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SOME SHARP INEQUALITIES RELATED TO
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DEDICATION

To my parents Fuliang Li and Guiling Wang.

To my husband Zhaowen and our children Jerry and Jasper.

Words cannot express how much I love you all.

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Abstract

In this dissertation, we focus on the study of sharp inequalities of Moser-Trudinger-Onofri type. We first establish the analog Bliss and Hardy inequalities with sharp constant involving exponential weight function. One special case of the inequalities (for $n = 2$) leads to a direct proof of Onofri inequality on S^2. Then we establish the sharp trace inequalities on any smooth bounded simply connected domain in $\mathbb{R}^2$.

Chapter 1

Introduction

The sharp Sobolev inequality on $\mathbb{R}^n$ $(n \geq 3)$ was obtained by Aubin [3] and Talenti [31], respectively, in 1976: for all $u(x)$ satisfying $u \in L^{2n/(n-2)}(\mathbb{R}^n)$ and $\nabla u \in L^2(\mathbb{R}^n)$

$$(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \leq S_1 \int_{\mathbb{R}^n} |\nabla u|^2, \tag{1.1}$$

where $1/S_1 = \pi n(n-2)(\Gamma(n/2)/\Gamma(n))^{2/n}$ is called the best constant of Sobolev inequality. Extremal functions were also given by them. By even reflection, One can have the sharp Sobolev inequality on the upper half space:

$$(\int_{\mathbb{R}^n_+} |u|^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \leq 2^{\frac{2}{n}} S_1 \int_{\mathbb{R}^n_+} |\nabla u|^2$$

for all u satisfying $u \in L^{2n/(n-2)}(\mathbb{R}^n_+)$ and $\nabla u \in L^2(\mathbb{R}^n_+)$, where $\mathbb{R}^n_+ = \{(x_1, ..., x_n) \mid x_n > 0\}$ is the upper half space. And (1.1) also implies the following local sharp Sobolev inequality: on any smooth bounded domain $\Omega \subset \mathbb{R}^n$,

$$(\int_{\Omega} |u|^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \leq S_1 \int_{\Omega} |\nabla u|^2, \quad \forall \ u \in H^1_0(\Omega).$$

Based on this local inequality, Aubin [2] was able to establish the sharp type Sobolev inequality on any compact Riemmanian manifolds via the partition of unity: if M is an n-dimensional compact Riemmanian manifold $(n \geq 3)$, then for every $\epsilon > 0$, there is a constant $C = C(\epsilon) > 0$, such that

$$(\int_M |u|^{\frac{2n}{n-2}} dv_g)^{\frac{n-2}{n}} \leq (S_1 + \epsilon) \int_M |\nabla_g u|^2 dv_g + C(\epsilon) \int_M u^2 dv_g. \qquad (1.2)$$

Inequality (1.2) is successfully used by Aubin to settle down the Yamabe problem for high dimensional ($n \geq 5$) and non-conformally flat manifolds. Later, Yamabe problem was completely solved through the work of Schoen [30].

It is certainly an interesting mathematical problem to study the behavior of $C(\epsilon)$ as ϵ tends to zero. In fact it was conjectured by Aubin in 1976 that on any smooth n-dimensional compact Riemannian manifold without boundary (M, g) ($n \geq 3$), there is a constant $C(M, g)$ such that $\forall u \in H^1(M)$

$$(\int_M |u|^{\frac{2n}{n-2}} dv_g)^{\frac{n-2}{n}} \leq S_1 \int_M |\nabla_g u|^2 dv_g + C(M, g) \int_M u^2 dv_g. \qquad (1.3)$$

Aubin's conjecture was later proved by Hebey and Vaugon [18] in 1996.

On the upper half space $\mathbb{R}^n_+$, the sharp trace inequality was proved by P. L. Lions [24]:

$$(\int_{\partial\mathbb{R}^n_+} |u|^{\frac{2(n-1)}{n-2}})^{\frac{n-2}{n-1}} \leq S \int_{\mathbb{R}^n_+} |\nabla u|^2$$

for all u satisfying $u \in L^{2(n-1)/(n-2)}(\partial\mathbb{R}^n_+)$ and $\nabla u \in L^2(\mathbb{R}^n_+)$, where $S = 2/(n-2) \cdot (2\pi^{n/2}/(\Gamma(n/2)))^{1/(1-n)}$ is called the best constant of trace inequality. Later the extremal functions were found by Escobar [14] and Beckner [4] independently.

Stimulated by the work of Hebey and Vaugon [18], Y.Y. Li and M. Zhu [21] established the corresponding sharp trace inequality on any compact Riemannian manifold with boundary. Namely, they showed that for any smooth $n-$dimensional compact Riemannian manifold (M, g) ($n \geq 3$) with smooth boundary, there exists a constant $A(M,g)>0$, such that $\forall u \in H^1(M)$

$$(\int_{\partial M} |u|^{\frac{2(n-1)}{n-2}} ds_g)^{\frac{n-2}{n-1}} \leq S \int_M |\nabla_g u|^2 dv_g + A(M,g) \int_{\partial M} u^2 ds_g, \tag{1.4}$$

where dv_g denotes the volume form of (M,g) and ds_g denotes the induced volume form on ∂M. Using the same method, they [22] established a sharp Sobolev inequality on any bounded domain with smooth boundary: For any smooth bounded open set $\Omega \subset \mathbb{R}^n (n \geq 3)$, there exists some constant $A(\Omega)>0$ such that

$$(\int_\Omega |u|^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \leq 2^{\frac{2}{n}} S_1 \int_\Omega |\nabla u|^2 + A(\Omega) \int_{\partial\Omega} u^2 \quad \forall u \in H^1(\Omega). \tag{1.5}$$

The two dimensional case is quite different.

Let Ω be a smooth bounded domain in $\mathbb{R}^2$. It is well-known that $L^\infty(\Omega)$ is not a subset of $H_0^1(\Omega)$ even if Ω is a two dimensional ball. Nevertheless, it was shown by Trudinger [32] that functions in $H_0^1(\Omega)$ are actually in the exponential class.

Theorem A (**Trudinger's inequality**) *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. There are two positive constants β_0 and C_1 (C_1 depending on the area of Ω) such that if $u \in H_0^1(\Omega)$ and $||\nabla u||_{L^2} \leq 1$, then*

$$\int_\Omega e^{\beta u^2} dx \leq C_1 \tag{1.6}$$

for all $\beta \leq \beta_0$. Moreover, if $\{u_i\}_{i-1}^\infty$ is bounded in $W_0^{1,n}(\Omega)$, then up to a subsequence of i, $u_i \rightharpoonup u_0$ in $W_0^{1,n}(\Omega)$ and $\exp(nu_i) \to \exp(nu_0)$ *in L^1 norm.*

The key observation in Trudinger's proof is that in the following embedding inequality

$$\left\{\frac{1}{|\Omega|}\int_\Omega |u|^p dx\right\}^{1/p} \leq c(p)\left\{\int_\Omega |\nabla u|^2 dx\right\}^{1/2}, \tag{1.7}$$

the coefficient satisfies $c(p) \leq Cp^{1-1/2}$. Adams in [1] pointed out that one can obtain the similar results involving higher order derivatives.

The best constant β_0 in Theorem A was later found by Moser [25], $\beta_0 = n\omega_{n-1}^{\frac{1}{n-1}}$ and ω_{n-1} is the $(n-1)-$dimensional surface of the unit sphere. Carleson and Chang [8] state that there is an extremal function which realizes the equality in Theorem A when $\beta = \beta_0$ and Ω is a unit ball in $\mathbb{R}^n$. For $n = 3$, Onofri derived the following inequality [28] (See also [19] and [29]):

Theorem B (**Onofri inequality**) *Let (S^2, g_0) be the standard unit sphere in $\mathbb{R}^3$. For any $u \in W^{1,2}(S^2)$,*

$$\ln\left(\frac{1}{4\pi}\int_{S^2} e^{2u}dx\right) \leq \frac{1}{4\pi}\int_{S^2}(|\nabla u|^2 + 2u)dx.$$

The equality holds if and only if the curvature under metric $e^{2u}g_0$ is constant.

Onofri proved this inequality in his study of the volume element in string theory integrals (see also Hong [19] for an analytic approach). It was re-discovered by Osgood, Phillips and Sarnak in their proof of the C^∞-compactness of isospectral metrics on compact surfaces. As another important application, Onofri inequality is used to derive the lower bound for the Liouville energy on a topological two-sphere, which can be used to derive the global existence of Ricci flow and Calabi flow on 2-sphere, see, for example, Chow [11] and Chen [9].

The proofs of Theorem B by Onofri and others rely on the highly nontrivial sharp inequalities of Moser mentioned above [25]. This makes their proofs more difficult to digest. In [23], J. Li and M. Zhu gave a more direct and simpler proof of Theorem B, which does not depend on Moser's sharp inequality, but based on the following sharp local inequality: Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, Ω^* be the ball in $\mathbb{R}^2$ that has the same area as Ω, and denote

$$D_{a,b}(\Omega) = \{f(y): \ f(y) - b \in H_0^1(\Omega), \ \int_\Omega e^{2f} dy = a\},$$

where $a(>0)$ and b are two constants. Then

$$\inf_{w \in D_{a,b}(\Omega)} \int_\Omega |\nabla w|^2 dx \geq 4\pi \cdot (\ln \frac{ae^{-2b}}{\pi r^2} + \frac{\pi r^2}{ae^{-2b}} - 1), \tag{1.8}$$

where r is the radius of Ω^*.

Comparing the proof of sharp Sobolev inequality (based on the Bliss Lemma) with that of Onofri inequality, we believe that there are undiscovered calculus inequalities, which turn out to be the main theorem of our paper[20]. We include this in Chapter 2. One of the inequalities we derived is the following:

Theorem 1.1. *(1). Let $n > 1$ be given. For any nonnegative function $u \in C^1[0, +\infty)$ with $u(0) = 0$ and $\int_0^\infty e^{nu-nr} dr = a > 1/n$, we have*

$$\int_0^\infty |u_r|^n dr \geq \left(\frac{n}{n-1}\right)^{n-1} \Big\{ \ln(na) - \int_{1/na}^1 \frac{1}{t}(1 - (1-t)^{n-[n]})\, dt - \sum_{i=1}^{[n]-1} \frac{(na-1)^{n-i}}{(n-i)(na)^{n-i}} \Big\}, \tag{1.9}$$

where $[n]$ is the integer part of n. The equality in (1.9) holds if and only if

$$u(r) = \ln \frac{na(na-1)^{-1}}{(na-1)^{-1} + e^{-nr/(n-1)}}.$$

(2). For any non-negative function $u \in C^1[0, +\infty)$ with $u(0) = 0$,

$$\ln \int_0^{+\infty} \frac{e^u}{e^r} dr \leq \int_0^{+\infty} |u_r| dr, \tag{1.10}$$

the equality holds if and only if $u(r) = 0$.

As a consequence of such sharp local inequality (1.8), one can obtain a sharp inequality on any smooth bounded domain $\Omega \subset \mathbb{R}^2$:

$$\int_\Omega e^{2u} \leq meas(\Omega) \cdot e \cdot exp\{\frac{1}{4\pi} \int_\Omega |\nabla u|^2\}, \quad \forall u \in H_0^1(\Omega). \tag{1.11}$$

Also, for any smooth Riemann surface (M, g), given $\epsilon > 0$, there are $r_0 > 0$ and $C(\epsilon) > 0$ such that if $\Omega \subset B(x_0, r_0)$ for some point $x_0 \in M$, then

$$\int_\Omega e^w dV_g \leq C(\epsilon) \exp\{(\frac{1}{16\pi} + \epsilon) \int_\Omega |\nabla_g w|^2 dV_g\}, \quad \forall w \in H_0^1(\Omega). \tag{1.12}$$

From inequalities (1.11) and (1.12), X. Chen and M. Zhu [10] derived that on Riemann surface (M, g), given $\epsilon > 0$, there are constants $C_1 = C_1(\epsilon)$ and $C_2 = C_2(\epsilon)$ such that

$$\int_M e^w dV_g \leq C_1 \exp\{(\frac{1}{16\pi} + \epsilon) \int_M |\nabla_g w|^2 dV_g + C_2 \int_M w^2\}, \quad \forall w \in H^1(M).$$

They followed the strategy to prove Hebey and Vaugon's inequality (1.3), without using the uniformization theorem and Onofri inequality, showed that on any topological two sphere (M, g), there is a constant $C(M, g) > 0$, such that, for all $u \in H^1(M)$,

$$\int_M (|\nabla_g u|^2 + 2R_g u) dV_g - \ln \int_M e^u dV_g \geq C(M^2, g), \tag{1.13}$$

where R_g is twice the Gaussian curvature with respect to metric g.

