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COMPLEXITY OF MODULES OVER LIE SUPERALGEBRAS

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Contents

Abstract

In this dissertation, a fair amount of work is dedicated to computing the complexity of modules over a classical Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ over the complex numbers $\mathbb{C}$. We will consider the category $\mathcal{F}$ of finite dimensional $\mathfrak{g}$-supermodules which are completely reducible as $\mathfrak{g}_{\bar{0}}$-modules. Every module $M \in \mathcal{F}$ admits a minimal projective resolution whose terms have dimensions which increase at a polynomial rate of growth. This rate of growth is called the *complexity* of M. In [2] the authors compute the complexity of the simple and the Kac modules over the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ of type A. A natural continuation to their work is computing the complexity of the same family of modules over the ortho-symplectic Lie superalgebra $\mathfrak{osp}(2|2n)$ of type C. The two Lie superalgebras are both of *Type I*, thus the Kac modules in the two cases are constructed by the same induction functor. This similarity will result in similar computations. In fact, our geometric interpretation of the complexity agrees with theirs. The complexity is not a categorical invariant. However, we compute a categorical invariant called the z-complexity, introduced in [2], and we interpret this invariant geometrically in terms of a specific detecting subsuperalgebra. In addition, we compute the complexity and the z-complexity of the simple modules over the *Type II* Lie superalgebras $\mathfrak{osp}(3|2)$, $D(2,1;\alpha)$, $G(3)$, and $F(4)$.

Chapter 1

Introduction

Let A be a finite group scheme and consider the category of rational modules for A. It is well known that an A-module is projective if and only if it is injective. Moreover, every rational A-module admits a minimal projective resolution which has polynomial rate of growth. This rate of growth, called the *complexity*, was first introduced by Alperin in 1977 (cf. [18]). Carlson [7] later introduced the idea of the support variety of a module over group algebras whose dimension coincides with the complexity of the module.

In this dissertation, we will be considering a classical Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ over the complex numbers $\mathbb{C}$. By definition there exists a connected reductive algebraic group $G_{\bar{0}}$ such that $\mathrm{Lie}(G_{\bar{0}}) = \mathfrak{g}_{\bar{0}}$. The simple classical Lie superalgebras were classified by Kac [15]. The authors in [4] used relative cohomology for the pair $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$ to investigate the combinatorics and representation theory of the category finite-dimensional representations of $\mathfrak{g}$. In fact, the cohomology ring $R = \mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ is finitely generated because $G_{\bar{0}}$ is reductive. In [4, 5] the authors initiated a study of support varieties for $\mathcal{F}$ using relative cohomology for the pair $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$. The construction of these support varieties provides a homological framework which encapsulates the important combinatorial notions of atypicality and defect as defined by Kac and Wakimoto.

We will be working with the category $\mathcal{F} := \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$ of finite-dimensional $\mathfrak{g}$-supermodules which are completely reducible over $\mathfrak{g}_{\bar{0}}$. The authors in [3] showed that $\mathcal{F}$ has enough projectives and it satisfies: (i) it is a self-injective category and (ii) every module in this category admits a projective resolution which has a polynomial rate of growth. For a module $M \in \mathcal{F}$, the complexity $c_{\mathcal{F}}(M)$ is the rate of growth of the minimal projective resolution of M. It should be noted that the support varieties (mentioned above) of a supermodule do not have their dimension equal to the complexity of the supermodule. For example, for the simple Lie superalgebra $\mathfrak{gl}(1|1)$, the complexity of the trivial supermodule is equal to 2, but the dimension of the $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$-support variety is 1 (which is also the atypicality).

This suggests that it is not sufficient to consider only these cohomological varieties when studying the category $\mathcal{F}$ and leads us to introduce new tools in the study of $\mathcal{F}$. Recently Duflo and Serganova introduced associated varieties for objects in $\mathcal{F}$. These varieties detect projectivity (that is, are trivial for a given supermodule if and only if the supermodule is projective). However, they do not have a direct connection to the cohomology of the category $\mathcal{F}$. For type I Lie superalgebras, the authors in [3] introduced new cohomologically defined varieties which are naturally subvarieties of the Duflo-Serganova associated varieties. Using these newly defined subvarieties together with support varieties we give a geometric interpretation of the complexity. This was first done in [2] for the type A Lie superalgebra $\mathfrak{gl}(m|n)$. This geometric interpretation of the complexity will be shown to hold over the type C Lie superalgebra $\mathfrak{osp}(2|2n)$, the type B Lie superalgebra $\mathfrak{osp}(3|2)$, and the three exceptional ones $D(2,1,\alpha)$, $G(3)$, and $F(4)$.

In chapter 2 we compute the complexity of the simple and the Kac modules for the orthosymplectic Lie superalgebra $\mathfrak{osp}(2|2n)$. Let $K(\lambda)$ (resp. $L(\lambda)$) be the Kac (resp. simple) module of highest weight λ. Let $\operatorname{atyp}(\lambda)$ denote the atypicality of λ (see section 2.5). We will see that $\operatorname{atyp}(\lambda)$ is either zero or one. For typical λ (i.e. $\operatorname{atyp}(\lambda) = 0$), the simple and the Kac modules are projective and hence they have a zero complexity. For atypical λ (i.e. $\mathrm{atyp}(\lambda)=1$), the complexity is computed in Theorems 3.5 and 3.13

$$c_{\mathcal{F}}(L(\lambda)) = 2n+1, \quad c_{\mathcal{F}}(K(\lambda)) = 2n.$$

These computations can be interpreted geometrically as follows. For a module M, let $\mathcal{X}_M$ denotes the associated variety defined by Duflo and Serganova [9], and $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$ the support variety as defined in [5]. Then, if $X(\lambda)$ is a Kac or a simple $\mathfrak{osp}(2|2n)$-module, we have the geometric interpretation of the complexity in Theorem 3.23 :

$$c_{\mathcal{F}}(X(\lambda)) = \dim \mathcal{X}_{X(\lambda)} + \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(X(\lambda)). \tag{1.1}$$

The authors in [2] introduced a categorical invariant called the z-complexity of modules and denoted by $z_{\mathcal{F}}(-)$ (see [2, Section 9]). They computed the z-complexity of the simple and the Kac modules over $\mathfrak{gl}(m|n)$ and then used a detecting subsuperalgebra $\mathfrak{f}$ to interpret their computations geometrically. In chapter 2, we carry these computations over $\mathfrak{osp}(2|2n)$ and conclude in Theorem 3.34 that if $X(\lambda)$ is a Kac or a simple module, we have

$$z_{\mathcal{F}}(X(\lambda)) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(X(\lambda)). \tag{1.2}$$

It is worth noting that our geometric interpretations of the complexity and the z-complexity over $\mathfrak{osp}(2|2n)$ agree with the results obtained in [2]. This was expected since both types A and C are *Type I* Lie superalgebras (Section 2.5). In particular, the Kac modules are defined similarly in the two cases and they are finite-dimensional. It was interesting to know if these interpretations would hold over *Type II* Lie superalgebras, hence we computed the complexity and the

z-complexity of the simple (finite-dimensional) modules over $\mathfrak{osp}(3|2)$, and the three exceptional Lie superalgebras $D(2,1;\alpha)$, $G(3)$, and $F(4)$. In chapters three through six, we show that equations (1.1) and (1.2) hold for the simple modules over these Lie superalgebras. The results in this paper lead us to believe that a more general result holds :

Conjecture 1.1. *Let* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *be a basic classical Lie superalgebra over the complex numbers and let* $\mathcal{F} := \mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ *be the category of finite-dimensional* $\mathfrak{g}$*-supermodules which are completely reducible over* $\mathfrak{g}_{\bar{0}}$*. If* $M \in \mathcal{F}$*, then*

(a) $c_{\mathcal{F}}(M) = \dim \mathcal{X}_M + \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$.

(b) $z_{\mathcal{F}}(M) = \dim \mathcal{V}_{\mathfrak{f}}(M)$.

Remark 1.2. *Through out this dissertation, we will be working over the complex numbers* $\mathbb{C}$*.*

Chapter 2

Preliminaries

In this chapter I give a quick introduction to Lie superalgebras and their representations. I also introduce the notions of complexity, support varieties, associated varieties, z-complexity, and detecting subsuperalgebras. Some examples will be provided from the work in [2] over the Lie superalgebra $\mathfrak{gl}(m|n)$. For more about Lie superalgebras, I refer the reader to [15], [23], and [24].

2.1 Lie superalgebras

Definition 2.1. *A superspace V is a $\mathbb{Z}_2$-graded vector space, i.e., a direct sum of two vector spaces $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Elements of $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) are called* even *(resp.* odd*). Nonzero elements of $V_{\bar{0}} \cup V_{\bar{1}}$ are* homogeneous *and for a homogeneous vector v, we write $\bar{v} \in \mathbb{Z}_2$ for the parity (or degree) of v.*

We will be using the following important convention throughout this dissertation: if $\bar{v}$ appears in some formula or expression, then v is assumed to be homogeneous.

Definition 2.2. *A superalgebra is a $\mathbb{Z}_2$-graded algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ which satisfies $A_r A_s \subseteq A_{r+s}$ for all r, $s \in \mathbb{Z}_2$. A* Lie superalgebra *is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ together with a bilinear map $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which preserves the $\mathbb{Z}_2$-grading and satisfies graded versions of the usual Lie bracket axioms. Explicitly, for all $a, b, c \in \mathfrak{g}$ we have*

(a) $[\mathfrak{g}_r, \mathfrak{g}_s] \subseteq \mathfrak{g}_{r+s}$ *for $r, s \in \mathbb{Z}_2$ (preserves the $\mathbb{Z}_2$-grading),*

(b) $[a,b] = -(-1)^{\bar{a}\bar{b}}[b,a]$ *(graded skew-symmetry),*

(c) $(-1)^{\bar{a}\bar{c}}[a,[b,c]] + (-1)^{\bar{a}\bar{b}}[b,[c,a]] + (-1)^{\bar{b}\bar{c}}[c,[a,b]] = 0$ *(graded Jacobi identity).*

Definition 2.3. *A Lie superalgebra* $\mathfrak{g}$ *is abelian if* $[\mathfrak{g},\mathfrak{g}] = 0$.

Definition 2.4. *A* $\mathbb{Z}_2$*-graded subspace* $\mathfrak{a}$ *of* $\mathfrak{g}$ *is an ideal if* $[\mathfrak{a},\mathfrak{g}] \subseteq \mathfrak{a}$.

2.2 Simple Lie superalgebras

Definition 2.5. *A Lie superalgebra* $\mathfrak{g}$ *is* simple *if it is not abelian and the only* $\mathbb{Z}_2$*-graded ideals of* $\mathfrak{g}$ *are* 0 *and* $\mathfrak{g}$.

The subspace $\mathfrak{g}_{\bar{0}}$ is a Lie algebra under the bracket and $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module. A finite dimensional Lie superalgebra $\mathfrak{g}$ is called *classical* if there is a connected reductive algebraic group $G_{\bar{0}}$ such that $\mathrm{Lie}(G_{\bar{0}}) = \mathfrak{g}_{\bar{0}}$, and an action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{0}}$ which differentiates to the adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$.[1] In particular, if $\mathfrak{g}$ is classical, then $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and $\mathfrak{g}_{\bar{1}}$ is semisimple as a $\mathfrak{g}_{\bar{0}}$-module. A *basic classical* Lie superalgebra is a classical Lie superalgebra with a nondegenerate invariant supersymmetric even bilinear form. The simple (basic) classical Lie superalgebras were classified by Kac [15]:

Theorem 2.6. *Let* $\mathfrak{g}$ *be a finite dimensional classical simple (basic) Lie superalgebra. Then either* $\mathfrak{g}$ *is a simple Lie algebra or* $\mathfrak{g}$ *is isomorphic to one of the following algebras:*

$$A(m,n) = \mathrm{sl}(m+1,n+1) \quad \text{with } m > n \geq 0,$$

[1] Unlike in Kac's original definition [15], we do not require a classical Lie superalgebra to be simple.

$$A(n,n) = \mathrm{psl}(n+1,n+1) \quad \textit{with } n \geq 1,$$

$$B(m,n) = \mathrm{osp}(2m+1,2n) \quad \textit{with } m \geq 0, n > 0,$$

$$C(n) = \mathrm{osp}(2,2(n-1)) \quad \textit{with } n \geq 2,$$

$$D(m,n) = \mathrm{osp}(2m,2n) \quad \textit{with } m \geq 2, n \geq 1,$$

$$D(2,1;\alpha) = \Gamma(1,-1-\alpha,\alpha) \quad \alpha \neq 0,-1,$$

$$\mathrm{p}(n), \quad n \geq 2; \quad \mathrm{q}(n), \quad n \geq 2,$$

$$G(3),\ \textit{a simple algebra of dimension } 31,$$

$$F(4),\ \textit{a simple algebra of dimension } 40.$$

Among the Lie superalgebras listed above, we have the following isomorphisms:

$$A(1,0) \cong C(2) \quad D(2,1) \cong D(2,1,1).$$

In addition, there are some isomorphisms between the various $D(2,1;\alpha)$. There are no further isomorphisms between the algebras listed in the theorem.

2.3 Roots

Let $\mathfrak{g}$ be any finite dimensional Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is reductive and $\mathfrak{g}_{\bar{1}}$ is a semisimple $\mathfrak{g}_{\bar{0}}$-module. Let $\mathfrak{h}_{\bar{0}}$ be a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$. For $\alpha \in \mathfrak{h}_{\bar{0}}^*$, set

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \,|\, [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}_{\bar{0}}\},$$

and let

$$\Phi = \{\alpha \in \mathfrak{h}_{\bar{0}}^* \,|\, \alpha \neq 0, \mathfrak{g}^\alpha \neq 0\}$$

be the set of *roots* of $\mathfrak{g}$. Since the action of $\mathfrak{h}_{\bar{0}}$ on any simple $\mathfrak{g}_{\bar{0}}$-modules is diagonalizable, it follows that the adjoint action of $\mathfrak{h}_{\bar{0}}$ on $\mathfrak{g}$ is diagonalizable. Thus there is a *root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}^{\alpha},$$

where $\mathfrak{h} = \mathfrak{g}^0$ is the centralizer of $\mathfrak{h}_{\bar{0}}$ in $\mathfrak{g}$.

Suppose $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. We say that $\mathfrak{b}$ is a *Borel subalgebra* of $\mathfrak{g}$ if

1. $\mathfrak{b}_{\bar{0}} = \mathfrak{b} \cap \mathfrak{g}$ is a Borel subalgebra of $\mathfrak{g}_{\bar{0}}$,

2. $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ with $\mathfrak{n}$ a nilpotent ideal of $\mathfrak{b}$,

and $\mathfrak{b}$ is maximal with these properties. The root vectors that lie in $\mathfrak{b}$ will be called *positive root vectors* and their corresponding roots $\lambda \in \Phi$ will be called the *positive roots*. The set of positive roots will be denoted by Φ^+. The *negative roots* are $\Phi^- := \Phi \setminus \Phi^+$.

2.4 Supermodules

Definition 2.7. *A $\mathfrak{g}$-supermodule M is a superspace together with an action $\mathfrak{g} \times M \to M$ such that $\mathfrak{g}_r.M_s \subseteq M_{r+s}$ for $r, s \in \mathbb{Z}_2$.*

If M and M' are two superspaces, then the space $\mathrm{Hom}_{\mathbb{C}}(M, M')$ is naturally $\mathbb{Z}_2$-graded by declaring $f \in \mathrm{Hom}_{\mathbb{C}}(M, M')_r$ $(r \in \mathbb{Z}_2)$ if $f(M_s) \subseteq M'_{r+s}$ for all $s \in \mathbb{Z}_2$.

Let $U(\mathfrak{g})$ be the universal enveloping superalgebra of $\mathfrak{g}$. Let us describe the category of $\mathfrak{g}$-supermodules. The objects are all left $U(\mathfrak{g})$-modules which are $\mathbb{Z}_2$-graded; that is, superspaces $M = M_{\bar{0}} \oplus M_{\bar{1}}$ satisfying $U(g)_r M_s \subseteq M_{r+s}$ for all r, $s \in \mathbb{Z}_2$. If M is a $\mathfrak{g}$-supermodule, then $N \subseteq M$ is a subsupermodule if it is a supermodule which inherits its grading from M in the sense that $M_r \cap N = N_r$ for $r \in \mathbb{Z}_2$. Given $\mathfrak{g}$-supermodules M and N one can define a $\mathfrak{g}$-supermodule struture on the contragradient dual M^* and the tensor product $M \otimes N$ using the antipode and coproduct of $U(\mathfrak{g})$. A morphism of $U(\mathfrak{g})$-supermodules is an element of $\mathrm{Hom}_{\mathbb{C}}(M, M')$ satisfying $f(xm) = (-1)^{\bar{f}\bar{x}} x f(m)$ for all $m \in M$ and $x \in U(\mathfrak{g})$. In this definition, f and x are assumed to be homogeneous. The general case can be obtained by linearity.

Let $\mathcal{F} := \mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ be the category of finite-dimensional $\mathfrak{g}$-supermodules which are completely reducible over $\mathfrak{g}_{\bar{0}}$. In the case $\mathfrak{g}$ is one of the Lie superalgebras $\mathfrak{osp}(3|2)$, $D(2,1;\alpha)$, $G(3)$, and $F(4)$, the even part $\mathfrak{g}_{\bar{0}}$ is semisimple, thus the category $\mathcal{F}$ is the category of finite-dimensional $\mathfrak{g}$-supermodules. Note that we allow all (not only graded) module homomorphisms, thus for a simple module S, it can happen that $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{F}}(S, S)$ is either one or two. This category decomposes into a direct sum of full subcategories called *blocks* $\mathcal{F}^{\chi}$, where $\mathcal{F}^{\chi}$ consists of all finite dimensional modules with generalized central character χ. A block having more than one element is called an *atypical* block. The interested reader can find more about central characters and blocks in [23, Section 8.2.4].

The category $\mathcal{F}$ has some properties that are of great importance to the computations in this work. To state these properties, we will need to use a fair amount of homological algebra. The interested reader can find the definitions of projective and injective modules, minimal projective resolutions, extensions, cohomology rings, etc. in any text on homological algebra.

Definition 2.8. *A module P is projective if for any modules B, C with module homomorphisms $f : B \twoheadrightarrow C$ and $g : P \to C$, there is a module homomorphism $h : P \to B$ such that $f \circ h = g$.*

Theorem 2.9. *Projective modules satisfy the following properties:*

- *A module P is projective iff the functor* $\mathrm{Hom}(P,\ )$ *is exact.*
- *A module P is projective iff it is a summand of a free module.*
- *The direct sum of projective modules is projective.*
- *Every summand of a projective module is projective.*

We can also define injective modules that satisfy similar (dual) properties. In our category $\mathcal{F}$, injective modules will not be used since:

Proposition 2.10. *[3, Prop. 2.2.2] $\mathcal{F}$ is self-injective; that is, a module $M \in \mathcal{F}$ is projective if and only if it is injective.*

Definition 2.11. *A minimal projective resolution (in $\mathcal{F}$) of a module $M \in \mathcal{F}$ is an exact sequence:*

$$\cdots \to P_d \to \cdots \to P_1 \to P_0 \to M \to 0,$$

*such that each P_d is projective with minimal dimension. The module P_0 is called the **projective cover** of M since it is the minimal projective module that surjects onto M.*

Remark 2.12. *In our consideration of projective resolutions, we always assume the maps to be even.*

Proposition 2.13. *[3, Theorem 2.5.1] Every module in $\mathcal{F}$ admits a minimal projective resolution whose terms have dimensions which increase at a polynomial rate of growth.*

2.5 Type I/II Lie superalgebras

A Lie superalgebra is said to be of *Type I* if it admits a $\mathbb{Z}$-grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

concentrated in degrees -1, 0, 1 with $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0$ and $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ and if the bracket respects this grading. Otherwise, $\mathfrak{g}$ is of *Type II*. Examples of Type I Lie superalgebras include: $\mathfrak{gl}(m|n)$ and the simple Lie superalgebras of types A(m, n), C(n) and p(n).

The simple modules for a Type I classical Lie superalgebra $\mathfrak{g}$ can be constructed as follows. Let $\mathfrak{t}_{\bar{0}}$ be a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ and $X_0^+ \subseteq \mathfrak{t}_{\bar{0}}^*$ be the set of dominant integral weights for $\mathfrak{g}_{\bar{0}}$ with respect to a fixed Borel subalgebra of $\mathfrak{g}_{\bar{0}}$. For $\lambda \in X_0^+$, let $L_{\bar{0}}(\lambda)$ be the simple finite dimensional module over the Lie algebra $\mathfrak{g}_{\bar{0}}$ of highest weight λ. Set

$$\mathfrak{p}^+ = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_1 \quad \mathfrak{p}^- = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{-1}.$$

Since $\mathfrak{g}$ is a Type I Lie superalgebra, $\mathfrak{g}_{\pm 1}$ is an abelian ideal of $\mathfrak{p}^\pm$. We can then view $L_{\bar{0}}(\lambda)$ as a simple finite dimensional $\mathfrak{p}^\pm$-module via inflation through the canonical quotient $\mathfrak{p}^\pm \twoheadrightarrow \mathfrak{g}_{\bar{0}}$. For $\lambda \in X_0^+$, the *Kac module* $K(\lambda)$ is defined by:

$$K(\lambda) := U(g) \otimes_{U(\mathfrak{p}^+)} L_{\bar{0}}(\lambda).$$

The Kac module $K(\lambda)$ has a unique maximal submodule. The head of $K(\lambda)$ is the simple finite dimensional $\mathfrak{g}$-module $L(\lambda)$. The set $\{L(\lambda) \mid \lambda \in X_0^+\}$ is a complete set of non-isomorphic simple modules in $\mathcal{F} = \mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}$. The projective cover of $L(\lambda)$ will be denoted by $P(\lambda)$. These are all finite dimensional.

Definition 2.14. *Let ρ be half the sum of the positive even roots minus half the sum of the positive odd roots. Define the atypicality of λ to be the maximal number of pairwise orthogonal positive isotropic (odd) roots which are also orthogonal to $\lambda + \rho$ with respect to the bilinear form on $\mathfrak{h}^*$. We will write* $\operatorname{atyp}(\lambda)$ *for the atypicality of λ. Note that* $\operatorname{atyp}(\lambda) \leq \operatorname{def}(\mathfrak{g})$ *for all λ. We say that λ is typical if* $\operatorname{atyp}(\lambda) = 0$, *otherwise λ is atypical. If $L(\lambda)$ is a simple $\mathfrak{g}$-modules of highest weight λ, then we define* $\operatorname{atyp}(L(\lambda)) := \operatorname{atyp}(\lambda)$.

If $L(\lambda)$ is a simple $\mathfrak{g}$-modules of highest weight λ, then we define $\operatorname{atyp}(L(\lambda)) := \operatorname{atyp}(\lambda)$. It is known that the atypicality of a simple module is independent of the choice of Cartan and Borel subalgebras and, furthermore, is the same for all simple modules in a given block. Hence it makes sense to refer to the atypicality of a block. The principal block is the block containing the trivial module. It is known that ([16, Theorem 1]) if $\operatorname{atyp}(\lambda) = 0$, then $P(\lambda) = L(\lambda)$ and hence $L(\lambda)$ is projective. If $\mathfrak{g}$ is of *Type I* and $\operatorname{atyp}(\lambda) = 0$, then $P(\lambda) = L(\lambda) = K(\lambda)$.

Definition 2.15. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra with a root system Φ. Kac and Wakimoto [17, Section 2] define the* defect *of $\mathfrak{g}$, denoted by* $\operatorname{def}(g)$, *to be the maximal atypicality.*

The defects for the various simple basic Lie superalgebras are given as follows [9, Section 4]:

$$\operatorname{def}(\mathfrak{sl}(m|n)) = \min(m, n), \quad \operatorname{def}(\mathfrak{psl}(n|n)) = \min(n, n),$$

$$\operatorname{def}(\mathfrak{osp}(2m|2n)) = \operatorname{def}(\mathfrak{osp}(2m+1|2n)) = \min(m, n),$$

and the exceptional Lie superalgebras $D(2, 1; \alpha)$, $G(3)$, and $F(4)$ all have defect 1.

