor's lecture.

Three qualities of the diagrams should be noted. First, all three diagrams look like arrows. Second, the vectors have bold dots at both ends, but the ray excludes the bold dot following the arrow tip. Third, both vectors began as line segments with 2 bold dots at either end and then the arrow tip was added; the ray, on the other hand, is drawn by extending out away from the initial dot, ends with an arrow tip, and excludes a second dot. Though not explicitly stated, the manner in which the ray and vectors are drawn and the lack of a dot at the terminal end seems to express the difference between the definitions of vectors and rays. The vectors depict their fixed lengths with direction, and the ray depicts its quality of extending in a direction without ending.

To summarize, the trigonometry instructor opens the unit with the definition for vectors, and immediately draws two examples. Based on a student's questions, she distinguishes between the definition and illustration of a ray and vector. As a result, the vectors have two bold dots at either end emphasizing their fixed lengths, and the ray excludes the bold dot following the arrow tip which seems to depict its quality of never ending. Both are drawn on white space.

Physics Instructor's Inception of the Visual Vector

Unlike the trigonometry instructor who begins the vector unit with a formal definition along with immediately embodying vectors visually, the physics instructor begins his unit on vectors by showing the inadequacy of a single number to answer all physical questions by contrasting two questions: "How many people are in the room?" and "Where are you?" While comparing these two questions, he uses the word *vector* seemingly interchangeably with the words *vector quantities,* and he does not define or visually symbolize a vector until his second slide. The following paragraphs describe his instruction building up to his introduction of the visual form of vectors.

He begins his instruction for the unit on his first slide by stating,

1. The way we chop up all physical quantities in physics,
2. In broad strokes we chop these things up into two categories based on whether we are talking about things that only have size,
3. or whether they have size and another parameter,
4. which we would call--
5. just,
6. uh,
7. direction.
8. in their description.
9. So I would like to attack this by taking a look at two different questions.

He then continues by explaining that some physical questions such as "How many people are in the room?" can be quantified by a single number, and he states that "This kind of a question is answerable using numerical values that we call scalars" (16). He then provides time, temperature, and mass as other examples of scalar quantities that require only a single number to be quantified and described. To contrast, he provides the question of "Where are you?" to show that a single number couldn't answer the question. He states, "You have to have some sort of an external reference set in order to answer that particular question" (74), and it requires two numerical values to be quantified and described. He explains,

75. So,
76. the position,
77. In this case, we're using position as the example,
78. This position will describe your particular location in terms of the coordinate system.
79. If you and I choose different coordinate systems,

80. obviously we're talking about the same place,
81. but the description would differ.
82. So if you and I choose different coordinate systems,
83. although we are describing the same exact physical quantity,
84. the descriptions that you and I use may not be exactly the same.
85. This is one of the features of vectors,
86. which is the stuff we're going to be talking about,
87. One of the features of vectors which provides a lot of people some conceptual difficulty.
88. n-K?

89. So these vector quantities,
90. notice,
91. have size.
92. There's some numerical parameters having to do with the answer to this question,
93. but you also have to provide some location,
94. or directional pieces of information to fully specify your answer.

95. Examples of vector quantities that we will use this semester is the position,
96. which is what we were just talking about.
97. We will be very specific in how we define these vector quantities, by the way.
98. Velocity,
99. which is related to the familiar concept of speed.
100. And acceleration,
101. which we have some colloquial understanding, for these terms.

In lines 79-84, the instructor indicates that the coordinate system is selected by the user and makes a difference in how the physical quantity is quantified and described. In lines 89-94, the instructor indicates two numerical values are necessary to quantify and describe the physical quantity, which he states are *size* and directional pieces of information.

Line 85 provides the instructor's first use of the word *vector*, he uses it again in lines 85 and 87, and he does not call these physical quantities *vector quantities* until line 89. Although his discussion during the first instructional segment is about scalar quantities and vector quantities, he uses the word *vector quantity* only 5 times; whereas, he uses the word *vector* 12 times. Also, he seems to use them interchangeably. For example, notice in line 83 he is talking about describing a *physical quantity* in different ways, and then he says through lines 85-87 that "*This* is one of the features…which provides a lot of people conceptual difficulty." Although he is talking about how describing physical quantities in different ways provides people conceptual difficulty, he does not state, "This is one of the features of *physical quantities*…," and he does not state, "This is one of the features of *vector quantities*…" which provides conceptual difficulty. Instead, he states, "This is one of the features of *vectors*…which provide a lot of people conceptual difficulty." So, although he is talking about physical quantities, he refers to them as *vectors*. Yet, in line 89, he now references the physical quantities of which he has been speaking with "So *these* vector quantities," and in lines 95-101, he provides "examples of *vector quantities.*" The result is that he seems to be using *vector* and *vector quantities* interchangeably.

During the second instructional segment, the physics instructor formally introduces vectors, and he provides three informal definitions. All three statements seemingly parallel each other in their latter halves, but the beginning of each is a little different. The title of the slide is "Vectors and Coordinate Systems," and just below the title in a smaller font size, the subtitle from the previous slide also appears reading in blue "Question 2: Where are you?" He begins by saying, "So let's take a closer look at this particular question. We've already posed this—We want to know where are you?" In the expanse of white space currently on the slide, a purple dot appears to represent where we are. He then adds a standard rectangular coordinate system "in order to understand where you are" (160). After discussing attributes of the coordinate system and explaining "We'll be talking about coordinate systems a lot in the course" (163), he begins discussing how a vector can be used to describe the location of the dot (see Table 4.4).

Notice the instructor first states that "a vector is just some sort of a construction that goes from one place to another in this coordinate plane" (181) while the slide behind him reads "Vectors extend from one point in the CS plane to another." In line 188 as he says, "I can simply create a vector," a red arrow appears stretching from the origin to the purple dot. Figure 4.5 has a snapshot of what the PowerPoint displayed.

Table 4.4

Physics Instructor Transcript, Day 1 – Defining Vectors

Written	**Verbal**	**Gesture**
177. Vectors extend from one point in the CS plane to another.	178. Ok. 179. If you would like to describe the position of this purple dot in the context of this coordinate plane,	179. cursor glides to rest at the purple dot and rolls around the coordinate plane
	180. you might be able to use a vector.	180. cursor glides to rest at written text
	181. a vector is just some sort of construction that goes from one place to another in this coordinate plane.	181. hand bounces from one spot to another spot in front of himself. Then points at screen behind him
182. Vectors are not required to originate at the CS origin.	182. It is not true, 183. that all vectors have to begin at the origin.	
	184. This is another thing that can-- 185. that can-- 186. that can give you conceptual difficulty.	184. hand holds an invisible “thing” and shakes it gently in agitation
	187. So if I'd like to describe where this position is,	187. cursor glides to rest at purple dot
188. (a red arrow appears stretching from origin to purple dot, arrowhead resting at purple dot)	188. I can simply create a vector,	
	189. which is a line pointing from one place to the other in this coordinate system.	189. cursor glides toward red arrow
	190. Notice that,	
	191. the vector I drew to describe the position of this purple dot, starts at the origin for my convenience.	191. cursor glides to point at origin,
	192. It didn't have to,	
	193. and it also ends at the location that we're interested in.	193. cursor glides to point at purple dot

Notice after stating in line 188 “I can simply create a vector” with the red vector appearing, he chooses to define what a vector is again by saying, “which is a line pointing from one place to the other in this coordinate system” (189). He then elaborates in lines 191-193 about how the given example vector *starts* at the origin and *ends* at the purple dot; he uses his cursor to glide from the origin to the purple dot to emphasize what he is saying. Notice in lines 182-183 and 192 that he explains that vectors do not have to begin at the origin but this one "starts at the origin for my convenience" (191). His discussion then continues as he explains that the vector can be described by its length in the horizontal and vertical directions, which he calls components.

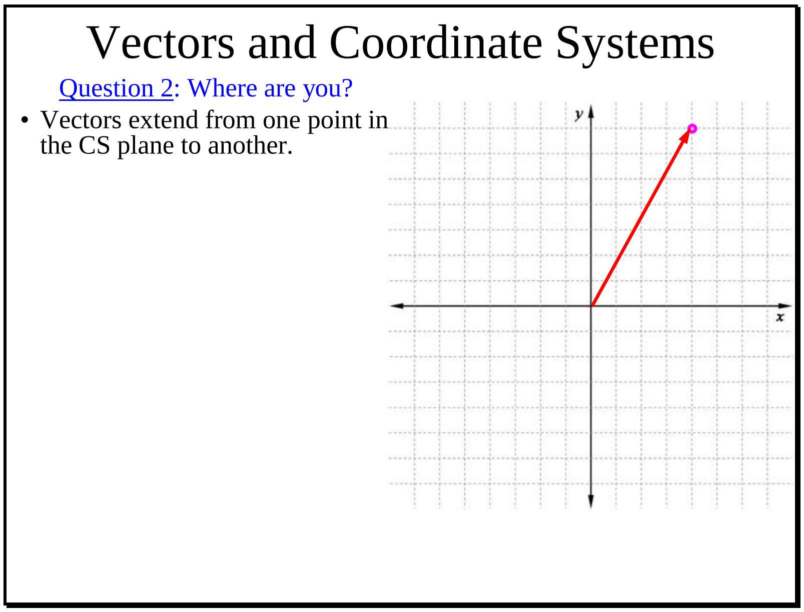


Figure 4.5. Excerpt from the physics instructor's PowerPoint depicting the initial vector.

In summary, the physics instructor uses the word *vector* throughout the first instructional segment as he discusses scalar and vector quantities and seemingly interchangeably with the words *vector quantities*. The first slide focuses on discussing vector quantities, but during the discussion, he uses the word *vector* 12 times and the words *vector quantities* only 5 times. Although he uses the word vector in his first instructional segment, he does not define it until his second instructional segment. At that time, he symbolizes a vector as an arrow going from the origin to the purple dot. He embodies the vector after he emphasizes the need for a coordinate system and briefly describing some of coordinate system's attributes.

Cross-Analysis of Instructor's Initial Diagrams

There are four similarities in how the instructors introduce vectors visually. First, both instructors define and create a vector as a geometric object that is similar to an arrow, which in this case is a straight stroke with one of its ends finishing in the middle of a v-like shape. Second, both instructors seem to create a geometric object that begins and ends at endpoints. The physics instructor emphasized that his vector started at the origin and ended at the purple dot. The trigonometry instructor emphasizes the beginning and ending with large dots at either end. Third, both instructors depict direction in such a way that the vector begins at one endpoint and ends at the other. Fourth, both instructors seem to create geometric objects with fixed lengths. The trigonometry instructor emphasized the bounds of the vector by sketching it first as a line segment with bold points at either end and then adding its arrow tip. Similarly, the vector created by the physics instructor is a bounded length between the origin and purple dot. Fifth, both seem to rely on the direction of the plane as necessary. The physics instructor adds the coordinate system to the slide before adding the vector and explicitly emphasizes the need for a coordinate system. He also emphasizes that all qualities of the vector rely and are determined by the axes. Though the trigonometry instructor does not explicitly draw a coordinate system, she does seem to rely on the direction of the plane because she emphasizes that left/right/up/down make the vectors unique. This distinction in the orientation of the vectors in making them unique is dissimilar to other mathematical situations; for example a square on its side is still a square. In drawing the initial vectors, the orientation of the vectors on the white board causes them to be unique. The direction of the plane seems to matter to both.

In comparing the initial visual embodiment of vectors, three differences are also apparent. First, although both seem to be relying on the direction of the plane, both do not draw a coordinate plane to accompany the vectors. The physics instructor emphasized the need for a coordinate system before a vector could be created, but the trigonometry instructor did not draw her initial vectors on a coordinate system, nor did she verbally reference the need for one. So, although both seem to be following the direction of a fixed plane, the trigonometry instructor does not explicitly draw it or specify that it is necessary, like the physics instructor does. Second, the two instructors differ in the process of drawing the vectors. The trigonometry instructor emphasizes that vectors are directed line segments by drawing them as line segments and then adding arrows to show direction, but the physics instructor draws it all as one object simultaneously. Third, the trigonometry instructor uses the large dots at either end of the vectors, but the physics instructor does not use the large dots at either end. His arrow simply extends from the origin to the purple dot. His endpoints seem separate from the vector, while her endpoints seem to be part of drawing the vector. The trigonometry instructor seems to include the large dots at either end as a way of conveying the vectors' fixed lengths, which is perceivable because she contrasts it to the ray. The ray only has one large black dot at its initial end in order to convey that the arrow at the other end extends boundlessly.

There are also differences in the manner in which the topic is developed within the instruction. First, notice the trigonometry instructor begins the unit with a definition and diagrams to illustrate. Everything in the unit develops and extends from the definition creating a geometric object. This differs from the physics instructor who begins his unit discussing the need for both scalar quantities and vector quantities to describe physical situations. His introduction to the visual embodiment of vector arrows occurs within the greater discussion of how to describe physical quantities, and it does not occur until his second slide. Second, the trigonometry instructor gives a very formal definition, and the physics instructor does not. Third, the trigonometry instructor introduces vectors as geometric objects separated from any context, but the physics instructor introduces the first vector arrow in a specific context. His first vector arrow is used as an example and a means of describing a location.

Trigonometry Textbook's Introduction to the Visual Vector

The opening paragraph of the trigonometry textbook defines vectors, magnitude and direction along with stating, "It is customary to represent a vector by using an arrow." The second paragraph provides an example. It states,

> Many quantities in physics can be represented by vectors. For example, the velocity of an aircraft can be represented by an arrow that points in the direction of movement; the length of the arrow represents speed. If the aircraft speeds up, we lengthen the arrow; if the aircraft changes direction, we introduce an arrow in the new direction. See Figure 43. (p. 372)

The trigonometry textbook states a vector is a *quantity*, and it states that the arrows in Figure 43 *represent* vectors—but are not vectors in and of themselves. The textbook's *Figure 43* is to the left of the two opening paragraphs and is provided as Figure 4.6. Notice the arrows accompany an aircraft, and both arrows are floating in white space with no large dots at either end.

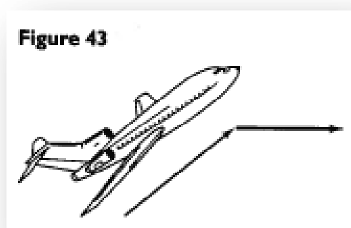


Figure 4.6. Excerpt from p. 372 of the trigonometry textbook "Figure 43" depicting arrows representing vectors.

Because the arrows in Figure 43 are not referenced as vectors and are only referenced as arrows that represent vectors, the second diagram has also been included as an initial depiction of a vector. The second diagram accompanies the instructional narrative that introduces geometric vectors. "Geometric Vectors" is the section title which immediately follows the opening two paragraphs. The instructional narrative is a discussion comparing lines, segments, and directed line segments, which are called geometric vectors. The paragraph reads, If P and Q are two distinct points in the xy-plane, there is exactly one line containing both P and Q [Figure 44(a)]. The points on that part of the line that joins P to Q, including P and Q, form what is called the **line segment** $\overline{PQ}$ [Figure 44(b)]. If we order the points so that they proceed from P to Q, we have a **directed line segment** from P to Q, or a **geometric vector**, which we denote by $\overrightarrow{PQ}$. In a directed line segment $\overrightarrow{PQ}$, we call P the **initial point** and Q the **terminal point**, as indicated in Figure 44(c).

Immediately following the paragraph are the following three diagrams.

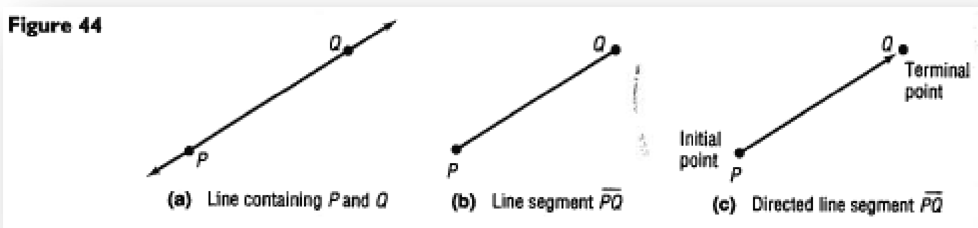


Figure 4.7. Excerpt from p. 372 of the trigonometry textbook "Figure 44" depicting geometric vectors.

Notice all three objects float in white space. Notice the gradual changes and adjustments made to each diagram illustrating their similarities and differences. Notice the geometric vector in the third diagram has large dots at both ends and the arrow stretches between the two dots. Because of the generic nature of the example in Figure 43 and the direct instruction surrounding the introduction of the vector in Figure 44, this diagram is considered to be the primary diagram in discussing the textbook's initial embodiment of a vector.

The geometric vector in Figure 43c follows all five of the similarities resulting from comparing the trigonometry and physics instructors' initial visual embodiment of vectors: vectors are drawn as arrows, the arrows began and ended at endpoints, the arrows depict direction by beginning at one endpoint and ending at the other endpoint, the arrows had a fixed length, and the arrows followed a fixed plane. The vector in Figure 43 is depicted as an arrow, begins and ends at particular places labeled points *P* and Q, and has a fixed length, just as the line segment did. Nothing in the narrative or diagram seems to depict the vector following a fixed plane, but since it follows the direction from *P* to *Q*, which are two points assumed to be fixed in a plane, its direction within the plane would be fixed.

The geometric vector in the trigonometry textbook followed the same pattern as the trigonometry instructor: being situated without coordinate axes, having two large dots at either end of the arrow, and being associated with a line segment. Much like the trigonometry instructor drew the initial vector as a line segment and then added an arrow, the sequence of diagrams depicts the line segment and then adds the arrow to depict the vector.

Physics Textbook's Introduction to the Visual Vector

Prior to the first time vectors are directly referenced in the physics textbook's narrative, two diagrams earlier on the page have arrows that have been labeled as "displacement vectors." After the narrative discusses the differences between scalar and vector quantities, it provides displacement as an example of a vector quantity. The text states, "In Figure 1.6a, we represent the object's change in position by an arrow that points from the starting position to the ending position." (p12). The narrative goes on to explain that displacement may not be the actual path of the object; rather, "Displacement is always a straight-line segment, directed from the starting point to the endpoint" (p. 12). The comment with the figure title calls both arrows "displacement vectors." Figure 4.8 provides the first two arrows provided by the textbook.

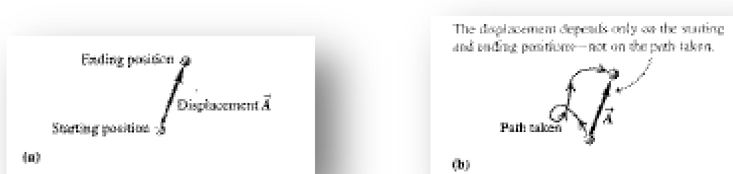


Figure 4.8. Excerpt from p. 12 of the physics textbook depicting the initial displacement vectors.

In the paragraph following this discussion, arrows are first referenced as vectors in the narrative. As stated earlier, the narrative does not provide a definition for vectors, but based on the discussion in the narrative, the arrows to the left of the narrative must be vectors because the paragraph opens, “The vector $\vec{\boldsymbol{A}}'$ from point 3 to point 4 in Figure 1.7 has the same length and direction as the vector $\vec{\boldsymbol{A}}$ from point 1 to point 2.”   Figure 4.9 provides the physics textbook’s “Figure 1.7.”

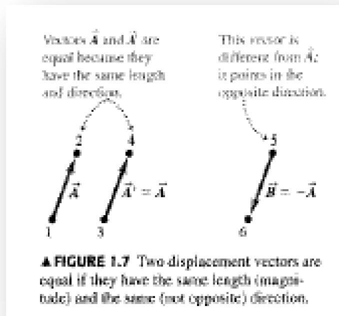


Figure 4.9. Excerpt from p. 12 of the physics textbook “Figure 1.7” depicting the initial vectors.

Similar to the two diagrams earlier on the page, all three vectors have large dots at their ends and the arrows stretch from one point to the other. Both comments above the arrows reference the arrows as vectors, and the comment accompanying the figure title, which is below the arrows, references the arrows as “displacement vectors.”

To summarize, all five of the textbook diagrams depict vectors as arrows that begin and end at endpoints, have a fixed length, show direction from the initial point to the terminal point, and follow the direction of the plane.   Similar to the physics instructor, these five vectors all depict displacement, but unlike the physics instructor, who did not include large dots at either end of the arrow and emphasized the necessity for coordinate axes, all five of these vectors do. The textbook’s drawings are pre-fixed so the manner in which the author created it cannot be compared to the instructors whose manner of creating the vector was done in real time.

Cross-Analysis of All 4 Sources' Initial Diagrams

In all four sources, the initial vectors are depicted as arrows. All four sources also have the vectors beginning and ending at endpoints, having a fixed length, showing direction by beginning at the initial point and ending at the terminal point, and following the direction of a fixed plane.

Although the two instructors differ in whether they include large dots at either end of the arrow, both textbooks include the dots. As a result, only the physics instructor's vector does not have the dots. Also, the physics instructor's vector is the only one situated on coordinate axes; the other three (trigonometry instructor, trigonometry textbook, and physics textbook) have their initial vectors floating in white space. In a later section, these similarities and differences will be noted and compared for the remaining vectors for each of the four sources.

CHAPTER V

ANALYSIS OF THE DATA: VECTORS IN USE

The previous chapter described the instruction that defined and initially symbolized vectors. Taken together, the previous chapter described the inception of vectors. This chapter describes the similarities and differences in practices by the instructors and textbooks in *using* vectors, in the broadest sense, throughout the remainder of the collected data.

Sfard (2003) writes, "The act of naming and symbolizing is, in a sense, the act of inception, and using the words and symbols is the activity of constructing meaning" (p. 374). Chapter IV offered a description of "the act of inception," by describing findings answering the following research questions:

1. In the act of defining vectors, what are the similarities and differences between trigonometry and physics instruction?
2. In the act of symbolizing vectors, what are the similarities and differences between trigonometry and physics instruction?

This chapter offers a description of "using the words and symbols" by describing findings answering the following research question:

3. In the activities of using vectors, what are the similarities and differences between trigonometry and physics instruction?

Chapter VI will weave the findings of Chapters IV and V to provide a description of the instruction that students use as "the activity of constructing meaning."