Comparing at the high dimensional sharp trace inequalities, it is quite natural to seek two-dimensional sharp trace inequality. In Chapter 3 we will derive such inequality on simply connected domain in $\mathbb{R}^2$.

Theorem 1.2. *Let Ω be a simply connected domain in $\mathbb{R}^2$ with bounded Geodesic curvature K_g. Then there exists a constant $C(\Omega)$ such that for any $u \in H^1(\Omega)$,*

$$\ln \int_{\partial\Omega} e^u dS \leq \frac{1}{4\pi} \int_{\Omega} |\nabla u|^2 dy + \frac{1}{2\pi} \int_{\partial\Omega} K_g \cdot u dS + C(\Omega).$$

Chapter 2

A Hardy Inequality and Its Applications

The classical Hardy inequality says that for any non-negative function $f(x)$ on $[0, +\infty)$, if $F(x) = \int_0^x f(t)dt$, then

$$\int_0^\infty \left(\frac{F}{x}\right)^k dx \leq \left(\frac{k}{k-1}\right)^k \int_0^\infty f^k dx,$$

where $k > 1$ is a given parameter. See, for example, Inequality 327 in the book by Hardy, Littlewood and Polya [17]. It is important to note that the constant $(\frac{k}{k-1})^k$ is the optimal one and the equality in the inequality never holds. Using Hölder inequality, Hardy and Littlewood [16] were able to derive that

$$\int_0^\infty \frac{F^l}{x^{l-\alpha}} dx \leq \left(\frac{k}{k-1}\right)^k \left(\int_0^\infty f^k dx\right)^{\frac{l}{k}},$$

where $l \geq k$ and $\alpha = l/k - 1$. It was quite clear to them that the constant is not optimal for $l > k$. Though they guessed what is the best constant, it was later proved by Bliss, who obtained nowadays the famous Bliss Lemma (see the interesting papers [16, 6]):

Bliss Lemma: *Let k, l be constants, such that $l > k > 1$, and let $f(x)$ be a non-negative measurable function in the interval $0 \leq x < \infty$, such that the integral $J = \int_0^\infty f^k dx$ is finite. Then the integral $y = \int_0^x f dx$ is finite for every* *x and*

$$I = \int_0^\infty \frac{y^l}{x^{l-\alpha}} dx \le C_b J^{l/k}, \tag{2.1}$$

where

$$\alpha = \frac{l}{k} - 1, \qquad C_b = \frac{1}{l-\alpha-1}\left[\frac{\alpha\Gamma(l/\alpha)}{\Gamma(1/\alpha)\Gamma((l-1)/\alpha)}\right]^\alpha.$$

The equality in (2.1) holds if and only if $f(x) = c/(1+dx^\alpha)^{(\alpha+1)/\alpha}$ for some positive constants c, d.

The Bliss Lemma later (after more than forty years) became a crucial ingredient in the proof of sharp Sobolev inequality by Aubin [3], and Talenti [31], respectively. The latter inequality has played an essential role in the resolution of the Yamabe problem, which mainly concerns about finding a canonical metric with constant scalar curvature on compact manifolds with dimension higher than or equal to three. Comparing the proof of sharp Sobolev inequality (based on the Bliss Lemma) with the prove of Onofri inequality in [23] directly from Trudinger's inequality, we believe that there are undiscovered calculus inequalities.

In this chapter, we establish the following analog Bliss and Hardy inequalities with sharp constant involving exponential weight function. One special case of the inequalities (for $n = 2$) leads to a direct proof of Onofri inequality on S^2.

Theorem 1.1. *(1). Let $n > 1$ be given. For any nonnegative function $u \in C^1[0, +\infty)$ with $u(0) = 0$ and $\int_0^\infty e^{nu-nr}dr = a > 1/n$, we have*

$$\begin{aligned}\int_0^\infty |u_r|^n dr \;&\ge \left(\tfrac{n}{n-1}\right)^{n-1}\Big\{ \ln(na) - \int_{1/na}^1 \tfrac{1}{t}(1-(1-t)^{n-[n]})\, dt \\ &\quad - \sum_{i=1}^{[n]-1} \tfrac{(na-1)^{n-i}}{(n-i)(na)^{n-i}} \Big\},\end{aligned}$$

where $[n]$ is the integer part of n. The equality in (1.9) holds if and only if

$$u(r) = \ln \frac{na(na-1)^{-1}}{(na-1)^{-1} + e^{-nr/(n-1)}}.$$

(2). For any non-negative function $u \in C^1[0, +\infty)$ with $u(0) = 0$,

$$\ln \int_0^{+\infty} \frac{e^u}{e^r} dr \leq \int_0^{+\infty} |u_r| dr,$$

the equality holds if and only if $u(r) = 0$.

Theorem 1.1 immediately yields the following sharp inequality:

Corollary 2.1. *Let $n > 1$ be given. For any nonnegative function $u \in C^1[0, +\infty)$ with $u(0) = 0$*

$$\ln \left\{ n \int_0^{+\infty} \frac{e^{nu}}{e^{nr}} dr \right\} \leq \left(\frac{n-1}{n} \right)^{n-1} \int_0^{+\infty} |u_r|^n dr + C_n, \tag{2.2}$$

where the constant C_n is given by

$$C_n = \int_0^1 \frac{1}{t} \left(1 - (1-t)^{n-[n]}\right) dt + \sum_{i=1}^{[n]-1} \frac{1}{(n-i)},$$

and $[n]$ is the integer part of n. Both constants $(\frac{n-1}{n})^{n-1}$ and C_n are optimal, and the equality never holds.

We first prove inequality (1.9) with a larger coefficient in Sec. 2.1 (Proposition 2.4 below). The argument is elementary and simple. It needs to be pointed out that for $n > 1$ being an integer, Theorem 1.1 can be read out from Theorem 1.3 in [23]. For general positive constant, it seems impossible to prove Theorem 1.1 from that theorem, rather, Theorem 1.1 provides an alternative proof of that theorem (Corollary 2.9 in this paper). Recall the original proof of Theorem 1.3 in [23] does rely on Trudinger's inequality. Quite interestingly, we also recall that Moser [25] used a similar argument to give a very simple proof of the improved Trundinger's inequality (with best constant):

Corollary 2.2. (Weak Moser's Inequality). *Let $\Omega \subset \mathbb{R}^n$ (for $n \geq 2$) be a smooth bounded domain. For any $\beta < n\omega_{n-1}^{1/(n-1)}$, there is a constant $C(\Omega, \beta)$ depending on the volume of Ω and β, such that for all $u \in W_0^{1,n}(\Omega)$ with $\int_\Omega |\nabla u|^n dx \leq 1$,*

$$\int_\Omega e^{\beta u^{\frac{n}{n-1}}} dx \leq C(\Omega, \beta).$$

Here and throughout this chapter, we use ω_n for the volume of unit sphere S^n in $\mathbb{R}^{n+1}$. This result is slightly weaker than Moser's inequality since it does not include the case of $\beta = n\omega_{n-1}^{1/(n-1)}$. It seems that one needs the argument due to Moser [25], or Carleson and Chang [8] to cover this extremal case.

In Sec. 2.2 we will show how to improve the rough inequality (Proposition 2.4) and complete the proof of Theorem 1.1. One particular reason that we can achieve this (but not for Moser's inequality) is that we can classify all extremal functions.

As Bliss Lemma yields a sharp Sobolev inequality, in Sec. 2.3 we will show that Theorem 1.1 can be used to give a more direct proof of the Onofri inequality (thus without even using Trudinger's inequality). In fact, let $B_r(0) \subset \mathbb{R}^n$ (now n is an integer greater than or equal to two) be a ball in $\mathbb{R}^n$ with radius r centered at the origin, and

$$D_{a,b}(B_r(0)) = \left\{ f(y) \ : \ f(y) - b \in W_0^{1,n}(B_r(0)), \ \int_{B_r(0)} e^{nf} dy = a \right\},$$

where a is a constant satisfying $a > \frac{\omega_{n-1} r^n e^{nb}}{n}$. We will show that Theorem 1.1 yields:

Corollary 2.3. (Local Sharp Inequality for $n = 2$).

$$\inf_{w \in D_{a,b}(B_r)} \int_{B_r} |\nabla w|^2 dy = 4\pi \cdot \left(\ln \frac{ae^{-2b}}{\pi r^2} + \frac{\pi r^2}{ae^{-2b}} - 1 \right).$$

It is known now that this corollary implies Onofri inequality on S^2, see [23]. For the readers' convenience, we include a complete proof of the Onofri inequality in Sec. 2.3.

In Sec. 2.3, we shall also discuss the applications of Theorem 1.1 to other geometric problems.

2.1 Rough inequality

We shall establish two elementary calculus inequalities ((2.3) and (2.6)) in this section. The first one will be used to prove Theorem 1.1, and the second one will be used to derive Corollary 2.2.

Proposition 2.4. *(1). Let $n > 1$ and $\beta_0 > (\frac{n-1}{n})^{\frac{n-1}{n}}$. There is a constant $c_1(\beta_0)$, such that for any $u(r) \in C^1[0, +\infty)$ satisfying $u(0) = 0$,*

$$\ln \int_0^\infty e^{n(u-r)} dr \leq \beta_0^n \int_0^\infty |u_r|^n dr + c_1(\beta_0). \tag{2.3}$$

(2). For $u(r) \in C^1[0, +\infty)$ satisfying $u(0) = 0$,

$$\ln \int_0^\infty \frac{e^u}{e^r} dr \leq \int_0^\infty |u_r| dr.$$

Equality holds if and only if $u(r) = 0$.

For $n \geq 2$ we will improve the inequality by variational method in next section.

Proof. Let $u(r)$ be any function in $C^1[0, +\infty)$ satisfying $u(0) = 0$. We have

$$u(r) \leq \int_0^\infty |u_r| dr,$$

thus

$$\int_0^\infty \frac{e^u}{e^r} dr \leq \exp\{\int_0^\infty |u_r| dr\},$$

which yields

$$\ln \int_0^\infty \frac{e^u}{e^r} dr \leq \int_0^\infty |u_r| dr.$$

It is easy to see that the equalities in the above inequalities hold if and only if $u_r = 0$, thus $u(r) = 0$.

Now, for given $n > 1$ and positive parameter $\beta > 0$, we have

$$\begin{aligned} u(r) &= \int_0^r u_r dr \leq (\int_0^r |u_r|^n dr)^{1/n} \cdot r^{\frac{n-1}{n}} \\ &\leq \frac{\beta^n \int_0^r |u_r|^n dr}{n} + \frac{\beta^{-\frac{n}{n-1}} r}{n/(n-1)}. \end{aligned}$$

Thus

$$\begin{aligned} \int_0^\infty \frac{e^{nu}}{e^{nr}} dr &\leq \int_0^\infty \frac{\exp\{\beta^n \int_0^\infty |u_r|^n dr + (n-1)\beta^{-\frac{n}{n-1}} r\}}{e^{nr}} dr \\ &= \exp\{\beta^n \int_0^\infty |u_r|^n dr\} \cdot \int_0^\infty e^{[(n-1)\beta^{-\frac{n}{n-1}} - n]r} dr. \end{aligned}$$

If we choose

$$\beta = \beta_0 > (\frac{n-1}{n})^{\frac{n-1}{n}}, \tag{2.4}$$

then

$$\int_0^\infty e^{[(n-1)\beta_0^{-\frac{n}{n-1}}-n]r}dr = c(\beta_0)$$

is a finite number depending on β_0. It follows that

$$\ln \int_0^\infty e^{n(u-r)}dr \le \beta_0^n \int_0^\infty |u_r|^n dr + c_1(\beta_0)$$

for $c_1(\beta_0) = \ln c(\beta_0)$.

□

It is obvious in the above proof that $c(\beta_0), c_1(\beta_0) \to +\infty$ as $\beta_0 \to (\frac{n-1}{n})^{\frac{n-1}{n}}$. We need another argument to derive the main theorem.

Remark 2.5. From (2.3) we can see that for β_0 satisfying (2.4),

$$\begin{aligned}\int_R^\infty \frac{e^{nu}}{e^{nr}}dr &\le \exp\{\beta_0^n \int_0^\infty |u_r|^n dr\} \cdot \int_R^\infty e^{[(n-1)\beta_0^{-\frac{n}{n-1}}-n]r}dr \\ &= o_R(1)\exp\{\beta_0^n \int_0^\infty |u_r|^n dr\},\end{aligned} \tag{2.5}$$

where $o_R(1) \to 0$ as $R \to \infty$.