Example 2.16. *The standard example of a* Type I *classical Lie superalgebra is* $\mathfrak{g} = \mathfrak{gl}(m|n)$. *The even component* $\mathfrak{g}_{\bar{0}}$ *is the span of the matrix units* $E_{i,j}$ *where* $1 \leq i,\, j \leq m$ *or* $m+1 \leq i,\, j \leq m+n$. *A basis for* $\mathfrak{g}_{\bar{1}}$ *is given by* $E_{i,j}$ *where* $1 \leq i \leq m$ *and* $m+1 \leq j \leq m+n$ *or* $m+1 \leq i \leq m+n$ *and* $1 \leq j \leq n$. *We can choose the Cartan subalgebra* $\mathfrak{h}$ *to be set of the diagonal matrices and the Borel subalgebra* $\mathfrak{b}$ *to be the set of all upper triangular matrices. The dual of the Cartan* $\mathfrak{h}^*$ *has basis* $\{\varepsilon_i \ : \ \mathfrak{h} \to \mathbb{C} \,|\, 1 \leq i \leq m+n\}$ *where* ε_i *is the linear functional which picks out the* i^{th} *diagonal entry. With respect to this basis we define a bilinear form on* $\mathfrak{h}$ *by*

$$(\varepsilon_i, \varepsilon_j) = \begin{cases} \delta_{i,j}, & 1 \leq i,\, j \leq m; \\ -\delta_{i,j}, & m+1 \leq i,\, j \leq m+n; \\ 0, & \textit{otherwise.} \end{cases}$$

The set of roots for $\mathfrak{g}$ *is*

$$\Phi = \{\varepsilon_i - \varepsilon_j \,|\, 1 \leq i,\, j \leq m+n,\, i \neq j\}$$

and the positive roots are

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j \,|\, 1 \leq i < j \leq m+n\}.$$

The even roots of $\mathfrak{g}$ *are:*

$$\Phi_{\bar{0}} = \{\varepsilon_i - \varepsilon_j \,|\, 1 \leq i \neq j \leq m \text{ or } m+1 \leq i \neq j \leq m+n\},$$

while the odd roots are $\Phi_{\bar{1}} = \Phi \setminus \Phi_{\bar{0}}$. *Using this data, we can compute*

$$\rho = \frac{m-n-2}{2}\varepsilon_1 + \frac{m-n-4}{2}\varepsilon_2 + \ldots + \frac{-m-n}{2}\varepsilon_m + \frac{m+n}{2}\varepsilon_{m+1} + \ldots + \frac{m-n+2}{2}\varepsilon_{m+n}.$$

The set of dominant integral weights for $\mathfrak{gl}(m|n)$ *is given in [2, Subsection 2.4] by*

$$X_0^+ = \left\{ \lambda = \sum_{i=1}^{m+n} \lambda_i \varepsilon_i \,|\, \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0} \text{ for } i \neq m,\, m+n \right\}.$$

For a weight $\lambda \in X_0^+$, $\operatorname{atyp}(\lambda) = k$ *is equivalent to having a set of* k *positive odd pairwise orthogonal roots*

$$\{\varepsilon_{i_t} - \varepsilon_{j_t} \,|\, t = 1, \ldots, k\},$$

such that $(\lambda + \rho, \varepsilon_{i_t} - \varepsilon_{j_t}) = 0$ *for all* $t = 1, \ldots, k$ *where* $1 \leq i_1, \ldots, i_k \leq m$ *and* $m + 1 \leq j_1, \ldots, j_k \leq m + n$. *Then the defect of* $\mathfrak{g}$ *is* $\min(m, n)$. *We can then define the* core *of* λ *as the pair of multisets*

$$\Big(\{(\lambda + \rho, \varepsilon_r)\}, \{(\lambda + \rho, \varepsilon_s)\} \Big),$$

where $r \in \{1, \ldots, m\} \setminus \{i_1, \ldots, i_k\}$ *and* $s \in \{m + 1, \ldots, m + n\} \setminus \{j_1, \ldots, j_k\}$. *From [12, Section 5] showed that two simple modules* $L(\lambda)$ *and* $L(\mu)$ *lie in the same block if and only if* $\operatorname{atyp}(\lambda) = \operatorname{atyp}(\mu)$ *and the core of* λ *equals the core of* μ.

The trivial module has weight $\lambda = 0$ *whose atypicality is the same as* $\operatorname{def}(\mathfrak{g})$. *If we take* $m = n = 1$, *the simple modules in the principal block are one dimensional and indexed by* $L(\lambda\varepsilon_1 - \lambda\varepsilon_2)$ *where* $\lambda \in \mathbb{Z}$.

2.6 Complexity

The authors in [3] showed that $\mathcal{F}$ has enough projectives, (i) it is a self-injective category and (ii) every module in this category admits a projective resolution which has a polynomial rate of growth. For a module $M \in \mathcal{F}$, define the complexity $c_{\mathcal{F}}(M)$ to be the rate of growth of the minimal projective resolution of M. More explicitly,

Definition 2.17. *Let M be a $\mathfrak{g}$-module in the category $\mathcal{F}$ and $\mathbf{P}. \twoheadrightarrow M$ be a minimal projective resolution. The rate of growth of $\mathbf{P}.$, $r(\mathbf{P}.)$, is the smallest nonnegative integer c such that there exists a constant $C > 0$ with $\dim P_d \leq Cd^{c-1}$ for all d. If no such integer exists, then $\mathbf{P}.$ is said to have infinite rate of growth. Define the complexity of M to be $c_{\mathcal{F}}(M) := r(\mathbf{P}.)$.*

Fortunately, the complexity of all modules in $\mathcal{F}$ is finite:

Theorem 2.18. *[3, Theorem 2.5.1] Let M be an object of $\mathcal{F}$. Then $c_{\mathcal{F}}(M) \leq$* $\dim \mathfrak{g}_{\bar{1}}$.

This concept of complexity was first introduced by Alperin for finite modules over groups [18] in 1977. One of the important features of the complexity is the fact that it measures how far the module is from being projective:

Lemma 2.19. *[3, Corollary 2.7.1] $c_{\mathcal{F}}(M) = 0$ if and only if M is projective.*

Corollary 2.20. *If λ is typical (i.e. $\operatorname{atyp}(\lambda) = 0$), then $c_{\mathcal{F}}(L(\lambda)) = 0$. If $\mathfrak{g}$ is of* Type I, *then $c_{\mathcal{F}}(K(\lambda)) = 0$.*

Example 2.21. *Let $\mathfrak{g} = \mathfrak{gl}(1|1)$. The simple modules in the principal block have highest weights $\lambda\varepsilon_1 - \lambda\varepsilon_2$, $\lambda \in \mathbb{Z}$. Boe, Nakano, and Kujawa showed in [3] that $c_{\mathcal{F}}(L(\lambda\varepsilon_1 - \lambda\varepsilon_2)) = 2$ $(\lambda \in \mathbb{Z})$. The projective cover $P(\lambda| -\lambda)$ of*

$L(\lambda|-\lambda)$ is four-dimensional with three radical layers. The head and socle of $P(\lambda|-\lambda)$ are both isomorphic to $L(\lambda|-\lambda)$, and the second layer is isomorphic to $L(\lambda+1|-\lambda-1) \oplus L(\lambda-1|-\lambda+1)$. For simplicity, let us choose $\lambda = 0$. They constructed an explicit minimal projective resolution of the trivial module $L(0|0)$:

$$\ldots \to P(2|-2) \oplus P(0|0) \oplus P(-2|2) \to P(1|-1) \oplus P(-1|1) \to P(0|0) \to L(0|0) \to 0.$$

Therefore, $\dim P_d = 4(d+1)$ *and* $c_{\mathcal{F}}(L(0|0)) = 2$.

More generally,

Theorem 2.22. *[2, Theorem 8.2.1] Let $L(\lambda)$ be a simple $\mathfrak{gl}(m|n)$-module of atypicality k. Then*

$$c_{\mathcal{F}}(L(\lambda)) = (m+n)k - k^2 + k.$$

Example 2.23. *Let $\mathfrak{g} = \mathfrak{gl}(1|1)$. Given the structure of the projective indecomposable modules given in the previous section, we can verify that the Kac module $K(0|0)$ admits a minimal projective resolution of the form*

$$\ldots \to P(2|-2) \to P(1|-1) \to P(0|0) \to K(0|0) \to 0.$$

Therefore, $\dim P_d = 4$ *and* $c_{\mathcal{F}}(K(0|0)) = 1$.

More generally,

Theorem 2.24. *[2, Theorem 6.3.1] Let $K(\lambda)$ be a Kac $\mathfrak{gl}(m|n)$-module of atypicality k. Then*

$$c_{\mathcal{F}}(K(\lambda)) = (m+n)k - k^2.$$

We write $\mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M,N)$ for the relative cohomology for the pair $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$. For

$M, N \in \mathcal{F}$, we will use the isomorphism:

$$\operatorname{Ext}^{\bullet}_{\mathcal{F}}(M, N) \cong \operatorname{Ext}^{\bullet}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}(M, N).$$

The following proposition provides a characterization of the complexity via rates of growth of these extension groups:

Proposition 2.25. *[3, Proposition 2.8.1] Let $\mathfrak{g}$ be a classical Lie superalgebra, and let M be an object in $\mathcal{F}$. Then*

$$c_{\mathcal{F}}(M) = r\Big(\operatorname{Ext}^{\bullet}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}(M, \bigoplus S^{\dim P(S)})\Big)$$

where the sum is over all the simple modules in $\mathcal{F}$, and $P(S)$ is the projective cover of S.

Proof. Let $\mathbf{P}. \twoheadrightarrow M$ be a minimal projective resolution of M, then we have

$$\operatorname{Ext}^{n}_{\mathcal{F}}\left(M, \bigoplus S^{\dim P(S)}\right) \cong \operatorname{Hom}_{\mathcal{F}}\left(P_n, \bigoplus S^{\dim P(S)}\right).$$

Let $[P_n : P(S)]$ be the number of times the projective cover $P(S)$ appears as a summand of the projective module P_n. Then we have,

$$\begin{aligned} \dim_{\mathbb{C}} \operatorname{Ext}^{n}_{\mathcal{F}}\left(M, \bigoplus S^{\dim P(S)}\right) &= \sum_{S} \dim_{\mathbb{C}} P(S) \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{F}}(P_n, S) \\ &= \sum_{S} \dim_{\mathbb{C}} P(S) \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{F}}(S, S)[P_n : P(S)] \end{aligned}$$

Then, by Schur's Lemma in the super setting, we have

$$\dim_{\mathbb{C}} P_n \le \dim_{\mathbb{C}} \operatorname{Ext}^{n}_{\mathcal{F}}\left(M, \bigoplus S^{\dim P(S)}\right) \le 2 \dim_{\mathbb{C}} P_n.$$

This proves the result. $\square$

In the group case, the cohomology ring of G is finitely generated and we can construct the support variety, $\mathcal{V}_G(M)$, of a module M, with $\dim \mathcal{V}_G(M) = c_G(M)$. However, the relative cohomology $H^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ for $\mathfrak{gl}(1|1)$ has Krull dimension one, and so is not large enough to use to construct a support variety theory which measures the complexity. This geometric interpretation will be completed by using both the cohomological variety and the associated variety of Duflo and Serganova [9].

2.7 Support varieties

Let $R = H^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ be the cohomology ring of $\mathfrak{g}$ and let $M_1,\ M_2 \in \mathcal{F}$. According to [4, Theorem 2.7], $\mathrm{Ext}^\bullet_{\mathcal{F}}(M_1, M_2)$ is a finitely generated R-module. Set

$$J_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M_1, M_2) := Ann_R(\mathrm{Ext}^\bullet_{\mathcal{F}}(M_1, M_2)).$$

The *relative support variety* of the pair (M_1, M_2) is defined by

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M_1, M_2) := MaxSpec(R/J_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M_1, M_2)).$$

In the case $M = M_1 = M_2$, set $J_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) = J_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, M)$, and

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) := \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, M).$$

The variety $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$ is called the *support variety* of M. Note that in this situation $J_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) = Ann_R(Id)$ where Id is the identity morphism in $\mathrm{Hom}_{\mathcal{F}}(M, M)$. Boe, Kujawa, and Nakano had computed the support varieties of modules over certain families of Lie superalgebras. For example,

Lemma 2.26. *[5, Corollary 3.2] Let $K(\lambda)$ be a Kac $\mathfrak{gl}(m|n)$-module. Then*

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda)) = 0.$$

Lemma 2.27. *[5, Theorem 4.7] Let $L(\lambda)$ be a simple $\mathfrak{gl}(m|n)$-module of atypicality k. Then $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(L(\lambda) \cong \mathbb{A}^k$. Hence* $\dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(L(\lambda)) = \mathrm{atyp}(\lambda)$.

2.8 Rank varieties for *Type I* Lie superalgebras

Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a *Type I* Lie superalgebra. Observe that $\mathfrak{g}_{\pm 1}$ is an abelian Lie superalgebra and, consequently,

$$R_{\pm} := H^{\bullet}(\mathfrak{g}_{\pm 1}, \{0\}; \mathbb{C}) \cong S(\mathfrak{g}_{\pm 1}^{*})$$

as graded algebras. Let $\mathcal{F}(\mathfrak{g}_{\pm 1})$ be the category of finite dimensional $\mathfrak{g}_{\pm 1}$-modules. Define the $\mathfrak{g}_{\pm 1}$ *support variety* of $M \in \mathcal{F}(\mathfrak{g}_{\pm 1})$ by:

$$\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M) := \mathcal{V}_{(\mathfrak{g}_{\pm 1},0)}(M).$$

From [5, Section 5], we have $\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M)$ is canonically isomorphic to the following rank variety:

$$\mathcal{V}_{\mathfrak{g}_{\pm 1}}^{rank}(M) := \{x \in \mathfrak{g}_{\pm 1} \mid \text{M is not projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\},$$

where $U(\langle x \rangle)$ denotes the enveloping algebra of the Lie superalgebra generated by $x \in \mathfrak{g}_{\pm 1}$. As a consequence of this isomorphism, we can use the various properties of a rank variety. For example, [5, Sections 5,6], it satisfies the tensor product rule

$$\mathcal{V}_{\mathfrak{g}_{\pm 1}}^{rank}(M \otimes N) = \mathcal{V}_{\mathfrak{g}_{\pm 1}}^{rank}(M) \cap \mathcal{V}_{\mathfrak{g}_{\pm 1}}^{rank}(N).$$

and detects projectivity:

Lemma 2.28. *[3, Theorem 3.5.1(a)] Let M be a $\mathfrak{g}$-module in the category $\mathcal{F}$. Then M is projective if and only if $\mathcal{V}_{\mathfrak{g}_1}^{rank}(M) = \mathcal{V}_{\mathfrak{g}_{-1}}^{rank}(M) = 0$.*

As a consequence, if $\lambda \in X_0^+$ is typical, $L(\lambda)$ and $K(\lambda)$ are projective and hence

$$\mathcal{V}_{\mathfrak{g}_1}^{rank}(L(\lambda)) = \mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda)) = 0.$$

2.9 Associated variety

Let

$$\mathcal{X} = \{x \in \mathfrak{g}_{\bar{1}} \mid [x, x] = 0\}.$$

Let $G_{\bar{0}}$ be an algebraic group with Lie algebra $\mathfrak{g}_{\bar{0}}$. Then $\mathcal{X}$ is invariant under the adjoint action of $G_{\bar{0}}$. It was shown in [9] that for any $x \in \mathcal{X}$, there exists $g \in G_{\bar{0}}$ and isotropic mutually orthogonal linearly independent roots $\alpha_1, \ldots, \alpha_k$ such that $Ad_g(x) = x_1 + \ldots + x_k$ with $x_i \in \mathfrak{g}_{\alpha_i}$. The number k does not depend on the choice of g and is called the *rank* of x (notation $rk(x) = k$). Let

$$X_k = \{x \in \mathcal{X} \mid rk(x) = k\}.$$

If s is the defect of $\mathfrak{g}$, then $\mathcal{X} = X_0 \cup \ldots \cup X_s$ is a stratification of $\mathcal{X}$ such that $\overline{X_k} = X_0 \cup \ldots \cup X_k$.

If $M \in \mathcal{F}$, then Duflo and Serganova [9] define an associated variety of M which is equivalent to:

$$\mathcal{X}_M = \{x \in \mathcal{X} \mid \text{M is not projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\}.$$

As an example, we consider the work done in [2]:

Theorem 2.29. *[2, Corollary 6.4.1] Let $K(\lambda)$ be a Kac $\mathfrak{gl}(m|n)$-module of atypicality k. Then $\mathcal{X}_{K(\lambda)} = \overline{(\mathfrak{g}_1)_k}$ where $\overline{(\mathfrak{g}_1)_k}$ is the closure of the $G_{\bar{0}}$-orbit, $(\mathfrak{g}_1)_k$ described in [2, Section 2.6]. Hence,* $\dim \mathcal{X}_{K(\lambda)} = (m+n)k - k^2$.

On the other hand,

Theorem 2.30. *[9, Theorems 4.5 and 5.4] Let $L(\lambda)$ be a simple $\mathfrak{gl}(m|n)$-module of atypicality k. Then $\mathcal{X}_{L(\lambda)} = \overline{(\mathfrak{g}_1)_k}$. Hence* $\dim \mathcal{X}_{L(\lambda)} = (m+n)k - k^2$.

Combining the results stated in the previous three sections, the following geometric interpretation of the complexity holds:

Theorem 2.31. *If $X(\lambda)$ is a Kac or a simple $\mathfrak{gl}(m|n)$-module, then*

$$c_{\mathcal{F}}(X(\lambda)) = \dim \mathcal{X}_{X(\lambda)} + \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(X(\lambda)).$$

2.10 z-complexity

Complexity is not a categorical invariant; however, the authors in [2] defined a closely related invariant, called the z-complexity:

Definition 2.32. *Let M be a module in $\mathcal{F}$. Consider*

$$z_{\mathcal{F}}(M) = r\Big(\operatorname{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, \bigoplus S)\Big),$$

where the direct sum runs over all simple modules of $\mathcal{F}$. Unlike complexity, $z_{\mathcal{F}}(-)$ has the advantage of being invariant under category equivalences.

Proposition 2.33. *Let $M \in F$ and $\mathbf{P}. \to M$ be a minimal projective resolution of M. Define $s(\mathbf{P}.)$ to be the rate of growth of the number of summands in P_d.*

Explicitly, $s(\mathbf{P}.)$ is the smallest nonnegative integer c such that there exists a constant $C > 0$ with

$$\text{number of summands in } P_d \leq Cd^{c-1}, \quad \text{for all } d.$$

We have $z_{\mathcal{F}}(M) = s(\mathbf{P}.)$.

Proof. The proof follows the same argument in the proof of Proposition 2.25 where $\dim P(S)$ is removed from the computations. □

Example 2.34. *Let $\mathfrak{g} = \mathfrak{gl}(1|1)$. Recall the explicit minimal projective resolution of the trivial module $L(0|0)$:*

$$\ldots \to P(2|-2) \oplus P(0|0) \oplus P(-2|2) \to P(1|-1) \oplus P(-1|1) \to P(0|0) \to L(0|0) \to 0.$$

The number of summands in each P_d is $d+1$, thus the rate of growth of this number is 2 and hence $z_{\mathcal{F}}(L(0|0)) = 2$.

More generally,

Theorem 2.35. *[2, Theorem 9.1.1] Let $L(\lambda)$ be a simple $\mathfrak{gl}(m|n)$-module of atypicality k. Then*

$$z_{\mathcal{F}}(L(\lambda)) = 2k.$$

Example 2.36. *Let $\mathfrak{g} = \mathfrak{gl}(1|1)$. The Kac module $K(0|0)$ admits a minimal projective resolution of the form*

$$\ldots \to P(2|-2) \to P(1|-1) \to P(0|0) \to K(0|0) \to 0.$$

The number of summands in each P_d is 1, thus the rate of growth of this number is 1 and hence $z_{\mathcal{F}}(K(0|0)) = 1$.

More generally,

Theorem 2.37. *[2, Theorem 9.1.1] Let $K(\lambda)$ be a Kac $\mathfrak{gl}(m|n)$-module of atypicality k. Then*

$$z_{\mathcal{F}}(K(\lambda)) = k.$$

This invariant can be also interpreted geometrically, this time in terms of detecting subalgebras which were introduced in [5]:

2.11 Detecting subsuperalgebras

Boe, Kujawa, and Nakano [4, Section 4] constructed two subsuperalgebras of $\mathfrak{g}$, namely $\mathfrak{f}$ and $\mathfrak{e}$, which will detect the relative cohomology ring of $\mathfrak{g}$ [4, Theorem 4.1.1]. We will only use the detecting subsuperalgebra $\mathfrak{f}$ throughout this work. An explicit definition of $\mathfrak{f}_{\bar{1}}$ is given in [4, Table 2] using root vectors. The even part is defined by $\mathfrak{f}_{\bar{0}} = [\mathfrak{f}_{\bar{1}}, \mathfrak{f}_{\bar{1}}]$. Then

$$\mathfrak{f} := \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}}.$$

Since $\mathfrak{g}$ is classical, the Lie superalgebra $\mathfrak{f}$ is also classical [4, Theorem 4.1.1]. We can then define the support variety of M relative to $\mathfrak{f}$, $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M)$, in the same way we defined $\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M)$. If $[\mathfrak{f}_{\bar{0}}, \mathfrak{f}_{\bar{1}}] = 0$, these varieties admit a rank variety description and, in particular, can be identified as subvarieties of $\mathfrak{f}_{\bar{1}}$, i.e.,

$$\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M) = \mathcal{V}_{\mathfrak{f}_{\bar{1}}}^{rank}(M) \quad = \{y \in \mathfrak{f}_{\bar{1}} \mid \text{M is not projective as } U(\langle y \rangle)\text{-module}\} \cup \{0\}.$$

Note that $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(\mathbb{C}) = \mathfrak{f}_{\bar{1}}$. Boe et al. computed the dimensions of $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M)$ when M is either a simple or a Kac $\mathfrak{gl}(m|n)$-module. They came to the conclusion:

Theorem 2.38. *[2, Theorem 9.2.1] If $X(\lambda)$ is a Kac or a simple $\mathfrak{gl}(m|n)$-module, then*

$$z_{\mathcal{F}}(X(\lambda)) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(X(\lambda)).$$

Chapter 3

Computing the complexity over $\mathfrak{osp}(2|2n)$

3.1 The Lie superalgebra $\mathfrak{osp}(m|n)$

We first discuss the orthosymplectic Lie superalgebra osp$(m|n)$. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded space, with $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$. Let F be a nondegenerate consistent supersymmetric bilinear form on V, so that $V_{\bar{0}}$ and $V_{\bar{1}}$ are orthogonal and the restriction of F to $V_{\bar{0}}$ is symmetric and to $V_{\bar{1}}$ a skew-symmetric form. Define the Lie superalgebra $\mathfrak{gl}(m|n) := \mathrm{End}(V)$ with the $\mathbb{Z}_2$-grading:

$$\mathrm{End}(V)_i = \{f \in \mathrm{End}\, V \mid f(V_s) \subseteq V_{i+s},\, s \in \mathbb{Z}_2\}, \quad i \in \mathbb{Z}_2,$$

and the bracket is given by $[f, f'] = ff' - (-1)^{\bar{f}\bar{f}'} f'f$. We define in $\mathfrak{gl}(m|n)$ the subalgebra $\mathfrak{osp}(m|n) = \mathfrak{osp}(m|n)_{\bar{0}} \oplus \mathfrak{osp}(m|n)_{\bar{1}}$ by setting

$$\mathfrak{osp}(m|n)_i = \{f \in l(m|n)_i \mid F(f(x), y) = -(-1)^{i\bar{x}} F(x, f(y))\}, \quad (i \in \mathbb{Z}_2).$$

3.2 The Lie superalgebra $\mathfrak{osp}(2|2n)$

In this chapter we will be working with the Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(2|2n)$ of type $C(n+1)$. The bilinear form F can be represented by the matrix:

$$F = \begin{bmatrix} G & 0 \\ 0 & H \end{bmatrix},$$

where

$$G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

where I_n is the identity $n \times n$ matrix. Then a matrix $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is in $\mathfrak{osp}(2|2n)$ if and only if

$$A^t G + GA = B^t G - HC = D^t H + HD = 0.$$

Using this equation, we can see that $\mathfrak{g}$ has a $\mathbb{Z}$-grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

given by the following matrix realization:

$$\mathfrak{g}_{-1} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & y & y_1 \\ \hline y_1^t & 0 & 0 & 0 \\ -y^t & 0 & 0 & 0 \end{array}\right],$$

$$\mathfrak{g}_0 = \left[\begin{array}{cc|cc} \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ \hline 0 & 0 & a & b \\ 0 & 0 & c & -a^t \end{array}\right],$$

and

$$\mathfrak{g}_1 = \left[\begin{array}{cc|cc} 0 & 0 & x & x_1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & x_1^t & 0 & 0 \\ 0 & -x^t & 0 & 0 \end{array}\right].$$

In the above, x, x_1, y, y_1 are $1 \times n$ matrices; a, b, c are $n \times n$ matrices where b and c are symmetric; α is a scalar in $\mathbb{C}$. The $\mathbb{Z}_2$-grading is given by: $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ and $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0$. The Lie super-bracket is defined by

$$[A, B] = AB - (-1)^{\bar{A}\bar{B}} BA,$$

for homogeneous elements A, $B \in \mathfrak{g}_{\bar{0}}$ or $\mathfrak{g}_{\bar{1}}$. We then extend the definition of the bracket to all of $\mathfrak{g}$ by bilinearity. In the above AB denotes the matrix multiplication of A and B. The degree $\bar{A}$ is

$$\bar{A} = \begin{cases} 0 & \text{if } A \in \mathfrak{g}_{\bar{0}}, \\ 1 & \text{if } A \in \mathfrak{g}_{\bar{1}}. \end{cases}$$

Using this bracket, we can easily check that $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are both abelian.