Making sense of the practices surrounding vectors evolved as a result of the researcher approaching the analyses inductively. The inception of vectors (Chapter IV) provided the signifier *vector*, the signified definitions, and the symbolization of vectors as arrow-like objects; in this chapter, the remaining data was combed for patterns in using these three aspects of vectors. This system allowed the practices to emerge from the relationship between signifiers and signifieds inductively, based on the specific contexts and instruction under investigation.

Categories of practices emerged from patterns found in the instruction. These categories include using the signifier *vector*, using notation, using vocabulary related to vectors, using the diagrammed vector arrows, using vectors in context-free mathematical activities, and using vectors in contextualized-physics activities. Observations of unity and disunity in the instruction within each of these categories were then noted using within case comparisons and across case comparisons. This method of comparing across and within the cases usually resulted in returning to further analyze and develop the cases with more specificity than first used. This continuous weaving within the cases, across the cases, and across categories of practice produced the descriptions provided in this chapter. The representation of that analysis compares what the instructors explicitly said against what they did, compares the instructors' instruction against their textbooks' instruction, and compares the trigonometry instruction to the physics instruction.

Unlike the earlier chapter that provided rich descriptions of each instructor's presentations along with two layers of cross-analyses, this section represents the descriptions pragmatically. Generally the final cross-analysis is provided, but in some cases, the instruction is individually described as well. The chapter summarizes the instructors' use of the signifier *vector*, a description of the notation explicitly introduced, the notation used, other related notation, explicitly related vocabulary and its definitions, the vectors in diagrams, the mathematical procedures introduced, and the mathematical procedures used in the study of physics.

The Use of the Word *Vector*

Borrowing Saussure's idea of recognizing a sign as having both a signifier and signified, an underlying assumption of this project is that there may be slippage between signifiers and signifieds across communities of practice. Said another way, the signifier might have different meanings in separate communities of practice. As a result, all occurrences/uses of the words *vector/vectors* were analyzed to determine if they related to objects matching the explicitly-stated definitions described in Chapter IV. While the trigonometry instructor, physics instructor, and physics book all define vectors to be geometric objects and initially depict them as geometric objects, the trigonometry textbook did not. The trigonometry textbook's explicit definition differs dramatically from the other three sources as it defines vectors to be a type of quantity. This section of Chapter V opens with a description of how the trigonometry textbook does not follow its own explicit definition in referencing vectors as quantities. The remainder of the section describes how the trigonometry instructor, physics instructor, and physics textbook do not always reference geometric arrows as vectors; sometimes they reference alphabetic letters, vector quantities, diagrams with no arrows in them, and components as vectors.

Trigonometry Textbook References Arrows—Not Quantities

Since the trigonometry textbook's explicit definitions distinguished between vectors and geometric vectors, and since the trigonometry textbook defined vectors as quantities, the analysis should gather and analyze vectors as quantities and treat vectors and geometric vectors as separate objects, but the textbook's instructional narrative does not seem to follow its explicit definitions for vectors and geometric vectors. In fact, by the end of the first segment of its chapter, the textbook seems to reject the distinctions between vectors and geometric vectors as the authors clarify, "we find it useful to think of a vector simply as an arrow" (p. 373). The following paragraphs elaborate.

As stated earlier, the trigonometry textbook begins its instruction by defining *vector* to be a type of quantity, by stating that vectors can be represented by arrows, and by accompanying the statements with a diagram showing two arrows representing velocity. Following the introduction, the opening section of instruction defines a geometric vector. The text reads, "If we order the points so that they proceed from P to Q, we have a **directed line segment** from P to Q, or a **geometric vector**, which we denote by $\overrightarrow{PQ}$" (emphasis in the original). Note the grammatical use of commas to indicate an appositive, creating the meaning that another name for a *directed line segment* is a *geometric vector*.

This distinction between vectors and geometric vectors is continued in the following paragraphs within the first instructional segment when the text introduces separate notation for each term. Vectors are notated in typed text using a bold-faced letter, such as $\boldsymbol{v}$, and in handwritten text using a letter with an arrow above it, in which no example is given. In contrast, geometric vectors are notated using the letters naming the initial point and terminal point sequenced side-by-side with an arrow above, such as $\overrightarrow{PQ}$. Despite the separate definitions and notational difference, the distinction collapses in the final paragraph of this segment. The textbook directly references the arrows in the accompanying drawing as vectors and states, "we find it useful to think of a vector simply as an arrow" (p. 373). After this statement, the text never again references geometric vectors and never again uses the notation for geometric vectors. After this point in the instructional narrative, all arrows are referenced as vectors, and the vector notation of using single, bold letters is utilized to reference them.

Surprisingly, this pattern continues even when quantities from physics are referenced. For example, in the sixth segment, the textbook opens by stating, "If a vector represents the speed and direction of an object, it is called a velocity vector. If a vector represents the direction and amount of a force acting on an object, it is called a force vector" (p. 379). Notice that the textbook has vectors *representing* the quantities of speed and force when accompanied by their direction. The sentence structure does not have quantities *being* vectors, as the reader might expect from the initial definition. Instead, the word *vector* is referencing the arrows. On the following page, the text states, "Because forces can be represented by vectors, two forces "combine" the way that vectors "add" (p. 380). Notice again the word "vector" is referencing arrows—not quantities, and notice that the forces are being *represented* by vectors—not that the quantities are themselves vectors. Both quotes directly conflict with the opening two paragraphs, which had the word *vector* referencing quantities and stating that the quantities are represented by arrows.

Even when introducing different types of vectors, the trigonometry text does not reference anything else throughout its remaining instruction as vectors except arrows. The textbook introduces unit vectors, algebraic vectors, position vectors, velocity vectors, and force vectors. Unit vectors, position vectors, velocity vectors, and force vectors are all names given to vectors with specific, particular qualities. Unit vectors have a length of one, position vectors begin at the origin, velocity vectors *represent* velocity, and force vectors *represent* force. All four of these types of vectors reference physical arrows and not quantities. A fifth type, algebraic vectors, seems at first not to reference physical arrows, but the textbook follows the definition with a sentence that designates that algebraic vectors do have physical initial points that may at times be at the origin. The textbook states, "If **v** = < a, b > is an algebraic vector whose initial point is at the origin, …" (p.375). Algebraic vectors have geometric qualities. Thus, all five different types of vectors seem to reference physical arrows and not quantities. As a result, there are no references to vectors other than those made to geometric arrows.

In conclusion, although the textbook opens its instruction by stating that vectors are quantities and separate from geometric vectors, the textbook's actual practice throughout its instruction is to use the word *vector* to reference geometric arrows. Because the textbook references the arrows as vectors—not geometric vectors—and never references quantities as vectors again in the remaining pages, the analysis of the textbook's vectors will continue in a manner similar to the physics textbook's and two instructors' analysis of vectors. Also, because the remaining portion of the instruction considers vectors as arrows, Diagram 43, which is the diagram accompanying the defining of geometric vectors, will serve as the initial diagram in the analysis by which all remaining diagrams will be compared.

While the trigonometry textbook references vectors only as arrows after the opening paragraphs, not all the trigonometry instructor's, physics instructor's nor physics textbook's verbal references to vectors are made regarding arrows. References to vectors are also made toward alphabetic letters, vector quantities, drawings without any arrow-like objects, and vectors' components.

Referencing Notation

The trigonometry instructor and physics instructor both reference alphabetic letters as vectors. For example, the instructor claims she has written vectors on the board as she says, "I just wrote some vectors on the board, u and v" (263-264), but she has not drawn arrow-like objects—instead, she has written just the letters *u* and *v*. As she continues, she adds half-arrows above the letters and states, "So that's our notation for vectors" (284), and the board reads, "$\vec{u} + \vec{v}$." Likewise, the physics instructor, for example, references letters with arrows above them as vectors when introducing a formula to calculate velocity. He has the equation $\vec{\mathrm{v}} = \frac{\Delta\vec{\mathrm{R}}}{\Delta \mathrm{t}} = \frac{\vec{\mathrm{R}}_{\mathrm{f}} - \vec{\mathrm{R}}_0}{\mathrm{t}_{\mathrm{f}} - \mathrm{t}_0}$ on his PowerPoint, and he uses his mouse to reference particular parts of the equation as vectors (see Table 5.1).

Notice he is speaking of vectors in lines 106 and 107, but his gestures are helping the listener focus on $\Delta\vec{R}$ when he speaks of a "displacement vector" in line 106 and focus on $\vec{R}_f - \vec{R}_0$ when speaking of the "difference of vectors" in line 107. In line 106 and 107, these letters with arrows above them are being referenced as vectors, but they are not written as geometric arrows. Notation becomes a third wheel to the signifier-signified situation. Notation is a signifier for a geometric vector; descriptions of the instruction and use of notation is discussed later in this chapter.

Table 5.1

Physics Instructor Transcript – Referencing letters as vectors

Written	Verbal	Gesture
$\vec{v} = \frac{\Delta\vec{R}}{\Delta t} = \frac{\vec{R}_f - \vec{R}_0}{t_f - t_0}$		
	106. This is nothing more than the change in the displacement vector with respect to time. 107. So, we essentially have the difference of vectors divided by difference of scalars.	106. He circles mouse around and around $\Delta\vec{R}$ the entire time he says "…change in displacement vector" 107. He slides mouse back & forth a few times under $\vec{R}_f - \vec{R}_0$ as he says "difference of vectors"

Referencing Vector Quantities

The physics instructor and physics book both reference vector quantities as vectors. As stated earlier, the physics instructor used the word *vector* prior to his introducing the word as the signifier for an arrow-like object. In his initial uses of the word, the word *vector* seems to function as a nickname for *vector quantities.* However, there are times when the physics instructor would directly state that particular vector quantities *were* vectors. For example, when introducing and defining velocity during the third lesson, he stated,

75. We also need to talk briefly about velocity and its definition.
76. Velocity,
77. generically we just say this is the V variable.
78. It's a vector.
79. The metric system units are meters per second.
80. The velocity,
81. if you would like to talk about what does it mean,
82. This is essentially the description of how the object's displacement--
83. it's a vector,
84. the displacement, changes a- over some time intervals.

Notice he explicitly states in line 78 that “It’s a vector” referencing velocity in line 76. Notice he interrupts himself in line 83 to state that displacement “it’s a vector,” too. These explicit statements are first made in the third lesson when instruction about the vector quantities is first addressed and are made at least once in each of the subsequent lessons. Likewise, the physics textbook has two cases in which the word *vector* is used to reference vector quantities. In chapter 2, the text states, “Speed is a scalar quantity, not a vector; it has no direction” (p. 37). Notice while the textbook distinguishes that speed is a scalar quantity and not a vector quantity in this sentence, it uses the word “vector” and not “vector quantity.” Because vector quantities are represented as vectors, often geometric vectors and algebraically notated vectors are verbally referenced as vector quantities only. The relationship between vectors and vector quantities is further developed later in this chapter.

Referencing Diagrams Without Arrows

The trigonometry instructor and physics instructor both had two diagrams in which no arrows were included but vectors and vector quantities were referenced. For two example problems, the trigonometry instructor has the handle of a wagon treated as a vector but not drawn as an arrow. The two drawings accompany two examples, contextual problems during the 12^{th} and 15^{th} instructional segments. She uses the same drawing of the wagon in both instructional segments and just adjusts the labeled values to match the given information for each problem. While labeling the diagram during the 12^{th} instructional segment, the instructor references the wagon handle as she says,

399. And we know that the force that the child is using is, 40 pounds.
400. That's our force, that's our vector.
401. So here's my x-axis, my y-axis. This is 30 degrees, and my vector is 40.
402. I need to come up with-I'm sorry my magnitude's 40.
403. I need to come up with these components again.
404. right? I need a, and I need b.

While the instructor says, “that’s our vector” in line 400, she points to the handle of the wagon, which she has just labeled as being the force of 40 pounds while speaking line 399. Following these statements, she superimposes the axes at the base of the handle, labels the handle's direction from the positive x-axis as 30 degrees, reiterates that the magnitude of the vector is 40, and incorporates the handle into the hypotenuse of a right triangle with components labeled *ai* and *bj*. The resulting diagram is pictured in the snapshot provided in Figure 5.1.

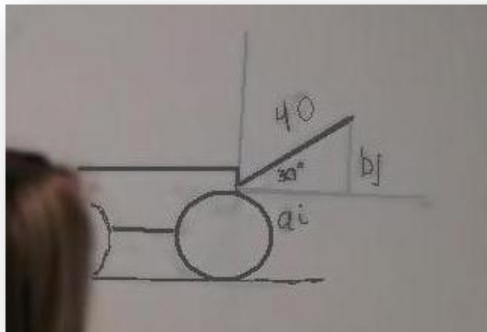


Figure 5.1. Excerpt from trigonometry lecture concerning referencing vectors but not drawing an arrow.

Notice that the wagon handle has not been drawn as an arrow, but it has been referenced directly in line 400 as a vector. Similarly, during the 15th segment, the handle is referenced as a vector, but the diagram does not depict an arrow. In both cases, the handle, though not drawn as an arrow, has been labeled and directly spoken of as a vector. These two drawings will be analyzed separately from the vectors that are drawn as arrows.

Likewise, the physics instructor has two drawings without arrows that are referenced as vectors or as vector quantities. Prior to the first situation, he begins by drawing a vector on a coordinate system along with its components. Upon completing the sketch, he declares, "Whenever I look at vector type problems, I generally speaking like to begin with some sort of a diagram of this vector" (P2: 256-257), and then redraws the situation again as a triangle below without the axes. A snapshot of the situation is in Figure 5.2.

Once drawn as a triangle, the hypotenuse, which is labeled as a vector, loses its arrow—it simply is a side of the triangle. When referencing the hypotenuse, he says, "here we have the r vector" (P2: 258). Because the vector was first drawn as an arrow and because the second drawing was drawn to highlight the triangular relationship between the vector and its components, this second diagram will not be included in the analysis as a vector—though it was explicitly referenced as a vector.

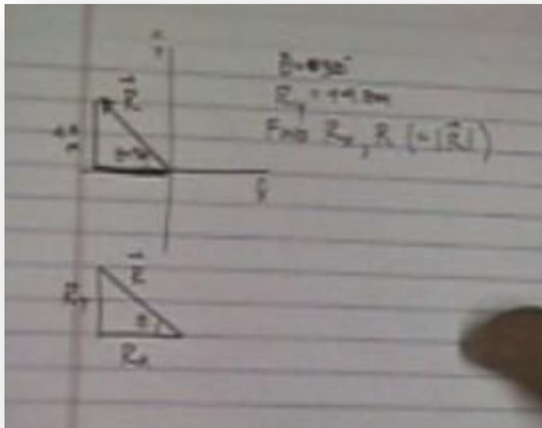


Figure 5.2. Excerpt from physics lecture concerning referencing vectors but not drawing an arrow.

In the second situation, he sketches a bullet leaving the barrel of a gun, and though he says he wants to answer several questions about the position, velocity and acceleration of the bullet and says, “These things are all vectors. Basically.” (day 4, line 149-150), he does not draw any arrows. He does draw the barrel, the bullet, and a coordinate system overlaying them both (see Figure 5.15). Because this drawing does not have an arrow, the drawing will not be analyzed with the remaining diagrams describing vector arrows.

Referencing Components

When components are first introduced by the physics instructor during the second instructional segment, they are referenced as vectors. He states,

193. Vectors are characterized by the fact that they have sort of sub-pieces that describe how much of them live--
194. or extend,
195. in these principle coordinate directions in our coordinate system.
196. So,
197. vectors can be broken into pieces,

198. Each of these pieces we call components,

He then provides a blue arrow-like object and names it the x-component, and he provides a green arrow-like object and says, “So the green *vector* is what we would call the Y component of the red vector” (P1-223, emphasis by researcher). He then follows by referencing both components as vectors by saying, “I'd also like to point out that this blue *vector* and this green *vector* form the legs of a right triangle” (P1-228, emphasis by researcher). Because the instructor states that they are vectors, the reader could assume that they are, but there seems to be inconsistencies in his overall instruction as to whether he always considers them vectors. The following paragraphs of the analysis elaborate the inconsistencies.

There are four reasons why the instructor does not seem to consider components as vectors despite reasons that they might. First, although they are introduced as vectors, they are never referenced as vectors again. This lack of referencing components as vectors may be inconsequential in meaning given the dynamic nature of teaching, but when all components are not even labeled as vectors, the inconsistency seems worth investigating. Thus, a second inconsistency is that components are not labeled in the manner in which vectors are notated. As described later in this chapter, vector notation has an arrow above the labeled letter or vectors are notated as paired coordinates in either polar or rectangular style. Components are not notated in either of these fashions; instead, they have their own separate notation and are always labeled with singular numerical values.

Third, components are not always drawn as arrows. For 21 of the components, they were drawn looking like arrows, but in one example problem, two components were not. They are drawn just as segments without an arrow. Fourth, components are considered sub-pieces of vectors (line193). The instructor states, “vectors can be broken into pieces, Each of these pieces we call components” (197-198). This informal definition for components by the instructor expresses the idea that as sub-pieces of vectors, components are elements of a larger thing and not the thing itself. This quality is echoed in how vectors are notated in rectangular coordinates using components as the pieces of the notation. The physics book distinguishes between components and component vectors, and it provides explicit definitions for each, which is described in the following section.

In conclusion, the trigonometry textbook references vectors only as arrows after the opening paragraphs, and the trigonometry instructor's, physics instructor's and physics textbook's verbal references to vectors are not always made as arrows. References to vectors are also found to be alphabetic letters, vector quantities, drawings without any arrow-like objects, and components.

Explicit Statements & Use of Notation

Both instructors and textbooks use notation for two purposes: as a means of naming vectors and as describing them. This section has four subsections. The first subsection describes the explicit notation introduced by the trigonometry instructor for naming vectors, algebraically describing vectors, and other related objects. Woven within the subsection are statements describing the trigonometry textbook's similar introduction to the notation and describing whether the notations are used by both the instructor and the textbook.

The second subsection of this part of the analysis describes the explicit notation introduced by the physics instructor for naming vectors, algebraically describing vectors, and other related objects. While describing how the physics instructor introduces notation to name vectors, statements are inserted in the analysis to make comparisons to what he says, what his textbook says, and what the trigonometry instructor and textbook said. The comparisons reveal that the explicit statements about notation are fairly consistent in both trigonometry and physics. Other related notations are introduced before the descriptions are given of how the physics instructor and textbook use notation differently than they explicitly stated they would.

The third subsection of this part of the analysis summarizes the notational practices in trigonometry and physics instruction. The fourth subsection provides a description of the purposes of naming vectors with notation used by both the trigonometry and physics instruction.

Trigonometry Notations

A description of the trigonometry instructor's and textbook's instruction concerning what notation they introduce to name vectors, what notation they actually use to name vectors throughout the instruction, and how they algebraically describe vectors is presented below. This section also includes other notations used during the vector unit as introduced by the instructor and the textbook.

Naming vectors. The trigonometry instructor and textbook name vectors primarily using lower-case letters. The instructor's example problems generally are selected from the book, so her use of the lower-case letters echoes the book's notation. She uses the letters *u*, *v*, and *w* exclusively, and the textbook does the same unless the vector represented a physics force or velocity. In those cases, the vectors were labeled **F** and **v**, respectively. In the textbook, the letters are in bold print, but the instructor explains she cannot bold print as she writes on the board so the notation of a line with half an arrow is used as a notation to reference the vectors. She says,

263. I just wrote some vectors on the board,
264. u and v.
265. in your book they bold print them.
266. I can't really bold print on the board.
267. Right?
268. You can't really bold print on your papers.
269. Um,
270. So,
271. Really and truly what we should do is have a notation that means vector then since we can't bold.
272. So if you can't bold,
273. we put this little...line above.
274. Now it's a line with half an arrow.
275. It's not a whole arrow because if it was a whole arrow it would look like a ray.
276. and we don't want it to look like a ray.
277. Because it's a vector.
278. Sometimes I forget and don't put 'em there,
279. so you just have to go "oh we're in the vector section and that's a vector."

280. Because,
281. but I can't bold print them,
282. and I can't remember to put the arrows there.
283. But that's vector u plus v.
284. So that's our notation for vectors. (T1)

Likewise, the book states, “Boldface letters will be used to denote vectors, to distinguish them from numbers. For handwritten work, an arrow is placed over the letter to signify a vector.” (p. 373). An example of the style of arrow is not provided. Just as the instructor predicted, she often did not include the arrow when she wrote a vector. The arrow was included about half the time, and, in one problem where she began using the arrows, she interrupted her lecture to say, “Ok I’m going to drop the little vector signs because they are slowing me down. We’ll just assume that all of these are just little vectors, Right?” (148-150).

A second manner of notation the trigonometry instructor and the textbook introduced to name vectors has two capital letters side-by-side with the vector arrow above them, for example $\overrightarrow{PQ}$. The capital letters echo the letters used to name the initial point and terminal point in that order. The textbook and instructor slightly differ in the details of the notation. When the textbook did use this style of notation, it used a full arrow ($\overrightarrow{PQ}$) above the letters; whereas, the trigonometry instructor used only a half arrow above the two capital letters ($\overset{\rightharpoonup}{PQ}$).

Both rarely used the second notation. The trigonometry instructor introduced the notation as $\overrightarrow{PQ}$ and used it only once later within the formula $W = \boldsymbol{F} \cdot \overrightarrow{AB}$, which calculates the physics quantity of work. The textbook used the notation in the same two circumstances as the instructor and in one other example problem, but in a second example problem, the notation is not used to name a vector. The notation was used to describe a vector instead. The textbook states “Find the position vector of the vector $\boldsymbol{v} = \overrightarrow{P_1P_2}$ if $P_1 = (-1, 2)$ and $P_2 = (4, 6)$” (p. 376). Notice the name of the vector is a single, lower-case letter and the double-letter notation is being used to define the vector as having an initial point P_1 and terminal point P_2. In this circumstance, the notation is used not as a means of naming the vector but more as a means of describing the vector.

Other notations. Other notations were also introduced by the trigonometry instructor and textbook for use in the vector unit. First, both the instructor and textbook "use the symbol $\|\boldsymbol{v}\|$ to represent the **magnitude** of **v**" (p. 375, emphasis in the original). Second, both the instructor and textbook use a bold-printed dot (**v · w**) at the same height and size as a multiplication sign in algebra (v · w) as representing the dot product.