We now compare this with Moser's proof of Trudinger's inequality

Lemma 2.6. *For $n > 1$, $a > 0$ and $\beta < na^{\frac{1}{1-n}}$, there is a constant $C_{\beta,a}$ depending only on β and a, such that for any nonnegative function $u \in C^1[0,+\infty)$ with $u(0) = 0$ and $\int_0^\infty |u_r|^n dr \le a$,*

$$\int_0^\infty \frac{e^{\beta u^{\frac{n}{n-1}}}}{e^{nr}}dr \le C_{\beta,a}. \tag{2.6}$$

Proof. For given $n > 1$ we have

$$u(r) = \int_0^r u_r dr \le (\int_0^r |u_r|^n dr)^{1/n} \cdot r^{\frac{n-1}{n}} \le a^{\frac{1}{n}} r^{\frac{n-1}{n}}.$$

Thus for any positive parameter $\tau > 0$,

$$\int_0^\infty \frac{e^{\tau u^{\frac{n}{n-1}}}}{e^{nr}} dr \le \int_0^\infty \exp\{\tau a^{\frac{1}{n-1}} - n\} r dr. \tag{2.7}$$

The right hand side of the above inequality is bounded if we choose $\tau = \beta < na^{\frac{1}{1-n}}$. □

Based on Lemma 2.6, one can verify Corollary 2.2 as follows.

Due to the rearrangement and rescaling, we only need to prove Corollary 2.2 when $\Omega = B_1(0)$ and $u \in C_0^1(B_1(0))$ is radially symmetric and nonnegative.

From $\int_{B_1} |\nabla u|^n dx \le 1$, we know that (let $r = -\ln s$)

$$1 \ge \int_{B_1} |\nabla u|^n dx = \omega_{n-1} \int_0^1 |u_s|^n s^{n-1} ds = \omega_{n-1} \int_0^\infty |u_r|^n dr.$$

Also,

$$\int_{B_1} e^{\beta u^{\frac{n}{n-1}}} dx = \omega_{n-1} \int_0^1 e^{\beta u^{\frac{n}{n-1}}} s^{n-1} ds = \omega_{n-1} \int_0^\infty \frac{e^{\beta u^{\frac{n}{n-1}}}}{e^{nr}} dr.$$

One immediately has Corollary 2.2 by using Lemma 2.6 with $a = \omega_{n-1}^{-1}$.

This result is slightly weaker than Moser's inequality since it does not include the case of $\beta = n\omega_{n-1}^{1/(n-1)}$. It seems that one needs the argument due to Moser [25], or Carleson and Chang [8] to cover this extremal case.

2.2 Sharp Inequality

We shall prove Theorem 1.1 in this section. Since the case of $n = 1$ has been settled by Proposition 2.4, we will focus on the case of $n > 1$. For given $a > 0$, define

$$D_a^n := \{u(r) \in W^{1,n}(\mathbb{R}^+) \ : \ u(0) = 0, \int_0^\infty \exp\{nu - nr\}dr = a\}. \qquad (2.8)$$

Lemma 2.7. *There is a $v \in D_a^n$ such that*

$$\int_0^\infty |v_r|^n dr = \inf_{u \in D_a^n} \int_0^\infty |u_r|^n dr := \xi.$$

Proof. Let $\{v^i\}$ be a minimizing sequence of $\inf_{u\in D_a^n} \int_0^\infty |u_r|^n dr$. Then

$$v^i \rightharpoonup v \quad \text{in } W^{1,n}(\mathbb{R}^+), \quad \text{and} \quad \int_0^\infty |v_r|^n dr \leq \underline{\lim}_{i\to\infty} \int_0^\infty |v_r^i|^n dr = \xi$$

for some $v \in W^{1,n}(\mathbb{R}^+)$. We need to verify $v \in D_a^n$.

First, from (2.5), we know that for $w = v^i$, or v:

$$\int_R^\infty \frac{e^{nw}}{e^{nr}} dr = o_R(1).$$

On the other hand, it follows from the embedding $H^1(0, R) \hookrightarrow C^{0,1/2}(0, R)$ and Arzela-Ascoli lemma that

$$\lim_{i\to\infty} \int_0^R \exp\{nv^i - nr\}dr = \int_0^R \exp\{nv - nr\}dr.$$

Letting $i,\ R \to \infty$, we have $\int_0^\infty \exp\{nv - nr\}dr = a$, that is $v \in D_a^n$. □

We now begin the proof of Theorem 1.1.

Proof. We only need to consider the case of $n > 1$ for nontrivial nonnegative functions. For $a > 1/n$, let v be the minimizer of $\inf_{u \in D_a^n} \int_0^\infty |u_r|^n dr$. It is easy to see that $v_r \geq 0$. So it satisfies the following Euler-Lagrange equation:

$$v_r^{n-2} v_{rr} = -\tau e^{nv-nr}, \qquad v(0) = 0 \tag{2.9}$$

for some $\tau > 0$. Though it is not obvious how to obtain the general solution from the uniqueness of the ordinary differential equation since v_r could be zero, one can follow the argument given by Carleson and Chang [8] to show that the general solution to (2.9) is given by

$$v(r) = \ln \frac{1}{\lambda_0 + e^{-nr/(n-1)}} - \frac{1}{n} \ln \frac{\tau}{(\frac{n}{n-1})^n \lambda_0},$$

where λ_0 is a positive constant and $\tau = \frac{(\frac{n}{n-1})^n \lambda_0}{(\lambda_0+1)^n}$. Thus

$$v(r) = \ln \frac{\lambda_0 + 1}{\lambda_0 + e^{-nr/(n-1)}}. \tag{2.10}$$

Since $a = \int_0^\infty e^{nv-nr}\, dr$, we have

$$\begin{aligned} a &= \int_0^\infty \left(\frac{\lambda_0 + 1}{\lambda_0 + e^{-nr/(n-1)}} \right)^n e^{-nr}\, dr \\ &= \int_0^1 \left(\frac{\lambda_0 + 1}{\lambda_0 + s^{n/(n-1)}} \right)^n s^n (\frac{1}{s})\, ds \\ &= (\lambda_0 + 1)^n \int_0^1 \frac{s^{n-1}}{(\lambda_0 + s^{n/(n-1)})^n}\, ds \\ &= (\lambda_0 + 1)^n \frac{s^n}{n\lambda_0 (\lambda_0 + s^{n/(n-1)})^{n-1}} \Big|_{s=0}^1 \\ &= \frac{\lambda_0 + 1}{n\lambda_0}, \end{aligned}$$

That is

$$\lambda_0 = \frac{1}{na-1}. \tag{2.11}$$

We compute

$$\begin{aligned}
\int_0^\infty |v_r|^n dr &= \int_0^\infty \left| \frac{\frac{n}{n-1} e^{-nr/(n-1)}}{\lambda_0 + e^{-nr/(n-1)}} \right|^n \ dr \\
&= (\frac{n}{n-1})^n \int_0^\infty \left(\frac{e^{-nr/(n-1)}}{\lambda_0 + e^{-nr/(n-1)}} \right)^n \ dr \\
&= (\frac{n}{n-1})^{n-1} \int_0^{1/\lambda_0} \frac{t^{n-1}}{(1+t)^n} \ dt \\
&= (\frac{n}{n-1})^{n-1} \int_1^{1/\lambda_0+1} \frac{(\tau-1)^{n-1}}{\tau^n} \ d\tau \\
&= -(\frac{n}{n-1})^{n-1} \int_1^{1/\lambda_0+1} (\tau-1)(1-\frac{1}{\tau})^{n-2} \ d\frac{1}{\tau} \\
&= (\frac{n}{n-1})^{n-1} \int_{\lambda_0/(\lambda_0+1)}^1 (\frac{1}{t}-1)(1-t)^{n-2} \ dt.
\end{aligned}$$

Using (2.11) we have: If $n \in \mathbf{N}$,

$$\begin{aligned}
\int_0^\infty |v_r|^n \ dr = (\frac{n}{n-1})^{n-1} \cdot \{ &- \int_{\lambda_0/(\lambda_0+1)}^1 (1-t)^{n-2} \ dt \\
&- \cdots - \int_{\lambda_0/(\lambda_0+1)}^1 (1-t) \ dt - \int_{\lambda_0/(\lambda_0+1)}^1 (1-\frac{1}{t}) \ dt \} \\
&= (\frac{n}{n-1})^{n-1} \{ \ln \frac{\lambda_0+1}{\lambda_0} - \frac{1}{\lambda_0+1} - \sum_{i=1}^{n-2} \frac{(\frac{1}{\lambda_0+1})^{n-i}}{n-i} \} \\
&= (\frac{n}{n-1})^{n-1} \{ \ln(na) - \sum_{i=1}^{n-1} \frac{(na-1)^{n-i}}{(n-i)(na)^{n-i}} \};
\end{aligned}$$

For general $n > 1$, we have

$$\int_0^\infty |v_r|^n\ dr = (\frac{n}{n-1})^{n-1} \cdot \{-\int_{\lambda_0/(\lambda_0+1)}^1 (1-t)^{n-2}\ dt$$
$$- \cdots - \int_{\lambda_0/(\lambda_0+1)}^1 (1-t)^{n-[n]}\ dt + \int_{\lambda_0/(\lambda_0+1)}^1 \frac{1}{t}(1-t)^{n-[n]}\ dt\}$$
$$= (\frac{n}{n-1})^{n-1}\{\int_{\lambda_0/(\lambda_0+1)}^1 \frac{1}{t}(1-t)^{n-[n]}\ dt - \sum_{i=1}^{[n]-1} \frac{(\frac{1}{\lambda_0+1})^{n-i}}{n-i}\}$$
$$= (\frac{n}{n-1})^{n-1}\{\int_{\lambda_0/(\lambda_0+1)}^1 \frac{1}{t} + \int_{\lambda_0/(\lambda_0+1)}^1 \frac{1}{t}((1-t)^{n-[n]} - 1)\ dt$$
$$- \sum_{i=1}^{[n]-1} \frac{(\frac{1}{\lambda_0+1})^{n-i}}{n-i}\}$$
$$= (\frac{n}{n-1})^{n-1}\{\ln(na) - \int_{\lambda_0/(\lambda_0+1)}^1 \frac{1}{t}(1-(1-t)^{n-[n]})\ dt$$
$$- \sum_{i=1}^{[n]-1} \frac{(na-1)^{n-i}}{(n-i)(na)^{n-i}}\},$$

where $[n]$ is the integer part of n. Let $a \to \infty$, then $\lambda_0 \to 0$ by (2.11). We know that C_n is optimal. The proof is completed. □

Remark 2.8. For negative function u, we can certainly improve the inequalities. In particular, similar argument will yield Theorem 1.3 (ii) in [23] for integer $n > 1$.

2.3 Applications

We first show that Theorem 1.1 implies:

Corollary 2.3 (Local sharp inequality for $n = 2$)

$$\inf_{w \in D_{a,b}(B_r)} \int_{B_r} |\nabla w|^2 dy = 4\pi \cdot (\ln \frac{ae^{-2b}}{\pi r^2} + \frac{\pi r^2}{ae^{-2b}} - 1).$$

Proof. Let $v \in D^2_\alpha$ (recalling the notation in (2.8)). We have, from Theorem 1.1, that

$$\inf_{v\in D^2_\alpha}\int_0^\infty |v_r|^2\ dr \geq 2\{\ln(2\alpha)+\frac{1}{2\alpha}-1\}, \tag{2.12}$$

where $\int_0^\infty e^{2v-2r}dr=\alpha$. For $w \in D_{a,0}(B_1(0))$,

$$\int_{B_1}|\nabla w|^2dx = 2\pi\int_0^1 |w_s|^n sds = 2\pi\int_0^\infty |w_r|^2dr.$$

and

$$\int_{B_1} e^{2w}dx = 2\pi\int_0^1 e^{2w}sds = 2\pi\int_0^\infty \frac{e^{2w}}{e^{2r}}dr.$$

Combing with (2.12), we have

$$\inf_{w\in D_{a,0}(B_1)}\int_{B_1}|\nabla w|^2dx = 4\pi\cdot(\ln\frac{\int_{B_1}e^{2w}dx}{\pi}+\frac{\pi}{\int_{B_1}e^{2w}dx}-1).$$

After rescaling and shifting, we get Corollary 2.3. □

In the same spirit, from (2.2) we easily obtain

Corollary 2.9. *Let $u \in C^1(B_1)$ be a nonnegative function satisfying $u = 0$ on ∂B_1*

$$\ln\frac{n\int_{B_1}e^{nu}}{\omega_{n-1}} < (\frac{n-1}{n})^{n-1}\omega_n^{-1}\int_{B_1}|\nabla u|^n + F(1).$$

where

$$F(1) = 1+\frac{1}{2}+\cdots+\frac{1}{n-1}.$$

The fact that the strict inequality holds on a bounded domain coincides with the one that the strict sharp Sobolev inequality holds on a bounded domain.