Note that $\mathfrak{g}_{\bar{0}} \cong \mathbb{C} \oplus \mathfrak{sp}(2n)$, hence the choice of the Cartan subalgebra:

$$\mathfrak{h} = \left[\begin{array}{cc|cc} \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ \hline 0 & 0 & a & 0 \\ 0 & 0 & 0 & -a \end{array}\right],$$

where a is $n \times n$ diagonal matrix. Let $\varepsilon_1 : \mathfrak{h} \to \mathbb{C}$ be the linear map that takes an element of $\mathfrak{h}$ to its first diagonal entry (α) and let $\delta_i : \mathfrak{h} \to \mathbb{C}$ be the linear map that takes an element of $\mathfrak{h}$ to the ith diagonal entry of the matrix a. The set $\{\varepsilon_1, \delta_1, \delta_2, \ldots, \delta_n\}$ forms a basis of $\mathfrak{h}^*$ and we define a bilinear form, $(\, , \,)$, on $\mathfrak{h}^*$ by setting

$$(\varepsilon_1, \varepsilon_1) = 1, \quad (\delta_i, \delta_j) = -\delta_{ij}, \quad (\varepsilon_1, \delta_i) = 0,$$

for all $1 \leq i \leq n$. The Borel subalgebra for $\mathfrak{g}_{\bar{0}}$ is chosen as follows:

$$\mathfrak{b}_{\bar{0}} = \left[\begin{array}{cc|cc} \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ \hline 0 & 0 & a & b \\ 0 & 0 & 0 & -a^t \end{array}\right],$$

where a is upper-triangular and b is symmetric. With our choice of the above Cartan subalgebra the root system of $\mathfrak{g}_{\bar{0}}$, that is the even roots of $\mathfrak{g}$, is

$$\Phi_{\bar{0}} = \{\pm\delta_i \pm \delta_j \mid 1 \leq i \neq j \leq n\} \cup \{\pm 2\delta_i \mid 1 \leq i \leq n\}.$$

The set of odd roots is

$$\Phi_{\bar{1}} = \{\pm\varepsilon_1 \pm \delta_i \mid 1 \leq i \leq n\}.$$

The Borel superalgebra for $\mathfrak{g}$ will be given by $\mathfrak{b} = \mathfrak{b}_{\bar{0}} \oplus \mathfrak{g}_1$. With this choice, the set of positive roots of $\mathfrak{g}$ is given by

$$\Phi^+ = \{\delta_i + \delta_j \mid 1 \leq i \leq j \leq n\} \cup \{\delta_i - \delta_j \mid 1 \leq i < j \leq n\} \cup \{\varepsilon_1 \pm \delta_i \mid 1 \leq i \leq n\},$$

and the set of simple roots is

$$\Delta = \{\delta_i - \delta_{i+1} \mid 1 \leq i \leq n\} \cup \{2\delta_n, \, \varepsilon_1 - \delta_1\}.$$

Let $X_0^+ \subseteq \mathfrak{h}^*$ be the parameterizing set of highest weights for the simple finite dimensional $\mathfrak{g}_{\bar{0}}$-modules with respect to the pair $(\mathfrak{h}, \mathfrak{b}_{\bar{0}})$. By [20, Section 21.1], we can show that the simple finite-dimensional $\mathfrak{sp}(2n)$-modules have integral weights. An explicit description of X_0^+ is

$$X_0^+ = \{\lambda = \lambda_{-1}\varepsilon_1 + \sum_{i=1}^{n} \lambda_i \delta_i \mid \lambda_{-1} \in \mathbb{C},\, \lambda_i \in \mathbb{Z},\, \forall i \geq 1;\; \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0\}.$$

For $\lambda \in X_0^+$, let $L_{\bar{0}}(\lambda)$ be the simple finite dimensional $\mathfrak{g}_{\bar{0}}$-module of highest weight λ. Since $\mathfrak{osp}(2|2n)_{\bar{0}} \cong \mathbb{C} \oplus \mathfrak{sp}(2n)$, the simple $\mathfrak{g}_{\bar{0}}$-modules are of the form

$$L_{\bar{0}}(\lambda) = \mathbb{C}_{\lambda_{-1}} \boxtimes L_0(\lambda),$$

where $L_0(\lambda)$ is the simple $\mathfrak{sp}(2n)$-module of weight $\sum_{i=1}^{n} \lambda_i \delta_i$. Note that there is a slight abuse of notation where we used λ for the simple $\mathfrak{sp}(2n)$-module even though we removed the λ_{-1}-part. The action of $\mathfrak{g}_{\bar{0}}$ on $L_{\bar{0}}(\lambda) = \mathbb{C}_{\lambda_{-1}} \boxtimes L_0(\lambda)$ is given by

$$(\alpha \oplus T).(c_{\lambda_{-1}} \otimes v) = \alpha.c_{\lambda_{-1}} \otimes v + c_{\lambda_{-1}} \otimes T.v,$$

for α, $c_{\lambda_{-1}} \in \mathbb{C}$, $T \in \mathfrak{sp}(2n)$ and $v \in L_0(\lambda)$. It is worth mentioning that if v_1 is the highest weight vector for $L_0(\lambda)$, then $w_1 = 1 \otimes v_1$ is the highest weight vector in $L_{\bar{0}}(\lambda)$.

Let ρ be half the sum of the positive even roots minus half the sum of the positive odd roots. Then

$$\rho = -n\varepsilon_1 + \sum_{i=1}^{n} (n - i + 1)\delta_i.$$

We can easily check that the atypicality of λ is either zero or one. In fact, $\mathrm{atyp}(\lambda) = 1$ if and only if there is an odd positive root $\varepsilon_1 \pm \delta_i$ such that

$(\lambda + \rho, \varepsilon_1 \pm \delta_i) = 0$ for some $1 \leq i \leq n$. A weight is called *atypical* if its atypicality is equal to 1, and is called *typical* otherwise. It is important to note that if $\lambda = \lambda_{-1}\varepsilon_1 + \sum_{i=1}^{n} \lambda_i \delta_i$ is atypical, then $\lambda_{-1} \in \mathbb{Z}$.

If $L(\lambda)$ is a simple $\mathfrak{g}$-module of highest weight λ, then we define $\operatorname{atyp}(L(\lambda)) := \operatorname{atyp}(\lambda)$. It is known that the atypicality of a simple module is independent of the choice of Cartan and Borel subalgebras and, furthermore, is the same for all simple modules in a given block. Hence it makes sense to refer to the atypicality of a block. It is known that ([16, Theorem 1]) if $\operatorname{atyp}(\lambda) = 0$, then $P(\lambda) = L(\lambda) = K(\lambda)$ hence $L(\lambda)$ and $K(\lambda)$ are projective.

3.3 Computing the complexity

For $\lambda \in X_0^+$, we want to find the complexity of $L(\lambda)$ and $K(\lambda)$. If λ is typical, the simple module $L(\lambda)$ is projective and hence it has zero complexity. We only need to consider the case when λ is atypical since the complexity of the typical simple and Kac modules is zero. In this section, we will refer to [8, Sections 1.2, 3.1] to get a description of the projective covers.

Let W be the Weyl group of $\mathfrak{g}$ which is, by definition, the Weyl group of the $\mathfrak{sp}(2n)$-subalgebra, generated by the reflections corresponding to the even simple roots of $\mathfrak{g}$. If $\lambda \in X_0^+$ is atypical with respect to an odd positive root γ, the authors in [8] defined an "L-operator" given by:

$$\lambda^L := \omega(\lambda + \rho - k\gamma) - \rho,$$

where k is the smallest positive integer such that $(\lambda + \rho - k\gamma, \alpha_i) \neq 0$ for all the even simple roots α_i and ω is the unique element in the Weyl group of $\mathfrak{sp}(2n)$

rendering λ^L dominant. Given an atypical $\lambda \in X_0^+$, we shall write

$$\lambda^{(0)} = \lambda, \quad \lambda^{(l+1)} = (\lambda^{(l)})^L, \quad l \geq 0. \tag{3.1}$$

Let ω_0 be the longest element in the Weyl group of $\mathfrak{sp}(2n)$ and let $\beta = 2n\varepsilon_1$ be the sum of all positive roots. Let λ be atypical, we can use [8, equation (4)] to get:

$$\lambda = \omega_0^{-1}\Big(\beta - (\beta - \omega_0\lambda^L)^L\Big),$$

which proves that the L-operator is one-to-one. We will prove that L is onto on the set of atypical weights in X_0^+ using a representation theoretical approach (Proposition 3.21). Using this we can define λ^{-L} to be the unique weight μ such that $\mu^L = \lambda$, hence we can extend the definition in (3.1) to any $l \in \mathbb{Z}$.

Consider the set $Y_+^{1|n}$ of $(n+1)$-tuples of integers:

$$Y_+^{1|n} = \{f = (f_{-1}|f_1, f_2, \ldots, f_n) \mid f_i \in \mathbb{Z},\ f_1 < f_2 < \cdots < f_n < 0\}.$$

There is a bijection $X_0^+ \to Y_+^{1|n}$ given by $\lambda \mapsto f_\lambda$, where

$$(f_\lambda)_{-1} = (\lambda + \rho, \varepsilon_1), \quad (f_\lambda)_i = (\lambda + \rho, \delta_i),\ i \geq 1.$$

Since λ is atypical, we have $(f_\lambda)_{-1} \in \mathbb{Z}$ and $|f_{-1}| = -f_i$ for some $i \geq 1$. We set $f^L := f_{\lambda^L}$. Note that we can compute λ^L for each atypical λ by passing over to the set $Y_+^{1|n}$. The description of f^L is divided into three cases as follows (cf. [8, Section 1.2]:

(I) $f_{-1} = f_i < 0$ for some $1 \leq i \leq n$. let c be the largest integer such that

$c < f_i$ and $d \notin \{f_1, f_2, \dots, f_n\}$. Then

$$f^L = (c|f_1, \dots \widehat{f_i}, \dots, f_n, c)^+,$$

where $\widehat{f_i}$ denotes the removal of f_i, and the + denotes the rearrangement of

$$f_1, \dots \widehat{f_i}, \dots, f_n, c$$

in an increasing order.

(II) $f_{-1} = -f_i > 0$ for some $1 \leq i \leq n$ and $\{f_1, f_2, \dots, f_n\}$ does not contain $\{-1, -2, \dots, f_i\}$ as a subset. Let c be the largest integer such that $f_i < c \leq -1$ and $-c \notin \{f_1, f_2, \dots, f_n\}$. Then

$$f^L = (c|f_1, \dots \widehat{f_i}, \dots, f_n, -c)^+.$$

(III) $f_{-1} = -f_i > 0$ for some $1 \leq i \leq n$ and $\{f_1, f_2, \dots, f_n\}$ contains $\{-1, -2, \dots, f_i\}$ as a subset. Then

$$f^L = (-f_{-1}|f_1, \dots, f_n).$$

Using these cases, we can show the following:

Lemma 3.1. *We have:*

1. *For* $d \geq 0$, $(-d|d, 0, \dots, 0)^L = (-d-1|d+1, 0, \dots, 0)$.

2. *For* $d \geq 1$, $(2n+d|d, 0, \dots, 0)^L = (2n+(d-1)|d-1, 0, \dots, 0)$.

3. $(2n|0, \dots, 0)^L = (0|0, \dots, 0)$.

4. For $\lambda = (0|0,\ldots,0) = 0$ and $d \geq 0$, we have

$$\lambda^{(d)} = (-d|d,0,\ldots,0) \text{ and } \lambda^{(-d-1)} = (2n+d|d,0,\ldots,0).$$

Proof. For (1), $f_\lambda = (-n-d|-(n+d),-(n-1),-(n-2),\ldots,-2,-1)$, hence $f_{-1} = f_1 < 0$, and case (I) applies. The required integer c is $-n-d-1$, then

$$f_\lambda^L = f_{\lambda^L} = (-n-d-1|-n-d-1,-(n-1),-(n-2),\ldots,-2,-1),$$

which gives $\lambda^L = (-d-1|d+1,0,\ldots,0)$.

For (2), $f_\lambda = (n+d|-(n+d),-(n-1),-(n-2),\ldots,-2,-1)$, hence $f_{-1} = -f_1 > 0$, and case (II) applies. The required integer c is $n+d-1$, then

$$f_\lambda^L = f_{\lambda^L} = (n+d-1|-n-d+1,-(n-1),-(n-2),\ldots,-2,-1),$$

which gives $\lambda^L = (2n+d-1|d-1,0,\ldots,0)$.

For (3), $f_\lambda = (n|-n,-(n-1),-(n-2),\ldots,-2,-1)$, hence $f_{-1} = -f_1 > 0$, and case (III) applies. Then

$$f_\lambda^L = f_{\lambda^L} = (-n|-n,-(n-1),-(n-2),\ldots,-2,-1),$$

which gives $\lambda^L = (0|0,0,\ldots,0)$.

The last part of the Lemma follows easily from the earlier parts. □

Let $P(\lambda)$ be the projective cover of $L(\lambda)$. From [8, Theorem 7], we have the following 2-step Kac flag:

$$0 \to K(\lambda) \to P(\lambda^L) \to K(\lambda^L) \to 0,$$

which implies that $\dim P(\lambda^L) = \dim K(\lambda) + \dim K(\lambda^L)$. Moreover, using [8, Corollary 8], there is a short exact sequence:

$$0 \to L(\lambda^L) \to K(\lambda) \to L(\lambda) \to 0.$$

This shows that the Kac module has two layers: The head of $K(\lambda)$ is $L(\lambda)$ and the socle is $L(\lambda^L)$. On the other hand, the projective module $P(\lambda^L)$ decomposes as follows: the head and socle of $P(\lambda^L)$ are both isomorphic to $L(\lambda^L)$ and the second layer is isomorphic to $L(\lambda) \oplus L(\lambda^{LL})$. It has the form:

$$\begin{array}{ccc} & L(\lambda^L) & \\ \diagup & & \diagdown \\ L(\lambda) & & L(\lambda^{LL}) \\ \diagdown & & \diagup \\ & L(\lambda^L) & \end{array}$$

Note that by the Kac filtration of $P(\lambda^L)$ and the composition factors of $K(\lambda)$ we know that the socle of $P(\lambda^L)$ has only one submodule, namely $L(\lambda^L)$. Similarly the head of $P(\lambda^L)$ is the head of $K(\lambda^L)$ which is $L(\lambda^L)$. We can also see this from the fact that the $P(\lambda^L)$ is the projective cover and injective hull of $L(\lambda^L)$, $L(\lambda^L)$ is the unique head and socle at the same time. We will prove later in (3.13) that

$$\dim \operatorname{Ext}^1(L(\lambda), L(\mu)) = 1 \Leftrightarrow \lambda = \mu^L \text{ or } \lambda^L = \mu.$$

Thus there is no indecomposable module M such that the following sequence is exact:

$$0 \to L(\lambda) \to M \to L(\lambda^{LL}) \to 0.$$

This shows that the middle layer can not be of the form

$$\begin{array}{c} L(\lambda^{LL}) \\ | \\ L(\lambda) \end{array}$$

3.3.1 Complexity of simple modules

To compute the complexity of the simple modules $L(\lambda)$, we need the following bounds on the dimension of the simple $\mathfrak{sp}(2n)$-module $L_0(r, 0, \ldots, 0)$:

Lemma 3.2. *There are positive constants C and C' that depend only on n such that*

$$Cr^{2n-1} \leq \dim L_0(r, 0, \ldots, 0) \leq C'r^{2n-1}.$$

Proof. The dimension of a simple module of highest weight λ is given by the Weyl-dimension formula ([20, Section 24.3]):

$$\dim L_0(\lambda) = \frac{\prod_{\alpha \in \Phi_{\bar{0}}^+}(\lambda + \delta, \alpha)}{\prod_{\alpha \in \Phi_{\bar{0}}^+}(\delta, \alpha)},$$

where δ is half the sum of the positive roots in $\mathfrak{sp}(2n)$, i.e., half the sum of the positive even roots in $\mathfrak{osp}(2|2n)$.

Recall

$$\Phi_{\bar{0}}^+ = \{\delta_i + \delta_j \mid 1 \leq i \leq j \leq n\} \cup \{\delta_i - \delta_j \mid 1 \leq i < j \leq n\},$$

then $\delta = \sum_{i=1}^{n}(n - i + 1)\delta_i$. Thus,

$$\dim L_0(r, 0, \ldots, 0) = \frac{(2n + r)\prod_{j=2}^{n}(r + j - 1)(2n + r - j + 1)}{(2n)\prod_{j=2}^{n}(j - 1)(2n - j + 2)}.$$

Let

$$C = \frac{1}{(2n)\prod_{j=2}^{n}(j-1)(2n-j+2)}.$$

Note that C is a positive constant depending only on n and $\dim L_0(r,0,\ldots,0)$ is a polynomial in r of degree $2n-1$ with a positive leading coefficient. Moreover,

$$\begin{aligned}
\dim L_0(r,0,\ldots,0) &= C_1(2n+2r)\prod_{j=2}^{n}(r+j-1)(2n+r-j+1)\\
&\geq C_1(2n+2r)(r+1)^{n-1}(2n+r-n+1)^{n-1}\\
&= C_1(2n+2r)(r+1)^{n-1}(n+r+1)^{n-1}\\
&\geq C_1 r^{2n-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\dim L_0(r,0,\ldots,0) &= C_1(2n+2r)\prod_{j=2}^{n}(r+j-1)(2n+r-j+1)\\
&\leq C_1(2n+2r)(r+n-1)^{n-1}(2n+r-2+1)^{n-1}\\
&= C_1(2n+2r)(r+n-1)^{n-1}(2n+r-1)^{n-1},
\end{aligned}$$

Let us pick positive constants C_2, C_3, C_4 depending only on n such that

$$2n+2r \leq C_2 r, \quad r+n-1 \leq C_3 r, \quad 2n+r-1 \leq C_4 r,$$

then

$$\dim L_0(r,0,\ldots,0) \leq C.C_2.{C_3}^{n-1}.{C_4}^{n-1} r^{2n-1} = C' r^{2n-1},$$

where C' is a positive constant that depends only on n. Therefore, there are positive constants C and C' that depend only on n such that

$$Cr^{2n-1} \leq \dim L_0(r, 0, \ldots, 0) \leq C' r^{2n-1}.$$

$\square$

Recall that

$$0^{(d)} = (-d|d, 0, \ldots, 0) \quad \text{and} \quad 0^{(-d-1)} = (2n + d|d, 0, \ldots, 0).$$

For $i \in \mathbb{Z}$, set

$$[i] = L(0^{(i)}) \quad P^{(i)} = P(0^{(i)}), \text{ and } K^{(i)} = K(0^{(i)}).$$

Using these notations, the projective covers $P^{(i)}$ have the following radical layer structure:

$$\begin{array}{ccccc} & & [i] & & \\ & \diagup & & \diagdown & \\ [i-1] & & & & [i+1] \\ & \diagdown & & \diagup & \\ & & [i] & & \end{array}$$

Lemma 3.3. *Let*

$$P_0 = P(0) = \begin{array}{ccccc} & & [-1] & & \\ & \diagup & & \diagdown & \\ [-2] & & & & [0] \\ & \diagdown & & \diagup & \\ & & [-1] & & \end{array}$$

$$P_1 = P^{(-1)} \oplus P^{(1)} \begin{array}{ccccc} & & [-1] & & \\ & \diagup & & \diagdown & \\ [-2] & & & & [0] \\ & \diagdown & & \diagup & \\ & & [-1] & & \end{array} \oplus \begin{array}{ccccc} & & [1] & & \\ & \diagup & & \diagdown & \\ [0] & & & & [2] \\ & \diagdown & & \diagup & \\ & & [1] & & \end{array}$$

$$P_2 = P^{(-2)} \oplus P(0) \oplus P^{(2)} = \begin{array}{ccccc} & & [-2] & & \\ & \diagup & & \diagdown & \\ [-3] & & & & [-1] \\ & \diagdown & & \diagup & \\ & & [-2] & & \end{array} \oplus \begin{array}{ccccc} & & [0] & & \\ & \diagup & & \diagdown & \\ [-1] & & & & [1] \\ & \diagdown & & \diagup & \\ & & [0] & & \end{array} \oplus \begin{array}{ccccc} & & [2] & & \\ & \diagup & & \diagdown & \\ [1] & & & & [3] \\ & \diagdown & & \diagup & \\ & & [2] & & \end{array}$$

The first three terms of a minimal projective resolution of $L(0)$ are:

$$P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} [0] \to 0$$

Proof. To prove this result we use the diagrammatic method for modular representations given in [1]. In particular, we use the description given in [1, Section 10.3] for the kernel of the surjective map $P(M) \twoheadrightarrow M$ where $P(M)$ is the projective cover of $M \in \mathcal{F}$. We also use the fact that $P(M)$ is the direct sum of the projective covers of the simple modules in the head of M.

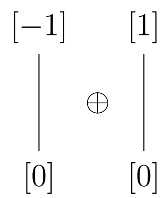
Let $f_0 : P(0) \twoheadrightarrow L(0) = [0]$ be the projection map from $P(0)$ onto its head. Let $Ker(f_0)$ be the kernel of f_0. Sine the map f_0 projects the head of $P(0)$ onto $L(0)$, we have $P(0)/Ker(f_0) \cong [0]$. Thus the kernel of f_0 is the submodule of $P(0)$ with the layer structure:

$$Ker(f_0) = \begin{array}{ccccc} [-1] & & & & [1] \\ & \diagdown & & \diagup & \\ & & [0] & & \end{array}$$

By taking a quotient map we have

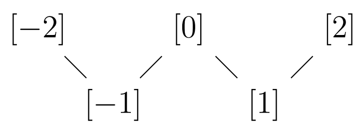
$$\begin{array}{ccccc} & & [-1] & & \\ & \diagup & & \diagdown & \\ [-2] & & & & [0] \\ & \diagdown & & \diagup & \\ & & [-1] & & \end{array} \oplus \begin{array}{ccccc} & & [1] & & \\ & \diagup & & \diagdown & \\ [0] & & & & [2] \\ & \diagdown & & \diagup & \\ & & [1] & & \end{array} \twoheadrightarrow \begin{array}{c} [-1] \\ | \\ [0] \end{array} \oplus \begin{array}{c} [1] \\ | \\ [0] \end{array} \overset{\phi_1}{\twoheadrightarrow} \begin{array}{ccccc} [-1] & & & & [1] \\ & \diagdown & & \diagup & \\ & & [0] & & \end{array} \hookrightarrow \begin{array}{ccccc} & & [0] & & \\ & \diagup & & \diagdown & \\ [-1] & & & & [1] \\ & \diagdown & & \diagup & \\ & & [0] & & \end{array}$$

The map ϕ_1 can be chosen to satisfy $\phi_1(x, y) = x - y$ for $x \in [0]$ in the first summand and $y \in [0]$ in the second. Define f_1 to be the composition of these maps. Note that to find the projective cover of a module M, we take the direct sum of the projective covers of the simple modules which appear in the top level of M. Thus P_1 is the projective cover of

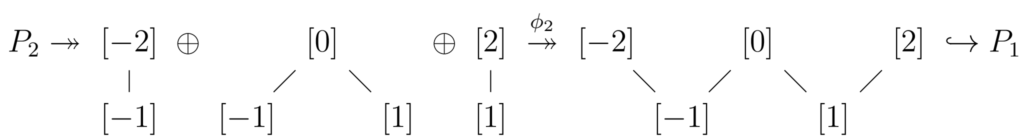


This means that P_1 is the minimal projective module that we can choose to make the sequence exact. The Kernel of f_1 is

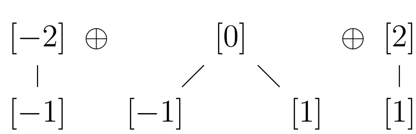
$Ker(f_1) =$



We again take a quotient map to get



The map ϕ_2 can be defined in a similar fashion as ϕ_1. Define f_2 to be the composition of these maps. As before, P_2 is the minimal projective module we could choose to make the sequence exact. This is the case because it is the projective cover of



The kernel of f_2 is

$$Ker(f_2) = \begin{array}{ccccccccccccc} [-3] & & & & [-1] & & & & [1] & & & & [3] \\ & \diagdown & & \diagup & & \diagdown & & \diagup & & \diagdown & & \diagup & \\ & & [-2] & & & & [0] & & & & [2] & & \end{array}$$

The exactness of the sequence follows from the construction of the maps f_i. □

Lemma 3.4. *The minimal projective resolution of* $L(0)$ *is*

$$\dots P_d \xrightarrow{f_d} \dots \xrightarrow{f_1} P(0) \xrightarrow{f_0} L(0) \to 0, \tag{3.2}$$

where the d^{th} *term in this resolution is given by: If* d *is even,*

$$P_d = P^{(-d)} \oplus P^{(-d+2)} \oplus \dots \oplus P^{(-2)} \oplus P^{(0)} \oplus P^{(d)} \oplus P^{(d-2)} \oplus \dots \oplus P^{(2)},$$

and if d *is odd,*

$$P_d = P^{(-d)} \oplus P^{(-d+2)} \oplus \dots \oplus P^{(-1)} \oplus P^{(d)} \oplus P^{(d-2)} \oplus \dots \oplus P^{(1)}.$$

The kernel of f_d *is*

$$Ker(f_d) = \begin{array}{ccccccccccccccccccccc} [-d-1] & & & & [-d+1] & & & & [-1] & & & & [1] & & & & [d-1] & & & & [d+1] \\ & \diagdown & & \diagup & & \diagdown & & \diagup & & \diagdown & & \diagup & & \diagdown & & \diagup & & \diagdown & & \diagup & \\ & & [-d] & & & & \dots & & & & [0] & & & & \dots & & & & [d] & & \end{array}$$

Proof. An inductive argument proves the result. The base case is proved in lemma 3.3. We again use the diagrammatic method for modular representations given in [1]. In particular, we use the description given in [1, Section 10.3] for the kernel of the surjective map $P(M) \twoheadrightarrow M$ where $P(M)$ is the projective cover of $M \in \mathcal{F}$. We also use the fact that $P(M)$ is the direct sum of the projective covers of the simple modules in the head of M. $\square$

Theorem 3.5. *For atypical* $\lambda \in X_0^+$, $c_{\mathcal{F}}(L(\lambda)) = 2n + 1$.