Describing vectors algebraically. Lastly, when a vector is not drawn, both the instructor and textbook introduced three manners of notating vectors' properties algebraically. The first manner of describing the vectors was P(-2, 4)Q(4, 7), in which the initial point and terminal point are listed in that order and named with a letter. This style of notating the description of the vector by the coordinates of its initial and terminal point is used only when it is introduced. The textbook uses a variation of this style of notation by just stating the initial and terminal points are given at the specified coordinates without linking the coordinates in a row as the instructor seemed to do. While this manner was introduced at the beginning of the unit, it was not used again.

The second manner of describing vectors was $\langle 6{,}3\rangle$, where the angular parentheses are used to pair the x-component with the y-component. This manner of describing the vector is paired with the notation naming the vector by an equal sign, and the instructor used both styles of notation to name vectors. The instructor wrote $\overrightarrow{PQ} = \langle 6{,}3\rangle$ and $\vec{v} = \langle 6{,}3\rangle$. Though this notation is introduced several times, it is not the notation primarily used. Similarly, the textbook introduces it, but phases it out primarily to use the third manner of notation.

The third manner of describing vectors was $6i + 3j$ or $6\vec{\imath} + 3\vec{\jmath}$, where the components are written as a sum. Notice that sometimes the *i* and *j* have vector arrows above them rather than their dots and sometimes they do not. While the trigonometry instructor does not introduce the notation with arrows above the *i* and *j* and generally does not include arrows above the *i* and *j*, she does remind the students in one circumstance not to forget to put the arrows there. The textbook always prints the *i* and *j* in bold print, for example $6\boldsymbol{i} + 3\boldsymbol{j}$**,** which conveys they should have arrows above them when written by hand. This third manner of describing the vector as a sum of its components is also paired with the notation naming the vector by an equal sign, and the instructor and textbook use both styles of notation used to name vectors. For example, the instructor wrote $\overrightarrow{PQ} = 6\vec{\imath} + 3\vec{\jmath}$ and $\vec{v} = 6i + 3j$. Almost all algebraic references to vectors are written using this style of notation both by the instructor and in the textbook.

Physics Notations

This section on notations describes the physics instructor's and the physics textbook's notation they introduced to name vectors, notation they actually used to name vectors throughout the instruction, notation they used to algebraically describe vectors, and notation they used related to vectors. The section opens describing what the physics instructor explicitly said concerning notation to name vectors with some nods comparing the practices in physics with the practices in trigonometry. The second part describes other explicitly introduced notations related to vectors. The third part then describes how the physics instructor and the physics textbook actually use notation to name vectors and the related notations that appear in the instruction. The final part addresses the notation used to algebraically describe vectors and combines the discussion of how the notation was explicitly introduced and then actually used.

Explicit instruction concerning naming vectors. While the trigonometry instructor and the trigonometry textbook introduced two ways of naming vectors, the physics instructor and the physics textbook only introduced an arrow over a single letter, for example $\vec{v}$. Both the physics instructor and the physics textbook did not introduce the style of notation utilizing the two capital letters naming the initial and terminal points. The physics instructor does not make explicit statements about the notation's arrow except to say the letter relates to the specific vector quantity being referenced. In this way, the letter itself becomes a referent for the specific vector quantity being referenced. The physics instructor explains if a vector represents velocity and acceleration, it is labeled with $\vec{v}$ and $\vec{a}$ respectively, and if a vector represents displacement or position, it is labeled with $\vec{R}$; and if a vector is "a single direction's displacement," it is written using $\vec{x}$, $\vec{y}$, and $\vec{z}$. Each time these labels are used, the physics instructor uses *R* as a capital letter, uses *v* in lower-case, and uses *a* in both lower and upper case.

In Chapters 2 and 3, the physics textbook used the same specific letters as the instructor used to represent specific vector quantities, but always used them as lower-case letters. Unlike the instructor, however, the physics textbook labels all the vectors in Chapter 1, whether they represent displacements or context-free situations, as letters from the beginning of the alphabet, such as $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$, $\vec{\boldsymbol{C}}$, or sometimes with $\vec{\boldsymbol{R}}$ when emphasizing a vector is the resultant vector of an operation. In Chapter 1, different letters differentiate between the various vectors.

While the physics instructor and the physics textbook used the same letters to represent vector quantities, the arrow style in the notation differed slightly. The instructor used a half arrow above the letter like the trigonometry instructor and trigonometry textbook, but the physics textbook used a full arrow above the letter. The physics textbook states,

> In this book, we always print vector symbols **in boldface type with an arrow above them**, to remind you that vector quantities have properties different from those of scalar quantities. In handwriting, vector symbols are usually written with an arrow above, as shown in Figure 1.6a, to indicate that they represent vector quantities. (p.12, emphasis in the original)

The figure shows a hand-written capital letter *A* with a full arrow above it. This practice contradicts the trigonometry instructor's statement that the notation of a full arrow is reserved as a way to reference rays.

While the trigonometry instructor and trigonometry textbook used different letters to differentiate between vectors in the same context, the physics instructor and physics textbook used subscripts. In some cases the subscripts are letters, for example in the expression $\vec{R}_f - \vec{R}_i$ where the initial vector is being subtracted from the final vector; in some cases the subscripts are full words, for example in a relative motion diagram the vectors were labeled $\vec{v}_{observer}$, $\vec{v}_{target}$, and $\vec{v}_{relative}$; and in some cases, the subscripts are numbers, for example a projectile motion diagram having four velocity vectors were labeled $\vec{v}_0, \vec{v}_1, \vec{v}_2$, and $\vec{v}_3$. The physics textbook exclusively used numbers in the subscript to differentiate between multiple, separate vectors representing the same vector quantity in the same context. In addition, the physics textbook differentiated between instantaneous and average velocity or instantaneous and average acceleration by abbreviating the word *average* to *av* to be included in the subscript as necessary. Instantaneous velocity and acceleration were to be assumed by the reader when the subscript did not include *av*.

Explicit instruction concerning other notations. Other notations were also introduced by the physics instructor and physics textbook. An additional notation used by the physics instructor was for unit vectors. He stated that unit vectors are notated as $\hat{x}$, $\hat{y}$, and $\hat{z}$, and he calls the angular roofs above the letters *hats*. As he introduces unit vectors, he states,

378. A unit vector is simply a vector who has unit size,
379. that is, whose magnitude is one,
380. and these guys principally serve as direction indicators.

His use of unit vector notation as a directional indicator occurs in both geometric and algebraic contexts. Geometrically, the physics instructor labels his coordinate axes with this notation. He says he is labeling the axes with “standard coordinates,” but he doesn’t label the axes with *x* and *y*; instead, he labels them as $\hat{x}$ and $\hat{y}$. Algebraically, the notation is used when vectors are described algebraically, for example $\vec{R}_1 = A\hat{x} + B\hat{y}$. This notation is further elaborated later in this section.

The physics textbook introduces the notation for a vector ($\vec{\boldsymbol{A}}$), a vector's magnitude (*A* or $|\vec{\boldsymbol{A}}|$), a component vector ($\vec{\boldsymbol{A}}_x$), and a component (A_x) all at the same time in order for the reader to compare and contrast the similarities and differences between them. A leading similarity is that all four notations use a letter to name the vector being referenced, and the differences in what accompanies or does not accompany the letter notates what specific characteristic of that vector is being referenced. If the vector itself or a vector component is being referenced, an arrow is used. If the vector's magnitude or one of its components is being referenced, then no arrow is used because they are considered scalars and not vectors. Component vectors and components use subscript *x* and *y*, and vectors and magnitude do not. The notation for magnitude is the same as the name of the vector without bold print or the arrow above it, unless the vector is notated between a set of single vertical bars.

Explicit statements are not made by the book concerning the differences in notation; the reader is expected to notice them as each notation is introduced within the flow of the instructional narrative. The physics instructor never introduces vector component notation, and he does not formally introduce component or vector component notation either. Component notation is introduced in the flow of his speech. As he writes, R_y for the first time, he says "y-component of the R vector" (224), and moments later as he says "x-component," he writes R_x.

The physics instructor does explicitly introduce the magnitude notation as he states, "A lot of times you just draw the name of the vector without the vector symbol with the assumption that people understand you're really talking about the size of the vector. The magnitude." (P2-line 244-245). As he says this, he writes *R*, which is the name of the vector without the arrow, and in parentheses he writes $|\vec{R}|$ as he says, "So in parenthesis I'd like to point out that whenever I write it like this, I'm really saying that 'This is the magnitude of the r vector.'" (P2-lines 246-247). Though both the physics textbook and instructor agree that the magnitude of a vector can be written as either *A* or $|\vec{\boldsymbol{A}}|$, the trigonometry textbook and trigonometry instructor state that magnitude is notated with double bars, as $\|\vec{v}\|$ by hand or $\|\boldsymbol{v}\|$ in type. The trigonometry textbook and trigonometry instructor do not introduce component or component vector notation.

Although these notational differences are not fully made explicit by the physics textbook and the physics instructor, the distinction between the notations seems clear. The importance of the distinction in notation also seems clear because the textbook breaks apart its instructional narrative to include two notes of warning. Both are indented and set apart spatially from the narrative, but are still within the column of the narrative—not in addition to the narrative within the margin, where most notes are stated. The first note is included immediately after the physics textbook introduces vector notation as being a letter with an arrow above it. The note of warning reads,

> When you write a symbol for a vector quantity, *always* put an arrow over it. If you don't distinguish between scalar and vector quantities in your notation, you probably won't make the distinction in your thinking either, and hopeless confusion will result. (p. 12, emphasis in the original)

Later when the notation for magnitude and components are introduced along with vector component notation, a second note of warning is included. It reads, "Be sure that you understand the relationship between $\vec{\boldsymbol{A}}$ (a vector), A or $|\vec{\boldsymbol{A}}|$ (the vector's magnitude), $\vec{\boldsymbol{A}}_x$ (a component vector), and A_x (a component)" (p17). Because vectors and component vectors have arrows and because components and magnitude, which are scalars, do not have arrows, the distinction seems clear; however, despite the physics textbook's explicit statements about notation and warnings concerning the problems that might arise in confusing the meanings for them, all but two of the vectors in Chapter 2 of the physics textbook are not notated with vector notation, and many in Chapter 3 do not bear vector notation. The following paragraphs describe the physics textbook's and instructor's actual practices.

Using notation to name vectors. The physics instructor equated position vectors and displacement vectors as being equivalent, and he stated that both could be notated with $\vec{R}$, but "a single direction's displacement" is written using $\vec{x}$, $\vec{y}$, and $\vec{z}$. He never used $\vec{x}$, $\vec{y}$, and $\vec{z}$ during the courses analyzed for this study; instead, throughout his instruction, the he notates displacement with $\vec{R}$, $\Delta\vec{R}$, Δx, or x and position with $\vec{R}$, x, x_0, x_1, x_i, or x_f.

The physics textbook seems to differentiate between position vectors and displacement vectors and references position vectors as $\vec{\boldsymbol{r}}$ and displacement vectors as Δx and $\Delta\vec{\boldsymbol{r}}$. In practice, both the physics instructor and physics textbook sometimes use these vector notations, but usually not because, like the instructor, the textbook notates position as scalar values using x, y, x_0, or y_0 as the notation.

Although the physics instructor states velocity is notated $\vec{v}$, he and the physics textbook notate it differently depending on whether the vector is one-dimensional or two-dimensional. In the physics textbook, when velocity is a one-dimensional vector, it is notated as $v_{av,x}$ for average velocity or v_x for instantaneous velocity. When multiple instantaneous velocity vectors are being referenced within the same context, the physics textbook adds numbers to their subscripts, for example v_{1x}, v_{2x}, or v_{0x}. For the instructor, one-dimensional velocity is notated in the form v_o, v_f, v_{0x}, or v_{fx}. When velocity is a two-dimensional vector, they notate with vector notation, either $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{v}}_{av}$. Similarly, the instructor and textbook differentiate in their practices between one- and two-dimensional accelerations. When acceleration is a one-dimensional vector, both the physics textbook and physics instructor notate it as a_y, $a_{av,x}$, or a_x. When acceleration is a two-dimensional vector, they notate it with vector notation, either $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{a}}_{av}$. The instructor and textbook seem to notate one-dimensional vectors using component notation instead of vector notation.

Describing vectors algebraically. Other than notation to name vectors, similar to the trigonometry instructor and trigonometry textbook, the physics instructor also had notation to describe vectors algebraically. The physics instructor and physics textbook do not introduce or use the notation introduced by the trigonometry instructor and trigonometry textbook in which the initial point and terminal point are listed in order and with a named letter, for example P(-2, 4)Q(4, 7). The physics instructor and physics textbook also do not introduce the angular notation, such as $\langle 6{,}3 \rangle$, where the angular parentheses arrange an ordered pair of the x-component with the y-component.

When the physics instructor provides instruction as to how vectors are described algebraically, he introduces two manners of describing vectors as ordered pairs. He states that one way of describing vectors is using rectangular coordinates and as an ordered pair of the x- and y-coordinates. He provides a generic example, (x, y). This matches the trigonometry instructor and textbook's notation except that he doesn't use angular parentheses; he uses standard parentheses to arrange the ordered pair.

His second way of describing vectors is using polar coordinates, which is an ordered pair of the vector's length and direction with respect to the positive x-axis arranged between standard parentheses. The physics instructor provides a generic example (R, θ). Although these two styles of describing vectors algebraically are introduced by the physics instructor, neither of these styles of notation were used in the instruction under analysis. The physics book does not introduce either notation.

Later, the physics instructor introduces another manner of describing vectors algebraically. The notation that is introduced is $\vec{R}_1 = A\hat{x} + B\hat{y}$, where the vector is expressed as a sum of its components. Notice three differences between the physics and trigonometry instructors' use of this notation. First, the trigonometry instructor used *i*'s and *j*'s to differentiate between the components, and the physics instructor uses *x*'s and *y*'s. Second, the trigonometry instructor said vector arrows should be above her *i*'s and *j*'s, but the physics instructor uses hats. Third, the trigonometry instructor references the *i*'s and *j*'s as part of the component sometimes, but the physics instructor calls them unit vectors and references only their coefficients as components. He states, "Notice that the x and the y pieces here, Have provided direction to each of the components" (P2-line 762-793). His cursor gestures on the PowerPoint that the *A* and *B* are the components. His gesture combined with his statement seems to communicate that the *A* is considered the x-component and the *B* is considered the y-component.

The physics instructor later used this style of notation to illustrate algebraic vector operations and to illustrate the operation of calculating displacement. The physics book does not introduce or use this style of notation; instead, when the textbook introduces components, it describes the vector as the sum of its vector components and writes $\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}_x + \vec{\boldsymbol{A}}_y$. Throughout the first three chapters, vectors are not written as a sum of their components or component vectors. Generally, components are labeled and listed separately from one another as algebraic variables would be.

Summarizing Notational Practices

In conclusion, both instructors and textbooks use notation for two purposes: as a means of naming vectors and describing them. While various manners of notation are introduced to name vectors, the main style of notating vectors in order to name them is very similar across physics and trigonometry and with both instructors and textbooks. Both instructors use a single letter with a half arrow above the letter as the notation for naming vectors, for example $\vec{v}$. Both textbooks state handwritten notation differs from printed notation within the textbook. The trigonometry book uses a bold-printed, single letter without an arrow above the letter; whereas, the physics textbook uses a full arrow above the bold-printed, single letter.

When vectors are not referencing vector quantities, the trigonometry instructor, trigonometry textbook, and physics textbook name each vector with different letters. When vectors are referencing vectors quantities, both instructors and both textbooks use specific letters to reference the desired vector quantity. The physics instructor and physics textbook state $\vec{v}$ is for velocity, $\vec{a}$ is for acceleration, and either $\vec{R}$ or $\Delta\vec{R}$ are for position or displacement. To differentiate different vectors representing the same quantities, subscripts are used, and the subscripts vary from numbers, letters, words, and partial words, depending on the circumstance.

However, in practice, the physics instructor and physics textbook generally do notate vectors and vector quantities without an arrow above the letter. The physics instructor and physics textbook generally use v for velocity, a for acceleration, x or y for position, and Δx for displacement. The trigonometry instructor states that sometimes she forgets to put the arrow above the letter, and indeed she does, but she states her students are to assume that it still means vector. In contrast, the physics instructor and physics textbook are not forgetting the arrow above the letter because lacking the arrow is a legitimate style of notation; they are referencing the vector's magnitude. This contrasts with the trigonometry instructor and trigonometry textbook that reference magnitude with two bars on either side of the vector name, for example $\|\boldsymbol{v}\|$.

The main style of notating vectors to describe them algebraically is also similar across the trigonometry instructor, trigonometry textbook, and physics instructor. While various styles are introduced to describe vectors, all three sources of instruction write vectors predominately as the sum of the components. Slight differences do occur in the practices because of how the two disciplines seem to notate unit vectors. The trigonometry instructor and trigonometry textbook notate unit vectors using vector notation (arrows above the single letters), but the physics instructor notates unit vectors using hats above the letters labeling the axes. The trigonometry instructor writes a vector in the algebraic form of $\vec{v} = a\vec{\imath} + b\vec{\jmath}$, the textbook writes it in the form $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$, and the physics instructor writes it in the form $\vec{v} = A\hat{x} + B\hat{y}$. The main difference in the manner of writing vectors algebraically is both the trigonometry instructor and trigonometry textbook use i's and j's, but the physics instructor uses $\hat{x}$'s and $\hat{y}$'s. The physics book does not introduce or use this style of notation; instead, the textbook describes the vector as the sum of its vector components and writes $\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}_x + \vec{\boldsymbol{A}}_y$, but it never uses it in the text that was analyzed.

The third section is a description across all of the functions for naming vectors with notation: to label diagrams, to label algebraic descriptions, and to reference either algebraic or geometric vectors in algebraic expressions. Also, sometimes the use of notation in an operational expression becomes objectified and used as the name of the resulting vector.

Naming Vectors for Three Purposes

For all four sources, there are three primary purposes for this use of notation. First, notation is used to label vectors in diagrams to differentiate them from each other. In this way, the instructional narrative can reference diagrammed vectors. Second, notation is used to label the algebraic description of a vector. Whether the vector occurs in a diagram or not, vectors are algebraically described. Notation names the algebraic description as a way to reference the vector in the narrative, and the notation differentiates between descriptions when multiple vectors are being referenced. For example, when the trigonometry instructor provided Problem 40 that stated $\vec{v} = 3\vec{\imath} - 5\vec{\jmath}$ and $\vec{w} = -2\vec{\imath} + 3\vec{\jmath}$, the $\vec{v}$ and $\vec{w}$ name the algebraic description of the vectors and the names differentiated which vector has the particular algebraic description. When the physics instructor wanted to express two vectors in algebraic form, he wrote $\vec{R}_1 = A\hat{x} + B\hat{y}$ and $\vec{R}_2 = C\hat{x} + D\hat{y}$, where $\vec{R}_1$ and $\vec{R}_2$ labeled the two vectors and served as their names. Third, the notation is also used to name vectors to be used in algebraic expressions with operations to be performed, such as in the trigonometry class which requested $3\vec{v} - 2\vec{w}$ to be determined and in the physics class which provided the formula defining velocity that reads $\vec{v} = \frac{\Delta\vec{R}}{\Delta t} = \frac{\vec{R}_f - \vec{R}_0}{t_f - t_0}$.

Also, sometimes the use of notation in an operational expression becomes the name of the resulting vector. For example, in the trigonometry class, the expression $3\vec{v} - 2\vec{w}$ is an operational expression calling for vector subtraction between two vectors to be completed after each is multiplied by a particular scalar. Once the operations are performed, a new vector is formed. When the instructor completed these operations, she used the expression as the name of the resultant vector by stating, $3\vec{v} - 2\vec{w} = 13i - 21j$. The operational expression was objectified to be the name of the resulting vector. The trigonometry book also does this. For example, if vector **v** and vector **w** are added together, the resulting vector is labeled **v** + **w**. Likewise, the physics instructor and textbook introduced $\Delta\vec{R}$ as the notation used to express the operation for finding the difference between two displacement vectors. While this notation is used to express an operation, it is also used to name the resulting vector.

Vocabulary Related to Vectors

The following section lists the vocabulary words provided by each source of instruction, and after comparing which vocabulary were introduced by all four sources, a description and comparison of their definitions is made.

Vocabulary words were provided during instruction by the instructors and textbooks through two techniques: using an explicit definition or without using an explicit definition. Sometimes when an explicit definition was not provided, the context of the sentence provided its meaning, and the vocabulary words were incorporated into the instructional narrative in a way that the context of the sentence provided enough clues to its meaning without the use of explicit statements. The researcher selected words from the textbooks to be vocabulary words if they were emphasized with bold or italic print. Instructors' verbal statements are sometimes subtle in their manner of introducing vocabulary resulting from intonation differences causing the words to be subtly emphasized. Table 5.2 lists mathematical vocabulary words both the instructors and the textbooks introduced.

Notice neither instructor nor textbook explicitly defines or describes *direction*. There seems to be a colloquial understanding of the word so it does not require a formal mathematical definition nor any explicit descriptions to be made about it. Notice all four sources of instruction introduced *vector*, *magnitude*, *scalar*, and *components*.

Table 5.2

Mathematical Vocabulary Explicitly Defined or Described

Trigonometry		Physics	
Textbook	Instructor	Instructor	Textbook
vector	vector	vector / vector quantities	vector quantity
magnitude	magnitude	magnitude	magnitude
directed line segment / geometric vector			
scalars	scalar	scalars/ scalar quantities	scalar quantity
			vector sum / resultant
initial/terminal point	initial/terminal points	tail / tip	tail / head
algebraic vector			
position vector	position vector		position vector
zero vector			
vertical/horizontal components	vertical/horizontal components	x- and y-components	components / component vectors
unit vector	unit vector	unit vector	
velocity vector			
force vector			
dot product	dot product		
scalar product			
parallel			
orthogonal	orthogonal		
vector projection			

Magnitude

Both instructors state that the length of a vector is called its *magnitude*. They both state that the words *length* and *magnitude* have the same meaning, and as they talk about vectors during their lectures, they use the terms interchangeably. In addition, the physics instructor explicitly states, “I will use the size, the length, or magnitude words in describing vectors interchangeably” (P1 line 246), which he does, and the word *size* is his most common way of referencing the magnitude of a vector. Similar to both instructors, the trigonometry textbook states that the magnitude of a vector “equals the length of a directed line segment” (p. 375).