Corollary 2.9 was first proved in [23] using Trudinger's inequality. The proof presented here does not rely on Trudinger's inequality. Inequality in Corollary 2.9 was refereed as *local sharp inequality* in [23], which is easily adapted for manifolds. See related topics in Chen and Zhu [10].

Finally, we shall show that one can prove the Onofri inequality using Corollary 2.3.

Due to the rearrangement, we only need to prove Onofri inequality for $u \in C^1(S^2)$ which depends only on x_3 and is monotonically decreasing in x_3. Also, we can assume that $u(x_3) \mid_{x_3=1} = 0$ (otherwise, we replace $u(x)$ by $u(x) - u(1)$). We can approximate u by a sequence of functions $u_i \in C^1(S^2)$ such that $u_i(x) = u_i(x_3)$ is monotonically decreasing in x_3, and $u_i(x) = 0$ in the geodesic ball $B_{1/i}(N)$ of the north pole N for $i \in \mathbf{N}$. Denote $S_i^2 := S^2 \setminus B_{1/i}(N)$.

Let Φ: $x \in S^2 \to y \in \mathbb{R}^2$ be a stereographic projection given by

$$x_i = \frac{2y_i}{1+|y|^2}, \qquad for \quad i = 1, 2;$$

and

$$x_3 = \frac{|y|^2 - 1}{|y|^2 + 1}.$$

Denote

$$g_0 = \sum_{i=1}^{3} dx_i^2 = (\frac{2}{1+|y|^2})^2 dy^2 := e^{2\varphi(y)} dy^2.$$

Thus

$$\varphi(y) = \ln \frac{2}{1+|y|^2}.$$

It is easy to check that $\varphi(y)$ satisfies

$$-\Delta\varphi = e^{2\varphi} \qquad in \quad \mathbb{R}^2. \tag{2.13}$$

Let $\Phi(S_i^2) = B_{R_i}$. It is obvious that $R_i \to +\infty$ as $i \to +\infty$. For

$$w_i(y) = u_i(x) + \varphi(y) = u_i(\Phi^{-1}(y)) + \varphi(y),$$

we have

$$\int_{B_{R_i}} e^{2w_i(y)} dy = \int_{S_i^2} e^{2u_i} dx := a_i,$$

and

$$\begin{aligned}
&\int_{B_{R_i}} |\nabla w_i|^2 dy \\
=& \int_{B_{R_i}} |\nabla(u_i \circ \Phi^{-1})|^2 dy + 2\int_{B_{R_i}} \nabla(u_i \circ \Phi^{-1}) \cdot \nabla\varphi dy + \int_{B_{R_i}} |\nabla\varphi|^2 dy \\
=& \int_{S_i^2} |\nabla u_i|^2 dx + 2\int_{S_i^2} u_i dx + \int_{B_{R_i}} |\nabla\varphi|^2 dy,
\end{aligned}$$

where we use the fact that φ satisfies (2.13). Since $w_i(y) = \ln \frac{2}{1+R_i^2}$ on ∂B_{R_i}, it follows from Corollary 2.3 that

$$\int_{B_{R_i}} |\nabla w_i|^2 dy \geq 4\pi(\ln \frac{a_i \cdot (\frac{1+R_i^2}{2})^2}{\pi R_i^2} + \frac{\pi R_i^2}{a_i \cdot (\frac{1+R_i^2}{2})^2} - 1).$$

Also, one can check that

$$\int_{B_{R_i}} |\nabla\varphi|^2 dy = 4\pi \left(\ln(1 + R_i^2) + \frac{1}{1 + R_i^2} - 1 \right).$$

We conclude

$$
\begin{aligned}
&\int_{S_i^2} |\nabla u_i|^2 dx + 2\int_{S_i^2} u_i dx \\
\geq &4\pi \left(\ln \frac{a_i \cdot \left(\frac{1+R_i^2}{2}\right)^2}{\pi R_i^2} + \frac{\pi R_i^2}{a_i \cdot \left(\frac{1+R_i^2}{2}\right)^2} - 1 \right) - 4\pi \left(\ln(1+R_i^2) + \frac{1}{1+R_i^2} - 1 \right) \\
= &4\pi \left(\ln \frac{a_i \cdot (1+R_i^2)}{4\pi R_i^2} + \frac{4\pi R_i^2}{a_i \cdot (1+R_i^2)^2} - \frac{1}{1+R_i^2} \right).
\end{aligned}
$$

Sending $i \to +\infty$, we have

$$
\int_{S^2} |\nabla u|^2 dx + 2\int_{S^2} u dx \geq 4\pi \left(\ln \frac{1}{4\pi} \int_{S^2} e^{2u} dx \right).
$$

Chapter 3

Two Dimensional Sharp Trace Inequalities

The main purpose of this chapter is to derive a global sharp trace inequality on two dimensional simply connected domain.

Theorem 1.2. *Let Ω be a simply connected domain in $\mathbb{R}^2$ with bounded Geo -desic curvature K_g. Then there exists a constant $C(\Omega)$ such that for any $u \in H^1(\Omega)$,*

$$\ln \int_{\partial\Omega} e^u dS \leq \frac{1}{4\pi} \int_{\Omega} |\nabla u|^2 dy + \frac{1}{2\pi} \int_{\partial\Omega} K_g \cdot u dS + C(\Omega).$$

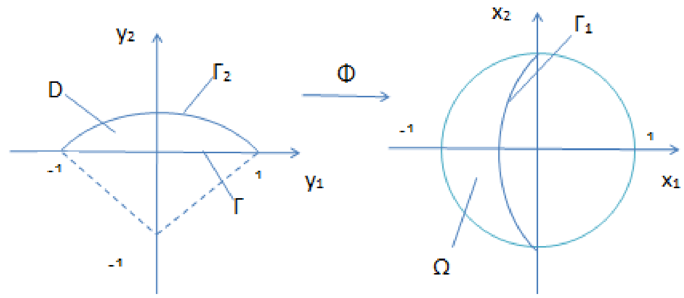
We start with a local sharp inequality, which allows us to obtain an $\epsilon-$level sharp trace inequality in Theorem 3.2. We then define a functional and study the behavior of the minimizing sequence as the parameter ϵ goes to 0. It will lead to the result in Theorem 1.2.

3.1 Sharp local trace inequality

Let $\Phi : y \in \mathbb{R}^2_+ \to x \in B_1$ be a conformal mapping given by

$$y_1 = \frac{-2x_2}{(x_1-1)^2 + x_2^2} \quad \text{and} \quad y_2 = \frac{1 - x_1^2 - x_2^2}{(x_1-1)^2 + x_2^2},$$

where B_1 is the unit disk in $\mathbb{R}^2$ centered at the origin.



Define

$$D = \{(y_1, y_2) \in \mathbb{R}^2_+ : y_1^2 + (y_2+1)^2 < 2\}$$

and

$$\Omega = \{(x_1, x_2) \in B_1 : (x_1 - 1)^2 + x_2^2 > 2\}.$$

Then $\Phi(D) = \Omega$ and $\Phi(\Gamma_2) = \Gamma_1$, where

$$\Gamma_1 = \{(x_1, x_2) \in B_1 : (x_1 - 1)^2 + x_2^2 = 2\}$$

and

$$\Gamma_2 = \{(y_1, y_2) \in \mathbb{R}^2_+ : y_1^2 + (y_2+1)^2 = 2\}.$$

Denote

$$g_x = \sum_{i=1}^{2} dx_i \wedge dx_i = \frac{4}{(y_1^2 + (y_2+1)^2)^2} \sum_{i=1}^{2} dy_i \wedge dy_i := e^{2\varphi} g_y.$$

Thus

$$\varphi(y) = \ln \frac{2}{y_1^2 + (y_2+1)^2}.$$

It is easy to check that φ satisfies

$$\begin{cases} -\Delta_y \varphi &= 0 \qquad \text{in } \mathbb{R}^2_+ \\ \frac{\partial \varphi}{\partial \nu} &= e^{\varphi} \qquad \text{on } \partial \mathbb{R}^2_+. \end{cases}$$

For $u \in H^1(D)$, we define

$$\tilde{u}(y) = \begin{cases} u(y) & \text{when } y \in D \\ \varphi(y) & \text{when } y \in \mathbb{R}^2_+ \backslash D. \end{cases}$$

Let $h(x) = \tilde{u}(\phi^{-1}(x)) - \varphi(\phi^{-1}(x))$. There is a well-known inequality

Theorem C (**Lebedev-Milin inequality**) *Let B_1 be unit disk in $\mathbb{R}^2$ centered at the origin. Then for any $u \in H^1(B_1)$*

$$\ln \int_{\partial B_1} e^u \frac{d\theta}{2\pi} \leq \int_{B_1} |\nabla u|^2 \frac{dx}{4\pi} + \int_{\partial B_1} u \frac{d\theta}{2\pi}.$$

Applying Lebedev-Milin inequality, we have

$$\ln \int_{\partial B_1} e^h \frac{d\theta}{2\pi} \leq \frac{1}{4} \int_{B_1} |\nabla h|^2 \frac{dx}{\pi} + \int_{\partial B_1} h \frac{d\theta}{2\pi}.$$

Since

$$\ln \int_{\partial B_1} e^h \frac{d\theta}{2\pi} = \ln \int_{\partial B_1} e^{\tilde{u}\circ\Phi^{-1} - \varphi\circ\Phi^{-1}} \frac{d\theta}{2\pi} = \ln (\int_{\mathbb{R}} e^{\tilde{u}} \frac{dy_1}{2\pi})$$

and

$$
\begin{aligned}
&\frac{1}{4}\int_{B_1}|\nabla h|^2\frac{dx}{\pi}+\int_{\partial B_1}h\frac{d\theta}{2\pi}\\
=&\frac{1}{4\pi}\int_{B_1}|\nabla_x(\tilde{u}\circ\Phi^{-1}-\varphi\circ\Phi^{-1})|^2dx+\frac{1}{2\pi}\int_{\partial B_1}(\tilde{u}\circ\Phi^{-1}-\varphi\circ\Phi^{-1})d\theta\\
=&\frac{1}{4\pi}[\int_{\Omega}|\nabla_x(\tilde{u}\circ\Phi^{-1})|^2dx+\int_{\Omega}|\nabla_x(\varphi\circ\Phi^{-1})|^2dx\\
&-2\int_{\Omega}\nabla_x(\tilde{u}\circ\Phi^{-1})\cdot\nabla_x(\varphi\circ\Phi^{-1})dx]+\frac{1}{2\pi}\int_{-1}^{1}(\tilde{u}-\varphi)e^{\varphi}dy_1\\
=&\frac{1}{4\pi}[\int_{D}|\nabla_y\tilde{u}|^2dy+\int_{D}|\nabla_y\varphi|^2dy+2\int_{\Omega}\tilde{u}\circ\Phi^{-1}\cdot\Delta_x(\varphi\circ\Phi^{-1})dx\\
&-2\int_{\partial\Omega}\tilde{u}\circ\Phi^{-1}\frac{\partial(\varphi\circ\Phi^{-1})}{\partial\nu}dS_x]+\frac{1}{2\pi}\int_{-1}^{1}(\tilde{u}(y_1,0)-\varphi(y_1,0))e^{\varphi}(y_1,0)dy_1\\
=&\frac{1}{4\pi}\int_{D}|\nabla u|^2dy+\frac{1}{4\pi}\int_{D}|\nabla\varphi|^2dy-\frac{1}{2\pi}\int_{-1}^{1}\varphi(y_1,0)e^{\varphi(y_1,0)}dy_1\\
\le&\frac{1}{4\pi}\int_{D}|\nabla u|^2dy+C,\quad\text{for some constant C,}
\end{aligned}
$$

we obtain a Lebedev-Milin type inequality on the special domain $D=\{(y_1,y_2)\in\mathbb{R}^2_+ : y_1^2+(y_2+1)^2<2\}$,

$$
\ln\int_{-1}^{1}e^u\frac{dy_1}{2\pi}\le\frac{1}{4\pi}\int_D|\nabla u|^2dy+C,\quad\forall u\in H^1(D).\tag{3.1}
$$

This inequality will be used to obtain $\epsilon-$ level sharp inequality on general domains.