Proof. Since $\dim P(\lambda^L) = \dim K(\lambda) + \dim K(\lambda^L)$, we have for $i \in \mathbb{Z}$,

$$\dim P^{(i)} = \dim K^{(i)} + \dim K^{(i-1)}.$$

By the PBW basis of $U(\mathfrak{g})$, we have $\dim K(\lambda) = 2^{\dim \mathfrak{g}_{-1}}. \dim L_0(\lambda)$ where $L_0(\lambda)$ is the simple $\mathfrak{sp}(2n)$-module of highest weight λ (where we omit the λ_{-1} from λ). Hence for $i \in \mathbb{Z}$,

$$\dim K^{(i)} = 2^{2n}. \dim L_0(i, 0, \ldots, 0).$$

Consider the case when d is even. Using the minimal projective resolution in (3.2) we have:

$$\begin{aligned}
\dim P_d &= \dim P^{(-d)} + \dim P^{(d)} + \cdots + \dim P^{(-2)} + \dim P^{(2)} + \dim P^{(0)} \\
&= \sum_{i=-d}^{d} \dim K^{(i)} \\
&= 2^{2n} \sum_{i=1}^{d} \Big(2 \dim L_0(i, 0, \ldots, 0) \Big) + \dim L_0(0, 0, \ldots, 0) \\
&\leq 2^{2n+1} \sum_{i=1}^{d} C_1' i^{2n-1} + 1 \qquad (\textit{Lemma } 3.2) \\
&\leq 2^{2n+1} \cdot C_1' d \cdot d^{2n-1} + 1 \\
&\leq 2^{2n+1} \cdot C_1' d^{2n} + d^{2n} \\
&= k' d^{2n},
\end{aligned}$$

where k' is a positive constant that depends only on n. On the other hand,

$$\begin{aligned}
\dim P_d &= 2^{2n} \sum_{i=1}^{d} \Big(2 \dim L_0(i, 0, \ldots, 0)\Big) + \dim L_0(0, 0, \ldots, 0) \\
&\geq 2^{2n+1} \sum_{i=1}^{d} C_1 i^{2n-1} + 1 \qquad (\textit{Lemma } 3.2) \\
&\geq 2^{2n+1} \cdot C_1 \sum_{i=d/2}^{d} i^{2n-1} \\
&\geq 2^{2n+1} \cdot C_1 \sum_{i=d/2}^{d} (\frac{d}{2})^{2n-1} \\
&= kd^{2n},
\end{aligned}$$

where k is a positive constant that depends only on n. Thus when d is even, there exist constants k, k' depending only on n such that:

$$kd^{2n} \leq \dim P_d \leq k' d^{2n}. \tag{3.3}$$

The same argument can be used to show that inequality (3.3) holds when d is odd. Therefore, $c_{\mathcal{F}}(L(0|0, \ldots, 0) = 2n + 1$. By [6, Theorem 4.1.1], all simple modules of the same atypicality have the same complexity. Thus the complexity of all atypical simple $\mathfrak{osp}(2|2n)$-modules is $2n + 1$. □

3.3.2 Complexity of Kac modules

Using the computations of the complexity of the trivial module, we can compute the complexity of the Kac module $K(0)$:

Proposition 3.6.

$$c_{\mathcal{F}}(K(0)) = 2n.$$

Proof. Using the notation developed in this section, the minimal projective resolution of $K(0)$ is given by:

$$\ldots \to P^{(-2)} \to P^{(-1)} \to P(0) \to K(0) \to 0. \tag{3.4}$$

For $d \geq 1$, we have

$$\begin{aligned}\dim P_d &= \dim P^{(-d)} \\ &= \dim K^{(-d)} + \dim K^{(-d-1)} \\ &= 2^{2n}\Big(\dim L_0(d,0,\ldots,0) + \dim L_0(d-1,0,\ldots,0)\Big),\end{aligned}$$

hence $\dim P_d$ is a polynomial in d of degree $2n-1$. Therefore,

$$c_{\mathcal{F}}(K(0)) = 2n.$$

□

We will use the complexity of $K(0)$ to compute the complexity of any atypical Kac module. First we show that that the L-operator on the weights does not change the complexity of the Kac modules. Then we give an explicit description of the block $\mathcal{F}^{\chi_0}$, where χ_0 is the central character corresponding to the weight $\lambda = (0|0,\ldots,0)$. This description will be obtained using the notion of weight diagrams given in [12, Section 6], and the characterization of blocks given in [12, Section 5]. We then use the fact that translation functors preserve the complexity to show that $c_{\mathcal{F}}(K(\lambda)) = 2n$.

Lemma 3.7. *For $\lambda \in X_0^+$, $c_{\mathcal{F}}(K(\lambda)) = c_{\mathcal{F}}(K(\lambda^{(l)}))$, $l \in \mathbb{Z}$.*

Proof. It is sufficient to prove that

$$c_{\mathcal{F}}(K(\lambda)) = c_{\mathcal{F}}(K(\lambda^{L})).$$

The complexity of any module $M \in \mathcal{F}$ is given by:

$$c_{\mathcal{F}}(M) = r(\mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, \bigoplus L(\mu)^{\dim P(\mu)})), \tag{3.5}$$

where the sum is over all simple modules in $\mathcal{F}$. Let $S = \bigoplus L(\mu)^{\dim P(\mu)}$. By applying the functor $\mathrm{Hom}(\ \ , S)$ to the following 2-step Kac flag:

$$0 \to K(\lambda) \to P(\lambda^L) \to K(\lambda^L) \to 0,$$

we get:

$$\ldots \to \mathrm{Ext}^{d+1}(P(\lambda^L), S) \to \mathrm{Ext}^{d+1}(K(\lambda), S) \to \mathrm{Ext}^{d}(K(\lambda^L), S) \to$$
$$\ldots \to \mathrm{Ext}^{1}(K(\lambda^L), S) \to \mathrm{Ext}^{1}(P(\lambda^L), S) \to \mathrm{Ext}^{1}(K(\lambda), S) \to$$
$$\mathrm{Hom}(K(\lambda^L), S) \to \mathrm{Hom}(P(\lambda^L), S) \to \mathrm{Hom}(K(\lambda), S) \to 0.$$

Since $P(\lambda^L)$ is projective, then $\mathrm{Ext}^d(P(\lambda^L), S) = 0$ for all $d \geq 1$, which gives:

$$\mathrm{Ext}^{d}(K(\lambda^L), S) = \mathrm{Ext}^{d+1}(K(\lambda), S).$$

This implies that $\mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda), S)$ and $\mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda^L), S)$ will have the same rate of growth, hence the theorem follows. □

3.3.3 Weight diagrams and translation functors

Assume λ is atypical

$$\lambda + \rho = a_1\varepsilon_1 + b_1\delta_1 + \ldots + b_n\delta_n,$$

The weight diagram of λ is the function $\tilde{f}_\lambda : \mathbb{Z}_{\geq 0} \to \{>, <, \times, 0\}$ represented by a diagram according to the following algorithm:

1. Put the symbol $>$ in position t if $t = |a_1|$.

2. Put the symbol $<$ in position t for all i such that $t = b_i$.

3. If there are both $>$ and $<$ in the same position replace them by the symbol $\times$, repeat if possible.

4. Put 0 otherwise.

For example, the weight diagram of $0 = (0|0, \ldots, 0)$ is given by:

$$0, <, <, \ldots, <, \times, \ldots,$$

where the first 0 is at position 0, the first $<$ is at position 1, the last $<$ is at position $n-1$, $\times$ is at position n, the dots after that stand for empty positions (or zeros). Note that the $\times$ at position n comes from having $|a_1| = b_1 = n$ in $0 + \rho$. In fact, the number of $\times$ in the weight diagram $\tilde{f}_\lambda$ equals the degree of atypicality of λ. In the case $\text{atyp}(\lambda) = 1$, we will not have any $>$ by itself in $\tilde{f}_\lambda$. Define the *core* of λ to have the same diagram as $\tilde{f}_\lambda$ but with replacing all $\times$'s by zeros. Thus the weight diagrams of λ and its core have $<, >$ at the same positions. For example, $\tilde{f}_{core(0)}$ is given by:

$$0, <, <, \ldots, <, 0, \ldots.$$

Remark 3.8. *It is important to note the relation with the notation developed in [8, Section 1.2] to compute λ^L. In fact, $f_i = (\lambda + \rho, \delta_i) = -b_i$.*

We have the following characterization of the blocks in $\mathcal{F}$ in [12, Section 5]:

Proposition 3.9. *Two dominant weights λ and μ belong to the same block if and only if they have the same atypicality and the same core.*

Corollary 3.10. *Let χ_λ be the central character corresponding to the weight λ and $\mathcal{F}^{\chi_\lambda}$ be the corresponding block. Then λ and $\lambda^{(l)}$, $l \in \mathbb{Z}$, belong to the same block $\mathcal{F}^{\chi_\lambda}$.*

Proof. It is sufficient to prove that λ and λ^L are in the same block. By the definition of the L-operator, λ^L and λ have the same atypicality. By proposition 3.9, we have to show that λ and λ^L have the same core. Consider the three cases that defined f^L in Section 3.3.1 and suppose $f_{-1} = \pm f_i$ for some $1 \leq i \leq n$. In the weight diagram of $core(\lambda)$, we will have $<$ at the positions $-f_j$, $j \neq i$. However, by the definition of f^L, we will have $<$ at the same $-f_j$, $j \neq i$ in the weight diagram of $core(\lambda^L)$. The corollary follows. $\square$

In the following we have an explicit description of the block $\mathcal{F}^{\chi_0}$:

Lemma 3.11. $\mathcal{F}^{\chi_0} = \{\lambda \in X_0^+ \mid \lambda = 0^{(l)}$ *for some* $l \in \mathbb{Z}\}$.

Proof. From Corollary 3.10, $0^{(l)} \in \mathcal{F}^{\chi_0}$, for all $l \in \mathbb{Z}$. Now assume $\lambda \in \mathcal{F}^{\chi_0}$. Then $\text{atyp}(\lambda) = 1$ and λ has the same core as 0. Thus the weight diagram of λ has one $\times$ and $n-1$ $<$'s at positions $1, 2, \ldots, n-1$ as in the weight diagram of 0. The $\times$ cannot be at the zero position because λ will not be dominant in that case, so it will be at some position $k \geq n$. The weight diagram of λ

$$0, <, <, \ldots, <, 0, \ldots, 0, \times, \ldots,$$

where the $\times$ is at position k, from which get:

$$\lambda + \rho = (-k|k, n-1, \ldots, 2, 1) \quad \text{or} \quad \lambda + \rho = (k|k, n-1, \ldots, 2, 1).$$

Thus we have two weights:

$$\lambda = (-k+n|k-n, 0, \ldots, 0) \quad \text{or} \quad \lambda = (k+n|k-n, 0, \ldots, 0).$$

Let $d = k - n \geq 0$. If $d = 0$, then $\lambda = 0$ or $\lambda = (2n|0, 0, \ldots, 0)$. By Lemma 3.1, $(2n|0, \ldots, 0)^L = 0$. Thus $\lambda = 0^{(0)}$ or $\lambda = 0^{(-1)}$.
If $d \geq 1$, then $\lambda = (-d|d, 0, \ldots, 0)$ or $\lambda = (2n + d|d, 0, \ldots, 0)$. By Lemma 3.1, $(-d|d, 0, \ldots, 0) = 0^{(d)}$ and $(2n + d|d, 0, \ldots, 0)^{(d+1)} = 0$. Thus $\lambda = 0^{(d)}$ or $\lambda = 0^{(-d-1)}$. This completes the proof. □

Moreover,

Lemma 3.12. *If* $\lambda \in \mathcal{F}^{\chi_\lambda}$ *with* $\operatorname{atyp}(\lambda) = 1$*, then*

$$c_{\mathcal{F}}(K(\lambda) = c_{\mathcal{F}}(K(\mu)),$$

for some $\mu \in \mathcal{F}^{\chi_0}$*.*

Proof. The translation functors defined in [12, Section 5] moves a simple module $L(\lambda) \in \mathcal{F}^{\chi}$ to $L(\mu) \in \mathcal{F}^{\tau}$ such that f_μ is obtained from f_λ by moving a symbol $<$ or $>$ at position t to the next right position or to the next left position. We can also exchange these two symbols with 0 and $\times$. We do not allow a symbol to move from or to the zero position.

Let χ_λ be the central character corresponding to the weight λ and $\mathcal{F}^{\chi_\lambda}$ be the corresponding block. Let $\mathcal{F}^{\chi_0}$ be the block containing the trivial module. The translation functors define an equivalence of blocks between $\mathcal{F}^{\chi_\lambda}$ and $\mathcal{F}^{\chi_0}$. To see this, we note that $core(\lambda)$ has $<$ at $n-1$ positions, the same as $core(0)$. Assume the $\times$ in f_λ is at position i for some $i \in \mathbb{N}$, then the translation functors allow us to move the $<$'s to the positions $1, 2, \ldots, n-1$ and move $\times$ to some position $k \geq n$. The diagram we get is f_μ where $\mu \in \mathcal{F}^{\chi_0}$. The same discussion as in [2, Section 6.3] shows that the translation functors preserve the complexity of any $\mathfrak{g}$-module. This completes the proof. □

Therefore,

Theorem 3.13. *For atypical $\lambda \in X_0^+$, $c_{\mathcal{F}}(K(\lambda)) = 2n$.*

Proof. For an atypical $\lambda \in X_0^+$, we use Lemmas 3.12, 3.7, and 3.11 to get

$$\begin{aligned} c_{\mathcal{F}}(K(\lambda)) &= c_{\mathcal{F}}(K(\mu)) \quad \text{for some } \mu \in \mathcal{F}^{\chi_0} \\ &= c_{\mathcal{F}}(K(0^{(l)})) \quad \text{for some } l \in \mathbb{Z} \\ &= c_{\mathcal{F}}(K(0)) = 2n. \end{aligned}$$

□

3.4 Support, rank and associated varieties

3.4.1 Support variety

Support varieties were defined in chapter 1. The dimensions of the support varieties of the simple modules and the Kac modules in $\mathcal{F}$ are known:

Lemma 3.14. *[6, Cor 4.4.2]*
For $\lambda \in X_0^+$, $\dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(L(\lambda)) = \operatorname{atyp}(\lambda)$.

Lemma 3.15. *[5, Cor 3.2]*
For $\lambda \in X_0^+$, $\dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda)) = 0$.

3.4.2 Rank variety

The rank variety will be introduced to give a geometric interpretation of the complexity of Kac modules. Furthermore, it will be used to compute the associated variety of the Kac module. For the original definition of the rank variety, and for more details, we refer the reader to [4, Subsection 6.3].

Let $\mathcal{F}(\mathfrak{g}_{\pm 1})$ be the category of finite dimensional $\mathfrak{g}_{\pm}$-modules. Let $M \in \mathcal{F}(\mathfrak{g}_{\pm 1})$ and consider the following rank variety of M:

$$\mathcal{V}_{\mathfrak{g}_{\pm 1}}^{rank}(M) := \{x \in \mathfrak{g}_{\pm 1} \mid \text{M is not projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\},$$

where $U(\langle x \rangle)$ denotes the enveloping algebra of the Lie superalgebra generated by $x \in \mathfrak{g}_{\pm 1}$. The rank variety has various properties such as detecting projectivity:

Proposition 3.16. *[3, Theorem 3.5.1(a)] Let M be a $\mathfrak{g}$-module in the category $\mathcal{F}$. Then M is projective if and only if $\mathcal{V}_{\mathfrak{g}_1}^{rank}(M) = \mathcal{V}_{\mathfrak{g}_{-1}}^{rank}(M) = 0$.*

As a consequence,

Corollary 3.17. *If $\lambda \in X_0^+$ is typical, then*

$$\mathcal{V}_{\mathfrak{g}_1}^{rank}(L(\lambda)) = \mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda)) = 0.$$

Proof. $L(\lambda)$ and $K(\lambda)$ are projective and the result follows from Proposition 3.16.

□

3.4.3 Rank variety of Kac modules

Consider the $\mathfrak{g}_{+1}$ case. We start by computing the rank variety, $\mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda))$, of an atypical Kac module in $\mathfrak{g} = \mathfrak{osp}(2|2)$.

Proposition 3.18. *For $\mathfrak{g} = \mathfrak{osp}(2|2)$ and for an atypical $\lambda \in X_0^+$, $\mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda)) = \mathfrak{g}_1$.*

Proof. Since $\mathfrak{g}_1$ is abelian, $[x, x] = 0$ for all $x \in \mathfrak{g}_1$. By [4, Proposition 5.4], the only simple $U(\langle x \rangle)$-module is the trivial module $\mathbb{C}$ and the indecomposable modules are $\mathbb{C}$ and $U(\langle x\rangle)$ where $U(\langle x\rangle)$ is two dimensional and it is the projective cover of $\mathbb{C}$. Hence, $K(\lambda)$ will be projective as a $U(\langle x\rangle)$-module if and only if it does not contain a copy of the trivial module. Thus, $x \in \mathfrak{g}_1$ is an element of $\mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda))$ if and only if $K(\lambda)$ contains a copy of the trivial module when viewed as a $U(\langle x\rangle)$-module.

By [3, 3.8.4], the action of $G_{\bar{0}} = \mathbb{C}^* \times Sp(2n)$ on $\mathfrak{g}_1$ has two orbits $\{0\}$ and $\mathfrak{g}_1 \setminus \{0\}$. Note that $\mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda))$ is a closed $G_{\bar{0}}$-invariant subvariety of $\mathcal{V}_{\mathfrak{g}_1}^{rank}(\mathbb{C}) = \mathfrak{g}_1$. Since the closed subvarieties of $\mathcal{V}_{\mathfrak{g}_1}^{rank}(\mathbb{C}) = \mathfrak{g}_1$ are the closure of those orbits, we have $\mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda)) = \{0\}$ or $\mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda)) = \mathfrak{g}_1$.

Hence, to prove the Proposition it is enough to find $0 \neq x \in \mathfrak{g}_1$ such that $K(\lambda)$ contains a copy of the trivial module when viewed as a $U(\langle x\rangle)$-module. Note that $\mathfrak{g}_1$ is spanned by the root vectors x_1 and x_2 with weights $wt(x_1) = \varepsilon_1 - \delta_1$ and $wt(x_2) = \varepsilon_1 + \delta_1$. On the other hand, $\mathfrak{g}_{-1}$ is spanned by y_1 and y_2 with weights $wt(y_1) = -wt(x_1) = -\varepsilon_1 + \delta_1$ and $wt(y_2) = -wt(x_2) = -\varepsilon_1 - \delta_1$. We will need the following elements of $\mathfrak{g}_{\bar{0}}$ as we proceed:

$$z_{11} = 1 \oplus h, \quad z_{22} = -1 \oplus h, \quad z_{12} = 0 \oplus -2f, \quad \text{and} \quad z_{21} = 0 \oplus 2e,$$

where $\{e, f, h\}$ is the standard basis of sl(2). Since $\mathfrak{osp}(2|2)_{\bar{0}} \cong \mathbb{C} \oplus \mathfrak{sl}_2$, the simple $\mathfrak{g}_{\bar{0}}$-modules are of the form $L_{\bar{0}}(\lambda) = \mathbb{C}_{\lambda_{-1}} \boxtimes L(d)$ where $L(d)$ is the simple $\mathfrak{sl}_2$-module of dimension $d+1$. Note that the action of $\mathfrak{g}_{\bar{0}}$ on $L_{\bar{0}}(\lambda) = \mathbb{C}_{\lambda_{-1}} \boxtimes L(d)$ is given by

$$(\alpha \oplus T).(c_{\lambda_{-1}} \otimes v) = \alpha.c_{\lambda_{-1}} \otimes v + c_{\lambda_{-1}} \otimes T.v,$$

for α, $c_{\lambda_{-1}} \in \mathbb{C}$, $T \in \mathrm{sl}(2)$ and $v \in L(d)$.

Let $\{v_k \mid 0 \leq k \leq d\}$ be the basis for $L(d)$ with v_0 being the highest weight vector given in [20, Subsection 7.2]. Then for $0 \le k \le d$, we have

$$h.v_k = (d-2k)v_k, \quad e.v_k = (d-k+1)v_{k-1}, \quad f.v_k = (k+1)v_{k+1},$$

with $v_{-1} = v_{d+1} = 0$.

Let $w_k = 1 \otimes v_k$ then $\{w_k \mid 0 \le k \le d\}$ is a basis of $L_{\bar{0}}(\lambda)$ with w_0 being the highest weight vector. We can easily show that for $0 \le k \le d$, we have

$$z_{11}.w_k = -2kw_k, \quad z_{12}.w_k = -2(k+1)w_{k+1}, \quad z_{21}.w_k = 2(d-k+1)w_{k-1},$$

with $w_{-1} = w_{d+1} = 0$. By the PBW-basis theorem for $U(\mathfrak{g})$, $\dim K(\lambda) = 4(d+1)$ with basis $\{y_1^a y_2^b \otimes w_k \mid a, b \in \{0,1\}, 1 \le k \le d\}$.

In the case of $\mathfrak{osp}(2|2)$, $\rho = (-1|1)$ then $\lambda = (\lambda_{-1}|\lambda_1) = (\lambda_{-1}|d)$ is atypical if and only if $(\rho + \lambda, \varepsilon_1 \pm \delta_1) = 0$. Thus we have two cases:

Case 1: Using the odd root $wt(x_1) = \varepsilon_1 - \delta_1$, we have $\lambda_{-1} = -\lambda_1$. We will write $\lambda = (-d|d)$.

we can check that $x_1(1 \otimes w_0) = 1 \otimes x_1 w_0 = 0$ since w_0 is annihilated by $\mathfrak{g}_1$. By degrees (or $\mathbb{Z}$-grading), $y_i \otimes w_k$ could be in the $U(\langle x_1 \rangle)$-submodule generated by $1 \otimes w_0$, but using weights, we can check that only $y_1 \otimes w_0$ could land in that submodule. However,

$$x_1(y_1 \otimes w_0) = -y_1 \otimes x_1 w_0 + 1 \otimes z_{11} w_0 = 0 + (-d+d)(1 \otimes w_0) = 0,$$

therefore, the $U(\langle x_1 \rangle)$-submodule generated by $1 \otimes w_0$ forms a trivial direct summand of $K(\lambda)$ when viewed as a $U(\langle x_1 \rangle)$-module. Thus $0 \ne x_1 \in \mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda))$, the result follows in this case.

Case 2: Using the odd root $wt(x_2) = \varepsilon_1 + \delta_1$, we have $\lambda_{-1} = \lambda_1 + 2$. We will write $\lambda = (d+2|d)$. Then we have:

$$\begin{aligned} x_2(y_1y_2 \otimes w_0) =& (-y_1x_1 + z_{21})y_2 \otimes w_0 \\ =& -y_1(-y_2x_2 + z_{22}) \otimes w_0 + (2y_1 + y_1z_{21}) \otimes w_0 \\ =& y_1y_2 \otimes x_2w_0 - y_1 \otimes z_{22}w_0 + 2y_1 \otimes w_0 + y_1 \otimes z_{21}w_0 \\ =& 0 - y_1 \otimes (-2w_0) + 2y_1 \otimes w_0 + 0 = 0. \end{aligned}$$

In the above we used the fact the x_2 and z_{21} both annihilate the highest weight vector w_0. Since $y_1y_2 \otimes w_0$ has the largest possible degree (i.e. -2), we know that the $U(\langle x_2 \rangle)$-submodule generated by $y_1y_2 \otimes w_0$ forms a trivial direct summand of $K(\lambda)$ when viewed as a $U(\langle x_2 \rangle)$-module. Thus $0 \neq x_2 \in \mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda))$, the result follows in this case. □

To generalize the above theorem, we need to use the equivalence of blocks stated in [12, Theorem 2]. Indeed, we have:

Theorem 3.19. *When λ is atypical, $\mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda)) = \mathfrak{g}_1$.*

Proof. Assume $\operatorname{atyp}(\lambda) = k = 1$. Let χ_λ be the central character corresponding to the weight λ and let $\mathcal{F}^{\chi_\lambda}$ be the corresponding block. [12, Theorem 2] implies that $\mathcal{F}^{\chi_\lambda}$ is equivalent to the maximal block of $\mathfrak{g}_k$ containing the trivial module, where $\mathfrak{g}_k = \mathfrak{osp}(2|2)$ or $\mathfrak{g}_k = \mathfrak{osp}(4|2)$. However, $\mathfrak{g}_k$ is chosen to correspond to a connected sub-Dynkin diagram containing the last node(s) of the diagram of $\mathfrak{g}$ and this is not the case if we use $\mathfrak{osp}(4|2)$ because it has the type $D(2,1)$ while $\mathfrak{osp}(2|2n)$ has type $C(n+1)$. Set $\mathfrak{g}' = \mathfrak{g}_k = \mathfrak{osp}(2|2)$. Using the discussion in [6, Section 4], $\mathfrak{g}'$ is the subalgebra of $\mathfrak{g}$ whose roots lie in the intersection of Φ with the $\mathbb{R}$-span of $\{\pm\varepsilon_1 \pm \delta_n, \pm 2\delta_n\}$.