The physics textbook has a slightly different meaning for *magnitude* because it relates it to a vector quantity and not a geometric vector. Although it does not define it formally, the physics textbook calls *magnitude* “the ‘how much’ or ‘how big’ part” (p. 12) of a vector quantity. Related to this idea, both the trigonometry instructor and trigonometry textbook state the reason for calling the length of a vector a *magnitude* since vectors can represent vector quantities, and vector quantities have magnitude—not length. So the use of vectors as representatives for vector quantities and the specific use of the lengths of vectors as representing the magnitude of the vector quantities results in the inclusive use of the word *magnitude* to describe both the length of a geometric vector and the size of a vector quantity.

Scalar

Just as *magnitude* is another word for *length*, the word *scalar* is used in the vector unit as another name for “numerical values” (P1-line 16) or “constants” (T1-293). Just like both instructors, the trigonometry textbook says the use of the words *scalar* is just a change in signifier for referencing real numbers. The text specifically states, “When dealing with vectors, we refer to real numbers as **scalars**.” As a reminder, a real number is any positive or negative quantity that can be written as a fraction. The trigonometry textbook continues in the following sentence to say, “Scalars are quantities that only have magnitude. Examples from physics of scalar quantities are temperature, speed, and time” (p. 374, emphasis in the original). Notice it states “scalars are quantities,” which is the physics textbook’s definition for scalar quantities. The physics textbook states “When a physical quantity is described by a single number, we call it a **scalar quantity**” (p. 12). The tight match between the definitions of *scalars* and *scalar quantities* is probably why the physics instructor seemed to use the word *scalar* as a nick name for a *scalar quantity*—just as he seemed to use the word *vector* as a nickname for a *vector quantity*.

Tip/Tail Verses Initial/Terminal Points

All four sources use different words to name the two ends of a vector. The trigonometry instructor and trigonometry textbook called them an initial point and terminal point; the physics instructor called them a tail and tip; and the physics textbook called them a tail and head. The trigonometry instructor stated that where a vector begins is called the initial point and where it ends is called the terminal point, and the physics instructor stated where a vector begins is called its tail and where it ends is called its tip. Although the physics textbook italicized the words *tail* and *head* as the vocabulary used to describe the ends of a vector, any reference following this initial statement referenced them as *tail* and *tip*. Even when the trigonometry instructor introduced what she called the tip-to-tale method of geometric vector addition, she referenced the tips and tails of the vectors as initial and terminal points.

Components

Neither instructor formally defined components, but both made explicit statements to describe components as the results of breaking a vector into lengths parallel to the x- and y-axes. The trigonometry instructor and textbook call them vertical and horizontal components, but the physics instructor and physics textbook called them x- and y-components. Neither textbook defined components, but in explicit statements about them, the trigonometry textbook states that the word *components* is the name for the scalars used to describe vectors algebraically, and the physics textbook states that components are the magnitudes of the component vectors along with a positive or negative sign to describe their directions. Unlike the physics instructor, the physics textbook first introduces *component vector* as new vocabulary, provides a description, and provides an accompanying diagram distinguishing the similarities and differences between component vectors and components as a means of helping describe each (see Figure 5.3).

> Now, what are components, and how are they used to add and subtract vectors? To define what we mean by components, let's start with a rectangular (Cartesian) coordinate axis system as in Figure 1.14. We can represent any vector lying in the x-y plane as the vector sum of a vector parallel to the x axis and a vector parallel to the y axis. These two vectors are labeled $\vec{A}_x$ and $\vec{A}_y$ in Figure 1.14a; they are called the **component vectors** of vector $\vec{A}$, and their vector sum is equal to $\vec{A}$. In symbols,
>
> $$\vec{A} = \vec{A}_x + \vec{A}_y. \quad \text{(vector } \vec{A} \text{ as a sum of component vectors)}$$

Figure 5.3. Excerpt from p. 16 of the physics textbook concerning defining component vectors.

Notice the words *component vectors* in bold print, which emphasizes the words are new vocabulary for the reader. The definition of component vectors comes in the sentence prior: they are two vectors "parallel to the x-axis and …y-axis" whose vector sum is the original vector. Notice they are being defined as vectors and they are being referenced as vectors. Also, notice that component vectors use vector notation, which is notated by using bold letters and an arrow above. The only notable difference between component vectors and vectors is the use or absence of subscripts. The use of the subscript letters *x* or *y* distinguishes which component is being referenced, and the absence of the subscript letter symbolizes that the bold-faced letter with the arrow above is the original, regular vector. To summarize, in the physics textbook, component vectors are defined as vectors, referenced as vectors, and given vector-style notation.

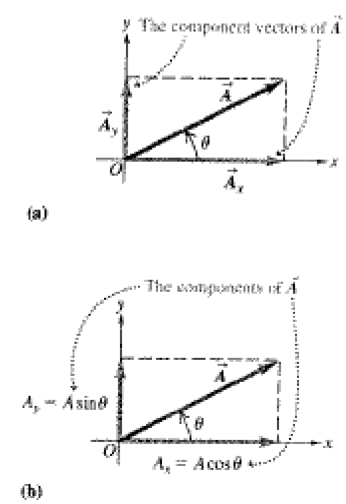
> By definition, each component vector lies along one of the two coordinate-axis directions. Thus, we need only a single number to describe each component. When the vector points in the positive axis direction, we define the number A_x to

> be the magnitude of $\vec{A}_x$. When $\vec{A}_x$ points in the $-x$ direction, we define the number A_x to be the *negative* of that magnitude, keeping in mind that the magnitude of a vector quantity is always positive. We define the number A_y the same way. The two numbers A_x and A_y are called the **components** of the vector $\vec{A}$. We haven't yet described what they're good for; we'll get to that soon!

Figure 5.4. Excerpt from p. 16 & continued onto p. 17 of the physics textbook concerning defining components.

To contrast, the vocabulary word *component* is introduced in bold-faced type in the following paragraph, which are provided in Figure 5.4. Notice the text states, “The two numbers A_x and A_y are called the **components** of the vector $\vec{\boldsymbol{A}}$.” In the sentences prior, components are described as numbers with two particular qualities. In the third sentence, notice the textbook describes one of the qualities, “we define the number A_x to be the magnitude of $\vec{\boldsymbol{A}}_x$.”

Thus, components are numbers that represent the magnitude of the component vectors. In the third and fourth sentence, notice the textbook describes the second of the qualities, which is the sign being either positive or negative represents which direction the arrow faces. As a result, components don’t seem to be vectors for two reasons: they are defined as numerical values, not vectors, and their notation doesn’t use vector-style notation, which is a bold-printed letter with an arrow above it, and does use a numerical style, which is a nonbold-print letter with no arrow.



▲**FIGURE 1.14** (a) The component vectors of $\vec{A}$ in the direction of the x and y axes. (b) The x and y components of $\vec{A}$.

Figure 5.5. Excerpt from p. 16 of the physics textbook “Figure 1.14” concerning comparing component vectors to components.

Beside the paragraphs provided in Figure 5.3 and 5.4 is a pair of diagrams to provide clarity to the written narrative. The first diagram illustrates component vectors, and the second illustrates components (see Figure 5.5). Notice the repetition between the two diagrams used to highlight the similarity and difference between component vectors and components. In both diagrams the component vectors are graphed in a lighter shade of gold than the original vectors, and both diagrams illustrate the component vectors as parallel to the x- and y-axes, as perpendicular to one another, and as the two vectors whose sum results in the original vector. The difference between the two diagrams is what is being labeled. In the first diagram, the notation $\vec{\boldsymbol{A}}_x$ and $\vec{\boldsymbol{A}}_y$ float beside the component vectors working as a means of "naming" them, and above the diagram, the words "The component vectors of $\vec{\mathrm{A}}$" have arrows pointing down to the geometric arrows. This first diagram highlights component vectors as being the arrows. In the second diagram, the words "The components of $\vec{\mathrm{A}}$" have arrows pointing down—not to the component vectors—to the right side of both labels beside the component vectors. The labels read "$A_y = Asin\theta$" and "$A_x = Acos\theta$." Thus, while the left side is notation in order to "name" the scalar value, the right side is being emphasized by the arrow and is an expression to produce a scalar value. This second diagram highlights the numerical values resulting from calculating the magnitude of the component vectors as components. Thus, the diagrams echo the instructional narrative in having component vectors as geometric vectors and components as scalar quantities—not as geometric objects.

After introducing the component vectors and components, the textbook interrupts its narrative to provide the reader with a special summary note of warning (see Figure 5.6). Notice the first sentence clarifies the distinctions between all four of the notations while also asking the reader to "understand the relationship" between them.

NOTE ➤ Be sure that you understand the relationship between $\vec{A}$ (a vector), A or $|\vec{A}|$ (the vector's magnitude), $\vec{A}_x$ (a component vector) and A_x (a component). The following tips are useful:

- A subscript x or y indicates that the quantity is a component vector or component along the x or y axis. Watch for these subscripts!
- A *magnitude* is always positive.
- A *component* is positive if the corresponding component vector points in the positive direction of the axis, negative if the component vector points in the negative direction of the axis. ◄

Figure 5.6. Excerpt from p. 17 of the physics textbook concerning offering a warning for confusing notation.

By this point in the narration, the distinctions seem obvious and clear, component vectors are geometric arrows, and components are the magnitudes of the component vectors along with a positive or negative sign to describe their directions. However, just 9 lines later, the distinction collapses as the textbook seems to call component vectors *components* as it says, "In Figure 1.15, the component B_x is negative because its direction is opposite that of the positive *x* axis." There are two problems with this sentence. First, this sentence is assuming the reader is looking at a figure to see the *component*. Component vectors are in figures—not components. Second, the pronoun *its* in the sentence has the antecedent "component B_x," but components are numbers and cannot have a "direction…opposite that of the positive *x* axis." Though the sentence references the component, the real antecedent for the pronoun *its* should be the geometric arrow in the diagram that is facing opposite of the x-axis, which is the component vector. In clarifying the sentence to maintain the distinctions, the sentence might have read, "In Figure 1.15, the component B_x is negative because the direction of the component vector $\vec{\boldsymbol{B}}_x$ is opposite that of the positive *x* axis." When the textbook is being careful with its language in the third bullet of Figure 5.6, the component vector faces the "negative direction of the axis" causing the component to be negative, but 9 lines later when the textbook authors are in full swing of the narrative again, their language says the component's "direction is opposite of the positive x-axis."

Even the physics instructor uses the word component to reference component vectors. When reintroducing a diagram illustrating projectile motion from the previous lesson, he adds an initial velocity vector at the origin of two axes as he says,

27. If you'll recall we had a projectile launched from the origin,
28. at some angle theta above the axis,
29. at some initial velocity v-naught,
30. We mentioned that a person can calculate the components,
31. of this initial velocity,
32. in terms of the coordinates system we're given.

As he states "a person can calculate the components" in line 30, the component vectors have appeared on the screen. He may be referencing the components but visually the hearer sees the component vectors added to the screen, which collapses the distinction between the two. Throughout the rest of this instructional segment, the physics instructor references x- and y-components as he adds the component vectors to the screen or as he references the component vectors to discuss them.

This collapse in distinction between the verbal references of component vectors and components is further complicated by the physics instructor's and physics textbook's practice of labeling one-dimensional vectors and vector quantities with component notation. The combination of these practices causes confusion between what a component *is* in practice and how it is explicitly defined to be. This discussion is further elaborated later in the chapter.

Summary

All four sources of instruction introduced *vector*, *magnitude*, *scalar*, and *components*. Their definitions for *magnitude* and *scalar* are nearly equivalent. Although none of the sources formally define components, the physics textbook provides explicit statements about components, but those statements do not match the remaining practices of the physics instructor nor the physics textbook. Components are defined to be the magnitudes of the component vectors along with a positive or negative sign to describe their orientation, but in practice, component vectors are labeled and referenced as components and their notation labels vectors as one-dimensional.

Another similarity is all four sources do not define *direction*; they all seem to use the word without providing an exact meaning. A difference between the sources is the trigonometry instructor and trigonometry textbook use the words *initial point* and *terminal point* to reference the endpoints of the vector; whereas, the physics instructor and physics textbook use the words *tail* and *tip*.

Remaining Diagrams

The following section analyzes the remaining diagrams against what the instructors and textbooks explicitly stated when defining and initially drawing vectors as arrows. The two visual differences between the initial vectors drawn by the textbooks and by the instructors were the use of bold dots at either end of the arrows and whether a coordinate axes was accompanied or not. The following section describes the pattern of these two visual attributes concerning the remaining diagrams and some of the results of what was being conveyed by the use of these attributes.

A difficulty in collecting vectors results from the chance that non-vectors may be included in the analysis. For example, arrows, rays, and vectors look very similar, as we have heard the trigonometry instructor clarify to her students in class. As a result, prior to describing the remaining diagrams, all diagrams for analysis were directly referenced as vectors in the instructional narrative, in the supporting comments beside diagrams, or in its accompanying notation directly as a vector or with vector notation. Only 1 of the 26 arrows created by the trigonometry instructor was not referenced as a vector. The only arrow which is not referenced as a vector was the arrow provided in the initial segment as an example of a ray. In all the remaining 25 cases, the trigonometry instructor references the arrow-like objects as vectors with statements like “so there’s my vector” (T2-317) or “there’s vector *v*” (T2-536).

For the trigonometry textbook, physics instructor, and physics textbook, there were arrows that were not referenced as vectors, and the majority of the arrows were referenced as components. While nothing said or done by the trigonometry instructor or trigonometry textbook would warrant classifying components as vectors, the physics instructor and physics textbook both seem to treat arrows as vectors and sometimes as a proxy for one-dimensional vectors. Because the physics textbook and instructor labels one-dimensional vectors with component notation, one-dimensional vectors will be included in the analysis and referenced simply as vectors along with the other vectors. Component vectors are not analyzed. Nothing said or done by the trigonometry instructor or trigonometry textbook warrants classifying components as vectors; therefore, they will not be included in the following analysis.

Use of Coordinate Axes

The trigonometry instructor's initial two vectors and the trigonometry textbook's initial drawing of a directed line segment were drawn floating in white space without being accompanied by a coordinate system, and they were drawn with large dots at either end. There were only 4 other vectors that followed its same description in the textbook and none by the instructor. In all other cases (92% of the trigonometry instructor's total and 95% of the trigonometry textbook's total), a variation of the description occurred. The physics instructor's original diagram of a vector did not have any bold ends and was drawn onto a coordinate system that had both the axes and a grid. Only the other two vectors in the first instructional segment followed the original vector's description. In all other cases (88% of total), a variation of the description occurred. In contrast to the physics instructor, the physics textbook's initial five vectors were similar to the trigonometry instructor's and trigonometry textbook's original drawings with the arrows floating in white space and having bold ends on either end. Only these 5 vectors followed this description; in all other cases (98% of total), a variation of the description occurred.

Variations in the diagrams illustrate flexibility in the depiction of the vectors. Much like the signifier *dog* encompasses animals that are similar in some elements and yet vary tremendously in size, shape, and color, the signifier *vector* encompasses arrows that are diagramed similarly in some elements and yet vary greatly in others. The variations are important to note because they expresses characteristics that are not essential for a diagrammed arrow to be a vector. The variations also reveal patterns in the beliefs and use of vectors in the separate disciplines. The initial diagrams varied whether large dots were used at the end of the vectors or when they accompanied the vectors with coordinate axes. The following paragraphs describe the patterns of the remaining diagrams concerning these two characteristics.

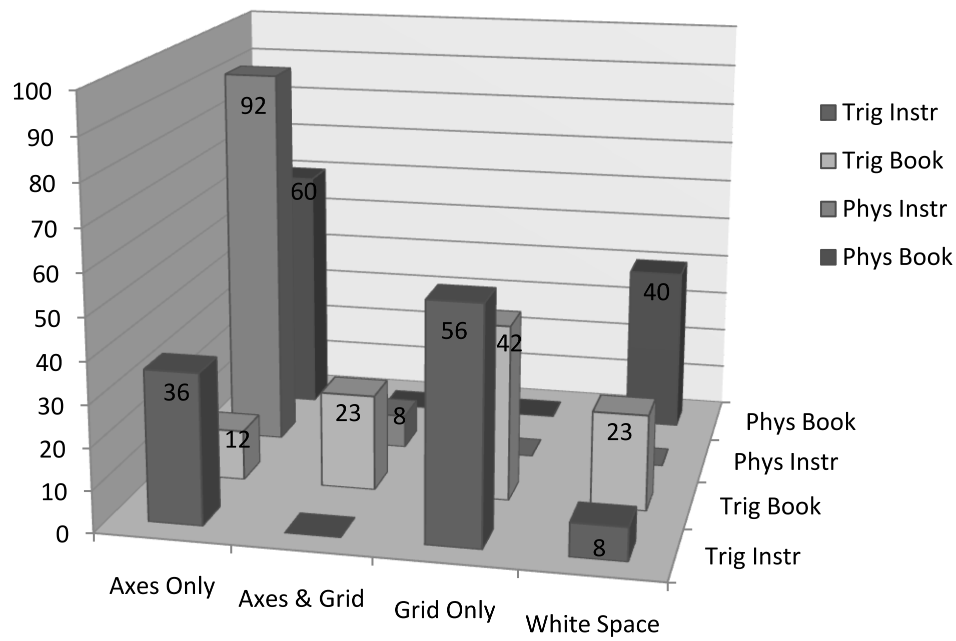


Figure 5.7. Percentages in variations of using axes and/or grids with the vectors.

Notice in Figure 5.7 that the trigonometry instructor and trigonometry textbook vary how they situate vectors in white space or with or without axes and grids. In contrast, notice the physics instructor accompanies 100% of the vectors with axes, and the physics textbook splits its vectors between accompanying them with axes or allowing them to float in white space.

The physics instructor's remaining diagrams match his explicit instruction. He emphasized several times the importance of selecting a coordinate system prior to sketching arrows, and all of his remaining vectors were accompanied by coordinate axes. With a few (8%), he used both axes and a grid. The physics book does not match his explicit statements because a large percentage of the vectors were situated in white space. The physics textbook had 40% of its vectors situated on white space, and 70% of the examples occur are in Chapter 1. These diagrams in Chapter 1 contained many examples of vectors and many of them illustrated geometric vector operations. Although 40% of the vectors are drawn in white space, they only account for 16% of the total diagrams. The most common way for any diagram to be drawn by the textbook was to situate it with axes.

Similar to the physics textbook, a large number of vectors were drawn in each diagram illustrating geometric vector operations. The trigonometry textbook and trigonometry instructor illustrate geometric vector operations on grids, which is why the greatest number of vectors for the trigonometry instructor and trigonometry textbook are situated with grids. However, similar to the physics textbook, the most common way for a diagram to be drawn by both the trigonometry instructor and trigonometry textbook was for the vectors to be situated with axes.

Interesting to note, the trigonometry instructor and trigonometry textbook followed a similar pattern in their illustrations of first depicting vectors in white space, then on grids, and then with axes. For the trigonometry instructor, only the original two vectors were drawn free-floating in white space. In the instructional segments immediately following, the vectors were situated on grids without axes, and all the remaining instructional segments had vectors drawn on a coordinate plane with axes. Likewise, the trigonometry textbook illustrates its original vectors on white space, with the following ones on grids, and then the remaining vectors with axes, but the textbook has a few scattered diagrams with vectors floating in whitespace.

Use of Bold Dots at the Ends

Figure 5.8 depicts the four sources of instruction and their use of bold points at the ends of the vectors. Notice the textbooks follow a similar pattern to the instructors', but there is a significant difference in patterns between physics and trigonometry. The physics instructor and physics textbook almost exclusively use a bold dot at just the initial end or at neither end. In contrast, the trigonometry instructor and trigonometry textbook generally emphasize both ends with large dots but have a few cases across the other categories.

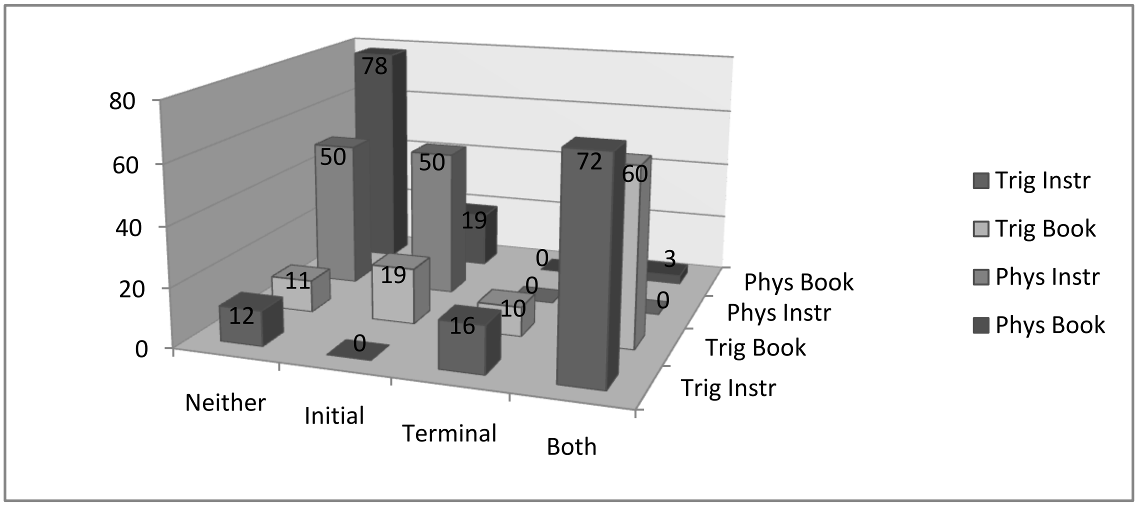


Figure 5.8. Percentages in variations of using bold dots at the ends of the vectors.