3.2 Global $\epsilon-$ level sharp trace inequality

Lemma 3.1. *Assume Ω is a smooth bounded domain in $\mathbb{R}^2$. For any given $\epsilon>0$, there are constants $C_1=C_1(\epsilon)$ and $C_2=C_2(\epsilon)$ such that*

$$
\ln\int_{\partial\Omega}e^u dS\le(\frac{1}{4\pi}+\epsilon)\int_{\Omega}|\nabla u|^2dy+C_1\int_{\Omega}u^2\ dy+C_2,\qquad\forall u\in H^1(\Omega).
$$

Proof. Step 1. For $\overline{y} \in \partial\Omega$, we first straighten the boundary $\partial\Omega$ near $\overline{y}$. Without loss of generality, we may assume that $\overline{y}$ is the coordinate origin and y_2-axis is the inner normal to $\partial\Omega$ at $\overline{y}$. Therefore, there exists a smooth function $\varphi(y_1) \geq 0$ defined for $|y_1| \leq \alpha$ for some small α, satisfying $\varphi(0) = 0$ and $\varphi'(0) = 0$, such that $\partial\Omega$ is given by $(y_1, \varphi(y_1))$ in a neighborhood of $\overline{y}$, written as $\partial\Omega(\overline{y}, \alpha) = (y_1, \varphi(y_1))$. Let $D_\alpha = \{(x_1, x_2) \in \mathbb{R}^2_+ \ : x_1^2 + (x_2 + \alpha)^2 \leq 2\alpha^2\}$. Define $y \ := \phi(x) = (y_1, y_2)$ by

$$\begin{cases} y_1 = x_1 \\ y_2 = x_2 + \varphi(x_1). \end{cases}$$

Since the differential map of ϕ satisfies $D\phi(0)$=Identity, we know that ϕ has the inverse in a neighborhood of $\{0\}$. If we denote $x = \psi(y) := \phi^{-1}(y)$ in a neighborhood of $\overline{y}$, then

$$\begin{cases} x_1 = y_1 \\ x_2 = y_2 - \varphi(y_1). \end{cases}$$

Thus

$$\begin{aligned} \int_{\partial\Omega(\overline{y},\alpha)} e^{u(y)} dS_y &= \int_{-\alpha}^{\alpha} e^{u\circ\Phi(x_1,0)} \sqrt{1 + \varphi\prime(x_1)^2} dx_1 \\ &\leq \int_{-\alpha}^{\alpha} e^{u\circ\Phi(x_1,0)} (1 + \varphi\prime(x_1)^2) dx_1. \end{aligned}$$

When α is small enough, we have

$$\int_{\partial\Omega(\overline{y},\alpha)} e^{u(y)} dS_y \leq \int_{-\alpha}^{\alpha} e^{u\circ\Phi(x_1,0)}(1+c_0^2\alpha^2)dx_1, \quad \text{for some } c_0>0. \tag{3.2}$$

Define $v(z) = u(\alpha z)$, then $v \in H^1(D)$. We have

$$\ln \int_{-1}^{1} e^v \frac{dz_1}{2\pi} \leq \frac{1}{4\pi}\int_D |\nabla v(z)|^2 dz + C,$$

where we use the fact that v satisfies (3.1). We know

$$\ln \int_{-1}^{1} e^v \frac{dz_1}{2\pi} = \ln \int_{-1}^{1} e^{u(\alpha z)} \frac{dz_1}{2\pi} = \ln \int_{-\alpha}^{\alpha} e^{u(y)} \frac{dy_1}{2\pi} - \ln \alpha$$

and

$$\frac{1}{4\pi}\int_D |\nabla v(z)|^2 dz = \frac{1}{4\pi}\int_D |\nabla_z u(\alpha z)|^2 dz = \frac{1}{4\pi}\int_{D_\alpha} |\nabla_y u(y)|^2 dy.$$

Hence

$$\ln \int_{-\alpha}^{\alpha} e^{u(y)} \frac{dy_1}{2\pi} \leq \frac{1}{4\pi}\int_{D_\alpha} |\nabla_y u(y)|^2 dy + \ln\alpha + c. \tag{3.3}$$

Applying inequalities (3.2) and (3.3), we can derive

$$\begin{aligned}
\int_{\partial\Omega(\overline{y},\alpha)} e^{u(y)} dS_y &\leq \int_{-\alpha}^{\alpha} e^{u\circ\Phi(x_1,0)}(1+c_0^2\alpha^2)dx_1 \\
&\leq e^c(1+c_0^2\alpha^2)2\pi\alpha\cdot\exp\{\frac{1}{4\pi}\int_{D_\alpha} |\nabla_x(u\circ\Phi(x))|^2 dx\} \\
&= e^c(1+c_0^2\alpha^2)2\pi\alpha\cdot\exp\{\frac{1}{4\pi}\int_{\Phi(D_\alpha)} |\nabla_x u(y)|^2 dy\}.
\end{aligned}$$

By the definition of Φ, we have

$$\begin{aligned}|\nabla_x u(y)|^2 &= |(\frac{\partial(u\circ\Phi)}{\partial x_1}, \frac{\partial(u\circ\Phi)}{\partial x_2})|^2 \\ &= |\nabla_y u(y)|^2 + 2\frac{\partial u}{\partial y_1}\frac{\partial u}{\partial y_2}\frac{d\varphi}{dx_1} + (\frac{\partial u}{\partial y_2})^2(\frac{d\varphi}{dx_1})^2 \\ &\leq |\nabla_y u(y)|^2[1 + \frac{d\varphi}{dx_1} + (\frac{d\varphi}{dx_1})^2] \\ &\leq |\nabla_y u(y)|^2(1 + c_0\alpha + c_0^2\alpha^2).\end{aligned}$$

Therefore

$$\ln \int_{\partial\Omega(\overline{y},\alpha)} e^{u(y)} dS_y \leq \frac{1 + c_0\alpha + c_0^2\alpha^2}{4\pi} \int_{\Phi(D_\alpha)} |\nabla_y u(y)|^2 dy + c + \ln[2\pi\alpha(1 + c_0^2\alpha^2)].$$

For any given $\epsilon_0 > 0$, choose α small enough such that $\frac{c_0\alpha + c_0^2\alpha^2}{4\pi} < \epsilon_0$, then

$$\ln \int_{\partial\Omega(\overline{y},\alpha)} e^u dS_y \leq (\frac{1}{4\pi} + \epsilon_0) \int_{\Phi(D_\alpha)} |\nabla_y u|^2 dy + c(\epsilon_0). \tag{3.4}$$

Step 2. Let α be the constant chosen above. Since Ω is a bounded domain in $\mathbb{R}^2$, we can find finitely many points $\overline{y_i} \in \partial\Omega$, where $0 < \alpha_i < \alpha$, and corresponding neighborhoods $\partial\Omega(\overline{y_i}, \frac{\alpha_i}{2})$ such that $\partial\Omega \subset \cup_{i=1}^{i=N} \partial\Omega(\overline{y_i}, \frac{\alpha_i}{2})$. Let ψ_i be smooth cutoff functions subordinate to this covering, satisfying

$$\psi_i(y) = \begin{cases} 1 \text{ when } & y \in \Phi(D_{\frac{\alpha_i}{2}}(\overline{y_i})) \\ 0 \text{ when } & y \notin \Phi(D_{\alpha_i}(\overline{y_i})), \end{cases}$$

and $0 \leq \psi_i^2 \leq 1$, $|\nabla\psi_i| < c$ some constant c.

According to (3.4),

$$
\begin{aligned}
\int_{\partial\Omega} e^{u(y)} dS_y \leq & \sum_{i=1}^{N} \int_{\partial\Omega(\overline{y_i}, \frac{\alpha_i}{2})} e^{u(y)} dS_y \\
\leq & \sum_{i=1}^{N} \int_{\partial\Omega(\overline{y_i}, \alpha)} e^{u\cdot\psi_i} dS_y \\
\leq & \sum_{i=1}^{N} \exp\{(\frac{1}{4\pi} + \epsilon_0) \int_{\Phi(D_\alpha(\overline{y_i}))} |\nabla_y (u \cdot \psi_i)|^2 dy + c(\epsilon_0)\} \\
= & \sum_{i=1}^{N} \exp\{(\frac{1}{4\pi} + \epsilon_0) \int_{\Phi(D_\alpha(\overline{y_i}))} |(\nabla_y u)\psi_i|^2 \\
& + 2(\nabla_y u)\psi_i \cdot u\nabla_y \psi_i + |u\nabla_y \psi_i|^2 dy + c(\epsilon_0)\} \\
\leq & \sum_{i=1}^{N} \exp\{(\frac{1}{4\pi} + \epsilon_0) \int_{\Phi(D_\alpha(\overline{y_i}))} |\nabla_y u|^2 \psi_i^2 + u^2 |\nabla_y \psi_i|^2 \\
& + \epsilon_0 |\nabla_y u|^2 \psi_i^2 + \frac{u^2 |\nabla_y \psi_i|^2}{\epsilon_0} dy + c(\epsilon_0)\} \\
\leq & \sum_{i=1}^{N} \exp\{(\frac{1}{4\pi} + \epsilon_0) \int_{\Phi(D_\alpha(\overline{y_i}))} (1+\epsilon_0)|\nabla_y u|^2 \\
& + c(1 + \frac{1}{\epsilon_0})u^2] dy + c(\epsilon_0)\} \\
\leq & N \cdot \exp\{(\frac{1}{4\pi} + \epsilon_0) \int_{\Omega} (1+\epsilon_0)|\nabla_y u|^2 + c(1 + \frac{1}{\epsilon_0})u^2 dy + c(\epsilon_0)\}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \ln \int_{\partial\Omega} e^{u(y)} dS_y \\
\leq & [\frac{1}{4\pi} + \epsilon_0(1 + \frac{1}{4\pi} + \epsilon_0)] \int_{\Omega} |\nabla_y u|^2 dy \\
& + c(\frac{1}{4\pi} + \epsilon_0)(1+\epsilon_0) \int_{\Omega} u^2 dy + \ln N + c(\epsilon_0).
\end{aligned}
$$

Hence, for any given $\epsilon > 0$, choose $\epsilon_0 > 0$ small enough such that $\epsilon_0(1 + \frac{1}{4\pi} + \epsilon_0) <$

ϵ, there exist constants $C_1(\epsilon)$ and $C_2(\epsilon)$ so that

$$\ln \int_{\partial\Omega} e^{u(y)} dS_y \leq (\frac{1}{4\pi} + \epsilon) \int_\Omega |\nabla_y u|^2 dy + C_1 \int_\Omega u^2 dy + C_2.$$

□

We are now ready to establish the following $\epsilon-$level inequality.

Theorem 3.2. *Let Ω be a simply connected domain in $\mathbb{R}^2$ with bounded Geodesic curvature K_g. Then for any given $\epsilon > 0$, there is a constant $C_3 = C_3(\epsilon)$ such that for any $u \in H^1(\Omega)$ with $\int_{\partial\Omega} K_g \cdot u dS_g = 0$,*

$$\ln \int_{\partial\Omega} e^u dS \leq (\frac{1}{4\pi} + \epsilon) \int_\Omega |\nabla u|^2 dy + C_3. \tag{3.5}$$

To prove this theorem, we need the following Poincaré-Sobolev type inequality.

Lemma 3.3. *Assume Ω is a simply connected domain in $\mathbb{R}^2$ with bounded Geodesic curvature K_g. Then for any $p \geq 1$, there is a constant c_p such that for any $u \in H^1(\Omega)$ with $\int_{\partial\Omega} K_g \cdot u dS = 0$,*

$$(\int_\Omega |u|^p)^{2/p} dy \leq c_p \int_\Omega |\nabla u|^2 dy.$$

Proof. We prove it by contradiction. Suppose this is not true, then there exists a sequence of functions $\{u_n\}$ in $H^1(\Omega)$ with $\int_{\partial\Omega} K_g \cdot u_n dS = 0$ such that

$$(\int_\Omega |u_n|^p)^{2/p} \geq n \int_\Omega |\nabla u_n|^2.$$

Let

$$v_n = \frac{u_n}{(\int_\Omega |u_n|^p)^{1/p}}.$$

Then $||v_n||_{L^p(\Omega)} = 1$, and by assumption $||\nabla v_n||_{L^2(\Omega)} = \frac{||\nabla u_n||_{L^2(\Omega)}}{||u_n||_{L^p(\Omega)}} \leq \frac{1}{\sqrt{n}}$. Thus

$$||v_n||_{H^1(\Omega)} \leq \sqrt{2}, \quad \text{and} \quad ||\nabla v_n||_{L^2(\Omega)} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, $v_n \rightharpoonup v_0$ in $H^1(\Omega)$ and v_0 is a constant in Ω, say $v_0 = c$ in Ω. Let $u_0 = v_0 - c$ in $\overline{\Omega}$, then $||u_0||_{H^1(\Omega)} = 0$. By trace inequality $||u_0||_{L^p(\partial\Omega)} \leq C||u_0||_{H^1(\Omega)}$, $u_0 = 0$ on $\partial\Omega$ almost everywhere and $v_0 = c$ in $\overline{\Omega}$. By the compact embedding, we know that $v_n \to v_0$ in $L^q(\Omega)$ for any $q < \infty$, thus $||v_0||_{L^p(\Omega)} = 1$. Hence $c \neq 0$.