As discussed in [9], the equivalence of blocks is a composition of translation functors between the blocks of $\mathcal{F}$, followed by a restriction functor from $\mathfrak{g}$ to $\mathfrak{g}'$.

Upon restricting to $\mathfrak{g}'$, we have:

$$K(\lambda) = K(\lambda') \oplus M,$$

where $K(\lambda')$ is a Kac $\mathfrak{g}'$-module (the above equivalence of blocks takes Kac modules to Kac modules) and M is a $\mathfrak{g}'$-module. By Serganova's equivalence, $\text{atyp}(\lambda') = \text{atyp}(\lambda) = 1$, thus there exists $0 \neq x' \in \mathcal{V}_{\mathfrak{g}_1'}^{rank}(K(\lambda')) = \mathfrak{g}_1'$, (3.18). By [4, Theorem 6.6] the rank variety distributes over the direct sum as a union , thus

$$\mathcal{V}_{\mathfrak{g}_1'}^{rank}(K(\lambda)) = \mathcal{V}_{\mathfrak{g}_1'}^{rank}(K(\lambda')) \cup \mathcal{V}_{\mathfrak{g}_1'}^{rank}(M).$$

This shows that there exists $0 \neq x' \in \mathfrak{g}' \subseteq \mathcal{V}_{\mathfrak{g}_1'}^{rank}(K(\lambda))$. Using the embedding $\mathfrak{g}' \hookrightarrow \mathfrak{g}$, we can find $0 \neq x \in \mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda))$ but the $G_{\bar{0}}$-orbits of $\mathfrak{g}_1$ are $\{0\}$ and $\mathfrak{g}_1 \setminus \{0\}$ which imply that $\mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda)) = \mathfrak{g}_1$. □

Thus we can interpret the complexity of Kac modules geometrically in the following sense:

Corollary 3.20. *For $\lambda \in X_0^+$, $c_{\mathcal{F}}(K(\lambda)) = \dim \mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda))$.*

Proof. If λ is typical, the Kac module $K(\lambda)$ is projective in which case we have

$$c_{\mathcal{F}}(K(\lambda)) = \dim \mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda)) = 0.$$

If λ is atypical, Theorem 3.13 and Theorem 3.19 imply that

$$c_{\mathcal{F}}(K(\lambda)) = \dim \mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda)) = \dim \mathfrak{g}_1 = 2n.$$

□

After establishing the equivalence of blocks between $\mathfrak{g}$ and $\mathfrak{g}'$, we can show that the operator L is onto without using its combinatorial definition:

Proposition 3.21. *The operator L is surjective on the set of atypical weights in X_0^+.*

Proof. First, we will show that L is onto in the case of $\mathfrak{g}' = \mathfrak{osp}(2|2)$. Let μ be an atypical weight, then $\mu = (d+2|d)$ or $\mu = (-d|d)$ for some $d \geq 0$. By Theorem 3.1, $(0|0) = (2|0)^L$, $(d+2|d) = (d+3|d+1)^L$ for any $d \geq 0$ and $(-d|d) = (-d+1|d-1)^L$ for any $d \geq 1$. Thus μ is the image under L of some atypical weight.

Now Let μ be an atypical weight in the case of $\mathfrak{g} = \mathfrak{osp}(2|2n)$. Let $L(\mu')$ be the image of the simple module $L(\mu)$ under the above equivalence of blocks between $\mathfrak{g}$ and $\mathfrak{g}'$. Then μ' is an atypical weight for $\mathfrak{g}'$. Thus there exists an atypical weight λ' with $\lambda'^L = \mu'$. The head of $K(\lambda')$ is $L(\lambda')$ and the socle is $L(\lambda'^L) = L(\mu')$. This Kac module corresponds to a Kac module $K(\lambda)$ (where λ corresponds to λ' under the same equivalence) which has $L(\lambda)$ as its head and $L(\mu)$ as its socle. But the socle of $K(\lambda)$ is $L(\lambda^L)$, thus $L(\lambda^L) \cong L(\mu)$ which shows $\mu = \lambda^L$ for some atypical λ. $\square$

3.4.4 Associated variety

Recall

$$\mathcal{X} = \{x \in \mathfrak{g}_{\bar{1}} \mid [x,x] = 0\}.$$

If $M \in \mathcal{F}$, then Duflo and Serganova [9] define an associated variety of M which is equivalent to:

$$\mathcal{X}_M = \{x \in \mathcal{X} \mid \text{M is not projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\}.$$

We will indicate how the support variety and the associated variety can be combined to measure the complexity of the simple and the Kac modules. First, we find the dimension of the associated variety of the simple and Kac modules:

Lemma 3.22. *For $\lambda \in X_0^+$, let $X(\lambda)$ be $L(\lambda)$ or $K(\lambda)$, then*

$$\dim \mathcal{X}_{X(\lambda)} = \begin{cases} 2n, & \textit{if } \operatorname{atyp}(\lambda) = 1; \\ 0, & \textit{if } \operatorname{atyp}(\lambda) = 0. \end{cases}$$

Proof. When λ is typical, $L(\lambda)$ and $K(\lambda)$ are projective, hence $\dim \mathcal{X}_{X(\lambda)} = 0$ by [9, Theorem 3.4]. For the rest of the proof, assume λ is atypical. [14, Cor 2.5] implies that $\mathcal{X}_{L(\lambda)} = \mathcal{X}$ when λ is atypical. In $\mathfrak{osp}(2|2n)$, $\mathcal{X}$ has two irreducible components each with dimension equals to $\dfrac{\dim g_1}{2} = 2n$. Thus $\dim \mathcal{X} = 2n$, ([9, Cor 4.8, Cor 4.9]).

Now assume $X(\lambda) = K(\lambda)$. From the definitions of the rank variety and the associated variety, we have

$$\mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda)) = \mathcal{X}_{K(\lambda)} \cap \mathfrak{g}_1 \subseteq \mathcal{X}_{K(\lambda)}.$$

Following the same proof of [2, Theorem 6.4.1], we show that the inclusion is in fact an equality. Thus $\dim \mathcal{X}_{K(\lambda)} = \dim \mathcal{V}_{\mathfrak{g}_1}^{rank}(K(\lambda)) = \dim \mathfrak{g}_{\bar{1}} = 2n$, (using Theorem 3.19).

□

3.4.5 Geometric interpretation of the complexity

Combining the computations about the complexity, support variety, and the associated variety of the simple and Kac modules, we can conclude that:

Theorem 3.23. *For $\lambda \in X_0^+$, let $X(\lambda)$ be $L(\lambda)$ or $K(\lambda)$, then*

$$c_{\mathcal{F}}(X(\lambda)) = \dim \mathcal{X}_{X(\lambda)} + \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(X(\lambda)).$$

3.5 z-complexity

The z-complexity of $M \in \mathcal{F}$ is defined in section 2.10. We will compute this categorical invariant for the simple and the Kac modules in $\mathcal{F}$. Recall that if $\mathbf{P}. \to M$ is a minimal projective resolution of M and $s(\mathbf{P}.)$ is the rate of growth of the number of projective summands in each P_d, then $z_{\mathcal{F}}(M) = s(\mathbf{P}.)$.

Lemma 3.24. *If $L(\lambda)$ is the trivial module, then $z_{\mathcal{F}}(L(\lambda)) = 2$.*

Proof. The proof follows directly by counting the number of summands in each P_d in the resolution (3.2) of the trivial module. By doing so, we can see that the number of summands in each P_d is a polynomial in d of degree 1. Hence, the rate of growth of this number is $s(\mathbf{P}.) = 2$. The result follows using Proposition 2.33. □

To find the z-complexity of all atypical simple modules, we will show that simple modules of the same atypicality have the same z-complexity.

Lemma 3.25. *For any module $X \in \mathcal{F}$ and a Kac module $K(\lambda)$, there exists a constant D_X depending only on X such that*

$$K(\lambda) \otimes X \cong \bigoplus_{\gamma \in I} K(\gamma),$$

where $|I| \leq D_X$.

Proof. By the definition of Kac modules, we have

$$K(\lambda) = U(g) \otimes_{U(\mathfrak{p}^+)} L_{\bar{0}}(\lambda).$$

As a $\mathfrak{g}_{\bar{0}}$-module, $L_{\bar{0}}(\lambda) \otimes X$ decomposes into a direct sum $\bigoplus_{\mu \in I_X} L_{\bar{0}}(\lambda) \otimes L_{\bar{0}}(\mu)$, where I_X is a finite indexing set depending only on X. By the generalized Littlewood-Richardson formula [21, Subsection A.4], each summand $L_{\bar{0}}(\lambda) \otimes L_{\bar{0}}(\mu)$ decomposes into a finite direct sum $\bigoplus_{\gamma \in I_{\lambda,\mu}} L_{\bar{0}}(\gamma)$ indexed by the $\mathfrak{sp}(2n)$-standard Young tableaux of shape $p(\mu)$ which are λ-dominant (see [21, Subsection A.4] for definitions). Let $l(\mu)$ be the number of boxes in $p(\mu)$, then

$$|I_{\lambda,\mu}| \leq (2n)^{l(\mu)}.$$

By using the functor $U(g) \otimes_{U(\mathfrak{p}^+)} -$, we have the following isomorphisms as $U(\mathfrak{g})$-modules:

$$K(\lambda) \otimes X \cong \bigoplus_{\mu \in I_X} \bigoplus_{\gamma \in I_{\lambda,\mu}} U(g) \otimes_{U(\mathfrak{p}^+)} L_{\bar{0}}(\gamma) \cong \bigoplus_{\mu \in I_X} \bigoplus_{\gamma \in I_{\lambda,\mu}} K(\gamma).$$

The number of summands is at most $D_X := \sum_{\mu \in I_X} (2n)^{l(\mu)}$. This completes the proof. $\square$

Corollary 3.26. *For any module $X \in \mathcal{F}$ and a projective cover $P(\lambda)$, there exists a constant E_X depending only on X such that*

$$P(\lambda) \otimes X \cong \bigoplus_{\gamma \in J} P(\gamma),$$

where $|J| \leq E_X$.

Proof. Recall the following 2-step Kac flag:

$$0 \to K(\lambda) \to P(\lambda^L) \to K(\lambda^L) \to 0,$$

which can be rewritten as

$$0 \to K(\lambda') \to P(\lambda) \to K(\lambda) \to 0,$$

with $\lambda'^L = \lambda$. This Kac filtration of $P(\lambda)$ together with the exactness of the tensor functor (over $\mathbb{C}$) imply the exact sequence:

$$0 \to K(\lambda') \otimes X \to P(\lambda) \otimes X \to K(\lambda) \otimes X \to 0.$$

The factors $K(\lambda)\otimes X$ and $K(\lambda')\otimes X$ decompose into Kac modules by the previous lemma. Thus $P(\lambda) \otimes X$ has a Kac filtration. Moreover, there are two numbers D_X and D'_X such that the number of Kac modules in this filtration is at most $E_X := D_X + D'_X$. On the other hand, the projective module $P(\lambda)\otimes X$ decomposes into a finite direct sum of projective indecomposables $\bigoplus_{\alpha\in R} P(\alpha)$ where each summand has a Kac filtration. However Kac filtrations of the same module will have the same number of Kac modules. In fact, by [5, proposition 3.3], the number of times $K(\lambda)$ appears in a kac filtration of a module M is equal to the $\dim \operatorname{Hom}(M, K'(\lambda))$, where K' is the dual Kac module (cf. [5, Subsection 3.4]). This dimension is not dependent on the choice of the filtration. Thus the number of the projective indecomposables, $P(\alpha)$, is bounded by E_X.

□

Lemma 3.27. *Let M, N, X, $T \in \mathcal{F}$ such that $M \otimes X \cong N \oplus T$. Then $z_{\mathcal{F}}(M) \geq z_{\mathcal{F}}(N)$.*

Proof. Let $\mathbf{P}. \to M \otimes X$ and $\mathbf{Q}. \to M$ be the minimal projective resolutions of $M \otimes X$ and M respectively. Then $\mathbf{Q}. \otimes X \to M \otimes X$ is a projective resolution of $M \otimes X$. Using the above corollary, the d^{th} term in this resolution decomposes as follows:

$$Q_d \otimes X = \left(\bigoplus_{\lambda \in R_d} P(\lambda) \right) \otimes X \cong \bigoplus_{\lambda \in R_d} \bigoplus_{\gamma \in I} P(\gamma),$$

where $|I| \leq D_X$. Thus, for each d, the number of summands in $Q_d \otimes X$ is at most $|R_d|.D_X$ which is a constant multiple of the number of summands in each Q_d. By minimality of the resolution $\mathbf{P}.$, we have $s(\mathbf{P}.) \leq s(\mathbf{Q}. \otimes X)$ where $s(\mathbf{P}.)$ is the rate of growth of the number of summands in the d^{th}-term of the resolution $\mathbf{P}.$. Then

$$z_{\mathcal{F}}(M \otimes X) = s(\mathbf{P}.) \leq s(\mathbf{Q}. \otimes X) \leq s(\mathbf{Q}.) = z_{\mathcal{F}}(M).$$

By using the definition of the z-complexity we can easily show that

$$z_{\mathcal{F}}(N \oplus T) = \max\Big(z_{\mathcal{F}}(N),\, z_{\mathcal{F}}(T)\Big).$$

Then

$$z_{\mathcal{F}}(M \otimes X) \geq z_{\mathcal{F}}(N),$$

thus

$$z_{\mathcal{F}}(M) \geq z_{\mathcal{F}}(M \otimes X) \geq z_{\mathcal{F}}(N).$$

□

Theorem 3.28. *Let $\lambda \in X_0^+$.*

(1) If λ is typical, then $z_{\mathcal{F}}(L(\lambda)) = 0$.

(2) If λ is atypical, $z_{\mathcal{F}}(L(\lambda)) = 2$.

Proof. If λ is typical, then $L(\lambda)$ is projective, and hence $z_{\mathcal{F}}(L(\lambda)) = 0$. Let λ be atypical, then [6, Corollary 3.2.2] implies that if λ and μ have the same atypicality, there are modules X_1 and X_2 such that $L(\mu)$ is a direct summand of $L(\lambda) \otimes X_1$ and $L(\lambda)$ is a direct summand of $L(\mu) \otimes X_2$. Lemma 3.27 shows that $z_{\mathcal{F}}(L(\lambda)) = z_{\mathcal{F}}(L(\mu))$. Thus, for an atypical λ, $z_{\mathcal{F}}(L(\lambda)) = z_{\mathcal{F}}(\mathbb{C}) = 2$. □

Lemma 3.29. *For $\lambda \in X_0^+$, $z_{\mathcal{F}}(K(\lambda)) = z_{\mathcal{F}}(K(\lambda^{(l)}))$, $l \in \mathbb{Z}$.*

Proof. By induction it is sufficient to prove that

$$z_{\mathcal{F}}(K(\lambda)) = z_{\mathcal{F}}(K(\lambda^L)).$$

The z-complexity of any module $M \in \mathcal{F}$ is given by:

$$z_{\mathcal{F}}(M) = r(\mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, \bigoplus L(\mu)), \tag{3.6}$$

where the sum is over all simple modules in $\mathcal{F}$. Let $S = \bigoplus L(\mu)$. By applying the functor $\mathrm{Hom}(\ \ , S)$ to the following 2-step Kac flag:

$$0 \to K(\lambda) \to P(\lambda^L) \to K(\lambda^L) \to 0,$$

we get:

$$\ldots \to \mathrm{Ext}^{d+1}(P(\lambda^L), S) \to \mathrm{Ext}^{d+1}(K(\lambda), S) \to \mathrm{Ext}^{d}(K(\lambda^L), S) \to$$
$$\ldots \to \mathrm{Ext}^{1}(K(\lambda^L), S) \to \mathrm{Ext}^{1}(P(\lambda^L), S) \to \mathrm{Ext}^{1}(K(\lambda), S) \to$$
$$\mathrm{Hom}(K(\lambda^L), S) \to \mathrm{Hom}(P(\lambda^L), S) \to \mathrm{Hom}(K(\lambda), S) \to 0.$$

Since $P(\lambda^L)$ is projective, then $\mathrm{Ext}^d(P(\lambda^L), S) = 0$ for all $d \geq 1$, which gives:

$$\mathrm{Ext}^d(K(\lambda^L), S) = \mathrm{Ext}^{d+1}(K(\lambda), S).$$

This implies that $\mathrm{Ext}^\bullet_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda), S)$ and $\mathrm{Ext}^\bullet_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda^L), S)$ will have the same rate of growth, hence the theorem follows. □

Assume $\mathrm{atyp}(\lambda) = 1$. Let χ_λ be the central character corresponding to the weight λ and $\mathcal{F}^{\chi_\lambda}$ be the corresponding block. Let $\mathcal{F}^{\chi_0}$ be the block containing the trivial module. The translation functors define an equivalence of blocks between $\mathcal{F}^{\chi_\lambda}$ and $\mathcal{F}^{\chi_0}$. The z-complexity is a categorical equivalence, hence the translation functors preserve the z-complexity of any $\mathfrak{g}$-module, thus

Lemma 3.30. *Let* $\lambda \in \mathcal{F}^{\chi_\lambda}$ *with* $\mathrm{atyp}(\lambda) = 1$. *Then*

$$z_{\mathcal{F}}(K(\lambda)) = z_{\mathcal{F}}(K(\mu)),$$

for some $\mu \in \mathcal{F}^{\chi_0}$.

These results combined show that for atypical $\lambda \in X_0^+$,

$$\begin{aligned} z_{\mathcal{F}}(K(\lambda)) &= z_{\mathcal{F}}(K(\mu)) \quad \text{for some } \mu \in \mathcal{F}^{\chi_0} \\ &= z_{\mathcal{F}}(K(0^{(l)})) \quad \text{for some } l \in \mathbb{Z} \\ &= z_{\mathcal{F}}(K(0)). \end{aligned}$$

Lemma 3.31. $z_{\mathcal{F}}(K(0)) = 1$.

Proof. The result follows directly from the minimal projective resolution of $K(0)$ in (3.4). □

Therefore,

Theorem 3.32. *Let $\lambda \in X_0^+$.*

(1) If λ is typical, then $z_{\mathcal{F}}(K(\lambda)) = 0$.

(2) If λ is atypical, then $z_{\mathcal{F}}(K(\lambda)) = 1$.

3.5.1 Detecting subsuperalgebra

As we interpreted the complexity of the simple and the Kac modules geometrically through the dimensions of the associated variety and the support variety, we can also find a geometric interpretation of the z-complexity. To do so, a detecting subalgebra is introduced. Let $\mathfrak{f}_{\bar{1}} \subseteq \mathfrak{g}_{\bar{1}}$ be the span of the root vectors $x_\alpha, x_{-\alpha}$ where $\alpha = \varepsilon_1 - \delta_1$. In the matrix realization, $x_\alpha = E_{1,3} - E_{n+3,2}$ and $x_{-\alpha} = E_{2,n+3} + E_{3,1}$. Set $\mathfrak{f}_{\bar{0}} = [\mathfrak{f}_{\bar{1}}, \mathfrak{f}_{\bar{1}}]$. Then $\mathfrak{f}_{\bar{0}}$ is spanned by the diagonal matrix $E_{1,1} - E_{2,2} + E_{3,3} - E_{n+2,n+2}$. We define a three-dimensional subalgebra of $\mathfrak{g}$ by

$$\mathfrak{f} := \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}}.$$

The Lie superalgebra $\mathfrak{f}$ is classical and of Type I and so has a support variety theory. Furthermore, as $[\mathfrak{f}_{\bar{0}}, \mathfrak{f}_{\bar{1}}] = 0$, it follows that these varieties admit a rank variety description and, in particular, can be identified as subvarieties of $\mathfrak{f}_{\bar{1}}$, i.e.,

$$\begin{aligned}\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M) &= \mathcal{V}_{\mathfrak{f}_{\bar{1}}}^{rank}(M) \\ &= \{y \in \mathfrak{f}_{\bar{1}} \mid \text{M is not projective as } U(\langle y \rangle)\text{-module}\} \cup \{0\}.\end{aligned}$$

For example, $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(\mathbb{C}) = \mathfrak{f}_{\bar{1}}$. If M is projective, then $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M) = 0$.

Proposition 3.33. *Let $\lambda \in X_0^+$. If $\lambda \in X_0^+$ is typical, then*

$$\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(K(\lambda)) = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(L(\lambda)) = 0.$$

If λ is atypical, then

(1) $\dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(K(\lambda)) = 1.$

(2) $\dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(L(\lambda)) = 2.$

Proof. If λ is typical, then $L(\lambda)$ and $K(\lambda)$ are projective. Let λ be an atypical weight. We can use the proofs of [2, Theorem 6.4.1,Theorem 9.2.1] to show

$$\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(K(\lambda)) \subseteq \Big(\mathfrak{f}_{\bar{1}} \cap \mathcal{V}_{\mathfrak{g}_{-1}}(K(\lambda))\Big) \times \Big(\mathfrak{f}_{\bar{1}} \cap \mathcal{V}_{\mathfrak{g}_{1}}(K(\lambda))\Big). \tag{3.7}$$

From the rank variety description we have

$$\Big(\mathfrak{f}_{\bar{1}} \cap \mathcal{V}_{\mathfrak{g}_{-1}}(K(\lambda))\Big) \cup \Big(\mathfrak{f}_{\bar{1}} \cap \mathcal{V}_{\mathfrak{g}_{1}}(K(\lambda))\Big) \subseteq \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(K(\lambda)). \tag{3.8}$$

We have $\mathcal{V}_{\mathfrak{g}_1}(K(\lambda)) = \mathfrak{g}_1$ and $\mathcal{V}_{\mathfrak{g}_{-1}}(K(\lambda)) = 0$ (cf. [3, Theorem 3.3.1]). Since $\mathfrak{f}_{\bar{1}} \cap \mathfrak{g}_1$ is one-dimensional, then equation (3.7) and equation (3.8) imply that

$$\dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(K(\lambda)) = 1,$$

and, in fact, $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(X(\lambda)) = \mathfrak{f}_{\bar{1}} \cap \mathfrak{g}_1$.

On the other hand, [6, Theorem 4.1.1] implies that if λ and μ have the same atypicality, then

$$\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(L(\lambda)) = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(L(\mu)).$$

In particular,

$$\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(L(\lambda)) = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(L(0|0,\ldots,0)) = \mathfrak{f}_{\bar{1}},$$

Thus

$$\dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(L(\lambda)) = 2.$$

□

Note that Theorem 3.28, Theorem 3.32, and Proposition 3.33 imply the following geometric interpretation of the z-complexity:

Theorem 3.34. *If $X(\lambda)$ is a simple of a Kac module over $\mathfrak{osp}(2|2n)$ then*

$$z_{\mathcal{F}}(X(\lambda)) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(X(\lambda)).$$

3.5.2 Kazhdan-Lusztig polynomials

The Kazhdan-Lusztig polynomial associated to a pair of weights, $\lambda, \mu \in X_0^+$, is defined by:

$$l_{\lambda,\mu}(-q^{-1}) = \sum_{i\geq 0} \dim \operatorname{Ext}^i_{\mathcal{F}}(K(\lambda), L(\mu))q^i.$$

Kazhdan-Lusztig Polynomials can be used to compute the z-complexity of the Kac modules. Moreover, by using these polynomials, we will show that

$$\dim \operatorname{Ext}^1(L(\lambda), L(\mu)) = 1 \Leftrightarrow \lambda = \mu^L \text{ or } \lambda^L = \mu,$$

which was used in determining the radical layer structure of the projective covers. These polynomials have been computed in [8, Theorem 5, Remark 1]. If μ is typical, then $l_{\lambda,\mu} = \delta_{\lambda,\mu}$. If μ is atypical, then

$$l_{\lambda,\mu}(q) = \begin{cases} (-q)^i, & \text{if } \lambda = \mu^{(i)} \text{ for some } i \geq 0; \\ 0, & \text{otherwise,} \end{cases}$$

where $\mu^{(i)}$ was defined in equation (3.1). If μ is typical then

$$\dim \operatorname{Ext}^i_{\mathcal{F}}(K(\lambda), L(\mu)) = \begin{cases} 1, & \text{if } \lambda = \mu \text{ and } i = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

However, if μ is atypical, we get

$$\dim \operatorname{Ext}^i_{\mathcal{F}}(K(\lambda), L(\mu)) = \begin{cases} 1, & \text{if } \lambda = \mu^{(i)}; \\ 0, & \text{otherwise.} \end{cases} \tag{3.10}$$

We will also get an alternative condition to get a non zero extension group. Assume λ (or μ) is atypical. Using the proof of [2, Theorem 4.2.1], we have:

$$\dim \operatorname{Ext}^d_{\mathcal{F}}(K(\lambda), L(\mu)) \leq \dim \operatorname{Hom}_{\mathfrak{g}_{\bar{0}}}(L_{\bar{0}}(\lambda), S^d(\mathfrak{g}_1^*) \otimes \Lambda^\bullet(\mathfrak{g}_{-1}) \otimes L_{\bar{0}}(\mu)). \tag{3.11}$$

Now consider the element $c := E_{1,1} - E_{2,2} \in \mathfrak{g}_{\bar{0}}$. Then c is central in the enveloping algebra of $\mathfrak{g}_{\bar{0}}$. Let $\lambda = (\lambda_{-1}|\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_{-1}|\mu_1, \ldots, \mu_n)$. Note that c acts on the simple module $L(\lambda)$ by λ_{-1}. However, it acts on $\mathfrak{g}_{\bar{1}}$ by 1 and on $\mathfrak{g}_{-1}$ by -1. Thus c acts on $S^d(\mathfrak{g}_1^*) \otimes \Lambda^\bullet(\mathfrak{g}_{-1}) \otimes L_{\bar{0}}(\mu)$ by $-d - b + \mu_{-1}$ where $b \in \{0, 1, \ldots, \dim \mathfrak{g}_{-1} = 2n\}$. Therefore,

$$\dim \operatorname{Hom}_{\mathfrak{g}_{\bar{0}}}(L_{\bar{0}}(\lambda), S^d(\mathfrak{g}_1^*) \otimes \Lambda^\bullet(\mathfrak{g}_{-1}) \otimes L_{\bar{0}}(\mu)) \neq 0 \Rightarrow \lambda_{-1} = -d - b + \mu_{-1},$$

for some $b \in \{0, 1, \ldots, 2n\}$. Hence

$$\dim \operatorname{Ext}^d_{\mathcal{F}}(K(\lambda), L(\mu)) \neq 0 \Rightarrow \mu_{-1} = \lambda_{-1} + d + b, \tag{3.12}$$

for some $b \in \{0, 1, \ldots, 2n\}$.