For the trigonometry instructor and textbook, 72% and 60%, respectively, of their vectors had their initial and terminal points accentuated with large dots. This use of large dots at both ends communicates the vectors' fixed lengths, the existence of the vectors' endpoints, and the location of the endpoints. These locations were not always coordinate specific if the vectors were situated on grids, but the location was still specific. When the trigonometry instructor and textbook illustrated a vector with just the terminal point accentuated with a large dot, the initial point was at the origin, providing a natural demarcation for the exact location of the initial point without the use of a dot. When the textbook emphasized just the initial point, 90% of the vectors had their initial point on the origin with the dot emphasizing a shared location for multiple vectors' initial points.

This almost rigid pattern of always depicting the location of the initial point and terminal points communicates the vectors' fixed lengths and the vectors' beginning and ending locations. The trigonometry instruction's practice of using no large dots on the vectors followed two patterns: depicting generic situations in which the location of the endpoints is not of interest for what is being diagrammed, and depicting vector quantities.

Figure 5.9 illustrates an example of a generic situation. In the diagram, the quality of two vectors being orthogonal is being illustrated. Notice that the fixed length and the location of the end points is not important for the concept being illustrated. The diagram illustrates vectors are orthogonal if their direction is 90° of each other. Important to the analysis, nothing in the diagram serves to question that if these two generic vectors were replaced by two specific vectors then the specific vectors would not have endpoints and would lack fixed lengths. As a result, the reader of the diagram probably will see the diagram in the same vein as the others: having endpoints, resulting in a fixed length, and showing direction.

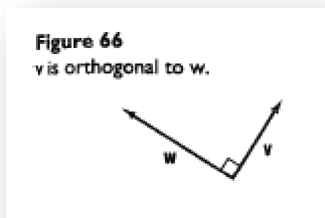


Figure 5.9. Excerpt from p. 387 of the trigonometry textbook depicting generic vectors illustrating orthogonality. The generic nature of the diagram does not necessitate the use of large dots on either end of the vectors.

In contrast, the physics instructor and physics textbook had 100% and 97%, respectively, of their vector arrows with neither end emphasized with large dots or had the tail of the vector emphasized with a large dot. The absence of dots is directly related to the fact that, based on their practices, the physics instructor and physics textbook would define vectors as vector quantities. The arrows represent the vector quantity's magnitude and direction. The lack of using large dots at both ends does not discredit the vectors' from communicating their fixed lengths because they represent vector quantities, which always have magnitude, but it does seem to communicate that the location of the vectors' endpoints are not fixed in the plane.

When the physics instructor and textbook began using arrows to represent vector quantities, the use of the dots was related to the definition and qualities of the vector quantity being represented. The physics book and physics instructor both label all axes as denoting distances, which can also be read as a measurement of positions. As a result, a displacement vector is the only vector quantity that can be defined to begin and end at particular locations in the plane or on axes. Displacement is defined by changes in positions; as a result, vectors representing displacements stretch from one position to another position in the plane or along the axes. The location of these positions exist and can be emphasized. They could be emphasized with bold dots, but the axes used are unscaled so the textbook depicts where displacement vectors begin and end using dotted-guide lines. Figure 5.10 depicts a displacement vector stretching from one position to the next with the dotted guide lines.

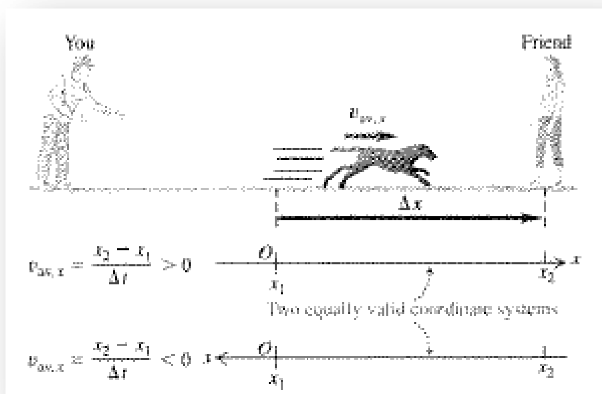


Figure 5.10. Excerpt from p. 33 of physics textbook offering an example of displacement vectors stretching between dotted-guide lines.

In contrast to displacement, velocity and acceleration vectors are not defined by their initial and terminal points existing at particular positions on the axes or in the plane. Average velocity vectors are often depicted with neither the tip nor the tail emphasized with large dots because the tips and tails do not relate to a particular location in the plane. In contrast, instantaneous velocity vectors were usually related to a particular position in the path of an object by floating near its related position or by attaching its tail to the dot at that location. Notice in

Figure 5.11 that the two instantaneous velocities $\vec{v}_1$ and $\vec{v}_2$ are depicted with a dot at the initial point communicating at what instant in the cars path the velocity is being related. Also notice the tips of the instantaneous velocity vectors are not defined to a particular location, which results in a dot not being used. Similarly, notice also in Figure 5.11 that the change in velocity vector and the average acceleration vector both have neither end emphasized with dots because neither end is related to a specific location.

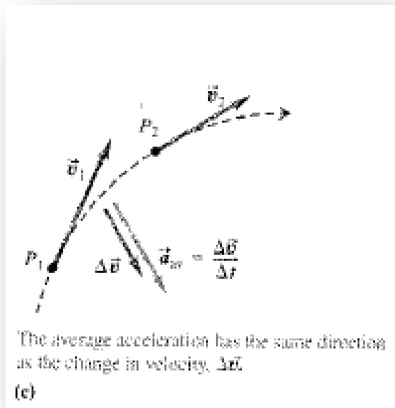


Figure 5.11. Excerpt from p. 72 of physics textbook offering an example of instantaneous velocity vectors.

Instantaneous velocity and acceleration are associated with a particular moment of time and, thereby, a particular location. Their geometric arrows representing the quantities sometimes float near the point highlighting the instant's location, and sometimes have their tails fixed to begin at the location, which is when the initial point has a large dot. Figure 5.12 depicts two examples.

Notice the first example has the two velocity arrows floating near the location being referenced. Generally, dots are used to mark the location of the object illustrated, which is done in both diagrams in Figure 5.12. Sometimes the diagram has the velocity vectors floating near the dots or the object being illustrated (in the first example, the cars), and sometimes the diagram has the velocity vectors attached to the location being referenced (in the second example, the projectile). To contrast with the velocity vectors, acceleration, which remains constant across time in both diagrams in Figure 5.12, does not have either end emphasized with dots because neither end is fixed in the plane or references a moment in the plane.

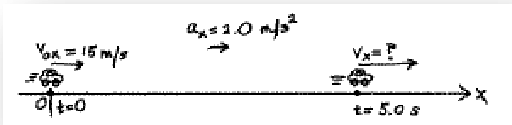
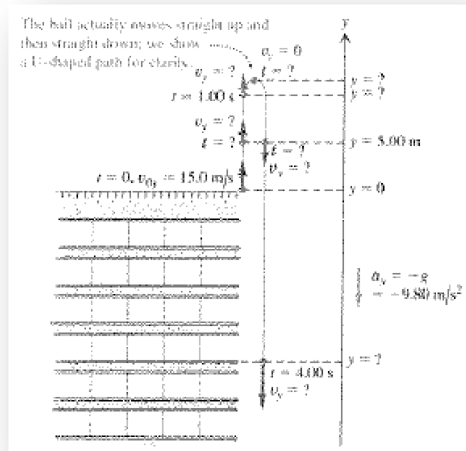
Velocity vectors not attached to “initial dots”	Velocity vectors attached to their “initial dots”
	

Figure 5.12. Excerpts from p. 43 & 53 from the physics textbook illustrating differences in the manner in which instantaneous velocity vectors are illustrated in respect to the related position.

After noticing the pattern of relating the type of vector quantity to its style of depicting the endpoints, returning to the trigonometry textbook for re-analysis resulted in noticing that most vectors in the trigonometry textbook are missing large dots at one or both endpoints represented vector quantities. The textbook has 17% of its vectors representing vector quantities, and they account for 44% of the vectors missing large dots on the ends.

Summary

The physical appearance of the remaining vectors (from 88-98% depending on the instructional source) varied from the initially diagrammed vectors. A repeated pattern in their appearance was the physics instructor and physics textbook almost exclusively use a bold dot at just the initial end or at neither end; but, in contrast, the trigonometry instructor and trigonometry textbook generally emphasize both ends with large dots with some cases across the other categories. A second pattern in their appearance was the physics instructor accompanied 100% of the remaining vectors with axes while the other instructional sources used axes most often but had the largest count either fixed on a grid on in white space. The use of dots and axes contribute to the qualities of the vector quantities. In trigonometry, both endpoints are generally are fixed in the plane, but velocity and acceleration vectors do not have both ends fixed in the plane. This pattern in visual appearance makes a difference in the meaning being displayed through the vector.

Explicitly Taught Math Procedures

The following section describes the mathematical content taught by the trigonometry instructor and by the physics instructor when he is providing mathematical instruction. The section describes what content was taught by one or both of the instructors.

For the first two days of instruction, the physics instructor primarily focuses on teaching the mathematics needed for the course. The topics of his lecture are similar to the topics selected by the trigonometry instructor. The only topics the trigonometry instructor included that are not mentioned by the physics instructor are how to move a vector algebraically and graphically to be a position vector, how to find a vector's unit vector using the formula $\boldsymbol{u} = \frac{v}{\|v\|}$, and how to find the dot product using the formula $\boldsymbol{v} \cdot \boldsymbol{w} = a_1 a_2 + b_1 b_2$. The physics instructor states that the calculus-based version of the course does calculate dot products.

The only mathematical content the physics instructor describes that is not taught by the trigonometry instructor is the manner of using a given velocity-verses-time graph to create the related acceleration-verses-time and displacement-verses-time graphs. This mathematical content is not included in a vector unit, which is the focus of this project, and it is generally not included in a trigonometry course but rather in a calculus course.

One of the topics both instructors describe is how both geometrically and algebraically to add and subtract vectors and to multiply them by a scalar. To add vectors geometrically, the trigonometry instructor described using both the tip-to-tale method and the parallelogram method, while the physics instructor only illustrated the parallelogram method. Other than this, there seems to be no significant differences between both instructors' manners of introducing geometric operations. Both instructors explain the parallelogram method places the vectors with their initial points together at the same location, treats the vectors as the first two sides of a parallelogram, and redraws each vector with its initial point at the terminal point of the other causing opposite parallel sides to be created. The resulting vector is the diagonal of the parallelogram beginning at the vectors' shared initial points and ending at the shared terminal points at the opposite angle. Both instructors introduce geometric vector subtraction as the same as adding a negative vector, which is accomplished by swapping the initial and terminal ends of the second vector before adding it to the first vector. Both instructors state that multiplying a vector by a scalar is the same as adding a series of the same vector multiple times to itself.

When teaching how to do it algebraically, differences between the instructors' instruction seem to be just notational. When the trigonometry instructor introduces algebraic addition, she solves problem 40. She begins,

133. So we're given 2 vectors, v is equal to 3 i minus 5 j. And w is equal to negative 2 i plus 3 j.
134. And in 40 they want us to find 3 times vector v, minus 2 times vector w.
135. So at first we're going to take vector v and multiply by the scalar.
136. Typically uh we would say we are multiplying by a constant, but using vector terminology, it becomes a scalar.

She then continues and her algebraic work is represented in Table 5.3. The physics instructor, on the other hand, introduces algebraic addition with a generic example. When the physics instructor introduces algebraic addition, he provides a generic example of adding $\vec{R}_1 = A\hat{x} + B\hat{y}$ and $\vec{R}_2 = C\hat{x} + D\hat{y}$, and states,

821. Wouldn't it be reasonable if these X hats acted like like terms?

822. You could just add the com- the actual coefficients together.
823. And that's exactly how it works.
824. So add the parts in the x direction together.
825. A plus C.
826. And add the parts in the Y direction together.
827. B plus D.
828. Add all this stuff together,
829. Keep in mind,
830. you still have to have the placeholders for directions for the components in place.

As he states this, he gestures toward his algebraic work written on his PowerPoint, which is represented in Table 5.3. Notice the similarity in the instructors' manners of adding the components together, although the notation is different. Both instructors replace the single-variable names of the vectors for the algebraic component expressions, both use parentheses to group their thoughts, and both add the coefficients of similar terms together to express the resulting vector in terms of the sum of the original components. The notable difference is the trigonometry instructor uses *i*'s and *j*'s with arrows above them, but the physics instructor uses *x*'s and *y*'s with hats above them.

Table 5.3

Trigonometry & Physics Instructors' Written Explanation of Algebraic Vector Addition

Trigonometry Instructor's Work	Physics Instructor's Work
40. given $\vec{v} = 3\vec{\imath} - 5\vec{\jmath}$ and $\vec{w} = -2\vec{\imath} + 3\vec{\jmath}$ find $3\vec{v} - 2\vec{w}$. Her board work: $3(3\vec{\imath} - 5\vec{\jmath}) - 2(-2\vec{\imath} + 3\vec{\jmath})$ $9\vec{\imath} - 15\vec{\jmath} - (-4\vec{\imath} + 6\vec{\jmath})$ $9i - 15j + 4i - 6j$ $3\vec{v} - 2\vec{w} = 13i - 21j$	Vectors are typically written: $\vec{R}_1 = A\hat{x} + B\hat{y}$ $\vec{R}_2 = C\hat{x} + D\hat{y}$ Vectors may be: -added: $\vec{R}_1 + \vec{R}_2 = (A\hat{x} + B\hat{y}) + (C\hat{x} + D\hat{y})$ $= (A + C)\hat{x} + (B + D)\hat{y}$

A second topic both instructors described was how to find the magnitude of a vector if given the length of its components. A slight distinction between their instruction is the physics instructor uses the Pythagorean Theorem ($a^2 + b^2 = c^2$), while the trigonometry instructor references Pythagorean Theorem while using a variation of it, namely $\|\vec{v}\| = \sqrt{a^2 + b^2}$.

Third, both instructors describe how to incorporate a vector into a right triangle using its components and employ the trigonometry functions of sine, cosine, and tangent to determine components from a given vector's magnitude and direction, to determine the magnitude and direction given a vector's components, or some combination of this information to find the missing information. A slight distinction in their instruction is that the trigonometry instructor uses the trig functions with the style of $sin\theta = \frac{opposite}{hypotnuse}$, $cos\theta = \frac{adjacent}{hypotnuse}$, $tan\theta = \frac{opposite}{adjacent}$, and while the physics instructor does state them this way, he states his preference is to use the style $opposite = hypotnuse \cdot sin\theta$, $adjacent = hypotnuse \ \cdot cos\theta$ and $\theta = tan^{-1}\frac{opposite}{adjacent}$, which he does when working example problems later.

Last, when working problems related to these trigonometry functions, the trigonometry instructor uses the given information to state the vector algebraically in terms of its components, such as $\boldsymbol{v} = a_1\boldsymbol{i} + b_1\boldsymbol{j}$. For example, one of her example problems asked for a vector to be found and written in the form a**i**+b**j** when its magnitude was 3 and its angle measure with the positive x-axis was 240°. The trigonometry instructor sketched the situation, and then proceeded to create a triangle from the vector and the segments serving to show the lengths of the x- and y-components. After using the sine and cosine functions to find the segments, she wrote the answer as $\vec{v} = \frac{-3}{2}\vec{\imath} - \frac{3\sqrt{3}}{2}\vec{\jmath}$. This same task was used by the physics instructor while working a projectile motion problem but in a different way. The problem asked "What's the ball's velocity and acceleration at its maximum height?" and while the physics instructor has been discussing velocity and acceleration as scalar quantities while working several angles of questions about this situation using kinematic equations, he now switches to wanting to write them as the sum of their component vectors.

To summarize, the physics instructor teaches most of the same topics as the trigonometry instructor. The only topics the trigonometry instructor included that are not mentioned by the physics instructor are how to move a vector algebraically and graphically to be a position vector, how to find a vector's unit vector using the formula $\boldsymbol{u} = \frac{v}{\|v\|}$, and how to find the dot product using the formula $\boldsymbol{v} \cdot \boldsymbol{w} = a_1 a_2 + b_1 b_2$. The only mathematical content the physics instructor describes that is not taught by the trigonometry instructor is the manner of using a given velocity-verses-time graph to create the related acceleration-verses-time and displacement-verses-time graphs, which is generally taught in a calculus course. The topics both instructors introduced were how to geometrically and algebraically add and subtract vectors and multiply them by a scalar, how to find the magnitude of a vector, and how to incorporate a vector and into a right triangle with its components and use the trigonometric functions of sine, cosine, and tangent to determine missing qualities of a vector from given qualities.

The topics both instructors introduced align with the goals the trigonometry instructor expressed during the interview. The trigonometry instructor expressed her desire to use her limited time in a vector unit to focus on what she called the "basic operations with vectors." These basic operations included how to add vectors geometrically and algebraically, how to multiply a vector by a scalar. The physics instructor expressed that her other two goals (dot and cross products) were needed for the calculus based course. As already stated, the only mathematical content the trigonometry instructor did not include that the physics instructor included was calculus-based (using a given velocity-verses-time graph to create the related acceleration-verses-time and displacement-verses-time graphs). For these two courses, the trigonometry curriculum seems aligned with the expected needs of the algebra-based physics course; however, the algebra-based physics course seems to be including aspects of mathematical thinking that are not introduced until calculus.

Mathematical Procedures Used When Doing Physics

The purpose of the next section is to compare the mathematical content described in the previous section with the mathematical content used by the physics instructor when teaching physics concepts.

The physics instructor spends almost two full class periods talking about vectors' properties, notation, and operations. In all cases the vectors were two-dimensional arrows and were expressed algebraically as paired information. The physics instructor explains at the close of the second lecture that the students have been learning "sorta the mathematical guts of what's going on" (P2-893), and now they will now start describing motion. Motion combines the study of the vector quantities of displacement, velocity, and acceleration. On the third day, he describes these three vector quantities, defines them, discusses the relationship between them, and introduces Kinematic equations. On the fourth day, he does an example problem using the kinematic equations in one dimension and introduces one- and two-dimensional motion along with relative motion. On the fifth day, he introduces projectile motion and models an example problem.

A surprising attribute of the mathematics used by the physics instructor and physics textbook when explaining physics problems is the lack of using any vector operations. The physics instructor does 3 example problems over the course of the 3 days of physics content. The first example problem is the velocity-verses-time graph that is used to create the related acceleration-verses-time and displacement-verses-time graphs. To solve this, an understanding of functions and their related derivatives was used, which is a mathematical process first introduced in calculus. The last two example problems use kinematic equations to determine unknown information from given information. This process of using kinematic equations requires decomposing the vectors into their separate components and working with the components as scalars. All calculations are scalar calculations. As a result, vector operations geometrically and algebraically are not used during this unit of physics. The mathematics needed to solve the problems is knowing a vector can be broken into two perpendicular components, the ability to use the right-triangle functions of sine, cosine, and tangent to find the sides or angles within the right triangle formed between a vector and its components, the ability to solve literal equations for a particular variable, and the ability to solve a system of two equations with two unknown variables. The following paragraphs elaborate his instruction.

On the third day, the physics instructor opens with a slide that introduces the notation and definitions of the vector quantities displacement, velocity, and acceleration. This includes discussing that both velocity and acceleration can have the property of being either average or instantaneous. During this slide, he makes purposeful contrasts between these three vector quantities as vectors and time as a scalar. Up to this point, students experienced vectors as arrows geometrically and as paired information algebraically; however, in the following three instructional segments, graphs do not have vector quantities expressed as geometric arrows nor as paired information. Instead, all the graphs label and utilize vector quantities as scalars.

During the first slide after the introduction of the vector quantities, the physics instructor introduces two graphs, using velocity as an example, to describe the similarities and differences between a quantity being average or instantaneous.   He states he believes the students need to know “the differences here between time intervals and specific instants in time. And how a person would go about calculating the slope of the graph that might not actually be linear itself” (P3-184).   His explanation describes how velocity is defined to be a ratio of the change in displacement compared to a change in time, how shorter changes in time better approximate the actual velocity of the object, and how the velocity of a particular instant is best described with the tangent line to the curve at that moment.

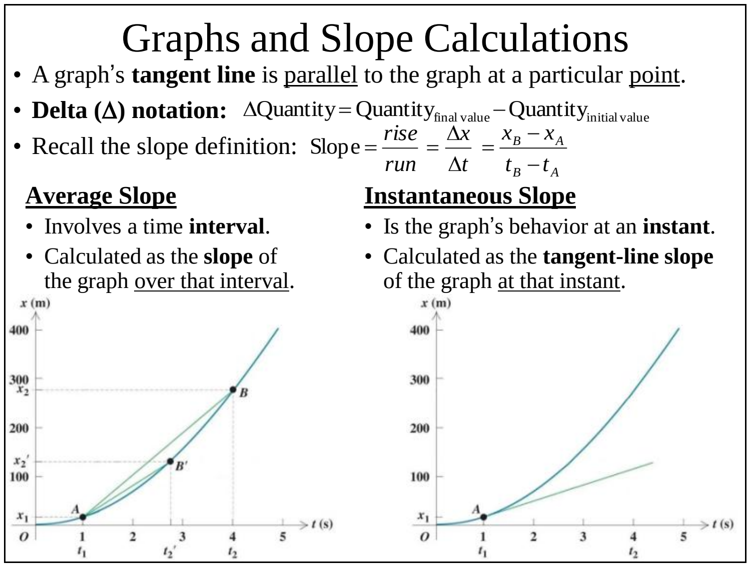


Figure 5.13. Excerpt from the physics instructor's lecture. PowerPoint slide 2 on day 3 of analysis does not depict vector quantities as arrows—rather they are numerical values paired with time in these displacement-verse-time graphs.

Notice on the graphs in Figure 5.13 that the y-axes are labeled with displacement and the x-axes are labeled with time. Time is a scalar quantity; therefore, it is reasonable to find an axis scaled to match time. Displacement, on the other hand, is a vector quantity, but this graph is illustrating displacement as just a numerical value. Displacement is not being graphed as an arrow, and it is not being graphed as paired information against time.

Likewise, in the next instructional segment, vector quantities are again discussed as being singular numerical values and not as paired values. Figure 5.14 provides the slide introducing physical-quantity-verses-time graphs by providing an example problem in which a velocity-verses-time graph is given and the related acceleration-verses-time and displacement-verses-time graphs are requested.