On the other hand, by the compact embedding $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\partial\Omega)$, we know that $v_n \to v_0$ in $L^q(\partial\Omega)$, thus $\int_{\partial\Omega} K_g \cdot v_0 dS = 0$. Since $\int_{\partial\Omega} K_g dS = 2\pi$, we conclude that $v_0 = 0$ on $\partial\Omega$. A contradiction. □

We now return to the proof of Theorem 3.2.

Proof. Due to Lemma 1 in [2] , we only need to prove the theorem for functions in $C^1(\Omega)$ with only non-degenerate critical points. Let $u(x)$ be such a function with $\int_{\partial\Omega} K_g \cdot u dS = 0$. For small enough $\eta > 0$, there exists a_η such that

$$Vol\{x \in \Omega \ : \ u(x) > a_\eta\} = \eta.$$

Using Lemma 3.1 we have for any $\epsilon_1 > 0$, there are constants $C_1(\epsilon_1)$ and $C_2(\epsilon_1)$

such that

$$
\begin{aligned}
&\ln \int_{\partial\Omega} e^u dS \\
\le& a_\eta + \ln \int_{\partial\Omega} e^{(u-a_\eta)} dS \\
\le& a_\eta + \ln \int_{\partial\Omega} e^{(u-a_\eta)_+} dS \\
\le& a_\eta + (\frac{1}{4\pi} + \epsilon_1) \int_\Omega |\nabla (u-a_\eta)_+|^2 dx \\
&+ C_1(\epsilon_1) \int_\Omega |(u-a_\eta)_+|^2 dx + C_2(\epsilon_1),
\end{aligned}
$$

where $f_+(x) = \max\{f(x), 0\}$. We know that

$$
a_\eta \cdot \eta \le \int_{\{x\in\Omega \ :u(x)>a_\eta\}} u dx \le \int_\Omega |u| dx \le c_1^{1/2} (\int_\Omega |\nabla u|^2)^{1/2},
$$

thus

$$
a_\eta \le \eta ||\nabla u||^2_{L^2(\Omega)} + \frac{c_1}{4\eta^3}.
$$

Also, applying Lemma 3.3, we obtain

$$
\begin{aligned}
\int_\Omega |(u-a_\eta)_+|^2 dx =& \int_{\{x\in\Omega \ :u(x)>a_\eta\}} |(u-a_\eta)_+|^2 dx \\
\le& (\int_{\{x\in\Omega \ :u(x)>a_\eta\}} |(u-a_\eta)_+|^4 dx)^{1/2} \cdot \eta^{1/2} \\
\le& (\int_\Omega |u|^4 dx)^{1/2} \cdot \eta^{1/2} \\
\le& c_4 ||\nabla u||^2_{L^2(\Omega)} \cdot \eta^{1/2}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&\ln \int_{\partial\Omega} e^u dS \\
\leq & a_\eta + (\frac{1}{4\pi} + \epsilon_1) \int_\Omega |\nabla(u - a_\eta)_+|^2 dx + C_1(\epsilon_1) \int_\Omega |(u - a_\eta)_+|^2 dx + C_2(\epsilon_1) \\
\leq & (\frac{1}{4\pi} + \epsilon_1 + \eta + C_1(\epsilon_1) \cdot c_4 \cdot \eta^{1/2}) ||\nabla u||^2_{L^2(\Omega)} + \frac{c}{4\eta^3} + C_2(\epsilon_1).
\end{aligned}$$

Choosing ϵ_1 and η sufficiently small such that $\eta + \epsilon_1 + C_1 \cdot c_4 \cdot \eta^{1/2} < \epsilon$, we obtain inequality (3.5). □

Notice that Theorem 3.2 can also be stated as the following.

Theorem 3.2′ Let Ω be a simply connected domain in $\mathbb{R}^2$ with bounded Geodesic curvature K_g. Then for any given $\epsilon > 0$, there is a constant $C_3 = C_3(\epsilon)$ such that for any $v \in H^1(\Omega)$,

$$\ln \int_{\partial\Omega} e^v dS \leq (\frac{1}{4\pi} + \epsilon) \int_\Omega |\nabla v|^2 dy + \frac{1}{2\pi} \int_{\partial\Omega} K_g \cdot v dS + C_3. \tag{3.6}$$

Proof. Let

$$u = v - \frac{\int_{\partial\Omega} K_g v dS}{\int_{\partial\Omega} K_g dS},$$

then

$$\int_{\partial\Omega} K_g u dS = 0.$$

According to Theorem 3.2, we obtain inequality (3.6). □

3.3 Global sharp trace inequality

In this section, we always assume that $\Omega \subset \mathbb{R}^2$ is simply connected.

For any $\epsilon > 0$, we define a functional

$$I_\epsilon(u) = \frac{1}{8}\int_\Omega |\nabla u|^2 + \frac{1-\epsilon}{4}\int_{\partial\Omega} K_g u - \frac{\pi(1-\epsilon)}{2}\ln\int_{\partial\Omega} e^u,$$

and denote

$$E_\epsilon = \inf_{u\in H^1(\Omega),\ \int_{\partial\Omega} K_g u dS = 0} I_\epsilon(u).$$

It is easy to see that $I_\epsilon(u+C) = I_\epsilon(u)$ for any constant C. Due to Theorem 3.2 we know that E_ϵ is a finite number. Moreover, we can show

Lemma 3.4. *E_ϵ is achieved by a function $u_\epsilon \in H^1(\Omega)$.*

Proof. Let $\{u_n\}$ be a minimizing sequence with $\int_{\partial\Omega} K_g u_n dS = 0$. Choose a small positive number ϵ_1 such that

$$\frac{1}{8} - \epsilon_1 > \frac{\pi(1-\epsilon_1)}{2}\cdot(\frac{1}{4\pi} + \epsilon_1).$$

Applying Theorem 3.2 we have

$$\begin{aligned}
E_\epsilon \leftarrow I_\epsilon(u_n) &= \frac{1}{8}\int_\Omega |\nabla u_n|^2 - \frac{\pi(1-\epsilon_1)}{2}\ln\int_{\partial\Omega} e^{u_n} \\
&= (\frac{1}{8} - \epsilon_1)\int_\Omega |\nabla u_n|^2 - \frac{\pi(1-\epsilon_1)}{2}\ln\int_{\partial\Omega} e^{u_n} + \epsilon_1\int_\Omega |\nabla u_n|^2 \\
&\geq (\frac{1}{8} - \epsilon_1)\int_\Omega |\nabla u_n|^2 \\
&\quad - \frac{\pi(1-\epsilon_1)}{2}\left[(\frac{1}{4\pi} + \epsilon_1)\int_\Omega |\nabla u_n|^2 dy + C_3(\epsilon_1)\right] + \epsilon_1\int_\Omega |\nabla u_n|^2 \\
&\geq C(\epsilon_1) + \epsilon_1\int_\Omega |\nabla u_n|^2,
\end{aligned}$$

thus $||\nabla u_n||_{L^2(\Omega)} < C$. It follows from Lemma 3.3 that $||u_n||_{H^1(\Omega)} < C$. Further, it implies that $u_n \rightharpoonup u_\epsilon$ in $H^1(\Omega)$ and $e^{\beta u_n} \to e^{\beta u_\epsilon}$ in $L^1(\Omega)$ for any positive number β by Trudinger's inequality. Thus, $||u_\epsilon||^2_{H^1(\Omega)} \leq \underline{\lim}_{n\to\infty}||u_n||^2_{H^1(\Omega)}$.

For any $1 < p < 2$,

$$\begin{aligned} \int_\Omega |\nabla e^{\beta u_n}|^p &= \int_\Omega |e^{\beta u_n}\beta\nabla u_n|^p \\ &\le \beta^p\left(\int_\Omega e^{\frac{2p\beta}{2-p}u_n}\right)^{(2-p)/2}\left(\int_\Omega |\nabla u_n|^2\right)^{p/2} \\ &\le c\left(\int_\Omega |\nabla u_n|^2\right)^{p/2} \\ &\le C, \end{aligned}$$

thus $\{e^{\beta u_n}\}$ is a bounded sequence in $W^{1,p}(\Omega)$. Therefore, for any $\beta > 0$ and $1 < p < 2$, by the compact embedding, $e^{\beta u_n} \to e^{\beta u_\epsilon}$ in $L^p(\partial\Omega)$ upto subsequence. Hence $e^{\beta u_n} \to e^{\beta u_\epsilon}$ in $L^1(\partial\Omega)$ for any $\beta > 0$. So, $K_g u_n \to K_g u_\epsilon$ and $\int_{\partial\Omega} K_g u_\epsilon = 0$. It follows that

$$u_\epsilon \in \{u \in H^1(\Omega) \mid \int_{\partial\Omega} K_g \cdot u = 0\}$$

and

$$\underline{\lim}_{n\to\infty} I_\epsilon(u_n) \ge I_\epsilon(u_\epsilon).$$

Thus u_ϵ is a minimizer. □

To prove Theorem 1.2, we only need to show that E_ϵ is uniformly bounded from below as $\epsilon \to 0$. Following the similar arguments given in [10], [12], [13] and [26], we argue by contradiction. Suppose that E_ϵ is not bounded from below, that is, up to a subsequence (due to the nature of the proof, for convenience we will not distinguish subsequence $\{\epsilon_i\}$ and the original sequence $\{\epsilon\}$ in this section),

$$\lim_{\epsilon\to 0} E_\epsilon = -\infty. \tag{3.7}$$

Let

$$v_\epsilon = u_\epsilon - \ln\int_{\partial\Omega} e^{u_\epsilon},$$

then v_ϵ is also a minimizer, which satisfies the following Euler-Lagrange equations

$$\begin{cases} -\Delta v_\epsilon &= 0 \quad \text{in } \Omega \\ \frac{\partial v_\epsilon}{\partial \nu} + (1-\epsilon)K_g &= 2\pi(1-\epsilon)e^{v_\epsilon} \quad \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Let $\lambda_\epsilon = v_\epsilon(x_\epsilon) := \max_{x\in\overline{\Omega}} v_\epsilon(x)$, then $\lambda_\epsilon = \max_{x\in\partial\Omega} v_\epsilon(x)$. We first claim

Lemma 3.5.

$$\lim_{\epsilon\to 0} \lambda_\epsilon = +\infty.$$

Proof. Let

$$v_\epsilon^a = \frac{\int_{\partial\Omega} v_\epsilon dS}{\int_{\partial\Omega} dS}.$$

If $\lambda_\epsilon < C$, then $e^{v_\epsilon} < C$. From (3.8), we have

$$0 = \int_\Omega -v_\epsilon \Delta v_\epsilon = \int_\Omega |\nabla v_\epsilon|^2 - \int_{\partial\Omega} v_\epsilon \frac{\partial v_\epsilon}{\partial \nu},$$

and

$$\int_\Omega |\nabla(v_\epsilon - v_\epsilon^a)|^2 = \int_{\partial\Omega} \frac{\partial v_\epsilon}{\partial\nu} v_\epsilon = \int_{\partial\Omega} [-(1-\epsilon)K_g + 2\pi(1-\epsilon)e^{v_\epsilon}]v_\epsilon.$$

$\|\nabla(v_\epsilon - v_\epsilon^a)\|_{L^2(\Omega)} < C$ and $\|v_\epsilon - v_\epsilon^a\|_{L^2(\Omega)} < C$. Since $\|\Delta(v_\epsilon - v_\epsilon^a)\|_{L^\infty(\Omega)} < C$, $v_\epsilon - v_\epsilon^a \in C^\alpha(\Omega)$. Also, $v_\epsilon - v_\epsilon^a \in C^\alpha(\partial\Omega)$ since $\|\frac{\partial(v_\epsilon - v_\epsilon^a)}{\partial\nu}\|_{L^\infty(\partial\Omega)} < C$.