From the last equation in [8, Subsection 3.1] we have

$$\sum_{i \geq 0} \dim \operatorname{Ext}^i_{\mathcal{F}}(L(\lambda), L(\mu)) q^i = \sum_{\sigma \in X_0^+} l_{\sigma,\lambda}(-q^{-1}) l_{\sigma,\mu}(-q^{-1}),$$

which implies that

$$\dim \operatorname{Ext}^d_{\mathcal{F}}(L(\lambda), L(\mu)) = \begin{cases} 1, & \text{if } \lambda^{(i)} = \mu^{(j)} \text{ for some } i, j \text{ with } i + j = d; \\ 0, & \text{otherwise.} \end{cases} \tag{3.13}$$

In other words,

$$\dim \operatorname{Ext}^d_{\mathcal{F}}(L(\lambda), L(\mu)) = \sum_{i+j=d} \sum_{\sigma \in \mathcal{B}} \dim \operatorname{Ext}^i_{\mathcal{F}}(K(\sigma), L(\lambda)) \dim \operatorname{Ext}^j_{\mathcal{F}}(K(\sigma), L(\mu)), \tag{3.14}$$

where $\mathcal{B}$ is the block containing $L(\lambda)$ and $L(\mu)$. We can use this equation to show that if $\dim \operatorname{Ext}^d_{\mathcal{F}}(L(\lambda), L(\mu)) \neq 0$, then there exists positive integers i, j with $i+j = d$ and σ in the block $\mathcal{B}$ containing λ, μ such that $\dim \operatorname{Ext}^i_{\mathcal{F}}(K(\sigma), L(\lambda)) \neq 0$ and $\dim \operatorname{Ext}^j_{\mathcal{F}}(K(\sigma), L(\mu)) \neq 0$. Then by equation (3.12), we have

$$\sigma_{-1} = \lambda_{-1} - i - b_1 \quad \text{and} \quad \sigma_{-1} = \mu_{-1} - j - b_2,$$

hence

$$\dim \operatorname{Ext}^d_{\mathcal{F}}(L(\lambda), L(\mu)) \neq 0 \Rightarrow \lambda_{-1} - \mu_{-1} = i - j + b_1 - b_2, \tag{3.15}$$

for some $b_1, b_2 \in \{0, 1, \ldots, 2n\}$ and some i, j such that $i + j = d$.

We have the following alternative proof of Theorem 3.32 using Kazhdan-Lusztig polynomials:

Proof. If λ is typical, then $K(\lambda)$ is projective, and hence $z_{\mathcal{F}}(K(\lambda)) = 0$. Assume λ is atypical. Equation (3.12) gives

$$\dim \operatorname{Ext}^d_{\mathcal{F}}(K(\lambda), L(\mu)) \neq 0 \Rightarrow \mu_{-1} = \lambda_{-1} + d + b, \tag{3.16}$$

for some $b \in \{0, 1, \ldots, 2n\}$, which implies that

$$\dim \operatorname{Ext}^d_{\mathcal{F}}(K(\lambda), \bigoplus L(\mu)) = \sum_{\mu} \sum_{b=0}^{2n} \dim \operatorname{Ext}^d_{\mathcal{F}}(K(\lambda), L(\mu)), \tag{3.17}$$

where the first sum runs over all μ that are in the same block of λ. Hence, μ and λ have the same atypicality (i.e, 1) and the same core. Let $\lambda = (\lambda_{-1}|\lambda_1, \ldots, \lambda_n)$ then $\mu = (\lambda_{-1} + d + b|\mu_1, \ldots, \mu_n)$. Let μ_k be the component that gives $\operatorname{atyp}(\mu) = 1$. Then μ_k is uniquely determined by μ_{-1}. However, the components $\mu_1, \mu_2, \ldots, \mu_{k-1}, \mu_{k+1}, \ldots, \mu_n$ are determined by λ since μ and λ have the same core. We have

$$\operatorname{Ext}^d_{\mathcal{F}}(K(\lambda), L(\mu)) = 1,$$

when it is not zero. Thus

$$\dim \operatorname{Ext}^d_{\mathcal{F}}(K(\lambda), \bigoplus L(\mu)) \leq 2n + 1 = (2n + 1)d^{1-1},$$

which proves $z_{\mathcal{F}}(K(\lambda)) \leq 1$. Assume $z_{\mathcal{F}}(K(\lambda)) = 0$, then there exists $N \in \mathbb{N}$ such that

$$\dim \operatorname{Ext}^d_{\mathcal{F}}(K(\lambda), \bigoplus L(\mu)) = 0,$$

for all $d > N$. Thus if $\mathbf{P}. \to K(\lambda)$ is a minimal projective resolution, then $P_d = 0$ for all $d > N$. Assuming the resolution has at least two projective modules, it can be rewritten as:

$$0 \to P_N \hookrightarrow P_{N-1} \to P_{N-2} \to \cdots \to K(\lambda) \to 0.$$

Since projective modules are injective in $\mathcal{F}$, the short exact sequence

$$0 \to P_N \hookrightarrow P_{N-1} \to P_{N-1}/P_N \to 0$$

splits, hence $P_{N-1} \cong P_N \oplus P_{N-1}/P_N$ which makes P_{N-1}/P_N projective. Thus the projective resolution can be written as:

$$0 \to P_N \hookrightarrow P_N \oplus P_{N-1}/P_N \to P_{N-2} \to \cdots \to K(\lambda) \to 0,$$

but the exactness of the original resolution implies the exactness of the following:

$$0 \to P_{N-1}/P_N \to P_{N-2} \to \cdots \to K(\lambda) \to 0,$$

which contradicts the minimality of the original resolution. Now if the resolution had only one projective module, then $K(\lambda)$ is projective which is not the case since λ is atypical. This shows that $z_{\mathcal{F}}(K(\lambda)) = 1$.

□

Chapter 4

Computing the complexity over $\mathfrak{osp}(3|2)$

After computing the complexity of the simple and Kac modules over the type C Lie superalgebra, we look at the type B Lie superalgebra $\mathfrak{osp}(3|2)$. This Lie superalgebra is of *Type II* and hence the Kac modules (as defined before) are not finite dimensional. The simple modules are characterized by highest weights. Using the description of the projective covers given in [10], we will compute the complexity and the z-complexity of the simple modules over $\mathfrak{osp}(3|2)$. We show these complexities have the same geometric interpretation as was shown for types A and C.

4.1 The Lie superalgebra $\mathfrak{osp}(3|2)$

Throughout this chapter, let $\mathfrak{g} = \mathfrak{osp}(3|2)$. The definition of the general orthosymplectic Lie superalgebra $\mathfrak{osp}(m|n)$ was given in Chapter 2. The bilinear form F given there can be represented by the matrix:

$$F = \begin{bmatrix} G & 0 \\ 0 & H \end{bmatrix},$$

where

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then $\mathfrak{g}$ consists of all the matrices

$$\left[\begin{array}{ccc|cc} 0 & -u & -v & x & x_1 \\ v & a & b & y & y_1 \\ u & c & -a & z & z_1 \\ \hline -x_1 & -z_1 & -y_1 & d & e \\ x & z & y & f & -d \end{array}\right].$$

The even part of $\mathfrak{g}$ is $\mathfrak{g}_{\bar{0}} \cong \mathfrak{so}_3 \oplus \mathfrak{sp}_2 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, hence $\dim \mathfrak{h}^* = 2$. We can choose a basis $\{\varepsilon, \delta\}$ of $\mathfrak{h}^*$ such that

$$\begin{aligned} &\Phi_{\bar{0}} = \{\pm\varepsilon; \pm 2\delta\}, \qquad \Phi_{\bar{1}} = \{\pm\varepsilon \pm \delta; \pm\delta\}, \\ &\text{and} \qquad (\varepsilon, \varepsilon) = -(\delta, \delta) = 1, \quad (\varepsilon, \delta) = 0. \end{aligned}$$

We choose the following distinguished set of positive roots : $\Phi^+ = \{\varepsilon; 2\delta\} \cup \{\pm\varepsilon + \delta; \delta\}$ which corresponds to the simple roots $-\varepsilon + \delta$ and ε. Here we have $2\rho = \varepsilon - \delta$. Using the basis $\{\varepsilon, \delta\}$, dominant weights for $\mathfrak{g}$ will be identified with $\mathbb{N}/2 \times \mathbb{N}$. By proposition 2.3 in [16], the set of dominant weights for $\mathfrak{g}$ is : $X^+ = \{(a, b) \in \mathbb{N}/2 \times \mathbb{N} \mid \ b = 0 \Rightarrow a = 0\}$. The atypical dominant weights are $\lambda_0 = (0, 0)$ and $\lambda_l = (l - 1, l)$ (for $l \in \mathbb{N}^*$). For example, $S(\lambda_0)$ is the trivial module, $S(\lambda_1)$ is the standard module.

Theorem 4.1. *[10, Theorem 2.1.1] Let* $\mathfrak{g} = \mathfrak{osp}(3|2)$.

(a) *Up to a shift of parity, the principal block* $\Gamma_0 = \{\lambda_l \mid \ l \in \mathbb{N}\}$ *is the unique atypical block.*

(b) *The projective indecomposable modules have the following radical layer structure :*

$$\begin{array}{ccccc}
S(\lambda_0) & S(\lambda_1) & S(\lambda_2) & S(\lambda_1) & \\
| & | & \diagup \;|\; \diagdown & \diagup \quad \diagdown & \\
S(\lambda_2) & S(\lambda_2) & S(\lambda_0)S(\lambda_1)S(\lambda_3) & S(\lambda_{l-1}) \quad S(\lambda_{l+1}) & (l \geq 3) \\
| & | & \diagdown \;|\; \diagup & \diagdown \quad \diagup & \\
S(\lambda_0) & S(\lambda_1) & S(\lambda_2) & S(\lambda_l) &
\end{array}$$

Remark 4.2. *It is worth mentioning that the author in [10] considers the category $\mathfrak{g}$-mod of finite-dimensional $\mathfrak{g}$-modules with even morphisms of representations. On the other hand, we take the category $\mathcal{F}$ of finite-dimensional $\mathfrak{g}$-modules with all morphisms of representations. However, his choice of morphisms does not affect the structure of the projective modules. We will be using his classification of the simple modules in $\mathfrak{g}$-mod which contains the simple modules in $\mathcal{F}$. This remark applies in the next chapters where we consider the three exceptional Lie superalgebras.*

We first relate the dimension of $P(\mu)$ to the dimension of $S(\mu)$. Note that $S_{\bar{0}}(\lambda_0)$ is the trivial $\mathfrak{g}_{\bar{0}}$-module and for $\lambda_l = (l-1, l) \in X^+$, we let $S_{\bar{0}}(\lambda_l) = V_{l-1} \otimes V_l$ where V_m denotes the simple $\mathfrak{sl}_2$-module of dimension $m+1$. As $\mathfrak{g}_{\bar{0}}$-modules, $S(\mu)$ contains $S_{\bar{0}}(\mu)$ as a composition factor. Thus

$$\dim P(\mu) \geq \dim S(\mu) \geq \dim S_{\bar{0}}(\mu).$$

On the other hand, by the PBW theorem for Lie superalgebras we see that $U(\mathfrak{g})$ is a free right $U(\mathfrak{g}_{\bar{0}})$-module and thus, $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} S_{\bar{0}}(\mu)$ is a projective $U(\mathfrak{g})$-module. Hence it decomposes into a direct sum of projective indecomposable modules. Set $P := U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} S_{\bar{0}}(\mu)$. Then by applying Frobenius reciprocity we have

$$\mathrm{Hom}_{\mathfrak{g}}(P, S(\gamma)) = \mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}(S_{\bar{0}}(\mu)), S(\gamma)\Big|_{\mathfrak{g}_{\bar{0}}}) \neq 0 \text{ if } \mu = \gamma.$$

Thus P surjects onto $L(\mu)$ which implies that $P(\mu)$ is a direct summand of P. However, by the PBW theorem,

$$P = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} S_{\bar{0}}(\mu) \cong \Lambda^{\bullet}(\mathfrak{g}_{\bar{1}}) \otimes S_{\bar{0}}(\mu)$$

as vector spaces. Therefore, we have

$$\dim S_{\bar{0}}(\mu) \leq \dim P(\mu) \leq 2^{\dim \mathfrak{g}_{\bar{1}}} \dim S_{\bar{0}}(\mu). \tag{4.1}$$

Thus

$$l(l+1) \leq \dim P(\lambda_l) \leq 2^6 l(l+1). \tag{4.2}$$

4.2 Complexity of simple $\mathfrak{osp}(3|2)$-modules

We will use the above structure of the projective covers to give a minimal projective resolution of the trivial module $S(\lambda_0)$, then we use the generalized Kac-Wakimoto conjecture to show that any atypical simple module will have the same complexity as the trivial module. Recall that if the simple module is typical, then it is projective and hence has zero complexity.

Theorem 4.3. *For an atypical* $\lambda_l \in X^+$, $c_{\mathcal{F}}(S(\lambda_l)) = 4$.

Proof. The minimal projective resolution of $S(\lambda_0)$ is given by:

$$\ldots \to P_d \to \ldots \to P(\lambda_0) \to S(\lambda_0) \to 0, \tag{4.3}$$

where the d^{th} term in this resolution is given by:

$$P_d = P(\lambda_{d+1}) \oplus P(\lambda_{d-1}) \oplus \cdots \oplus P(\lambda_r),$$

where

$$r = \begin{cases} 2 & \text{if } d \text{ is odd}, \\ 0 & \text{if } d \equiv 0 \mod 4, \\ 1 & \text{if } d \equiv 2 \mod 4. \end{cases}$$

The construction of this minimal projective resolution follows our construction in Theorem 3.5. We use the same diagrammatic approach that was established in [1, Section 10.3]. Using equation (4.2) we get for all d:

$$\dim P_d \leq 2^6 \sum_{l=1}^{d+1} \Big(l(l+1) \Big) \leq C.d^3,$$

for some positive constant C. On the other hand,

$$\dim P_d \geq \frac{1}{2} \sum_{l=1}^{d+1} l^2 \geq C'.d^3,$$

for some positive constant C'. This shows that $c_{\mathcal{F}}(S(\lambda_0)) = 4$. By [6, Theorem 4.1.1], all simple modules of the same atypicality have the same complexity. Thus the complexity of all atypical simple $\mathfrak{osp}(3|2)$-modules is 4. □

4.2.1 A goemetric interpretation of the complexity

We can then interpret the complexity of simple modules geometrically:

Theorem 4.4. *If S is a simple $\mathfrak{g}$-module, then*

$$c_{\mathcal{F}}(S) = \dim \mathcal{X}_S + \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(S).$$

Proof. If S is typical then S is projective and both sides are zero. Let S be an atypical simple module (i.e. $\text{atyp}(S) = \text{def}(\mathfrak{g}) = 1$). Let

$$\mathcal{X} = \{x \in \mathfrak{g}_{\bar{1}} \mid [x, x] = 0\}.$$

Let $G_{\bar{0}}$ be an algebraic group with Lie algebra $\mathfrak{g}_{\bar{0}}$. Then $\mathcal{X}$ is invariant under the adjoint action of $G_{\bar{0}}$. It was shown that for any $x \in \mathcal{X}$, there exists $g \in G_{\bar{0}}$ and isotropic mutually orthogonal linearly independent roots $\alpha_1, \ldots, \alpha_k$ such that $Ad_g(x) = x_1 + \ldots + x_k$ with $x_i \in \mathfrak{g}_{\alpha_i}$. The number k does not depend on the choice of g and is called the *rank* of x (notation $rk(x) = k$). Let

$$X_k = \{x \in \mathcal{X} \mid rk(x) = k\}.$$

Corollary 2.5 in [14] implies that for an atypical simple module S, we have $\mathcal{X}_S = \overline{X_1} = \mathcal{X}$ by the above stratification. Using [9, Theorem 4.5, Cor 4.8], $\dim \mathcal{X} = 3$ in case $\mathfrak{g} = \mathfrak{osp}(3|2)$. On the other hand, by [6, Cor 4.4.2] we have $\dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(S) = \text{atyp}(S) = 1$. Thus

$$c_{\mathcal{F}}(S) = \dim \mathcal{X}_S + \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(S).$$

□

4.3 z-complexity

Using the projective resolution given in the proof of Theorem 4.3, we have

Proposition 4.5. $z_{\mathcal{F}}(S(\lambda_0)) = 2$.

Proof. The z-complexity of a module is the rate of growth of the number of direct summands in a minimal projective resolution of the module. In the proof of Theorem 4.3, the number of summands in P_d is a either $\frac{d}{2}+1$ if d is even or $\frac{d+1}{2}$ if d is odd. Thus the rate of growth of this number is 2. $\square$

We will also show that:

Proposition 4.6. *For $l \geq 1$, we have $z_{\mathcal{F}}(S(\lambda_l)) = 2$.*

Proof. Since the projective cover of $S(\lambda_1)$ has the same structure as the projective cover of $S(\lambda_0)$, it follows that $z_{\mathcal{F}}(S(\lambda_1)) = 2$. The minimal projective resolution of $S(\lambda_2)$ is given by:

$$\ldots \to P_d \to \ldots \to P(\lambda_2) \to S(\lambda_2) \to 0, \tag{4.4}$$

where the d^{th} term in this resolution is given by:

$$P_d = \begin{cases} P(\lambda_{d+2}) \oplus 2.P(\lambda_d) \oplus \cdots \oplus 2.P(\lambda_2) & \text{if } d \geq 2 \text{ is even,} \\ P(\lambda_{d+2}) \oplus 2.P(\lambda_d) \oplus \cdots \oplus 2.P(\lambda_3) \oplus P(\lambda_1) \oplus P(\lambda_0) & \text{if } d \geq 1 \text{ is odd.} \end{cases}$$

In the above, $2.P(\lambda)$ means $P(\lambda) \oplus P(\lambda)$. Thus the number of summands in P_d is either $d+1$ if $d \geq 2$ is even or it is $d+2$ if $d \geq 1$ is odd. This shows that $z_{\mathcal{F}}(S(\lambda_l)) = 2$.

The d^{th} term in the minimal projective resolution of $S(\lambda_3)$ is given by:

$$P_d = \begin{cases} P(\lambda_{d+3}) \oplus P(\lambda_{d+1}) & \text{if } d = 1, \\ P(\lambda_{d+3}) \oplus P(\lambda_{d+1}) \oplus P(\lambda_1) \oplus P(\lambda_0) & \text{if } d = 2, \\ P(\lambda_{d+3}) \oplus P(\lambda_{d+1}) \oplus 2.P(\lambda_{d-1}) \oplus \cdots \oplus 2.P(\lambda_2) & \text{if } d \geq 3 \text{ is odd,} \\ P(\lambda_{d+3}) \oplus P(\lambda_{d+1}) \oplus 2.P(\lambda_{d-1}) \oplus \cdots \oplus 2.P(\lambda_3) \oplus P(\lambda_1) \oplus P(\lambda_0) & \text{if } d \geq 4 \text{ is even.} \end{cases}$$

Thus the number of summands in P_d is either $d+2$ if $d \geq 2$ is even or it is $d+1$

if $d \geq 1$ is odd. This shows that $z_{\mathcal{F}}(S(\lambda_l)) = 2$.

For $l \geq 3$, the projective cover of $S(\lambda_l)$ has the same structure as the projective cover of $S(\lambda_3)$. This gives $z_{\mathcal{F}}(S(\lambda_l)) = 2$ □

4.3.1 Detecting subsuperalgebra

Let $\mathfrak{f}_{\bar{1}} \subseteq \mathfrak{g}_{\bar{1}}$ be the span of the root vectors $x_\alpha, x_{-\alpha}$ where $\alpha = \varepsilon_1 + \delta$. In the matrix realization, $x_\alpha = E_{1,4} - E_{5,2}$ and $x_{-\alpha} = E_{2,5} + E_{3,1}$. Set $\mathfrak{f}_{\bar{0}} = [\mathfrak{f}_{\bar{1}}, \mathfrak{f}_{\bar{1}}]$. Then $\mathfrak{f}_{\bar{0}}$ is spanned by the diagonal matrix $E_{1,1} - E_{2,2} + E_{4,4} - E_{5,5}$. We define a three-dimensional subalgebra of $\mathfrak{g}$ by

$$\mathfrak{f} := \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}}.$$

The Lie superalgebra $\mathfrak{f}$ is classical and so has a support variety theory. Furthermore, as $[\mathfrak{f}_{\bar{0}}, \mathfrak{f}_{\bar{1}}] = 0$, it follows that these varieties admit a rank variety description and, in particular, can be identified as subvarieties of $\mathfrak{f}_{\bar{1}}$, i.e.,

$$\begin{aligned} \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M) &= \mathcal{V}^{rank}_{\mathfrak{f}_{\bar{1}}}(M) \\ &= \{y \in \mathfrak{f}_{\bar{1}} \mid \text{M is not projective as } U(\langle y \rangle)\text{-module}\} \cup \{0\}. \end{aligned}$$

Note that $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S(\lambda_0)) = \mathfrak{f}_{\bar{1}}$. We can use this detecting subsuperalgebra to give a geometric interpretation of the z-complexity. For instance,

Corollary 4.7. $z_{\mathcal{F}}(S(\lambda_0)) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S(\lambda_0)) = 2$.

In general:

Theorem 4.8.

(1) *If S is a typical simple module, then*

$$z_{\mathcal{F}}(S) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S) = 0.$$

(2) *If S is an atypical simple module, then*

$$z_{\mathcal{F}}(S) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S) = 2.$$

Proof. If S is a typical simple module, then S is projective and the first part follows. If S is atypical, then by [6, Theorem 4.1.1],

$$\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S) = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S(\lambda_0)),$$

thus $\dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S) = 2$. The second part follows. □

Chapter 5

Computing the complexity over $D(2,1;\alpha)$

In this chapter we give the structure of the exceptional Lie superalgebra $D(2,1;\alpha)$ as discussed in [23]. In [10] the author establishes an equivalence between the principal blocks of $D(2,1;\alpha)$ and $\mathfrak{osp}(3|2)$. We use this equivalence to compute the complexity and the z-complexity of simple modules over $D(2,1;\alpha)$. We also give these complexities a geometric interpretation as in the earlier chapters.

5.1 The Lie superalgebra $D(2,1;\alpha)$

Let $\mathfrak{g}$ be the basic classical Lie superalgebra $D(2,1;\alpha)$ (see [16]). Scheunert [24] denotes these algebras by $\Gamma(\sigma_1,\sigma_2,\sigma_3)$ where $\sigma_i \in \mathbb{C}$ for $i=1,2,3$. We begin with Scheunert's construction.