Notice that the given graph has the y-axis labeled with velocity, but velocity is not being depicted as an arrow or as paired information in the graph. The velocity is being depicted as a simple scalar quantity to be paired with time, which is also a scalar quantity. As the problem is worked, all three vector quantities continue to be discussed as just numerical values without referencing them as paired values.

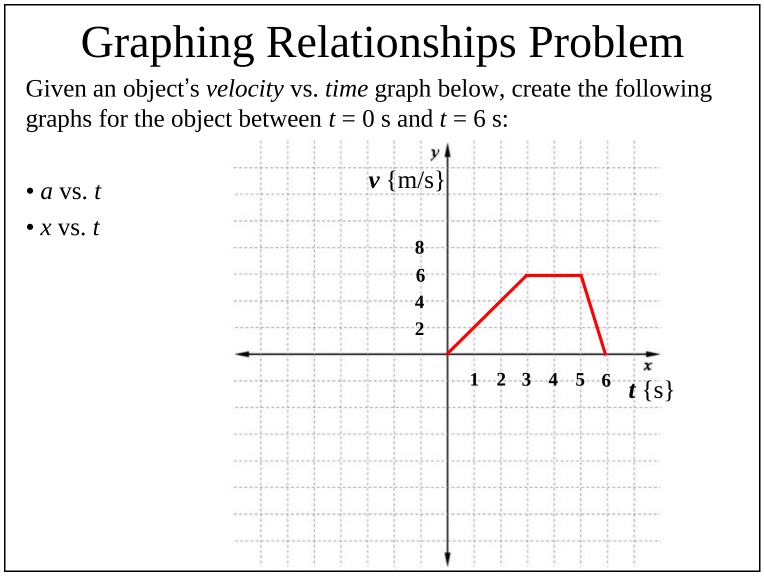


Figure 5.14. Excerpt from the physics instructor's lecture. PowerPoint slide 5 on day 3 of analysis does not depict vector quantities as arrows—rather they are numerical values paired with time in the velocity-verse-time graph and labeled along the side.

At the close of class that day, the physics instructor introduces kinematic equations, and he opens his lecture the following day with an example problem using them to discuss motion in a straight line. Just like the two previous instructional segments, for this example problem, he did not draw arrows, use vector operations, nor denote the vector quantities as paired values. The example problem concerns the motion of a bullet while in a gun, and kinematic equations are used to find the bullet's acceleration in the gun and the amount of time the bullet was in the barrel. The sketch, provided in Figure 5.15, uses a rectangle as the barrel of the gun, a triangle as the bullet, a horizontal arrow along the inside of the barrel as the horizontal axis of motion, and a vertical arrow along the end of the barrel as the vertical axis of motion, in which there is none. No vector arrow is depicted.

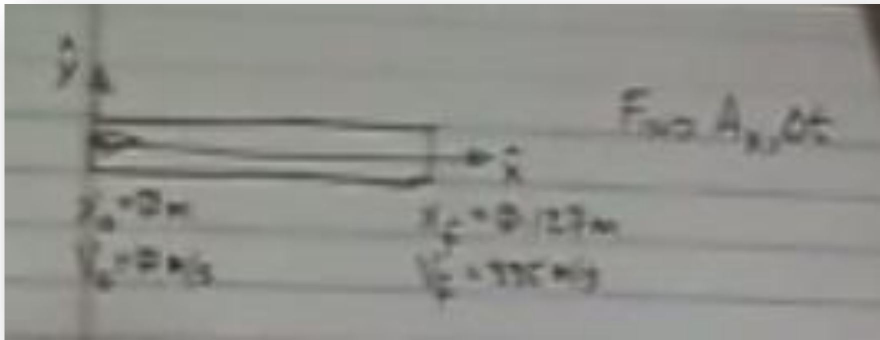


Figure 5.15. Excerpt from the physics instructor's lecture illustrating one-dimensional motion without an arrow.

Notice the sketch has been labeled with values, and each of the values is labeled with a variable with subscripts. The x_0 and x_f label displacement values, and the v_0 and v_f label velocity values. Notice displacement and velocity are not being depicted as paired values but just singular values. These values are then used within the kinematic equations to solve for the missing information, and the kinematic equations work with singular values and produce singular values. Though the instructor is discussing displacement, velocity, and acceleration, singular values are being written and scalar arithmetic and algebra are being used. The notation for displacement and velocity lack the *x* and *y* in the subscripts (using x_0, x_f, v_0 and v_f) to match the textbook's instruction stating they are components. Without the *x* and *y* subscripts, they can be assumed to be the magnitude of a vector; however, the physics instructor's verbal language depicts them as the "initial velocity *in the X direction*" (P4-282, emphasis by researcher) or "where the object ends up *in the X direction*" (P4-272, emphasis by researcher). This type of verbal reference makes the notation read as components rather than the magnitude of a vector.

For example, to find the acceleration of the bullet in the barrel, the kinematic equation $v_f^2 = v_0^2 + 2A\Delta x$, which he rewrites as $v_f^2 = 2Ax_f$ because $v_0^2 = 0$ and $\Delta x = \left(x_f - x_0\right) = \left(x_f - 0\right) = x_f$ as a result of the given information. He then algebraically rearranges $v_f^2 = 2Ax_f$

to be $A_x = \frac{v_f^2}{2x_f}$, which he leaves as his answer. During his lectures, the physics instructor leaves the last step of number crunching within the calculator for the students to do later. In this case, since $v_f = 335\ m/s$ and $x_f = .127\ m$, $A_x \approx 441830.7\ m/s^2$. Notice the initial and final velocities, the initial and final displacements, and the acceleration are all discussed as singular values and the Kinematic equation uses them as scalar quantities and produces a scalar quantity.

His lecture that day concluded with a slide contrasting one- and two-dimensional motion. The goal of the slide is to show the numerical definition of vector quantities in one-dimension is the same as in two-dimensions even though the diagrams may look different. In this slide along with the opening slide of the next class which introduces projectile motion, vector quantities *are* again denoted with vector notation and depicted as arrows, but the slide only introduces the topics; it doesn't display the actual "doing" while using the topics.

As the instructor works to solve an example projectile-motion problem the following day (example problems illustrate the actual "doing"), he again is discussing velocity and acceleration with none of the variables having vector notation, the variables representing single values and not ordered pairs, and the algebraic work not using vector operations but using scalar operations resulting from the use of Kinematic equations. The drawing has vectors drawn as arrows, but each arrow is splintered into the two separate component vectors and then discussed, labeled, notated, and operated upon as scalar quantities.

The textbook echoes the instructor's practices. Though Chapter 2 works discusses displacement, velocity, and acceleration, only one of the thirty-three diagrams has arrows notated as vectors. All of the other diagrams either do not have arrows or have arrows notated with component notation. Likewise, the instructional narrative and the example problems do not use vector notation as they discuss the vector quantities; in all cases, component notation is used.

Also similar to the instructor, as the textbook solves example problems, the solutions show that the authors do not substitute given values into multiple-variable equations and then solve for a one-variable equation. Instead, their solutions rearrange multiple-variable equations, such as a Kinematic equation or even the quadratic equation, for the desired variable while maintaining all the variables as variables, and only after the equation is solved for the desired variable are the given values substituted into the rearranged equation. This practice probably occurs because each variable has units of measure associated with it, and this type of solution strategy helps justify the solution's units.

Overall, the mathematics needed to solve the problems is knowing a vector can be broken into two perpendicular components, the ability to use the right-triangle functions of sine, cosine, and tangent to find the sides or angles within the right triangle formed between a vector and its components, the ability to solve literal equations for a particular variable, and the ability to solve a system of two equations with two unknown variables. Being able to solve literal equations and solving a system of two equations with two unknowns are processes taught in algebra classes. The other two processes (breaking a vector into two perpendicular components, and using the right-triangle functions of sine, cosine, and tangent to find either the sides or angles within a right triangle formed between a vector and its components) were taught by the instructors. The physics instructor also mentioned in the interviews that students needed to know these processes.

Because vector geometric and algebraic vector operations are not used during this unit of physics, three unexpected practices develop. First, component vectors and one-dimensional vectors are treated as equivalent. Because component vectors and one-dimensional vectors are visually equivalent when they are graphed, the physics instructor and physics textbook treat the components as one-dimensional vectors and vice versa. Second, component vectors are not labeled using component vector notation; they are labeled with component notation or magnitude notation. These two practices combine resulting in one-dimensional vectors being represented and treated as scalar quantities. In addition, when two-dimensional vectors are broken into their components, their components are treated as one-dimensional vectors, labeled with component notation, and operated as if they are scalar quantities. Although the physics instructor mentions in the interview the importance of students being able to break a vector into its components, he does not mention the need for students to view components as one-dimensional vectors nor the slippage of the notation.

The fluency of the physics instructor in seeing components and one-dimensional vectors as equivalent and in seeing magnitude and component notation as appropriate for one-dimensional vectors may be the reason notation was informally used by the physics instructor, which is the third unexpected practice. While the textbook is careful always to label component notation with necessary *x* and *y* subscripts that are part of the notation, the physics instructor does not use the component subscripts as he solves a one-dimensional motion problem. His verbal language references them as, for example, "in the x-direction" allows the listener who catches the phrasing to share the understanding that they are x-components.

In conclusion, a surprising attribute of the mathematics used by the physics instructor and physics textbook when explaining physics problems is the lack of using any vector operations. Over the course of the 3 days of physics content, the physics instructor treats vectors as arrows on two slides, but all operations with vectors during the three days do not treat vectors as paired information and do not use vector operations. The first two slides of the unit treat displacement, velocity, and acceleration as scalar quantities and pair them with time to make velocity-verses-time, acceleration-verses-time, and displacement-verses-time graphs. The last two example problems use kinematic equations to determine unknown information from given information, and in order to use kinematic equations, vectors are decomposed into their separate components, labeled as components, and mathematically operated with scalar operations. Surprisingly, all calculations in the unit are scalar calculations—not the vector operations described in the first two days of class. As a result, vector operations geometrically and algebraically are not used during this unit of physics. The lack of using these operations and decomposing all vectors into their components to use them as scalar values was not foreshadowed in the interviews or other moments of instruction; significantly, they seem to go unnoticed.

Chapter Summary

There seems to be a broad consensus in the data introduced in Chapter IV that vectors are geometric objects across five of the six definition-like statements: the trigonometry instructor's primary definition, the physics instructor's three definition-like statements, and the physics textbook statements. The broad consensus of these five definitions seems to be in conflict with the trigonometry textbook, which states that a vector is a quantity, an amount of something. This definition of *vector* from the trigonometry textbook is equivalent to the physics textbook's definition and the physics instructor's definition of a *vector quantity*. The physics textbook states, "a **vector quantity** has both a magnitude and a direction in space" (emphasis in the original). The similarity between these definitions seems to say that quantities that have magnitude and direction have different labels (signifiers) depending on the particular community of practice. Specifically, the trigonometry textbook seems to label quantities with magnitude and direction as being *vectors*; whereas, the physics instructor and textbook seems to label quantities with magnitude and direction as *vector quantities*. In practice, the physics instructor states that vector quantities *are* vectors. Also, in practice, the trigonometry textbook always references arrows as vectors, and never references vector quantities as vectors.

Over the course of the analysis, there are several practices that are the same for the two disciplines. Concerning the mathematical activities using vectors, other than one element which is commonly introduced in calculus, the content of the physics instructor's lectures focusing on the mathematics of vectors was equivalent to the trigonometry instructor's selection of content within her lectures.

Second, concerning notation, both instructors and both textbooks use notation for two purposes: as a means of naming vectors and as a means of describing vectors. While various manners of notation are introduced to name vectors, the main style of notating vectors is very similar across both physics and trigonometry instructors and both textbooks. Both instructors use a single letter with a half arrow above the letter as the notation for naming vectors, for example $\vec{v}$, and both textbooks state handwritten notation is different than printed notation within the textbook. Also, both courses use specific letters to always serve to represent specific vector quantities. Both courses' main style of notating vectors to algebraically describe them is by writing vectors as the sum of their components.

Third, concerning related vocabulary, all four sources of instruction introduced the vocabulary words *vector*, *magnitude*, *scalar*, and provided similar definitions. None of the sources defined *direction*, and while all four sources also introduced *components* as a vocabulary word, only the physics textbook explicitly described what they are. Despite the physics textbook's explicit description, there seems to be slippage in the physics course between the manner in which components are used and the manner in which they are formally defined, and components seem to be considered differently across the two courses, which is summarized more below.

Fourth, concerning the diagrams, both instructors and both textbooks seemed to show that vectors always show direction, always have fixed lengths, and always follow the direction of a fixed plane. Also, analyzing the diagrams revealed that accompanying vectors with either a coordinate axis or axes was the most common way for vectors to be illustrated.

Over the course of the analysis, there were also several practices that are different for the two disciplines. For example, concerning notation, the physics instructor and textbook have a specific notation for components, but the trigonometry instructor and textbook do not. The physics textbook seems to express that a component's notation is its related vector's notation without the arrow above the letter, without the bold type, and with the addition of a letter, for example x or y, in the subscript to express what type of component it is.

A second difference between the disciplines was the trigonometry instructor and book seem to treat components as horizontal and vertical lengths stretching between the initial and terminal point with a nod to the axes to express the distances as either positive or negative, but the physics instructor and textbook treat components as vectors. Components are often drawn as arrows, and the instructor references them as vectors and uses them to represent vector quantities. The physics textbook distinguishes between components and component vectors, but its practice is the same as the physics instructor in which component vector notation is not used, and the arrows are always labeled and referenced as just components and not component vectors. In addition, the physics instructor and textbook treat one-dimensional vectors as component vectors and, therefore, label them with component notation.

A third differences between the disciplines was the trigonometry instructor's and textbook's use of two bars on either side of the vector's name to notate magnitude, for example $\|\boldsymbol{v}\|$, while the physics instructor's and textbook's primary manner of notating magnitude is to adjust the notation of the vector's name by removing the arrow above the letter and the bold print. This style of notation for magnitude is equivalent to the notation for components except that components have the additional quality of adding a letter to the subscript to express which axes the component references. Sometimes the addition of the letter to the subscript was not included by the instructor because the context provided contextual clues that components were being referenced. This style of notation contrasts with the trigonometry instructor's practice of forgetting the arrow above the letter and wanting her students to assume that it still means a vector is being notated.

Concerning diagrams, the trigonometry instructor and textbook generally emphasize both ends with large dots with some cases of putting a dot on just one of the ends or at neither end. In contrast, the physics instructor and physics textbook generally emphasize the initial end with a large dot or neither end with dots. This becomes significant because the dots often serve to express the location of the endpoints of the vector. When vector arrows are illustrated by the trigonometry instructor and textbook without referencing vector quantities, they seem to have endpoints, but when they use a vector arrow to represent a vector quantity, the vectors do not always to have endpoints. In analyzing the physics instructor and textbook, displacement and position vectors are defined by an initial point and a terminal point, but velocity and acceleration do not have both endpoints defined by a location in the coordinate system. Not all vector quantities seem to be capable to be embodied in arrows that *"go/point/extend from one point/place to another/other in the coordinate system/plane,"* as the physics instructor and textbook seemed to claim.

While the main style of notating vectors to describe them algebraically is similar across the trigonometry instructor's, trigonometry textbook's, and physics instructor's practices, slight differences do occur across the practices because of how the two disciplines seem to notate unit vectors. The trigonometry instructor and textbook notate unit vectors using vector notation (arrows above the single letters), but the physics instructor notates unit vectors using hats above the letters labeling the axes. As a result, the trigonometry instructor writes a vector in the algebraic form of $\vec{v} = a\vec{\imath} + b\vec{\jmath}$, the textbook writes it in the form $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$, and the physics instructor writes it in the form $\vec{v} = A\hat{x} + B\hat{y}$. The main difference in the manner of writing them is both trigonometry instructions use *i*'s and *j*'s, but the physics instructor uses $\hat{x}$'s and $\hat{y}$'s.

While many of the related vocabulary words were similar, a difference in vocabulary was the trigonometry instructor's and textbook's use the words *initial point* and *terminal point* to reference the endpoints of the vector; whereas, the physics instructor and textbook use the words *tail* and *tip*. This change in vocabulary seems inconsequential until analyzing the velocity and acceleration vectors that do not always have initial and terminal points, but they always have tails and tips.

A final difference between the disciplines is the manner in which vectors are used. The trigonometry instructor always referenced and wrote vectors as paired information. For example, she would describe a vector in the form $\vec{v} = a\vec{\imath} + b\vec{\jmath}$, where the components were paired to describe the vector, but as the physics instructor and textbook worked motion problems, they would write vectors as separate variables in separate columns to be used independent of each other. For example, projectile motion diagrams and algebraic solutions would not describe the acceleration together as paired information using the notation taught, such as $\vec{\boldsymbol{a}} = 0\hat{x} - g\hat{y}$ or $\vec{\boldsymbol{a}} = -g\hat{y}$, but would describe the acceleration as two independent variables as $a_x = 0$ and $a_y = -g$. The majority of all the problems by the physics textbook and instructor referenced acceleration just as $a_y = -g$ with no reference to the x-component at all.

CHAPTER VI

DISCUSSION, CONCLUSIONS, & RECOMMENDATIONS

This project seeks to investigate a viable reason why students struggle applying their vector knowledge from a trigonometry course to a physics course based on a different set of assumptions than traditionally used in math and science research. As stated in Chapter III, the majority of the research in mathematics, science, and engineering is conducted under a positivist or post-positivist paradigm in which the purpose of research is to contribute to a collection of hypothesized-and-tested, universally-verifiable statements to explain a reality. In contrast, this particular educational study is positioned using constructionism as an epistemology and interpretivism as a theoretical perspective in which the purpose of research is to understand and reconstruct a description of a phenomena from a novel perspective for the overall objective of accumulating a more informed and sophisticated description of the possible multiple realities under study (Crotty, 2003). Briefly stated, this positioning results in the aim of this inquiry to be toward understanding the mathematical practices concerning vectors in the two courses and reconstructing descriptions of these practices for the reader. Chapters IV and V provided descriptions of these practices aligned with Sfard's (2003) statement, "The act of naming and symbolizing is, in a sense, the act of inception, and using the words and symbols is the activity of constructing meaning" (p. 374). Chapter IV offered a description of "the act of inception," by describing findings answering the following research questions:

1. In the act of defining vectors, what are the similarities and differences between trigonometry and physics instruction?
2. In the act of symbolizing vectors, what are the similarities and differences between trigonometry and physics instruction?

Chapter V offered a description of "using the words and symbols" by a third research question:

3. In the activities of using vectors, what are the similarities and differences between trigonometry and physics instruction?

Chapter VI reflects on the analysis in Chapters IV and V and has three parts. The first part describes the results of the analysis process concerning the use of Saussure's idea of signifiers and signifieds. While analyzing the data, Saussure's idea of signifier/signified was borrowed to sensitize the researcher to viewing the meaning of symbols and drawings as separate from the visual representations of the symbols and drawings. The first part of Chapter VI describes how the analytic process of using Saussure's binary division of the sign was insufficient for describing the multiple representations mathematical objects often have.

The second part of the chapter provides an overview of the comparisons and contrasts in practice across the two courses. These comparisons and contrasts focus on observable practices. The overview is intended to aid instructors and curriculum designers in their discussions for multiple-course curriculum alignment. In addition, the overview provides a description of the practices that when taken together provide the *meaning* the courses give to vectors, which is addressed in the third part of Chapter VI.

In the third part, a description is provided of the instruction that students use as "the activity of constructing meaning" that reflects upon the *meaning* the courses give to vectors. The meaning of vectors for the two courses cannot be described by strictly observing the practices; the meaning of vectors requires the researcher to weave and develop the meanings each course constructs, which is possible with the project's positioning having constructionism as an epistemology and interpretivism as a theoretical perspective (Crotty, 2003). With such positioning for the project, the researcher assumed multiple meanings were possible for vectors and for vectors' uses, and the researcher's goal during the inquiry was to re-construct the instructors' and textbook authors' meanings for vectors across its multiple signifiers and resulting from their use in the course activities. In each of the three parts, the researcher weaves discussion about the findings and recommendations for future research.

Reflections on Using Signifier/Signified

The analysis for the project began by searching the data corpus for the first use of the word *vector* and for its explicitly stated definition. The decision to examine the definition for *vector* as the initial step of analysis resulted from borrowing the idea from Saussure that a sign has two parts: the word and the sound concept, which he calls the *signifier* and *signified*, respectively. The word *vector* was the preselected signifier for the focus of the study; therefore, the analysis began by looking in the instruction for its related, explicitly-stated signified. Answering the first research question provided a description of both instructors' and both textbooks' acts of defining the concept of vectors. These descriptions serve as the explicitly-stated signified by the separate instructors and textbooks for the signifier *vector*.

In Chapter IV, a description of the explicitly-stated signified and the signifier *vectors* is provided; however, the object that is being named and classified does not exist in actual reality like a tree, a jacket, or an airplane might. Vectors are virtual objects; therefore, the second research question described the inception of the object visually. For example, the trigonometry instructor introduced the signifier *vector* and linked it with a diagram of an arrow, and the arrow serves to signify the concept of being a "directed line segment." The arrow is not a "directed line segment" in and of itself. An arrow can also represent a ray, which is a mathematical object that infinitely extends in one direction or it can provide directions without any mathematical meanings. Thus, the arrow is a signifier of the concept/definition, too.

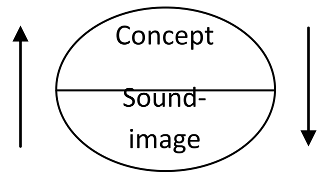


Figure 6.1. Saussure's diagram (1959) of the sign, signifier, and signified.

Because the data indicated that the arrow is also a signifier for the concept, the researcher extended Saussure's diagram to include a second signifier. Figure 6.1 and Figure 6.2 provide two diagrams. The first is Saussure's diagram depicting the relationship between the signifier and signified as the two parts of the one sign. The second provides a diagram that represents the varied signifiers after Chapter IV's analysis. The second diagram depicts the link between the two signifiers identified in the data with the one signified.