If v_ϵ^a is bounded, then $v_\epsilon = (v_\epsilon - v_\epsilon^a) + v_\epsilon^a$ is bounded, thus E_ϵ is bounded from below, this contradicts our assumption $\lim_{\epsilon\to 0} E_\epsilon = -\infty$.

If $v_\epsilon^a \to -\infty$ is unbounded, then $v_\epsilon - v_\epsilon^a \to w$, where w satisfies

$$\begin{cases} -\Delta w &= 0 \qquad \text{in } \ \Omega \\ \frac{\partial w}{\partial \nu} &= -K_g \quad \text{on } \ \partial\Omega. \end{cases}$$

Thus, $0 = \int_\Omega -\Delta w = \int_{\partial\Omega} \frac{\partial w}{\partial \nu} = \int_{\partial\Omega} -K_g = -2\pi$, a contradiction.

Hence $\lim_{\epsilon\to 0} \lambda_\epsilon = +\infty$. □

Assume that, up to a subsequence, $x_\epsilon \to \bar{x} \in \partial\Omega$. In a neighborhood of $\bar{x}$, we choose a normal coordinate system, and define

$$\varphi_\epsilon(x) = v_\epsilon(\tau_\epsilon^{-1} x + x_\epsilon) - \ln \tau_\epsilon,$$

where

$$\tau_\epsilon := e^{\lambda_\epsilon}.$$

For any given $R > 0$, if ϵ is sufficiently small, φ_ϵ satisfies

$$\begin{cases} -\Delta \varphi_\epsilon &= 0 \qquad\qquad\qquad \text{in } \ B_{2R}(0, -\frac{R}{\sqrt{2}})_+ \\ \frac{\partial \varphi_\epsilon}{\partial \nu} + \frac{1-\epsilon}{\tau_\epsilon} K_g &= 2\pi(1-\epsilon)e^{\varphi_\epsilon} \quad \text{on } \ \Gamma_2, \end{cases}$$

where $\Gamma_2 := \{(y_1, 0) | -\frac{\sqrt{7}}{2}R \le y_1 \le \frac{\sqrt{7}}{2}R\}$ and $B_{2R}(0, -\frac{R}{\sqrt{2}})_+ := \{(y_1, y_2) \in \mathbb{R}^2_+ | y_1^2 + (y_2 + \frac{R}{\sqrt{2}})^2 < (2R)^2\}$. The behavior of v_ϵ in a tiny neighborhood of $\bar{x}$ can be described by the behavior of φ_ϵ in a large set $B_{2R}(0, -\frac{R}{\sqrt{2}})_+$.

Lemma 3.6. *For a fixed $R > 0$, there is a constant $C(R)$ such that*

$$|\varphi_\epsilon(x)| < C(R) \quad \forall \ x \in \overline{B_R(0, -\frac{R}{\sqrt{2}})_+}.$$

Proof. Let $\varphi_\epsilon^{(1)}$ be the unique solution of

$$\begin{cases} -\Delta\varphi_\epsilon^{(1)} &= 0 \qquad \text{in } \; B_{2R}(0,-\frac{R}{\sqrt{2}})_+ \\ \varphi_\epsilon^{(1)} &= 0 \qquad \text{on } \; \Gamma_1 \\ \frac{\partial\varphi_\epsilon^{(1)}}{\partial\nu} + \frac{1-\epsilon}{\tau_\epsilon}K_g &= 2\pi(1-\epsilon)e^{\varphi_\epsilon} \qquad \text{on } \; \Gamma_2, \end{cases}$$

where $\Gamma_1 := \partial B_{2R}(0,-\frac{R}{\sqrt{2}})_+ \cap \mathbb{R}^2_+$.

Since $e^{\varphi_\epsilon} \le 1$, we have $|\varphi_\epsilon^{(1)}| < C_1(R)$ for all $x \in \overline{B_R(0,-\frac{R}{\sqrt{2}})_+}$.

Let

$$\varphi_\epsilon^{(2)}(x) = \varphi_\epsilon^{(2)}(x_1,x_2) := \begin{cases} \varphi_\epsilon(x) - \varphi_\epsilon^{(1)}(x) & \text{if } \; x_2 \ge 0 \\ \varphi_\epsilon^{(2)}(x_1,-x_2) & \text{if } \; x_2 < 0, \end{cases}$$

then

$$\begin{cases} -\Delta\varphi_\epsilon^{(2)} &= 0 \quad \text{in } \; B_{2R}(0,-\frac{R}{\sqrt{2}})_+ \cup B_{2R}(0,+\frac{R}{\sqrt{2}})_- \\ \varphi_\epsilon^{(2)} &= 0 \quad \text{on } \; \Gamma_1 \\ \frac{\partial\varphi_\epsilon^{(2)}}{\partial\nu} &= 0 \quad \text{on } \; \Gamma_2, \end{cases}$$

where $B_{2R}(0,+\frac{R}{\sqrt{2}})_- := \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 - \frac{R}{\sqrt{2}})^2 < 4R^2 \; and \; x_2 < 0\}$.

By weak solution, we check that

$$-\Delta\varphi_\epsilon^{(2)} = 0 \;\; in \;\; B_{2R}(0,-\frac{R}{\sqrt{2}})_+ \cup B_{2R}(0,+\frac{R}{\sqrt{2}})_- \cup \Gamma_2;$$

$\forall x \in \Gamma_2$, choose δ small enough such that

$$B_\delta(x) \subset \left[B_{2R}(0,-\frac{R}{\sqrt{2}})_+ \cup B_{2R}(0,+\frac{R}{\sqrt{2}})_- \cup \Gamma_2\right].$$

Let ψ be a test function with 0 on $\partial B_\delta(x)$, then

$$\begin{aligned}
-\int_{B_\delta(x)} \psi \Delta \varphi_\epsilon^{(2)} &= \int_{B_\delta(x)} \nabla\psi \nabla\varphi_\epsilon^{(2)} \\
&= \int_{B_\delta(x)_+} \nabla\psi \nabla\varphi_\epsilon^{(2)} + \int_{B_\delta(x)_-} \nabla\psi \nabla\varphi_\epsilon^{(2)} \\
&= -\int_{B_\delta(x)_+} \psi \Delta \varphi_\epsilon^{(2)} + \int_{\partial(B_\delta(x)_+)} \frac{\partial \varphi_\epsilon^{(2)}}{\partial \nu}\psi \\
&\quad -\int_{B_\delta(x)_-} \psi \Delta \varphi_\epsilon^{(2)} + \int_{\partial(B_\delta(x)_-)} \frac{\partial \varphi_\epsilon^{(2)}}{\partial \nu}\psi \\
&= 0.
\end{aligned}$$

Thus $-\Delta\varphi_\epsilon^{(2)} = 0$ in $B_{2R}(0, -\frac{R}{\sqrt{2}})_+ \cup B_{2R}(0, +\frac{R}{\sqrt{2}})_- \cup \Gamma_2$. Also, $\varphi_\epsilon^{(2)}(x)$ is bounded from above by $C_1(R)$. Applying Harnack inequality to $2C_1(R) - \varphi_\epsilon^{(2)}$, we have

$$\frac{1}{C} \le \frac{2C_1(R) - \varphi_\epsilon^{(2)}(x)}{2C_1(R) - \varphi_\epsilon^{(2)}(0)} \le C \qquad \forall x \in \overline{B_R(0, -\frac{R}{\sqrt{2}})_+ \cup B_R(0, +\frac{R}{\sqrt{2}})_-}.$$

This and the bound of $\varphi_\epsilon^{(1)}$ yield Lemma 3.6. $\square$

Since φ_ϵ is uniformly bounded in any fixed $\overline{B_R(0, -\frac{R}{\sqrt{2}})_+}$, based on the standard elliptic estimates, we have

$$\varphi_\epsilon \to \varphi_0 \quad \text{ in } C^{1,\alpha}(\overline{B_R(0, -\frac{R}{\sqrt{2}})_+}), \tag{3.9}$$

for some $\alpha \in (0, 1)$, where φ_0 satisfies

$$\begin{cases} -\Delta\varphi_0 &= 0 \qquad \text{in } \mathbb{R}^2_+ \\ \frac{\partial \varphi_0}{\partial \nu} &= 2\pi e^{\varphi_0} \quad \text{on } \partial(\mathbb{R}^2_+), \end{cases}$$

and

$$\begin{aligned}
\int_{\partial\mathbb{R}^2_+} e^{2\varphi_0} &= \lim_{R\to\infty}\overline{\lim_{\epsilon\to 0}}\int_{\partial B_R(0,-\frac{R}{\sqrt{2}})_+\cap\partial\mathbb{R}^2_+} e^{2\varphi_\epsilon} \\
&= \lim_{R\to\infty}\overline{\lim_{\epsilon\to 0}}\int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})_+\cap\partial\mathbb{R}^2_+} e^{2v_\epsilon} \\
&\le [\overline{\lim_{\epsilon\to 0}}\int_{\partial\Omega} e^{v_\epsilon}]^2 \\
&\le 1.
\end{aligned}$$

From a uniqueness theorem for harmonic functions on upper-half plane in [27] we know that

$$\varphi_0(x) = \varphi_0(x_1, x_2) = \ln\frac{2\tilde{x}_2}{(x_1-\tilde{x}_1)^2 + (x_2+\tilde{x}_2)^2} - \ln(2\pi), \tag{3.10}$$

where $\tilde{x}_1$ is any real number and $\tilde{x}_2$ is any positive number, are fundamental solutions.

Away from the singular point $\bar{x}$, we have the following global estimate.

Lemma 3.7. *For any compact domain $K \subset\subset \bar{\Omega}\setminus\{\bar{x}\}$, there is a constant $C(K)$ such that*

$$||v_\epsilon - v^a_\epsilon||_{L^\infty(K)} < C(K).$$

In order to prove Lemma 3.7, we need the following inequality.

Lemma 3.8. *Assume $K \subset \Omega$ is a bounded domain in $\mathbb{R}^2$ and u is a solution* *of*

$$\begin{cases} -\Delta u &= 0 \qquad in \;\; K \\ u &= 0 \qquad on \;\; \partial K \cap \Omega \\ \frac{\partial u}{\partial \nu} &= f \qquad on \;\; \partial K \cap \partial\Omega. \end{cases}$$

If $f \in L^1(\partial K \cap \partial\Omega)$*, then for every* $0 < \delta < 1$*, there is a constant* $C(K, \delta)$ *such that*

$$\int_K \exp\left\{ \frac{\pi(1-\delta)|u(y)|}{||f||_{L^1(\partial K \cap \partial\Omega)}} \right\} \leq C(K, \delta).$$

Proof. The proof is similar to Theorem 1. in [7] by Brezis and Merle.

Let $R = diam K$ such that $K \subset B_R^+$ for some upper-half ball of radius R in $\mathbb{R}^2$. Extend f to be zero outside K. Let

$$\tilde{u}(y) = \frac{1}{2\pi} \int_{\partial\mathbb{R}^2_+ \cap \partial B_R^+} (\log \frac{2R}{|x-y|} + \log \frac{2R}{|x-\bar{y}|}) |f(x)| dx, \;\; y \in \overline{B_R^+},$$

where $\bar{y}$ is the reflection point of y about the horizontal axis. A direct computation yields

$$\begin{cases} -\Delta \tilde{u} &= 0 \qquad \text{in} \;\; B_R^+ \\ \frac{\partial \tilde{u}}{\partial \nu} &= |f| \qquad \text{on} \;\; \partial B_R^+ \cap \partial\mathbb{R}^2_+. \end{cases}$$

Note that $\tilde{u}(y) \geq 0$ for $y \in \overline{B_R^+}$, and

$$\begin{cases} -\Delta(\tilde{u} \pm u) &= 0 \qquad \text{in} \;\; B_R^+ \\ \frac{\partial(\tilde{u} \pm u)}{\partial \nu} &\geq 0 \qquad \text{on} \;\; \partial B_R^+ \cap \partial\mathbb{R}^2_+. \end{cases}$$

It follows from the maximum principle that $|u| \leq \tilde{u}$ on K. Thus

$$\int_K \exp\{\frac{\pi(1-\delta)|u(y)|}{||f||_{L^1(\partial K \cap \partial\Omega)}}\} dy \leq \int_{B_R^+} \exp\{\frac{\pi(1-\delta)\tilde{u}(y)}{||f||_{L^1(\partial K \cap \partial\Omega)}}\} dy.$$

We estimate the right-hand side of the above inequality using Jensen's inequality

$$F(\int w(x)\varphi(x)dx) \leq \int w(x)F(\varphi(x))dx$$

with $F(t) = \exp t$, $w(x) = \frac{|f(x)|}{\|f\|_{L^1(\partial K\cap\partial\Omega)}}$ and $\varphi(x) = \frac{1-\delta}{2}(\log\frac{2R}{|x-y|} + \log\frac{2R}{|x-\bar{y}|})$. We have

$$\begin{aligned}
&\int_{B_R^+} \exp\{\frac{\pi(1-\delta)\tilde{u}(y)}{||f||_{L^1(\partial K\cap\partial\Omega)}}\}dy \\
\leq &\int_{B_R^+}\int_{\partial B_R^+\cap\partial\mathbb{R}^2_+} \frac{|f(x)|}{\|f\|_{L^1(\partial K\cap\partial\Omega)}} \exp\{\frac{1-\delta}{2}(\log\frac{2R}{|x-y|} + \log\frac{2R}{|x-\bar{y}|})dxdy\} \\
= &\int_{\partial B_R^+\cap\partial\mathbb{R}^2_+} \frac{|f(x)|}{\|f\|_{L^1(\partial K\cap\partial\Omega)}} \int_{B_R^+} (\frac{2R}{|x-y|})^{\frac{1-\delta}{2}} (\frac{2R}{|x-\bar{y}|})^{\frac{1-\delta}{2}} dydx \\
\leq &\frac{2^{2-\delta}}{\delta}(diamK)^2 = C(K,\delta).
\end{aligned}$$

□

Now we are ready to prove Lemma 3.7.