Let V be a two-dimensional vector space over $\mathbb{C}$ with basis $\{u_1,\, u_{-1}\}$ and let ψ be a non-degenerate skew symmetric bilinear form on V defined by $\psi(u_1,u_{-1})=1$. Then $\mathfrak{sl}(V)=\mathfrak{sp}(\psi)\cong\mathfrak{sl}_2$ is the algebra of linear transformations on V preserving ψ. Now for $i=1,2,3$ take copies (V_i,ψ_i) of (V,ψ) and set

$$\mathfrak{g}_{\bar{0}} \cong \mathfrak{sp}(\psi_1)\oplus\mathfrak{sp}(\psi_2)\oplus\mathfrak{sp}(\psi_3)\cong\mathfrak{sl}_2\oplus\mathfrak{sl}_2\oplus\mathfrak{sl}_2,$$

$$\mathfrak{g}_{\bar{1}} = V_1\boxtimes V_2\boxtimes V_3.$$

Then $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module. Define the bilinear map $p_i : V_i\times V_i\to\mathfrak{sp}(\psi_i)$ by

$$p_i(x,y)z=\psi_i(y,z)x-\psi_i(z,x)y$$

for $x, y, z \in V_i$. We then define a symmetric $\mathfrak{g}_{\bar{0}}$-invariant bilinear form $p : \mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ by

$$p(x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3) = \sum \sigma_k \psi_i(x_i, y_i)\psi_j(x_j, y_j)p_k(x_k, y_k), \tag{5.1}$$

where the sum is over all even permutations of $\{1, 2, 3\}$.

Lemma 5.1. *[23, Lemma 4.2.1] p defines a Lie super bracket if and only if $\sigma_1 + \sigma_2 + \sigma_3 = 0$.*

Thus if $(\sigma_1, \sigma_2, \sigma_3)$ satisfies $\sigma_1 + \sigma_2 + \sigma_3 = 0$, then $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra which we denote by $\Gamma(\sigma_1, \sigma_2, \sigma_3)$. Let $\alpha \in \mathbb{C} \setminus \{0, -1\}$, and set $D(2, 1; \alpha) = \Gamma(-(1 + \alpha)/2, 1/2, \alpha/2)$. It can be shown that if $\alpha = 1$, then $D(2, 1; \alpha)$ is isomorphic to $\mathfrak{osp}(4|2)$, and that the algebras corresponding to α, $-1 - \alpha^{-1}$ and α^{-1} are isomorphic (hence if $\alpha \in \mathbb{Q}$ we can assume $\alpha > 0$, 114 or even $\alpha > 1$).

Let

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

be the usual basis of $\mathfrak{sl}_2$ and identify V with the vector space of column vectors with basis

$$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u_{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Denote by H_i, E_i, F_i the copies of H, E, F in the i^{th} copy of $\mathfrak{sl}_2$. Then

$$p_i(u_1, u_{-1}) = -H_i, \; p_i(u_1, u_1) = 2E_i, \; p_i(u_{-1}, u_{-1}) = -2F_i.$$

The root system of $\mathfrak{g}$ can be realized in the space $\mathfrak{h}^* = \mathbb{C}^3$ endowed with a basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and with the bilinear whose matrix is $\text{diag}(-(1+\alpha)/2, 1/2, \alpha/2)$, the roots are given by :

$$\Phi_{\bar{0}} = \{\pm 2\varepsilon_i \mid \ 1 \leq i \leq 3\} \quad \text{and} \quad \Phi_{\bar{1}} = \{\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}.$$

We choose as simple roots $\beta_4 = \varepsilon_1 - \varepsilon_2 - \varepsilon_3$, $2\varepsilon_2$, and $2\varepsilon_3$. The non-simple positive odd roots are : $\beta_1 = \varepsilon_1 + \varepsilon_2 - \varepsilon_3$, $\beta_2 = \varepsilon_1 - \varepsilon_2 + \varepsilon_3$ and $\beta_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. We have $\rho = -\beta_4$.

The corresponding root vectors are

$$e_1 = E_2, \quad e_2 = u_1 \otimes u_{-1} \otimes u_{-1}, \quad e_3 = E_3$$

and

$$f_1 = F_2, \quad f_2 = u_{-1} \otimes u_1 \otimes u_1, \quad f_3 = F_3.$$

Set

$$h_1 = H_2, \quad h_2 = -\sigma_1 H_1 + \sigma_2 H_2 + \sigma_3 H_3, \quad h_3 = H_3.$$

These elements satisfy the relations

$$[e_i, f_j] = \delta_{ij} h_i, \ [h_i, h_j] = 0, \ [h_i, e_j] = a_{ij} e_j, \ [h_i, f_j] = -a_{ij} f_j,$$

where

$$A = (a_{ij}) = \begin{bmatrix} 2 & -1 & 0 \\ 2\sigma_2 & 0 & 2\sigma_3 \\ 0 & -1 & 2 \end{bmatrix}.$$

We identify the set of dominant weights for $\mathfrak{g}_{\bar{0}}$ with $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. By proposition 2.2 in [16], the set of dominant weights for $\mathfrak{g}$ is : $X^+ = \{(a, b, c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \mid \ a = 0 \Rightarrow b = c = 0 \text{ and } a = 1 \Rightarrow (b+1) = \pm\alpha(c+1)\}$. Besides, the eigenvalue of the

Casimir element is :

$$\mathrm{Cas}(a,b,c) = -\frac{1+\alpha}{2}(a-1)^2 + \frac{1}{2}(b+1)^2 + \frac{\alpha}{2}(c+1)^2.$$

A dominant weight $\lambda = (a,b,c)$ is atypical if and only if one of the following scalar products is zero :

$$\begin{aligned}
2(\lambda+\rho,\beta_1) &= -(1+\alpha)(a-1) + (b+1) - \alpha(c+1)\\
2(\lambda+\rho,\beta_2) &= -(1+\alpha)(a-1) - (b+1) + \alpha(c+1)\\
2(\lambda+\rho,\beta_3) &= -(1+\alpha)(a-1) + (b+1) + \alpha(c+1)\\
2(\lambda+\rho,\beta_4) &= -(1+\alpha)(a-1) - (b+1) - \alpha(c+1)
\end{aligned}$$

Notice that if $\alpha \notin \mathbb{Q}$, the only atypical dominant weights are $\lambda_0 = (0,0,0)$ (corresponding to the trivial module) and $\lambda_l = (l+1, l-1, l-1)$ for $l \geq 1$. For example, λ_1 is the highest root of $\mathfrak{g}$, *i.e.* $S(\lambda_1)$ is the adjoint representation.

If $\alpha \in \mathbb{Q}$, we assume $\alpha = p/q$, with p and q relatively prime positive integers. For $k \in \mathbb{N}$, let Γ_k be the set of atypical simple modules $\lambda \in X^+$ such that $\mathrm{Cas}(\lambda) = p(p+q)k^2/2$. For $l \in \mathbb{Z}$ we set moreover :

$$\lambda_{k,l} = \begin{cases} (-l+2, -l-kp, -l+kq) & \text{if } \; l \leq -kp,\\ (-l+1, l+kp-1, -l+kq-1) & \text{if } \; -kp+1 \leq l \leq 0,\\ (l+1, l+kp-1, -l+kq-1) & \text{if } \; 0 \leq l \leq kq-1,\\ (l+2, l+kp, l-kq) & \text{if } \; kq \leq l. \end{cases}$$

Theorem 5.2. *[10, Theorem 3.1.1] Let $\alpha \in \mathbb{C} \setminus \{-1,0\}$ and let $\mathfrak{g} = D(2,1;\alpha)$.* (i) *The principal block of $\mathfrak{g}$ is : $\Gamma_0 = \{\lambda_l \mid \; l \in \mathbb{N}\}$. This block is equivalent to the principal block of $\mathfrak{osp}(3|2)$: statement* (b) *in Theorem 4.1 holds for the principal block of $D(2,1;\alpha)$.*

(ii) *If* $\alpha \notin \mathbb{Q}$, Γ_0 *is the unique atypical block.*

(iii) *Let* $\alpha \in \mathbb{Q}$, $\alpha > 0$, *written as* $\alpha = p/q$ *with* p *and* q *relatively prime. Then any atypical block is one of the* Γ_k. *Besides, let* $k \geq 1$.

(a) *The map* $\mathbb{Z} \to \Gamma_k$, $l \mapsto \lambda_{k,l}$ *is a bijection ;*

(b) *The projective cover of* $S(\lambda_{k,l})$ $(l \in \mathbb{Z})$ *has the following radical layer structure :*

$$\begin{array}{ccc} & S(\lambda_{k,l}) & \\ \swarrow & & \searrow \\ S(\lambda_{k,l-1}) & & S(\lambda_{k,l+1}) \\ \searrow & & \swarrow \\ & S(\lambda_{k,l}) & \end{array}$$

Recall that we have the following bounds on the dimensions of the projective covers:

$$\dim S_{\bar{0}}(\mu) \leq \dim P(\mu) \leq 2^{\dim \mathfrak{g}_{\bar{1}}} \dim S_{\bar{0}}(\mu), \quad (\mu \in X^+), \tag{5.2}$$

where $S_{\bar{0}}(\mu)$ is the simple $\mathfrak{g}_{\bar{0}}$-module of weight μ. Note that $S_{\bar{0}}(\lambda_0)$ is the trivial $\mathfrak{g}_{\bar{0}}$-module and for $\lambda_l = (l+1, l-l, l) \in X^+$, $S_{\bar{0}}(\lambda_l) = V_{l+1} \otimes V_{l-1} \otimes V_l$ where V_m denotes the simple $\mathfrak{sl}_2$-module of dimension $m+1$. Thus

$$(l+2)(l)(l+1) \leq \dim P(\lambda_l) \leq 2^8 (l+2)(l)(l+1). \tag{5.3}$$

Remark 5.3. *In [17, Example 3.4], the authors compute the super-dimensions of the finite-dimensional simple atypical modules over the exceptional Lie superalgebras. Their computations show that the super-dimension of these modules is not zero which shows that the Kac-Wakimoto conjecture*

$$\operatorname{sdim} V = 0 \Leftrightarrow \operatorname{atyp}(V) < \operatorname{def}(\mathfrak{g})$$

holds over these Lie superalgebras. Since the conjecture is valid, then [6, Theorem 4.1.1] holds.

5.2 Complexity of simple $D(2,1;\alpha)$-modules

Since the principal block $\Gamma_0 = \{\lambda_l \mid \ l \in \mathbb{N}\}$ is equivalent to the principal block of $\mathfrak{osp}(3|2)$ and statement *(b)* in Theorem 4.1 holds for Γ_0, $S(\lambda_0)$ will have the same minimal projective resolution (4.4) as in Chapter 4. Then,

Theorem 5.4. *If S is an atypical simple $D(2,1;\alpha)$-module, we have $c_{\mathcal{F}}(S) = 5$.*

Proof. Using the projective resolution (4.4) and the bounds in equation (5.3), we have

$$C'.d^4 \leq \dim P_d \leq C.d^4,$$

for some positive constants C, C'. This shows that $c_{\mathcal{F}}(S(\lambda_0)) = 5$. Since [6, Theorem 4.1.1] holds over $D(2,1;\alpha)$, all simple modules of the same atypicality have the same complexity and the same support variety. Thus the complexity of all atypical simple $D(2,1;\alpha)$-modules is 5. □

5.2.1 A geometric interpretation of the complexity

The atypical simple module, S, over $D(2,1;\alpha)$ has $\operatorname{atyp}(S) = \operatorname{def}(D(2,1;\alpha)) = 1$. Using the proof of Proposition 4.4, we have $\mathcal{X}_S = \mathcal{X}$. Using [9, Theorem 4.5, Cor 4.8], $\dim \mathcal{X} = 4$. By [6, Theorem 4.1.1] we have

$$\dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(S) = \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(S(\lambda_0)) = 1.$$

Theorem 5.5. *Let S be a simple $D(2,1;\alpha)$-module. Then*

$$c_{\mathcal{F}}(S) = \dim \mathcal{X}_S + \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(S).$$

5.3 z-complexity

The z-complexity is a categorical invariant, thus using the equivalence between the principal blocks of $\mathfrak{osp}(3|2)$ and $D(2,1;\alpha)$ we have:

Proposition 5.6. *For all the simple modules $S(\lambda)$ in the principal block of $D(2,1;\alpha)$, we have $z_{\mathcal{F}}(S(\lambda)) = 2$.*

For the simple modules in the other atypical blocks Γ_k, we can compute the z-complexity by writing an explicit minimal projective resolution. We then have:

Theorem 5.7. *For $\lambda_{k,l} \in \Gamma_k$, we have $z_{\mathcal{F}}(S(\lambda_{k,l})) = 2$.*

Proof. The d^{th} term in the minimal projective resolution of $S(\lambda_{k,l})$ is given by:

$$P_d = \begin{cases} P(\lambda_{k,l\pm d}) \oplus P(\lambda_{k,l\pm(d-2)}) \oplus \cdots \oplus P(\lambda_{k,l\pm 1}) & \text{if } d \geq 1 \text{ is odd,} \\ P(\lambda_{k,l\pm d}) \oplus P(\lambda_{k,l\pm(d-2)}) \oplus \cdots \oplus P(\lambda_{k,l}) & \text{if } d \geq 2 \text{ is even.} \end{cases}$$

In the above, $P(\lambda_{k,l\pm i})$ means $P(\lambda_{k,l+i}) \oplus P(\lambda_{k,l-i})$. Thus the number of summands in P_d is $d+1$. This shows that $z_{\mathcal{F}}(S(\lambda_{k,l})) = 2$. □

5.3.1 Detecting subsuperalgebra

Let $\mathfrak{f}_{\bar{1}} \subseteq \mathfrak{g}_{\bar{1}}$ be the span of the root vectors $x_\alpha, x_{-\alpha}$ where $\alpha = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. Set $\mathfrak{f}_{\bar{0}} = [\mathfrak{f}_{\bar{1}}, \mathfrak{f}_{\bar{1}}]$. We define a subalgebra of $\mathfrak{g}$ by

$$\mathfrak{f} := \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}}.$$

The Lie superalgebra $\mathfrak{f}$ is classical and so has a support variety theory. Furthermore,

Lemma 5.8. $\mathfrak{f}$ *is three-dimensional. Moreover, $[\mathfrak{f}_{\bar{0}}, \mathfrak{f}_{\bar{1}}] = 0$.*

Proof. The root vectors x_α and $x_{-\alpha}$ are in fact given by

$$x_\alpha = u_1 \otimes u_1 \otimes u_1 \quad x_{-\alpha} = u_{-1} \otimes u_{\bar{1}} \otimes u_{-1},$$

Then by equation (5.1) we have

$$[x_\alpha, x_{-\alpha}] = \sum \sigma_k \psi_i(u_i, u_{-i})\psi_j(u_j, u_{-j})p_k(u_k, u_{-k})$$

where the sum is over the even permutations of $\{1, 2, 3\}$. We have

$$\psi_i(u_i, u_{-i}) = \psi_j(u_j, u_{-j}) = 1$$

thus

$$\begin{aligned}
[x_\alpha, x_{-\alpha}] &= \sigma_1 p_1(u_1, u_{-1}) + \sigma_2 p_2(u_1, u_{-1}) + \sigma_3 p_3(u_1, u_{-1}) \\
&= \sigma_1(-H_1) + \sigma_2(-H_2) + \sigma_3(-H_3) \\
&= -(\sigma_1 H_1 + \sigma_2 H_2 + \sigma_3 H_3) \in \mathfrak{g}_{\bar{0}}.
\end{aligned}$$

Having $[x_\alpha, x_\alpha] = [x_{-\alpha}, x_{-\alpha}] = 0$ shows that $\mathfrak{f}$ is three-dimensional. Moreover,

$$\begin{aligned}
[[x_\alpha, x_{-\alpha}], x_\alpha] &= -[\sigma_1 H_1 + \sigma_2 H_2 + \sigma_3 H_3, u_1 \otimes u_1 \otimes u_1] \\
&= -(\sigma_1 u_1 \otimes u_1 \otimes u_1 + \sigma_2 u_1 \otimes u_1 \otimes u_1 + \sigma_3 u_1 \otimes u_1 \otimes u_1) \\
&= -(\sigma_1 + \sigma_2 + \sigma_3) u_1 \otimes u_1 \otimes u_1 \\
&= 0.
\end{aligned}$$

The last equality follows from the definition of $D(2, 1; \alpha)$ (Lemma 5.1). Similarly we can show

$$[[x_\alpha, x_{-\alpha}], x_{-\alpha}] = 0.$$

This shows that $[\mathfrak{f}_{\bar{0}}, \mathfrak{f}_{\bar{1}}] = 0$. $\square$

It follows that these varieties admit a rank variety description and, in particular, can be identified as subvarieties of $\mathfrak{f}_{\bar{1}}$, i.e.,

$$\begin{aligned}\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M) &= \mathcal{V}_{\mathfrak{f}_{\bar{1}}}^{rank}(M) \\ &= \{y \in \mathfrak{f}_{\bar{1}} \mid \text{M is not projective as } U(\langle y \rangle)\text{-module}\} \cup \{0\}.\end{aligned}$$

Note that $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S(\lambda_0)) = \mathfrak{f}_{\bar{1}}$. We can use this detecting subsuperalgebra to give a geometric interpretation of the z-complexity. For instance,

Corollary 5.9. $z_{\mathcal{F}}(S(\lambda_0)) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S(\lambda_0)) = 2.$

In general:

Theorem 5.10.

(1) *If S is a typical simple module, then*

$$z_{\mathcal{F}}(S) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S) = 0.$$

(2) *If S is an atypical simple module, then*

$$z_{\mathcal{F}}(S) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S) = 2.$$

Proof. If S is a typical simple module, then S is projective and the first part follows. If S is atypical, then by [6, Theorem 4.1.1],

$$\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S) = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S(\lambda_0)),$$

thus $\dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S) = 2$. The second part follows. $\square$

Chapter 6

Computing the complexity over $G(3)$

In this chapter we calculate the complexity of simple modules over the exceptional Lie superalgebra $G(3)$. We introduce the structure of $G(3)$ as given in [23]. We use the characterization of the simple modules and their projective covers given in [10] to find the complexity and the z-complexity. We then verify the geometric interpretation of these complexities in this case.

6.1 The Lie superalgebra $G(3)$

Let $\mathfrak{g}$ be the basic classical Lie superalgebra $G(3)$ (see [16]). We will consider the construction introduced in [23, Section 4.4]. Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the quaternion algebra over $\mathbb{R}$, with $i^2 = j^2 = k^2 = -1$ and $ij = k$, $jk = i$, $ki = j$. If $x = ai + bj + ck + dk$, we set $\bar{x} = a - bj - cj - dk$. Let $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ be the octonion algebra with multiplication

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}),$$

for $a, b, c, d \in \mathbb{H}$. For $x = (a, b) \in \mathbb{O}$, define $\bar{x} = (\bar{a}, -b)$ and the trace of x by $t(x) = x + \bar{x} = (a + \bar{a}, 0) \in \mathbb{R}$. Set $\mathbb{O}_{(0)} = \{x \in \mathbb{O} \,|\, t(x) = 0\}$, the space of traceless octonions. A derivation D on $\mathbb{O}$ is a linear map satisfying

$$D(xy) = D(x)y + xD(y).$$

It is easy to check that $D(\mathbb{O}) \subseteq \mathbb{O}_{(0)}$. Let $\operatorname{Der}\mathbb{O}$ be the set of all derivations on $\mathbb{O}$. We define the associator (x, y, z) on $\mathbb{O}$ by

$$(x, y, x) = (xy)z - x(yz),$$

and the commutator

$$[x, y] = xy - yx.$$

For $x, y, z \in \mathbb{O}$, set $D_{x,z}(y) = [y, [x, z]] - 3(x, y, z)$. Now we can give an explicit construction of $G(3)$:

Theorem 6.1. *[23, Theorem 4.4.5] Let V_1 be a two-dimensional vector space over $\mathbb{C}$. Let $\psi_1 : V_1 \times V_1 \to \mathbb{C}$ be a nondegenerate skew symmetric bilinear form defined by $\psi_1(u_1, u_{-1}) = 1$ if we let $\{u_1, u_{-1}\}$ as a basis of V_1. Define $p_1 : V_1 \times V_1 \to \mathfrak{sl}_2$ by*

$$p_1(x, y)(z) = 4\{\psi_1(y, z)x - \psi_1(z, x)y\}$$

for $x, y, z \in V_1$.

Let $G_2 = \operatorname{Der}\mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$, $V_2 = \mathbb{O}_{(0)} \otimes_{\mathbb{R}} \mathbb{C}$ and let ψ_2 be the bilinear map $\psi_2 : V_2 \times V_2 \to \mathbb{C}$ given by $\psi_2(x, y) = t(x\bar{y})/2$. Define $p_2 : V_2 \times V_2 \to G_2$ by $p_2(x, y) = D_{x,y}$.

Let $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}_2 \oplus G_2$ and $\mathfrak{g}_{\bar{1}} = V_1 \boxtimes V_2$. Then $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ becomes a Lie superalgebra when we define

$$[x_1 \otimes x_2, y_1 \otimes y_2] = \psi_1(x_1, y_1)p_2(x_2, y_2) + \psi_2(x_2, y_2)p_1(x_1, y_1).$$

Consider the following basis of $\mathbb{O}_{(0)}$:

$$v_3 = (0, -i),\ v_2 = (0, -j),\ v_1 = (0, -k),\ v_0 = (0, 1)$$
$$v_{-1} = (k, 0),\ v_{-2} = (j, 0),\ v_{-3} = (i, 0).$$

Together with the identity $1 = (1, 0)$ these elements form a basis for $\mathbb{O}_{(0)}$. We will write $D_{j,k}$ for D_{v_j,v_k}. Define a basis for $V_2 = \mathbb{O}_{(0)} \otimes_{\mathbb{R}} \mathbb{C}$ by

$$e_3 = v_3 + iv_{-3},\ e_2 = v_1 - iv_{-1},\ e_1 = v_2 - iv_{-2},\ e_0 = v_0,$$

$$e_{-1} = v_2 + iv_{-2},\ e_{-2} = v_1 + iv_{-1},\ e_{-3} = v_3 - iv_{-3}.$$

Using this basis we identify G_2 with its image in $\mathfrak{gl}_7(\mathbb{C})$. The following computations are taken from [23, Section 4.7] :

Theorem 6.2.

1. *Using the basis $\{e_3, \ldots, e_{-3}\}$, the derivations $D_{1,-1}$ and $D_{2,-2}$ are represented by diagonal matrices with entries*

$$(-2i, -4i, 2i, 0, -2i, 4i, 2i) \quad \text{and} \quad (-2i, 2i, -4i, 0, 4i, -2i, 2i)$$

   respectively.

2. $D_{1,-1} + D_{2,-2} + D_{3,-3} = 0.$

3. *Set $h_1 = iD_{2,-2}/2$ and $h_2 = i(D_{1,-1} - D_{2,-2})/2$. Then $\{h_1, h_2\}$ is a basis for a Cartan subalgebra of G_2.*

4. *Define α_1, α_2 in the dual of this Cartan subalgebra by $\alpha_j(h_k) = a_{jk}$ where $A = (a_{jk}) = \begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}$. Then $e_{\pm 3}$, $e_{\pm 2}$, $e_{\pm 1}$, e_0 are weight vectors of*

weights $\pm(2\alpha_1 + \alpha_2)$, $\pm(\alpha_1 + \alpha_2)$, $\pm\alpha_1$, *and* 0 *respectively. As a set of positive roots of* G_2 *we can take* $\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$.

5. *The derivation* $\Delta_1 = D_{1,3} - iD_{1,-3} - iD_{-1,3} - D_{-1,-3}$ *is represented by the matrix* $x_1 := -4E_{1,2} + 4E_{3,4} - 8E_{4,5} + 4E_{6,7}$.

6. *The derivation* $\Delta_2 = D_{1,2} - iD_{1,-2} - iD_{-1,2} - D_{-1,-2}$ *is represented by the matrix* $x_1 := 12E_{2,3} - 12E_{5,6}$.

7. *For every weight* μ *of* V_2 *we have* $x_j.V^\mu \subseteq V^{\mu+\alpha_j}$ *and hence* $x_j \in G_2^{\alpha_j}$ *for* $j = 1, 2$.

8. *The remaining generators of* G_2 *as a Lie algebra are:*

$$y_1 = -k(x_1)/2 \quad \textit{and} \quad y_2 = -k(x_2)/48,$$

where $K(x_i)$ *is the complex conjugate of* x_i. *Then* $y_j \in G_2^{-\alpha_j}$ *for* $j = 1, 2$.

9. *For* $j = 1, 2$ *and* h *in the Cartan subalgebra of* G_2 *we have*

$$[h_i, h_j] = 0,\ [x_i, y_j] = \delta_{ij} h_i,\ [h, x_i] = \alpha_i(h) x_i,\ [h, y_i] = -\alpha_i(h) y_i.$$

10. *Let* $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ *and consider the basis* $\{u_1, u_{-1}\}$ *for* V_1. *In addition to the generators* x_i, y_i, h_i *for* $i = 1, 2$, $G(3)$ *has generators*

$$x_0 = (u_1 \otimes e_{-3})/2, \quad y_0 = (u_{-1} \otimes e_3)/2, \quad \textit{and} \quad h_0 = -2H - 2h_1 - h_2.$$

11. Let ψ_1, ψ_2, p_1, p_2 be as defined in Theorem 6.1. Then

$$p_1(u_1, u_{-1}) = -4H, \quad p_2(e_{-3}, e_3) = -8h_1 - 4h_2, \quad \psi_2(e_{-3}, e_3) = 2.$$

12. Let $\mathfrak{h}$ be the Cartan subalgebra of $G(3)$ with basis $\{h_0, h_1, h_2\}$ and define $\alpha_j \in \mathfrak{h}^*$ by $\alpha_j(h_k) = a_{jk}$ where

$$A = (a_{jk}) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -3 & 6 \end{bmatrix}.$$

Then for $j = 0, 1, 2$ and $h \in \mathfrak{h}$ we have

$$[h_i, h_j] = 0,\ [x_i, y_j] = \delta_{ij} h_i,\ [h, x_i] = \alpha_i(h) x_i,\ [h, y_i] = -\alpha_i(h) y_i.$$

13. Let 2δ be the positive root of $\mathfrak{sl}_2$ and set $\alpha_0 = \delta - 2\alpha_1 - \alpha_2$. Then $\{\alpha_0, \alpha_1, \alpha_2\}$ is a system of simple roots for $G(3)$.