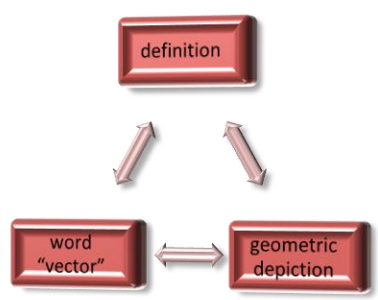


Figure 6.2. The signifier/signified relationship from Chapter IV

As analysis continued, the arrow was not the only additional signifier for a vector. As described in Chapter V, the word *vector* was also a signifier for algebraic notation, for vector quantities, and irregularly for components. Additional signifiers mean there were additional moments of instruction in "the act of symbolizing" a vector for inception. I questioned whether these additional moments of instruction in "the act of symbolizing" a vector for inception should be included while answering research question 2, but these later acts of inception did not receive the emphasis that embodying a vector as an arrow did. Although they were moments of inception, they were woven into the instructional narrative as ways of *using* vectors, which is why I selected to describe them in Chapter V while answering research question three rather than while answering research question two. As a result, Chapter V provided an extension of Chapter IV's analysis by describing other signifiers and their use during the instruction.

Two other signifiers provided by the instructors and textbooks is the algebraic notation for naming and describing vectors. The use of algebraic notation to signify vectors provides two more forms of signifying them. To name vectors, the algebraic notation might use a single letter, for example $\vec{v}$. To describe a vector algebraically, the algebraic notation might use the sum of the components, for example a**i** + b**j** or $a\hat{x} + b\hat{y}$. This addition of two more signifiers complicates the diagram representing signifiers/signification in the case-study data further (see Figure 6.3).

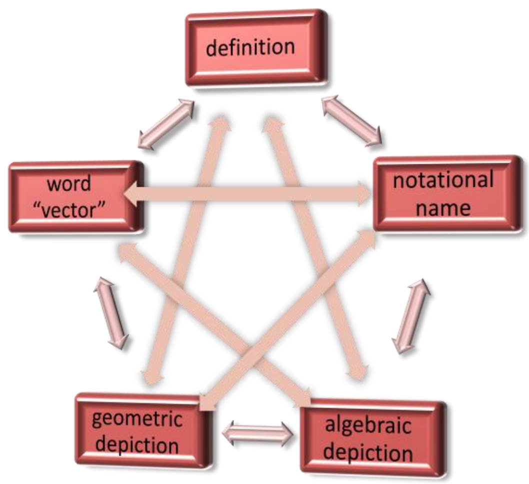


Figure 6.3. The context-free signifier/signified relationship from Chapter V

During the analysis in Chapter V, I operated as if the word *vector*, the geometric depiction, the algebraic description, and the notational name are four signifiers for the one signified definition; however, two problems develop with such clear distinctions. First, the signifiers sometimes signify one another, which means the signifiers sometimes switch to being the signified. A second problem develops from my belief that the definition is a signifier for the concept, too. A definition can be understood as instructors' construction to express the concept—but it's not the concept itself. This distinction is often well recognized in literacy theory. Barton

(1994) writes, "There is not in fact a meaning *in the text*, there are only the meanings which a reader *takes from the text*" (p. 68, emphasis in original). In considering the definition a signifier also, Figure 6.3 ceases to be an illustration relating signifiers and signifieds as separate and related objects and instead seems to relate the interrelationship among the signifiers to signify one another. Figure 6.3, as a result, no longer depicts the overarching signified that the 5 signifiers represent.

These two problems of the signifiers being both signifiers and signifieds and of the definition being itself a signifier resulted in my desire to find a way to discuss these ideas with clearer vocabulary. I recommend future research investigating the use of Tall and Vinner's (1981) ideas of *concept image* and *concept definition*. Currently the terms seem to provide access to the needed clarification and resolution.

Because I use the word *definition* for the words provided by the instructor to define a concept and because I consider a *definition* a signifier, I need a word to describe the actual concept's "definition," and I feel Tall and Vinner's term *concept definition* matches the idea I'm attempting to express. My only hesitation for the use of this wording is Tall and Vinner's seeming belief that there is only *one* concept definition. I argue that in the field of mathematics we generally do operate under the belief there is only one definition agreed upon by the mathematics community as a whole, but in actuality, I argue that multiple definitions may be used by pockets of differing fields of mathematics or in related fields using the mathematics.

Saussure mapped *concept* and *sound-image* to *signified* and *signifier* because the latter pair of words relate to each other and "have the advantage of indicating the opposition that separates them from each other and from the whole of which they are parts" (1959, p. 67). For those same reasons, I reached for *concept image* as a way of indicating the separation and linkage of the multiple signifiers with their *concept definition.* Tall and Vinner introduce the term *concept image* as a way of discussing "any mental picture, be it pictorial, symbolic or otherwise" (p. 151); the term encompass all possible signifiers, properties, and processes associated with the concept. My hesitation is its conceptualization as describing only mental pictures that are part of the cognitive structure. I wonder if the word could be loosened to include mathematics outside the mind because some theory recognizes that thinking is only possible with the use of language and tools outside the mind (e.g. Wertsch, 1991). Further research and theorizing is necessary to explore the implications of revisiting these terms and broadening their use.

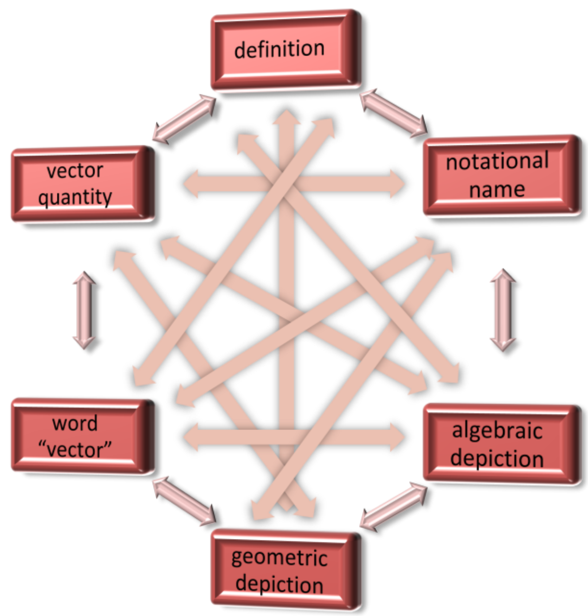


Figure 6.4. The contextualized signifier/signified relationship of vectors when representing vector quantities from Chapter V

In addition to the two notational signifiers, when vectors signify vector quantities, the geometric depiction and the algebraic descriptions signify the vector quantity, too. This means the vector quantity is signified by the arrows and the notation and results in vector quantities being an additional signified for these signifiers other than just the definitions. However, sometimes vector quantities also are signifiers for the geometric depictions and notations in situations when the arrows or notations are referenced as "velocity" or "acceleration." Since vector quantities serve as both signifiers and signifieds, a further complication of Figure 6.3 is necessary to describe situations in which vectors are not context-free and are actively representing vector quantities (see Figure 6.4).

In conclusion, Figures 6.3 and 6.4 provide possible depictions of a manner of conceptualizing the numerous connections between the signifiers/signifieds resulting from the analysis of the data collected in this project. Saussure's description stratified language into two parts, the signifier and the signified, and this idea of separating the signifier from the signified sensitized my view of the data to value the idea that vectors have multiple signifieds and in various contexts also switch to being signifieds. Thus, the potential range of signification that emerged from the data discussed here cannot be exhaustive. The web of connections between the signifiers is developed each time a signifier is used in the instruction, and with each use, patterns form that shape the meaning of vectors—a meaning that may not be incorporated into the formally stated definition. Just as research showed that activities using the equal sign shaped students' beliefs that the symbol signified "do something" instead of its formal meaning of "equivalence," the use of vectors shapes a meaning not provided in the explicit definition. The third part of Chapter VI describes the meaning the course instructors and textbooks give to vectors.

The next section provides an overview of the comparisons and contrasts in practice between the two courses. These comparisons and contrasts focus on observable practices. The overview is intended to aid instructors and curriculum designers in their discussions for multiple-course curriculum alignment. In addition, the overview provides a description of the practices that when taken together provide the *meaning* the courses give to vectors, which is addressed in the third part of Chapter VI.

Overview of Observable Practices

Without the analysis focusing on *meaning*, the analysis revealed key consistencies across the major observable practices of both courses concerning vectors. Other than one element that had an underlying, calculus-based content (the manner of using a given velocity-verses-time graph to create to related acceleration-verses-time and displacement-verses-time graphs), the procedural content of the physics instructor's lectures focusing on the mathematics of vectors was equivalent to the trigonometry instructor's selection of content within her lectures. The courses did not introduce inconsistent procedures. The topics both instructors introduced were how to geometrically and algebraically add and subtract vectors and multiply them by a scalar, how to find the magnitude of a vector, and how to incorporate a vector and into a right triangle with its components and use the trigonometric functions of sine, cosine, and tangent to determine missing qualities of a vector from given qualities. In addition, both courses generally depicted vectors as arrows, as showing direction, as having a fixed length, as following the fixed plane, and as commonly accompanied by coordinate axes. Both courses generally used the main notation for naming vectors as $\vec{v}$, the main form of algebraically describing vectors as the sum of their components, and represented specific vector quantities with the same specific letters.

Without the analysis focusing on *meaning*, many of the differences between the two courses seem trivial. For example, the physics instructor and textbook have a specific notation for components, but the trigonometry instructor and textbook do not. The trigonometry uses two bars, for example $\|\boldsymbol{v}\|$, to notate magnitude, but the physics instructor and textbook primary use the letter from the vector's name without the arrow above the letter. The trigonometry instructor and textbook use large dots at the ends of the vectors continuously, but the physics instructor and textbook, on the other hand, generally emphasize the initial end with a large dot or neither end. The trigonometry instructor and textbook notate unit vectors with arrows above the single letters, but the physics instructor notates unit vectors with hats above the letters. Each source of instruction writes the sum of the components slightly differently: the trigonometry instructor as $\vec{v} = a\vec{\imath} + b\vec{\jmath}$, the trigonometry textbook as $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$, the physics instructor as $\vec{v} = A\hat{x} + B\hat{y}$, and the physics textbook as $\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}_x + \vec{\boldsymbol{A}}_y$. The trigonometry instruction used the words *initial* *point* and *terminal point* to reference the endpoints of the vector; whereas, the physics instruction used the words *tail* and *tip*.

Without still including *meaning* in the analysis, one difference between the two courses did not seem trivial: the manner in which vectors were used as scalars and not as paired information. The trigonometry instructor and trigonometry textbook always referenced and wrote vectors as paired information. For example, the trigonometry instructor would describe a vector in the form $\vec{v} = a\vec{\imath} + b\vec{\jmath}$ or $\vec{v} = < a, b >$, where the components were paired to describe the vector. The physics instructor and physics textbook emphasized vectors as paired information, too, but as the physics instructor and textbook worked motion problems, they would write vectors as separate variables in separate columns to be used independent of each other. For example, the physics instruction described the acceleration in motion problems as two independent variables as $a_x = 0$ and $a_y = -g$. Each variable is labeled with component notation. Based on explicit instruction by the trigonometry instructor and general instruction by the physics instructor while he taught mathematics, the listener would expect it would be written as (0, -g), or $\vec{\boldsymbol{a}} = 0\hat{x} - g\hat{y}$ or $\vec{\boldsymbol{a}} = -g\hat{y}$. Yet, the majority of all the problems by the physics textbook and instructor referenced acceleration just as $a_y = -g$ without using vector notation and without referencing the partnered x-component at all. A seasoned member of the physics community may “read” the components as still paired, but the pairing is not visually obvious in the board work. Two-dimensional vectors visually look like two different, independent scalar values, and one-dimensional vector quantities are scalar numbers and labeled with component notation. The manner in which vectors were used as scalars and not as paired information may have significance for student success in learning during the Kinematic unit.

When analysis considers meaning, these differences and similarities become significant to the overall meaning each course gives to vectors in and through their instruction. The following section provides a description of the meaning each course gives to vectors resulting from their use. If these practices are common in instruction in varied contexts nationally, this may be one piece of the puzzle for understanding why students struggle using their trigonometry knowledge in physics.

The Activity of Constructing Meaning

Despite the uniformity across all four sources of instruction in their definitions, their initial diagrammed vectors, and their most commonly used notation, the differences in practices between physics and trigonometry instructor, when taken together, seem to indicate vectors are different objects in different contexts. Consider Figure 6.5 in which two images are incorporated as one. The eye is only able to see one at a time, but both images are still there. I'd like to use this image as a metaphor for how vectors are seen as two different images when comparing the practices of the physics and trigonometry instruction in this study. The following paragraphs serve to summarize findings across the three research questions in a way to consider what a vector *is*, based on the summation of all the practices taken together.

Figure 6.5. Old woman or young lady? At any one moment, the illustration allows the viewer to see one of two images: a young lady looking away into the page or to see the silhouette of an old woman as she looks toward the viewer's left.

As I begin this task, I recognized the impossibility of achieving the purpose for which I strive. While analysis can separate and sort instruction into practices, the summation of those practices will not re-create the instruction. Sfard (2000) eloquently and powerfully explains,

> As Lotman observed while referring to the exclusive use of linguistic analysis, "If we put together lots of veal cutlets, we do not obtain a calf. But if we cut up a calf, we obtain

> lots of veal cutlets" (quoted in Eco, 1990). To say it differently (and in a less "bloody" way!), understanding the parts of a whole, which is the kind of understanding we gain while analyzing *use*, does not translate automatically into an understanding of the whole. The collection of pieces is not enough to reconstruct the "living creature," this unique experience of "seeing" a meaning of a symbol. (p. 47, emphasis added by researcher)

Yet, the purpose of this study was to begin to characterize the various practices of the two disciplines with respect to vectors and describe the differences so math and physics instructors, curriculum designers, and policy makers begin to recognize any instructional differences between a trigonometry course and a physics course in order for these vested interest groups could intervene. If the meaning is in the *use*, then I offer a description of what a vector *is* for these two courses, based on the summation of all the practices described in Chapters IV and V when taken together.

In the trigonometry course, the instructor stated "vectors are directed line segments" and the textbook referenced the geometric vectors as vectors throughout its instruction. None of the practices by the trigonometry instructor and textbook were found to contradict this definition. In fact, the practices echoed it, and this is significant because the *meaning* vectors have when being used in the course activities aligns with the description of vectors provided in the explicit definition. I offer three examples of how the practices echoed the definition. First, the trigonometry instructor and textbook had 72% and 60%, respectively, of their vectors with their initial and terminal points accentuated with large dots. This almost rigid pattern of always depicting the location of the initial point and terminal point conveys their similarity to line segments, which, as the instructor reminds, "we know a line segment has end points, and then it's a fixed length" (26-27).

A second practice that seemed to echo the trigonometry instructor's statement that "vectors are directed line segments" was the language of saying vectors *have* initial and terminal *points* also contribute to vectors' association with line segments. Vectors stretch between two points like line segments do, and vectors *have* the end points, like line segments do.

A third practice was the manner in which the instructor illustrated vectors to relate them to line segments. The initial geometric vector in the trigonometry textbook, depicted in Figure 4.7, followed the same pattern as the trigonometry instructor: it situated the initial vector without coordinate axes, with two large dots at either end of the arrow, and was drawn after a line segment was already drawn. The trigonometry instructor drew all but 2 vectors in her discussions using this same manner throughout her instruction.

In contrast, although the physics instructor stated vectors *"go/point/extend from one point/place to another/other in the coordinate system/plane,"* many of the practices by the physics instructor and textbook contradicted the statement. For example, the physics instructor and textbook had 100% and 97%, respectively, of their vector arrows with neither end emphasized with large dots or had the tail of the vector emphasized with a large dot. This lack of dots is directly related to the practice of using the arrows as avatars for vector quantities, and velocity and acceleration cannot be said to "go from one point to another in the coordinate plane." Because vector quantities are defined to have magnitude and direction, the arrows representing them carry that meaning.

I claim the meaning given to vectors by the trigonometry instruction during this study is *directed line segments.* On the other hand, I claim the meaning given to vectors by the physics instruction during this study is *objects with magnitude and direction*. Return to Figure 6.5 and its metaphor for meanings. Just as the optical illusion conveys a young lady or an old woman, so vectors convey *a directed line segment* during the trigonometry instruction and *an object with magnitude and direction* during physics instruction. The two descriptions/definitions of vectors are the same (much like the young lady and old woman are the same illustration), but the focus is different. The following paragraphs describe a summation of the physics instruction.

First, the trigonometry instructor stated vectors are "defined by an initial point and a terminal point" (T1-427), but velocity and acceleration do not have initial and terminal points—they have tips and tails. While this change in vocabulary may have seemed inconsequential in the analysis earlier, the actual definition of what an instantaneous velocity and acceleration vector depicts does not allow for the difference in vocabulary to be inconsequential. When vectors represent velocity and acceleration, they are not always able to *have* initial and terminal points. They only *have* tips and a tails.

Second, all the vectors have a magnitude. Whether known or unknown at the time of the sketch, the magnitude of vectors is discussed and referenced in nearly every problem. As a result, vectors always have a fixed length.

Third, the physics instructor's manner of drawing vectors differs from the trigonometry instructor's manner of sketching a directed line segment first and then adding an arrow. The physics instructor, on the other hand, had vectors appear all at once as fixed-length arrows during his PowerPoint or drew them as magnitudes radiating from a location.

Fourth, the physics instructor and textbook always seem to depict magnitude and direction with vectors as they work with Kinematic equations. In mathematics the transitive property states, "If a = b and b = c, then a = c." Said again with different variables, "If v = p and a = v, then a = p." This statement could be written in words as such:

vectors are written in terms of ordered **p**airs …**v** = **p**

acceleration is a **v**ector…**a** = **v**

Therefore, **a**cceleration is written in terms of ordered **p**airs…**a** = **p.**

Acceleration, however, is not usually written in terms of ordered pairs in the chapters using Kinematic equations. Most often, acceleration is written a_y = -9.8 m/s^2; acceleration is written as a scalar quantity. Vector quantities are written as scalar quantities whenever they are one dimensional. When vectors are written this way, they still depict magnitude and direction. One-dimensional vectors are generally labeled with component notation, and because component notation references part of its direction, the scalar quantity can provide both magnitude and direction. When using Kinematic equations, all two-dimensional vectors are dismembered into one-dimensional component vectors, and the problems continue speaking only of the components.

Finally, the physics instructor and textbook reference vector quantities as vectors. Vector quantities have magnitude and direction, and their avatars (notational symbols and arrows) represent the magnitude and direction. Because notation and arrows serve as avatars for vector quantities, they change meaning from when they are used in context-free situations. Consider Figure 6.6. In context-free situations, the word *vector* is the signifier for the mathematical arrow-like object, and the arrow-like object is the signifier for the definition "directed line segment." The notation $\vec{v}$ and a**i** + b**j** serve as notational signifiers for the mathematical arrow-like objects. In contrast, when vectors represent vector quantities, the word *vector* is a signifier for vector quantities and the mathematical arrow-like object. The arrows and various notations are signifiers for vector quantities, which are defined to have a magnitude and direction. As a result, the arrows and various notations represent magnitude and direction.

Context-free Vectors

Signifier	Signified
vector	$\longrightarrow$
$\longrightarrow$	"directed line segment"
$\vec{v}$	$\longrightarrow$
a**i** + b**j**	$\longrightarrow$ or $\vec{v}$

Vectors Representing Vector Quantities

Signifier	Signified
vector	vector quantity or $\longrightarrow$
$\longrightarrow$	vector quantity
$\vec{v}$	vector quantity
a**i** + b**j**	vector quantity
v_x (component)	vector quantity

Figure 6.6. Comparing vectors' signifiers and signifieds in contextual or context-free situations

I would like to pause at this point and clarify for my reader that I am fully aware that a multitudinous-page dissertation was not necessary to make the statement that vectors represent vector quantities. Rather, the point that I am trying to make is the very nature of what vectors *are* and what the signifiers represent are completely distinct. This matters because the distinction may not be obvious to students. Just as the eye cannot see both the young lady and old woman simultaneously, so the eye must refocus to see vectors in both lights of its two meanings. Would the distinction cause students difficulty in applying their mathematical knowledge in the physics context? Do students think about "directed line segments" and "going from one place to another in the coordinate plane" when they view, for example, an instantaneous velocity vector that does not go "from one place to another in the coordinate plane?" If so, would this idea create a problem for them? The dual meaning of vectors within a physics course may cause difficulty in learning and should be investigated further.

Because the meaning changes whether the vectors are in contextual or context-free situations in physics, there seems to be nothing that needs to be fixed in the curriculum alignment between these two courses. However, acknowledgement of these differences and intentional instruction addressing such slippages in meaning by instructors for students may be helpful in supporting student learning.

Ideas for Further Research

In the trigonometry lecture, vectors are introduced and defined as "directed line segments" and drawn as arrows. Following their inception, the researcher found nothing in the remaining instruction that conflicts with this meaning for vectors. When vectors are introduced by the physics instructor and textbook, they represent the vector quantity of displacement. Displacement vector does "go from one place to another in the coordinate plane" and, as a result, could be considered to be represented by "a directed line segment." As the physics instructor and textbook teach the “mathematical guts” of vectors needed to do physics, the researcher found nothing in the instruction that conflicted with vectors being “directed line segments.”

The moment vectors began to represent vector quantities other than displacement, the graphing practices of the trigonometry instructor, trigonometry textbook, physics instructor, and physic textbook changed not to match earlier explicit statements and the expectations that vectors are “directed line segments.” When vectors first represented a vector quantity during the trigonometry lecture, the trigonometry instructor did not draw any arrows to represent the force of pulling a wagon handle. The handle of the wagon was used to represent the force (see Figure 5.1). When velocity is first graphed by the physics instructor, it is mapped with time to create a velocity-verses-time graph (see Figure 5.14), followed by an acceleration-verses-time and displacement-verses-time graphs. These graphs do not treat the vector quantities as scalars, and they do not graph arrows. The next day, the first example problem using Kinematics does not graph an arrow. The barrel of a gun, a bullet, and the coordinate system are drawn—but not the velocity vector (see Figure 5.15). In the physics textbook, the arrows in all the diagrams, except the last one, are labeled with component notation—not with vector notation. Neither does the instructional narrative label and reference vector quantities with vector notation. The trigonometry textbook’s practice of not using large dots when vectors represent vector quantities is less obtrusive than these other examples of change in practice by the trigonometry instructor and physics textbook and instructor. Further research should explore whether these changes between what the instruction explicitly stated and what the instruction did affected student learning.

Not only do the graphing practices change once vectors begin to represent vector quantities, so do the operations. The trigonometry and physics instruction all illustrate how to do geometric and algebraic vector operations. As the physics instructor explains, “Vectors do not work the same way as scalars do” (P1-102) because the manner of adding and subtracting vectors requires an adjustment to the procedures resulting from vectors being paired information. Yet, once the physics instructor and textbook begin using Kinematic equations, all operations are with scalar quantities. The data sampled for this study is limited to just the use of vectors within Kinematics. Further research should explore whether students stumble in working Kinematic equations because of their expectation about vector operations.

Though the operations appear to be scalar operations when working with Kinematic equations, the operations may be considered one-dimensional vector operations. The trigonometry instructor and textbook have very few examples including one-dimensional vectors. When they do, they are still operated in two-dimensions. The trigonometry textbook and instructor do not provide examples of one-dimensional vector operations. The physics instructor also does not include examples in his mathematical instruction displaying one-dimensional vector operations. Yet, Chapter 2 concerning motion in a line works entirely with one-dimensional vectors, and Chapter 3 concerning motion in two-dimensions always dismembers the vectors into their two component vectors in order to use the Kinematic equations. Research should investigate whether students understand how to do one-dimensional graphical and algebraic vector operations. An interesting question to investigate is whether physics instructors view using Kinematic equations as working with scalar quantities or working with one-dimensional vectors.

The use of component notation to label one-dimensional vectors is another change from what the physics instructor and textbook state and what they actually do. The physics instructor and textbook generally do notate vectors and vector quantities with an arrow above the letter during the instruction analyzed in this study. They generally use v for velocity, a for acceleration, x or y for position, and Δx for displacement. The trigonometry instructor states that sometimes she forgets to put the arrow above the letter, and indeed she does, but she states her students are to assume that it still means vector. In contrast, the physics instructor and textbook do not forget the arrow above the letter because lacking the arrow is a legitimate style of notation; they are referencing the vector's magnitude. Will students have a difficult time understanding what is being referenced by notation? Does students misunderstanding of physics instruction result from a lack of clarity in knowing the meaning of notation or confusion in using the concepts?

Finally, the meaning of direction seems to change over the course of the unit resulting from using or not using coordinate axes. Direction was not defined by all four sources, resulting in the sense that its colloquial definition is sufficient. For the trigonometry instructor and textbook and physics textbook that began their instruction situating the vectors in white space and with large dots at both ends, direction can be described as pointing from the initial point to the terminal point allowing the orientation of up/down/left/right to be sufficient to differentiate differences in direction. As the trigonometry instructor and textbook shift to using grids during their instruction on geometric vector operations, the rise and run between the initial point and terminal point are used to maintain or gauge the direction of the vectors. When the sources' instruction shits to using coordinate axes, direction is measured either as the angle between the vector and the horizontal axis or as the sum of the components.

All vectors from all four sources always depict direction in one of these manners. All vectors seem to depict direction result from having an arrow face a heading that allows the arrow to start from the tail or initial point and end at the tip or terminal point. Because velocity and acceleration do not stretch between points in the plane or on the axis, they cannot be said to go from one place to another in the coordinate plane, but they do have a specific orientation in the plane. The invisible qualities of up/down/left/right provided orientation to vectors even if the coordinate axes were not explicitly illustrated. Will students understand the multiple meanings that direction seems to take? Does their understanding of direction cause difficulties understanding other facets of vector instruction?

Looking Back at the Literature

Many of the questions I have presented in the previous section have the purpose of calling for research that will continue to identify the problem with mathematics in physics. Just because differences occur between the trigonometry and physics practices does not mean that they contribute to the "math problem" between the communities of practice.

Varsavsky (1995) reported "engineering lecturers are not fully aware of the mathematical background of their students" (p. 343). In this study an algebra-based physics course was selected for analysis. Yet, concepts from calculus were incorporated into the lecture twice. Once concerning the difference between average and instantaneous slope, using the concept of limits that is introduced in Calculus I. The other is concerning the transformations between velocity-verses-time, acceleration-verses-time, and displacement-verses-time graphs. The mathematics necessary to complete these transformations is first explored in Calculus I.

Varsavsky (1995) also reported "Engineering lecturers do not use the same techniques as the ones taught by the mathematics department" (p. 343). During most of my analysis, I began to believe that understanding vectors may not be the gate-keeper for success in physics, which is what I assumed when I started this study. As I observed the mathematical work of the physics instructor and textbook, I believe having taught algebra at the high school and collegiate levels for years, that the main gate-keeper for success in physics probably is students' ability to solve literal equations. The physics instructor and textbook will rearrange every multi-variable equation for the desired variable before substituting in the given numerical values. Even to the point where the textbook rearranged the quadratic formula for a variable before substituting in the given numerical values. In my experience as an algebra teacher, students rarely have the strength to accomplish these algebraic manipulations. Another slight distinction between the techniques of doing math is the trigonometry instructor uses the trig functions with the style of $sin\theta = \frac{opposite}{hypotnuse}, cos\theta = \frac{adjacent}{hypotnuse}, tan\theta = \frac{opposite}{adjacent}$, while the physics instructor uses them with the style $opposite = hypotnuse \cdot sin\theta, adjacent = hypotnuse \cdot cos\theta$ and $\theta = tan^{-1}\frac{opposite}{adjacent}$. This algebraic adjustment may be inconsequential to students, but it is worth exploring because my hunch is that it may be problematic for some students.

Clement, Lochhead, and Monk (1981) concluded "students should have definite opportunities to work on a precision translation from one language—verbal, algebraic, or diagrammatic—to another, and vice versa" (p. 726). With the physics instruction using component notation to represent components and one-dimensional vectors, I wonder if translating between vector quantities, diagrams, and notation is problematic.

There are three points about Knight's instrument (1995) that should be highlighted. Borrowing from literacy literature, I would consider these three points differences in mathematical register. First, he asks students if they are familiar with unit vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}},$ and $\widehat{\boldsymbol{k}}$. If students have not taken physics yet, students may not have seen hats as notation for unit vectors before. This study was limited to just one instructor and one textbook; therefore, further study is needed. Second, when the trigonometry instructor introduces the algebraic description for vectors, she first writes $\vec{v} = a\vec{x} + b\vec{y}$, but then she changes it to be *i*'s and *j*'s because the students book has it that way. These notational differences may have been problematic for understanding the problems Knight's questions presented. Third, he asks students to answer questions about three-dimensional vectors. Students may not have had experience with three-dimensional vectors yet. Recently I attended a conference in which a room of mathematics faculty who do not usually teach trigonometry were discussing students' understanding of three-dimensional vectors. I am not convinced that most trigonometry courses teach three-dimensional vector operations. Further study focusing on what aspects of the vector unit are usually included in trigonometry courses should be done.

Nguyen and Meltzer state, "It seems that the bulk of students' basic geometrical understanding of vectors was brought with them to the beginning of their university physics course and was little changed by their experiences in that course, at least during the first semester" (p. 635). If physics continues to use vectors in the manner this first unit has done, this may be the reason for the students' understanding resulting from prior to the course. The Nguyen and Meltzer (2003) instrument is very similar to the instruction provided by the trigonometry instruction in this study, and it is different than the way vectors are used within the Kinematic chapters.

Nguyen and Meltzer also observed during their study that

> many of the students' errors could perhaps be traced to a single general misunderstanding, that is, of the concept that vectors may be moved in space in order to combine them as long as their magnitudes and directions are exactly preserved. We suspect that, to some extent, this misunderstanding results in part from lack of a clear concept of how to determine operationally a vector's direction (through slope, angle, ect.). (p. 635)

The authors support future projects investigating whether introducing vectors on and off grids would help develop the concepts.

This study showed the trigonometry instructor and textbook ranged in their use illustrating vectors in white space, with grids, and later just on coordinate planes. The physics instruction, on the other hand, generally situated vectors on coordinate planes or in white space. The trigonometry instructor did spend time in her first unit teaching the students how to preserve the magnitude and direction of vectors when moving them from one grid to another grid. However, all four sources of instruction did not define direction and did not make explicit statements to teach about direction. The colloquial use of the word was assumed to be sufficient for instruction. Further analysis should investigate students' understanding of direction and explore how to best develop the ideas.

Implications for Teaching

This study has both conceptual and concrete implications for teaching. A broad conceptual implication for practice is simply the value of increasing instructors' awareness of the possibility of fluctuating signifiers/signifieds in math use. This awareness of the possibility of fluctuating signifiers/signifieds recognizes the impossibility of fully freezing the meaning of vectors within and across all communities of practice. Instructors and textbook authors are seasoned members of their community of practices, and they may be mobilizing an array of signifiers/signifieds that seem identical or nuanced in meaning for them but are not similarly clear to students. For example, definitions and explicit statements are simply signifiers for the meaning they wish to convey, and, as such, definitions and statements should be viewed as not holding fixed meanings. It is productive for instructors to view all visual signifiers whether in word, diagram, or notation and their meanings as fluctuating. Instructors who recognize signifiers do not hold their fixed meanings become increasingly aware of the multiple meanings that signifiers may have.

Other implications for teaching may require conversations between trigonometry and physics instructors based on the detailed descriptions within this study. I will make three recommendations based on the current practices that hopefully can be considered easily incorporated into our instruction to benefit students. For the trigonometry instructors, I think we should include operations of vectors in one-dimension. Physics instruction repeatedly uses vectors in one dimension when they study motion in a straight line and when they dismember two-dimensional vectors into one-dimensional vector components. Students will benefit from our including operations of vectors in one-dimension and not just in the plane. For the physics instructors, many of your notational practices are not explicitly introduced to your students. I think that students would benefit from explicit instruction concerning manners of notating components, magnitude, one-dimensional vectors, and two-dimensional vectors. Also for the physics instructors, I recommend making explicit statement that velocity and acceleration vectors do not have both ends fixed within the plane. Students need to think “magnitude and direction” as they “read” these vectors, and limit themselves to “reading” them as “directed line segments.”

Concluding Comments

Using literacy as a metaphor for students participating in communities of practice to develop the ways of speaking, reading, and doing mathematics seemed to elucidate some differences that may have gone unseen by earlier research. A limitation of the study is the brief view of physics instruction on vectors. This study only observed their use during the first two chapters of instruction when motion along a straight line and in a plane was being studied. The actual impact of these differences for students as they learn physics requires further research—either by you or me. I wish you happy exploring.

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APPENDICE 1

INTERVIEW QUESTIONS

Questions about Course Assumptions & Intended Curriculum

- So, what do you think is important for students to learn about vectors?
- What will you teach about vectors during your lecture tomorrow?
- Describe what you assume your students have learned about vectors prior to your course.
- In general, how do you prepare for your classes? What did you do to prepare for your class tomorrow?

APPENDICE 2

DEMOGRAPHIC EMAIL

Email for Demographics

Greetings, Instructor _______.

Thanks for letting me come video your class later this week and ask you a few questions. Before I come, would you be willing to answer a few demographic questions by email?

- Where and when did you get your degrees, and what are they in?
- What was your dissertation title?
- How long have you been teaching at ______?
- What courses have you taught while at ______?
- Have you taught somewhere other than ______? If so, when and what?
- Is teaching part of your permanent career goals?

APPENDICE 3

INFORMED CONSENT DOCUMENT: Trigonometry Instructor

Project Title: A Qualitative Study on the Use of Vectors in the Discourses of Physics and Trigonometry Courses: Possibility for Separate Literacies

Investigators:

Wendy James, M.S.
Professional Education Ph.D. Student
245 Willard Hall, OSU
Stillwater, OK 74078
wmjames@okstate.edu
(405) 744-9505

Juliana Utley, Ph.D.
Professor of Education
233 Willard Hall, OSU
Stillwater, OK 74078
juliana.utley @okstate.edu
(405) 744-8111

Purpose:

The purpose of this research project is analyze features of the course discourse and activities in order to identify how a physicist explains and uses vectors compared to how a mathematician explains and uses vectors.

Procedures:

You will be asked to be video recorded during your class instruction on any days when your instruction includes vector use. The video recording will only focus on you and any media devices you use during instruction without capturing student behavior. The video may capture student voices, but their voices will not be included in the analysis. In addition, you will be asked to participate in 3 types of interviews: pre-instruction, post-instruction, and post-analysis of the entire research project. The pre-post interviews are short (about 15 minutes each) and provide your reflections on the content of your instruction. They will occur prior to initial instruction and then again after the unit of lessons. The third type of interview provides you an opportunity to offer any positive or negative recognition of the themes that may have surfaced during analysis of the videos. The first two types of interviews will be tape recorded, and the third type of interview will be video recorded.

Risks of Participation:

There are no known risks associated with this research project which are greater than those ordinarily encountered in daily life.

Benefits:

The benefits of this research project are to provide awareness about one possible reason for students' lack of success in applying their mathematical knowledge during their physics courses. This knowledge will benefit both physics and mathematical instructors and their coordinating curriculum designers. This research project works as a means of communication between two separate fields of study in order for coordinated efforts to be made from both sides that will facilitate greater student learning and success for both fields.

Confidentiality:

As the researcher, I agree to meet the following conditions:

- I will video tape your class lectures with your permission and transcribe the video for the purpose of analysis. I will give you a copy of the transcript so that you may see that I have captured your words correctly. The video recording will display your face and voice, but when video clips used during research presentations, a pseudonym will be used. The video data itself will not be labeled or stored with your name or position in order to identify you.

- The videos will be recorded digitally and will *not* be made available for public use (U-Tube, Google, ect). The digital recordings will be stored in my computer, which is non-networked and guarded by password access made known only to me.
- The first two interviews will be tape recorded and the third will be video recorded with your permission. A transcription will be made for the purpose of analysis. I will give you a copy of the transcript so that you may see I have captured your words correctly. The tapes and transcriptions will not be labeled or stored with your name or position in order to identify you.
- Transcriptions, tape-recordings, and video recordings will be stored in a file cabinet accessible only by the investigator and will not be labeled or stored with your name or position in order to identify you.
- At the completion of the research project, the video data will be transitioned to storage on a CD. The CD and all paper data will be stored in a locked file cabinet accessible only by the investigator. Clips of the videos may be shown for the use of academic presentations, but all pseudonyms will be strictly enforced. All data will be retained for the length of time it is useful for future research projects designed to further analyze themes which surface during this project. On completion of the usefulness of the data, it will be destroyed.

Compensation:

No compensation will be provided for participation.

Contacts:

Feel free to ask questions about the nature of the research and methods I am using. Your suggestions and concerns are important to me. Please contact me or Dr. Utley at the address/email provided above. If you have questions about your rights as a research volunteer, you may contact Dr. Shelia Kennison, IRB Chair, 219 Cordell North, Stillwater, OK 74078, 405-744-1676 or **irb@okstate.edu.**

Participant Rights:

As a participant in this research, you are entitled to know the nature of my research. You are free to decline to participate, and you are free to stop the interview or withdraw from the study at any time. No penalty exists for withdrawing your participation.

Signatures:

I have read and fully understand the consent form. I sign it freely and voluntarily. A copy of this form has been given to me.

____________________________ __________________

Signature of Participant Date

I certify that I have personally explained this document before requesting that the participant sign it.

____________________________ __________________

Signature of Researcher Date

INFORMED CONSENT DOCUMENT: Physics Instructor

Project Title: A Qualitative Study on the Use of Vectors in the Discourses of Physics and Trigonometry Courses: Possibility for Separate Literacies

Investigators:

Wendy James, M.S.
Professional Education Ph.D. Student
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Professor of Education
233 Willard Hall, OSU
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juliana.utley @okstate.edu
(405) 744-8111

Purpose:

The purpose of this project is analyze features of the course discourse and activities in order to identify how a physicist explains and uses vectors compared to how a mathematician explains and uses vectors.

Procedures:

You will be asked to be video recorded during your class instruction on any days when your instruction includes vector use. The video recording will only focus on you and any media devices you use during instruction without capturing student behavior. The video may capture student voices, but their voices will not be included in the analysis. In addition, you will be asked to participate in 3 types of interviews: pre-instruction, post-instruction, and post-analysis of the entire research project. The pre-post interviews are short (about 15 minutes each) and provide your reflections on the content of your instruction. They will occur prior to initial instruction and then again after every third lesson. Thus, the cycle repeats once a week for six weeks. The third type of interview provides you an opportunity to offer any positive or negative recognition of the themes that may have surfaced during analysis of the videos. The first two types of interviews will be tape recorded, and the third type of interview will be video recorded.

Risks of Participation:

There are no known risks associated with this research project which are greater than those ordinarily encountered in daily life.

Benefits:

The benefits of this research project are to provide awareness about one possible reason for students' lack of success in applying their mathematical knowledge during their physics courses. This knowledge will benefit both physics and mathematical instructors and their coordinating curriculum designers. This research project works as a means of communication between two separate fields of study in order for coordinated efforts to be made from both sides that will facilitate greater student learning and success for both fields.

Confidentiality:

As the researcher, I agree to meet the following conditions:

- I will video tape your class lectures with your permission and transcribe the video for the purpose of analysis. I will give you a copy of the transcript so that you may see that I have captured your words correctly. The video recording will display your face and voice, but when video clips used during research presentations, a pseudonym will be used. The video data itself will not be labeled or stored with your name or position in order to identify you.

- The videos will be recorded digitally and will *not* be made available for public use (U-Tube, Google, ect). The digital recordings will be stored in my computer, which is non-networked and guarded by password access made known only to me.
- The first two interviews will be tape recorded and the third will be video recorded with your permission. A transcription will be made for the purpose of analysis. I will give you a copy of the transcript so that you may see I have captured your words correctly. The tapes and transcriptions will not be labeled or stored with your name or position in order to identify you.
- Transcriptions, tape-recordings, and video recordings will be stored in a file cabinet accessible only by the investigator and will not be labeled or stored with your name or position in order to identify you.
- At the completion of the research project, the video data will be transitioned to storage on a CD. The CD and all paper data will be stored in a locked file cabinet accessible only by the investigator. Clips of the videos may be shown for the use of academic presentations, but all pseudonyms will be strictly enforced. All data will be retained for the length of time it is useful for future research projects designed to further analyze themes which surface during this project. On completion of the usefulness of the data, it will be destroyed.

Compensation:

No compensation will be provided for participation.

Contacts:

Feel free to ask questions about the nature of the research and methods I am using. Your suggestions and concerns are important to me. Please contact me or Dr. Utley at the address/email provided above. If you have questions about your rights as a research volunteer, you may contact Dr. Shelia Kennison, IRB Chair, 219 Cordell North, Stillwater, OK 74078, 405-744-1676 or **irb@okstate.edu.**

Participant Rights:

As a participant in this research, you are entitled to know the nature of my research. You are free to decline to participate, and you are free to stop the interview or withdraw from the study at any time. No penalty exists for withdrawing your participation.

Signatures:

I have read and fully understand the consent form. I sign it freely and voluntarily. A copy of this form has been given to me.

__________________________ ________________

Signature of Participant Date

I certify that I have personally explained this document before requesting that the participant sign it.

__________________________ ________________

Signature of Researcher Date

Oklahoma State University Institutional Review Board

Date:	Friday, April 10, 2009
IRB Application No	ED0965
Proposal Title:	A Comparison Between a Physics and Trigonometry Course on the Teaching of Vectors
Reviewed and Processed as:	Exempt

Status Recommended by Reviewer(s): Approved Protocol Expires: 4/9/2010

Principal Investigator(s):

Wendy James
245 Willard
Stillwater, OK 74078

Juliana Utley
245 Willard
Stillwater, OK 74078

The IRB application referenced above has been approved. It is the judgment of the reviewers that the rights and welfare of individuals who may be asked to participate in this study will be respected, and that the research will be conducted in a manner consistent with the IRB requirements as outlined in section 45 CFR 46.

☒ The final versions of any printed recruitment, consent and assent documents bearing the IRB approval stamp are attached to this letter. These are the versions that must be used during the study.

As Principal Investigator, it is your responsibility to do the following:

1. Conduct this study exactly as it has been approved. Any modifications to the research protocol must be submitted with the appropriate signatures for IRB approval.
2. Submit a request for continuation if the study extends beyond the approval period of one calendar year. This continuation must receive IRB review and approval before the research can continue.
3. Report any adverse events to the IRB Chair promptly. Adverse events are those which are unanticipated and impact the subjects during the course of this research; and
4. Notify the IRB office in writing when your research project is complete.

Please note that approved protocols are subject to monitoring by the IRB and that the IRB office has the authority to inspect research records associated with this protocol at any time. If you have questions about the IRB procedures or need any assistance from the Board, please contact Beth McTernan in 219 Cordell North (phone: 405-744-5700, beth.mcternan@okstate.edu).

Sincerely,

Shelia Kennison, Chair
Institutional Review Board

VITA

Wendy Michelle James

Candidate for the Degree of

Doctor of Philosophy

Thesis: A QUALITATIVE STUDY COMPARING THE INSTRUCTION ON VECTORS BETWEEN A PHYSICS COURSE AND A TRIGONOMETRY COURSE

Major Field: Mathematics Education

Biographical:

Education:

Completed the requirements for the Doctor of Philosophy in Professional Educational Studies focusing on Mathematics Education at Oklahoma State University, Stillwater, Oklahoma in July, 2013.

Completed the requirements for the Master of Science in Teaching, Learning, and Leadership focusing on Mathematics Education at Oklahoma State University, Stillwater, Oklahoma in May, 2006.

Completed the requirements for the Bachelor of Behavioral Science in Mathematics at Hardin-Simmons University, Abilene, Texas in May 1999.

Experience:

University of Central Oklahoma (Fall 2011 – present), Lecturer
Oklahoma State University (Fall 2003 – Spring 2011), Graduate Teaching Associate, Graduate Research Associate, and Graduate Staff
Oklahoma Baptist University (Fall 2009), Adjunct
Upward Bound Program (Summers 2005 – 2007), Instructor
Cisco Junior College (Fall 2002 – Summer 2003), Adjunct
Abilene High School (Fall 1999 – Summer 2003), Teacher

Professional Memberships:

Research Council for Mathematics Learning
School Science and Mathematics Association
National Council of Teachers of Mathematics