If we set

$$\alpha_0 = \delta + \varepsilon_3, \quad \alpha_1 = \varepsilon_1, \quad \alpha_2 = \varepsilon_2 - \varepsilon_1,$$

the root system in Theorem 6.2 can be realized in the space $\mathbf{H}^* = \mathbb{C}^3$ endowed with a basis $\{\delta, \varepsilon_1, \varepsilon_2\}$ and with the bilinear defined by :

$$(\varepsilon_1, \varepsilon_1) = (\varepsilon_2, \varepsilon_2) = -2(\varepsilon_1, \varepsilon_2) = -(\delta, \delta) = 2.$$

We can then take as simple roots ε_1, $\varepsilon_2 - \varepsilon_1$ and $-\varepsilon_1 - \varepsilon_2 + \delta$. Then positive even roots are : $\Phi_{\bar{0}}^+ = \{\varepsilon_1, 2\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + 2\varepsilon_2, \varepsilon_2, \varepsilon_2 - \varepsilon_1, 2\delta\}$ and positive odd roots are : $\Phi_{\bar{1}}^+ = \{(\pm(u\varepsilon_1 + v\varepsilon_2) + \delta \mid\ (u, v) \in \{0, 1\}^2\}$. We have

$\rho = 2\varepsilon_1 + 3\varepsilon_2 - 5\delta/2$. We identify the set of dominant weights for $\mathfrak{g}_{\bar{0}}$ with $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ by means of the fundamental weights ($\omega_1 = \varepsilon_1 + \varepsilon_2; \omega_2 = \varepsilon_1 + 2\varepsilon_2; \omega_3 = \delta$). For $\lambda = m_1\omega_1 + m_2\omega_2$, the dimension of the simple $G(2)$-module, $L(\lambda)$, can be computed by the Weyl-dimension formula [20, Section 24.3]:

$$\dim L(\lambda) = \frac{1}{5!}(m_1+1)(m_2+1)(m_1+m_2+2)(m_1+2m_2+3)$$
$$(m_1+3m_2+4)(m_1+3m_2+4)(2m_1+3m_2+5). \quad (6.1)$$

By proposition 2.2 in [16], the set of dominant weights for $\mathfrak{g}$ is : $X^+ = \{(a,b,c) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} \mid \ a = 0 \Rightarrow b = c = 0,\ a \neq 1 \text{ and } a = 2 \Rightarrow b = 0\}$. Besides, the eigenvalue of the Casimir element is :

$$\mathrm{Cas}(a,b,c) = 2b^2 + 6bc + 10b + 6c^2 + 18c - 2a^2 + 10a.$$

For $k \in \mathbb{N}$, we denote by Γ_k the set of dominant weights $\lambda \in X^+$ such that $\mathrm{Cas}(\lambda) = 6k(k+1)$. For $l \in \mathbb{N}$, we set :

$$\begin{aligned}
\lambda_{0,0} &= (0,0,0),\\
\lambda_{0,1} &= (5,0,0),\\
\lambda_{k,0} &= (2,0,k-1) \quad \text{if } k \geq 1,\\
\lambda_{k,1} &= (3,0,k-1) \quad \text{if } k \geq 1,\\
\lambda_{k,l} &= \begin{cases} (l+2, 2l-2, k-l) & \text{if } 2 \leq l \leq k,\\ (l+3, 3k-l, l-k-1) & \text{if } k+1 \leq l \leq 3k,\\ (l+4, l-3k-1, 2k) & \text{if } 3k+1 \leq l. \end{cases}
\end{aligned}$$

Theorem 6.3. *([10, Theorem 4.1.1]) Let $\mathfrak{g} = G(3)$. Every atypical block of $\mathfrak{g}$ is one of the Γ_k. For every $k \in \mathbb{N}$, Γ_k is equivalent to the principal block of $\mathfrak{osp}(3,2)$.*

The map $\mathbb{N} \to \Gamma_k$, $l \mapsto \lambda_{k,l}$ *is bijective ; statements* (b) *and* (c) *of Theorem 4.1 apply to* Γ_k.

6.2 Complexity of simple $G(3)$-modules

Since the principal block $\Gamma_0 = \{\lambda_{0,l} \mid \ l \in \mathbb{N}\}$ is equivalent to the principal block of $\mathfrak{osp}(3|2)$ and statement *(b)* in Theorem 4.1 holds for Γ_0 ([10, Lemma 4.3.1]), we will have the same minimal projective resolution (4.4) for $S(\lambda_{0,0})$. Then

Theorem 6.4. *If* S *is an atypical simple* $G(3)$*-module, we have* $c_{\mathcal{F}}(S) = 8$.

Proof. Using the projective resolution (4.4), the bounds in equation (5.2), and the dimension formula given in (6.1), we have

$$C'.d^7 \leq \dim P_d \leq C.d^7,$$

for some positive constants C, C'. These bounds are obtained by multiplying the dimension formula given in (6.1) by an extra factor from the $\mathfrak{sl}_2$-part. This shows that $c_{\mathcal{F}}(S(\lambda_0)) = 8$. Since [6, Theorem 4.1.1] holds over $G(3)$, then all simple modules of the same atypicality have the same complexity and the same support variety. Thus the complexity of all atypical simple $G(3)$-modules is 8. □

6.2.1 A geometric interpretation of the complexity

The same discussion as in 5.2.1 shows that we have the following geometric interpretation in this case: If S is a simple $G(3)$-module, then

$$c_{\mathcal{F}}(S) = \dim \mathcal{X}_S + \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(S).$$

6.3 z-complexity

The z-complexity is a categorical invariant, thus using the equivalence between the principal block of $\mathfrak{osp}(3|2)$ and the atypical blocks $G(3)$ we have:

Proposition 6.5. *If S is an atypical simple $G(3)$-module, then $z_{\mathcal{F}}(S) = 2$.*

6.3.1 Detecting subsuperalgebra

Let $\mathfrak{f}_{\bar{1}} \subseteq \mathfrak{g}_{\bar{1}}$ be the span of the root vectors $x_0 = x_{\alpha_0}, y_0 = x_{-\alpha_0}$ where $\alpha_0 = \varepsilon_3 + \delta$. Set $\mathfrak{f}_{\bar{0}} = [\mathfrak{f}_{\bar{1}}, \mathfrak{f}_{\bar{1}}]$. We define a three-dimensional subalgebra of $\mathfrak{g}$ by

$$\mathfrak{f} := \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}}.$$

The Lie superalgebra $\mathfrak{f}$ is classical and so has a support variety theory. Furthermore,

Lemma 6.6. $[\mathfrak{f}_{\bar{0}}, \mathfrak{f}_{\bar{1}}] = 0$.

Proof. The even part $\mathfrak{f}_{\bar{0}}$ is spanned by the vector $[x_0, y_0] = h_0$. From Theorem 6.2 we have $[h_0, x_0] = \alpha_0(h_0)x_0 = 0$ and $[h_0, y_0] = \alpha_0(h_0)y_0 = 0$. The result follows. □

It follows that these varieties admit a rank variety description and, in particular, can be identified as subvarieties of $\mathfrak{f}_{\bar{1}}$, i.e.,

$$\begin{aligned}\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M) &= \mathcal{V}^{rank}_{\mathfrak{f}_{\bar{1}}}(M) \\ &= \{y \in \mathfrak{f}_{\bar{1}} \mid \text{M is not projective as } U(\langle y \rangle)\text{-module}\} \cup \{0\}.\end{aligned}$$

Note that $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S(\lambda_{0,0})) = \mathfrak{f}_{\bar{1}}$. We can use this detecting subsuperalgebra to give a geometric interpretation of the z-complexity. For instance,

Corollary 6.7. $z_{\mathcal{F}}(S(\lambda_{0,0})) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S(\lambda_{0,0})) = 2.$

In general:

Theorem 6.8.

(1) *If S is a typical simple module, then*

$$z_{\mathcal{F}}(S) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S) = 0.$$

(2) *If S is an atypical simple module, then*

$$z_{\mathcal{F}}(S) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S) = 2.$$

Proof. If S is a typical simple module, then S is projective and the first part follows. If S is atypical, then by [6, Theorem 4.1.1],

$$\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S) = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S(\lambda_{0,0})),$$

thus $\dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S) = 2$. The second part follows. □

Chapter 7

Computing the complexity over $F(4)$

In this chapter we calculate the complexity of simple modules over the exceptional Lie superalgebra $F(4)$. We introduce the structure of $F(4)$ as given in [23]. We use the characterization of the simple modules and their projective covers given in [22] to find the complexity and the z-complexity. The geometric interpretation of these complexities in this case agrees with our earlier results.

7.1 The Lie superalgebra $F(4)$

The construction of the classical Lie superalgebra $F(4)$ involves the use of Clifford algebras and spinor representations.

Definition 7.1. *Let $\mathbb{C}^{2n}$ be the vector space of column vectors with $2n$ entries and let β be the bilinear form on $\mathbb{C}^{2n}$ defined by $\beta(x,y) = x^t y$. Let $T(\mathbb{C}^{2n})$ be the tensor algebra on $\mathbb{C}^{2n}$ and J the ideal of this tensor algebra generated by the elements*

$$xy + yx - \beta(x,y)$$

for $x, y \in \mathbb{C}^{2n}$. The Clifford algebra associated to β is $C(\beta) \ := T(\mathbb{C}^{2n}/J$.

Proposition 7.2. *[23, Proposition A.3.5] If $\{x_1, \ldots, x_{2n}\}$ is a basis for $\mathbb{C}^{2n}$ then $C(\beta)$ has a basis consisting of the monomials $x_1^{r_1} x_2^{r_2} \cdots x_{2n}^{r_{2n}}$ where $r_i = 0$ or 1 for $i = 1, 2, \ldots, 2n$.*

Theorem 7.3. *[23, Theorems A.3.7 and A.3.8] $C(\beta)$ has a unique simple module S with $\dim_{\mathbb{C}} S = 2^n$. S is called the* ***space of spinors****.*

Let $R : C(\beta) \to \mathrm{End}(S)$ be the corresponding representation. Set $\Gamma_i = R(x_i)$ for $1 \leq i \leq 2n$. Then

$$\Gamma_k \Gamma_+ \Gamma_j \Gamma_k = 2\delta_{jk} \tag{7.1}$$

for $1 \leq j, k \leq 2n$. If we set $\Gamma_{2n+1} = i^n \Gamma_1 \cdots \Gamma_{2n}$ then equation (7.1) holds for $1 \leq j, k \leq 2n+1$.

Define $\Gamma_j^t \in \mathrm{End}\, S^*$ by $(\Gamma_j^t f)(v) = f(\Gamma_j v)$, for $f \in S^*$, $v \in S$. The relation (7.1) also holds if we replace the elements $\Gamma_j \in \mathrm{End}\, S$ by $(-1)^n \Gamma_j^t \in \mathrm{End}\, S^*$. By the uniqueness of S there is an isomorphism $f : S \to S^*$ such that

$$(-1)^n \Gamma_j^t \circ f - f \circ \Gamma_j$$

for $1 \leq j \leq 2n$. Define a nondegenerate bilinear form ψ on S by $\psi(u, v) = (f(u))(v)$. Then

$$\psi(\Gamma_j u, v) = (-1)^n \psi(u, \Gamma_j v) \tag{7.2}$$

for all $u, v \in S$ and $1 \leq j \leq 2n+1$. Let $\mathfrak{so}(2n+1)$ be the Lie algebra of all skew-symmetric $2n+1 \times 2n+1$ matrices. The spin representation of $\mathfrak{so}(2n+1)$ is the Lie algebra homomorphism $\mathfrak{so}(2n+1) \to \mathrm{End}\, S$ sending $E_{jk} - E_{kj}$ to $\frac{1}{2}\Gamma_j \Gamma_k$ when $j \neq k$. We identify $\mathfrak{so}(2n+1)$ with its image in $\mathrm{End}\, S$. Then S is a simple $\mathfrak{so}(2n+1)$-module and ψ is an $\mathfrak{so}(2n+1)$-invariant bilinear form on S. It is known that ψ is symmetric if $n \equiv 0, 3 \mod 4$ and it is skew-symmetric otherwise.

The following theorem gives an explicit construction of $F(4)$:

Theorem 7.4. *[23, Theorem 4.5.5] Let V_1 be a two-dimensional vector space over $\mathbb{C}$. Let $\psi_1 : V_1 \times V_1 \to \mathbb{C}$ be a nondegenerate skew symmetric bilinear form defined by $\psi_1(u_1, u_{-1}) = 1$ if we let $\{u_1, u_{-1}\}$ as a basis of V_1. Define*

$p_1 : V_1 \times V_1 \to \mathfrak{sl}_2$ by

$$p_1(x,y)(z) = 3\{\psi_1(y,z)x - \psi_1(z,x)y\}$$

for $x, y, z \in V_1$.

Let $V_2 = S$ be the spin representation of $\mathfrak{so}_7$ and let $\psi_2 : V_2 \times V_2 \to \mathbb{C}$ be a nondegenerate bilinear form such that $\psi_2(\Gamma_j u, v) = -\psi_2(u, \Gamma_j v)$ for $u, v \in V_2$. Identify $\mathfrak{so}_7$ with its image in End S *and define $p_2 : V_2 \times V_2 \to \mathfrak{so}_7$ by*

$$p_2(u,v) = \sum_{1 \le j < k \le 7} \psi_2(u, \Gamma_j \Gamma_k v) \Gamma_j \Gamma_k.$$

Let $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}_2 \oplus \mathfrak{so}_7$ and $\mathfrak{g}_{\bar{1}} = V_1 \boxtimes V_2$. Then $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ becomes a Lie superalgebra when we define

$$[x_1 \otimes x_2, y_1 \otimes y_2] = \psi_1(x_1, y_1) p_2(x_2, y_2) + \psi_2(x_2, y_2) p_1(x_1, y_1).$$

The root system can be realized in the space $\mathbf{H}^* = \mathbb{C}^4$ endowed with a basis $\{\delta, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and with the bilinear defined by : $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, $(\delta, \delta) = -3$, $(\varepsilon_i, \delta) = 0$ for all i, j. We take as simple roots $\alpha_1 = \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \delta)$, $\alpha_2 = \varepsilon_3$, $\alpha_3 = -\varepsilon_2 - \varepsilon_3$, and $\alpha_4 = \varepsilon_1 - \varepsilon_2$. Then positive even roots are : $\Phi_{\bar{0}}^+ = \{\delta, \varepsilon_i, \varepsilon_i \pm \varepsilon_j \,|\, i < j\}$ and positive odd roots are : $\Phi_{\bar{1}}^+ = \{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 + \delta)\}$. We have $\rho = \frac{1}{2}(5\varepsilon_1 + 3\varepsilon_2 + \varepsilon_3 - 3\delta)$.

The even part of $\mathfrak{g}$ is $\mathfrak{g}_{\bar{0}} \cong \mathfrak{sl}_2 \oplus \mathfrak{so}_7$ of type $A_1 \oplus B_3$. The set of dominant weights and the atypical blocks are described in [22]. The maximal atypicality is 1 as the defect of $F(4)$ is 1. The fundamental weights are $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1 + \varepsilon_2$, $\omega_3 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$, and $\omega_4 = \frac{1}{2}\delta$. For $\lambda = m_1\omega_1 + m_2\omega_2 + m_3\omega_3$, the dimension of the simple $\mathfrak{so}(7)$-module, $L(\lambda)$, can be computed by the Weyl-dimension formula [20, section 24.3]:

$$\dim L(\lambda) = \frac{1}{720}(m_1+1)(m_2+1)(m_3+1)(m_1+m_2+2)(m_2+m_3+2)$$
$$(2m_2+m_3+3)(m_1+m_2+m_3+3)(m_1+2m_2+m_3+4)(2m_1+2m_2+m_3+5). \tag{7.3}$$

Note that the spin representation of $\mathfrak{so}_7$ has the following explicit construction. Let W be a three-dimensional vector space over $\mathbb{C}$, then $\bigwedge^\bullet W$ is an eight-dimensional vector space over $\mathbb{C}$:

Proposition 7.5. *[11, prop 20.20] The vector space $\bigwedge^\bullet W$ is the irreducible representation of $\mathfrak{so}_7$ with highest weight $\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3)$. Moreover, $e_I = e_{i_1}\wedge e_{i_2}\wedge\ldots\wedge e_{i_k}\in\bigwedge^\bullet W$ has the weight $\frac{1}{2}(\sum_{i\in I}\varepsilon_i - \sum_{j\notin I}\varepsilon_j)$.*

7.2 Complexity of simple $F(4)$-modules

The description of the projective indecomposable modules over $F(4)$ given in [22, Lemma 11.1] is the same as the one given in [10] over $G(3)$, except for a small difference in the notation. For example, in [22], λ_1 corresponds to the trivial module, while in [10] λ_0 corresponds to the trivial module. This similarity means that the projective resolution (4.4) over $\mathfrak{osp}(3|2)$ will carry over to $F(4)$.

Theorem 7.6. *If S is an atypical simple $F(4)$-module, we have $c_{\mathcal{F}}(S) = 9$.*

Proof. Using the projective resolution (4.4) and the bounds in equation (5.2), and the dimension formula in equation (7.3), we have

$$C'.d^8 \le \dim P_d \le C.d^8,$$

for some positive constants C, C'. These bounds are obtained by multiplying the dimension formula in equation (7.3) by an extra factor from the $\mathfrak{sl}_2$-part. This shows that $c_{\mathcal{F}}(S(\lambda_1)) = 9$. Since [6, Theorem 4.1.1] holds over $F(4)$, then all simple modules of the same atypicality have the same complexity and the same support variety. Thus the complexity of all atypical simple $F(4)$-modules is 9. □

7.2.1 A geometric interpretation of the complexity

The same discussion as in 5.2.1 shows that we have the following geometric interpretation in this case: If S is an atypical simple $F(4)$-module, then

$$c_{\mathcal{F}}(S) = \dim \mathcal{X}_S + \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(S).$$

7.3 z-complexity

The radical layer structure of the projective indecomposable modules over $F(4)$ is the same as that over $G(3)$. This will give the same projective resolutions over $F(4)$. Thus

Proposition 7.7. *If S is an atypical simple $F(4)$-module, then $z_{\mathcal{F}}(S) = 2$.*

7.3.1 Detecting subsuperalgebra

Let $\mathfrak{f}_{\bar{1}} \subseteq \mathfrak{g}_{\bar{1}}$ be the span of the root vectors $x_\alpha, x_{-\alpha}$ where $\alpha = 1/2(\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta)$. Set $\mathfrak{f}_{\bar{0}} = [\mathfrak{f}_{\bar{1}}, \mathfrak{f}_{\bar{1}}]$. We define a subalgebra of $\mathfrak{g}$ by

$$\mathfrak{f} := \mathfrak{f}_{\bar{0}} \oplus \mathfrak{f}_{\bar{1}}.$$

The Lie superalgebra $\mathfrak{f}$ is classical and so has a support variety theory. Furthermore,

Lemma 7.8. *$\mathfrak{f}$ is three-dimensional and satisfies* $[\mathfrak{f}_{\bar{0}}, \mathfrak{f}_{\bar{1}}] = 0$.

Proof. The root vectors x_α and $x_{-\alpha}$ are given by

$$x_\alpha = u_1 \otimes (e_1 \wedge e_2 \wedge e_3), \quad x_{-\alpha} = u_{-1} \otimes 1,$$

where $\{u_1, u_{-1}\}$ is a basis of V_1 and $\{e_1, e_2, e_3\}$ is a basis of W, with $\bigwedge^\bullet W = V_2$. Recall that we have $\psi_1(u_1, u_{-1}) = 1$ in Theorem 7.4. We can easily show that

$$p_1(u_1, u_{-1}) = \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix} \in \mathfrak{sl}_2.$$

Since $e_1 \wedge e_2 \wedge e_3$ and 1 have opposite weights, then $\psi_2(e_1 \wedge e_2 \wedge e_3, 1) \neq 0$. Choose ψ_2 to be normalized in the sense that $\psi_2(e_1 \wedge e_2 \wedge e_3, 1) = 1$. To compute the bracket $[x_\alpha, x_{-\alpha}]$ we need to find all the elements, $\Gamma_j\Gamma_k$ in $\mathfrak{so}_7$ such that

$$\psi_2(e_1 \wedge e_2 \wedge e_3, \Gamma_j\Gamma_j.1) \neq 0$$

which is equivalent to having $\Gamma_j\Gamma_k$ as a Cartan element. These elements are

$$\Gamma_2\Gamma_5 = 2(E_{25} - E_{52}) \quad \Gamma_3\Gamma_6 = 2(E_{36} - E_{63}) \quad \Gamma_4\Gamma_7 = 2(E_{47} - E_{74}),$$

and they act on the eigenvector 1 by $-i$. Thus

$$\begin{aligned} p_2(e_1 \wedge e_2 \wedge e_3, 1) &= \psi_2(e_1 \wedge e_2 \wedge e_3, -i.1)\Gamma_2\Gamma_5 + \psi_2(e_1 \wedge e_2 \wedge e_3, -i.1)\Gamma_3\Gamma_6 \\ &\quad + \psi_2(e_1 \wedge e_2 \wedge e_3, -i.1)\Gamma_4\Gamma_7 \\ &= -i(\Gamma_2\Gamma_5 + \Gamma_3\Gamma_6 + \Gamma_4\Gamma_7) \\ &= -2i(E_{25} - E_{52} + E_{36} - E_{63} + E_{47} - E_{74}) \in \mathfrak{so}_7. \end{aligned}$$

If we use the definition of $\mathfrak{so}_7$ given in [20], then

$$p_2(e_1 \wedge e_2 \wedge e_3, 1) = 2(h_1 + h_2 + h_3),$$

where $h_i = E_{i+1,i+1} - E_{i+4,i+4}$ for $i = 1, 2, 3$. Let $h = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathfrak{sl}_2$, then

$$[x_\alpha, x_{-\alpha}] = \Big(-3h, 2(h_1 + h_2 + h_3)\Big) \in \mathfrak{sl}_2 \oplus \mathfrak{so}_7.$$

Having $[x_\alpha, x_\alpha] = [x_{-\alpha}, x_{-\alpha}] = 0$ shows that $\mathfrak{f}$ is three-dimensional. Moreover,

$$\begin{aligned} [[x_\alpha, x_{-\alpha}], x_\alpha] &= [\Big(-3h, 2(h_1 + h_2 + h_3)\Big), u_1 \otimes (e_1 \wedge e_2 \wedge e_3)] \\ &= -3h.u_1 \otimes (e_1 \wedge e_2 \wedge e_3) + u_1 \otimes (2h_1 + 2h_2 + 2h_3)(e_1 \wedge e_2 \wedge e_3) \\ &= -3x_\alpha + u_1 \otimes (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)(h_1 + h_2 + h_3)(e_1 \wedge e_2 \wedge e_3) \\ &= -3x_\alpha + 3x_\alpha = 0 \end{aligned}$$

Similarly we can show $[[x_\alpha, x_{-\alpha}], x_{-\alpha}] = 0$. This shows that $[\mathfrak{f}_{\bar{0}}, \mathfrak{f}_{\bar{1}}] = 0$. □

It follows that these varieties admit a rank variety description and, in particular, can be identified as subvarieties of $\mathfrak{f}_{\bar{1}}$, i.e.,

$$\begin{aligned}\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(M) &= \mathcal{V}_{\mathfrak{f}_{\bar{1}}}^{rank}(M) \\ &= \{y \in \mathfrak{f}_{\bar{1}} \mid \text{M is not projective as } U(\langle y \rangle)\text{-module}\} \cup \{0\}.\end{aligned}$$

Note that $\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S(\lambda_1)) = \mathfrak{f}_{\bar{1}}$. We can use this detecting subsuperalgebra to give a geometric interpretation of the z-complexity. For instance,

Corollary 7.9. $z_{\mathcal{F}}(S(\lambda_1)) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S(\lambda_1)) = 2.$

In general:

Theorem 7.10.

(1) *If S is a typical simple module, then*

$$z_{\mathcal{F}}(S) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S) = 0.$$

(2) *If S is an atypical simple module, then*

$$z_{\mathcal{F}}(S) = \dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S) = 2.$$

Proof. If S is a typical simple module, then S is projective and the first part follows. If S is atypical, then by [6, Theorem 4.1.1],

$$\mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S) = \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S(\lambda_1)),$$

thus $\dim \mathcal{V}_{(\mathfrak{f},\mathfrak{f}_{\bar{0}})}(S) = 2$. The second part follows. □

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