Proof. Let $K \subset\subset \Omega\setminus\{\bar{x}\}$ and v_ϵ^1 be a solution of

$$\begin{cases} -\Delta v_\epsilon^1 &= 0 \quad \text{in } K \\ v_\epsilon^1 &= 0 \quad \text{on } \partial K\cap\Omega \\ \frac{\partial v_\epsilon^1}{\partial\nu} &= 2\pi(1-\epsilon)e^{v_\epsilon} \quad \text{on } \partial K\cap\partial\Omega. \end{cases} \tag{3.11}$$

From Lemma 3.8, we know that $e^{|v_\epsilon^1|} \in L^p(K)$ for some $1 < p < 2$, thus $||v_\epsilon^1||_{L^p(K)} \leq C$.

Let $v_\epsilon^2 = v_\epsilon - v_\epsilon^1 - v_\epsilon^a$, then

$$\begin{cases} -\Delta v_\epsilon^2 &= 0 \quad \text{in} \ \ K \\ \frac{\partial v_\epsilon^2}{\partial \nu} + (1-\epsilon)K_g &= 0 \quad \text{on} \ \ \partial K \cap \partial\Omega. \end{cases}$$

We have, from the $W^{2,p}$ estimates,

$$\begin{aligned} ||v_\epsilon^2||_{L^\infty(K)} &\leq C\{||(1-\epsilon)K_g||_{W^{1-\frac{1}{p},p}(\partial K\cap\partial\Omega)} + ||v_\epsilon^2||_{L^p(K)}\} \\ &\leq C(||v_\epsilon - v_\epsilon^a||_{L^p(K)} + ||v_\epsilon^1|_{L^p(K)}) \\ &\leq C(||\nabla v_\epsilon||_{L^p(K)} + ||v_\epsilon^1|_{L^p(K_2)}). \end{aligned} \tag{3.12}$$

Note that for any $1 < p < 2$,

$$||\nabla v_\epsilon||_{L^p(\Omega)} < C_q. \tag{3.13}$$

This is because for any $\varphi \in W^{1,p/(p-1)}(\Omega)$ with $\int_\Omega \varphi = 0$ and $||\varphi||_{W^{1,p/(p-1)}(\Omega)} = 1$ (thus $\varphi \in L^\infty(\Omega)$),

$$\begin{aligned} &|\int_\Omega \nabla v_\epsilon \nabla\varphi| \\ =&|-\int_\Omega \Delta v_\epsilon \varphi + \int_{\partial\Omega} \frac{\partial v_\epsilon}{\partial\nu}\varphi| \\ =&|\int_{\partial\Omega} (2\pi(1-\epsilon)e^{v_\epsilon} - (1-\epsilon)K_g)\varphi| \\ \leq& C||\varphi||_{L^{\frac{p}{p-1}}(\partial\Omega)} \\ \leq& C||\varphi||_{W^{1,\frac{p}{p-1}}(\Omega)} \\ \leq& C. \end{aligned}$$

We derive from the above and (3.7) that

$$\begin{aligned} 2\pi v_\epsilon^a = & \int_{\partial\Omega} K_g v_\epsilon - \int_{\partial\Omega} K_g (v_\epsilon - v_\epsilon^a) \\ \leq & \int_{\partial\Omega} K_g v_\epsilon - C \to -\infty. \end{aligned} \tag{3.14}$$

From (3.13) and (3.12) we have $||v_\epsilon^2||_{L^\infty(K)} \leq C$. Thus

$$\int_K e^{pv_\epsilon} \leq \int_K e^{pv_\epsilon^a} \cdot e^{pv_\epsilon^1} \cdot e^{pv_\epsilon^2} \leq C,$$

we then know from (3.11) that $||v_\epsilon^1||_{L^\infty(K)} \leq C$, thus $||v_\epsilon - v_\epsilon^a||_{L^\infty(K)} \leq C$. □

From Lemma 3.7, we conclude, via the standard elliptic estimates, that

$$v_\epsilon - v_\epsilon^a \to G(x, \bar{x}) \quad in \quad C^{1,\alpha}(K) \tag{3.15}$$

for any compact domain K in $\bar{\Omega} \setminus \{\bar{x}\}$, where $G(x, \bar{x})$ satisfies

$$\begin{cases} -\Delta G & = 0 \qquad \text{in } \ \Omega \\ \frac{\partial G}{\partial \nu} + K_g & = 2\pi \delta_{\bar{x}} \qquad \text{on } \ \partial\Omega \\ \int_{\partial\Omega} G & = 0. \end{cases} \tag{3.16}$$

Thus

$$G(x, \bar{x}) = -\ln|x - \bar{x}| + A(\bar{x}) + o(1), \tag{3.17}$$

where $A(\bar{x})$ is a constant depending on the location of $\bar{x}$, $o(1) \to 0$ as $|x| \to 0$.

To complete the proof of the main theorem, we also need a lower bound for v_ϵ away from the singular point $\bar{x}$. For convenience, we choose a normal coordinate system centered at x_ϵ.

Lemma 3.9. *For any fixed $R > 0$, let $r_\epsilon = \tau_\epsilon^{-1} R$. Then*

$$v_\epsilon(x) \geq G(x, x_\epsilon) - \ln \frac{2\pi}{(\sqrt{2}-1)} - A(\bar{x}) + o_\epsilon(1) \qquad \forall x \in \Omega \setminus B_{r_\epsilon}(0, -\frac{r_\epsilon}{\sqrt{2}}), \tag{3.18}$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$.

Proof. On $\partial B_{r_\epsilon}(0, -\frac{r_\epsilon}{\sqrt{2}}) \bigcap \Omega$, we define $C_\epsilon^* := (v_\epsilon - G)|_{|x-(0,-\frac{r_\epsilon}{\sqrt{2}})|=r_\epsilon}$. From (3.9) and (3.10), we know that

$$C_\epsilon^* = \lambda_\epsilon + \ln \frac{\sqrt{2}}{2\pi R} + \ln|x| - A(\bar{x}) + o_\epsilon(1) \geq -\ln \frac{2\pi}{\sqrt{2}-1} - A(\bar{x}) + o_\epsilon(1).$$

Let

$$\tilde{K}_g(x) = \begin{cases} K_g(x) & \text{if } K_g(x) \leq 0 \\ \phi(x) K_g(x) & \text{if } K_g(x) > 0, \end{cases}$$

where $0 \leq \phi \leq 1$ is a measurable function such that $\int_{\partial\Omega} \tilde{K}_g = 0$. Let h satisfy

$$\begin{cases} -\Delta h = 0 & \text{in } \Omega \\ \frac{\partial h}{\partial \nu} + \tilde{K}_g = 0 & \text{on } \partial\Omega. \end{cases}$$

In $\Omega \setminus B_{r_\epsilon}(0, -\frac{r_\epsilon}{\sqrt{2}})$, we consider $v_\epsilon(x) - G(x, x_\epsilon) - C_\epsilon^* + \epsilon h(x)$. Since

$$-\Delta(v_\epsilon(x) - G(x, x_\epsilon) - C_\epsilon^* + \epsilon h(x)) = 0 \text{ in } \Omega \setminus B_{r_\epsilon}(0, -\frac{r_\epsilon}{\sqrt{2}})$$

and

$$\frac{\partial}{\partial \nu}(v_\epsilon(x) - G(x, x_\epsilon) - C_\epsilon^* + \epsilon h(x)) \geq 0 \text{ on } \partial(\Omega \setminus B_{r_\epsilon}(0, -\frac{r_\epsilon}{\sqrt{2}})),$$

we have (3.18) via the maximum principle. □

We are now ready to complete the proof of the main theorem.

We need to estimate $E_\epsilon = I_\epsilon(v_\epsilon)$. For any fixed small $\delta > 0$, we assume that ϵ is sufficiently small so that $\delta > r_\epsilon$. Then

$$\begin{aligned}
&\int_\Omega |\nabla v_\epsilon|^2 \\
=&\int_{\Omega\setminus B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})} |\nabla v_\epsilon|^2 + \int_{\Omega\bigcap B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\setminus B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})} |\nabla v_\epsilon|^2 + \int_{\Omega\bigcap B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})} |\nabla v_\epsilon|^2 \\
:=&\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3.
\end{aligned}$$

To estimate $\mathbf{I}_3$, we first let $\tilde{x}_1 = 0$ and $\tilde{x}_2 = \frac{R}{\sqrt{2}}$, then from (3.9) and (3.10)

$$\begin{aligned}
\mathbf{I}_3 = &\int_{B_R(0,-\frac{R}{\sqrt{2}})_+} |\nabla\varphi_0|^2 dx + o_\epsilon(1) \\
= &\int_{B_R(0,-\frac{R}{\sqrt{2}})_+} \left|\nabla\left(\ln\frac{\sqrt{2}R}{x_1^2+(x_2+\frac{R}{\sqrt{2}})^2} - \ln 2\pi\right)\right|^2 dx + o_\epsilon(1) \\
= &\int_{B_R(0,-\frac{R}{\sqrt{2}})_+} \frac{4}{x_1^2+(x_2+\frac{R}{\sqrt{2}})^2} + o_\epsilon(1) \\
= &\ 8\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{\frac{R}{\sqrt{2}\sin\theta}}^{R} \frac{1}{r^2} r\,dr\,d\theta + o_\epsilon(1) \\
= &\ 2\pi\ln\sqrt{2} + c + o_\epsilon(1).
\end{aligned} \tag{3.19}$$

From (3.15) and (3.16), we obtain

$$\begin{aligned}
\mathbf{I}_1 = &\int_{\Omega\setminus B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})} |\nabla v_\epsilon|^2 \\
= &\int_{\Omega\setminus B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})} |\nabla G|^2 + o_\epsilon(1) \\
= &-\int_{\Omega\setminus B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})} G\Delta G + \int_{(\partial\Omega)\setminus B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})} G\frac{\partial G}{\partial\nu} \\
&-\int_{(\partial B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}}))\bigcap\Omega} G\frac{\partial G}{\partial\nu} + o_\epsilon(1) \\
= &-\int_{(\partial\Omega)\setminus B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})} GK_g - \int_{(\partial B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}}))\bigcap\Omega} G\frac{\partial G}{\partial\nu} + o_\epsilon(1).
\end{aligned} \tag{3.20}$$

To estimate $\mathbf{I}_2$, we first use (3.8) to get

$$
\begin{aligned}
\mathbf{I}_2 =& \int_{\Omega \bigcap (B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}}) \backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}))} |\nabla v_\epsilon|^2 \\
=& - \int_{\Omega \bigcap (B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}}) \backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}))} \Delta v_\epsilon \cdot v_\epsilon + \int_{\partial(\Omega \bigcap (B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}}) \backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})))} v_\epsilon \frac{\partial v_\epsilon}{\partial \nu} \\
=& \int_{\partial B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}}) \bigcap \Omega} v_\epsilon \frac{\partial v_\epsilon}{\partial \nu} - \int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}) \bigcap \Omega} v_\epsilon \frac{\partial v_\epsilon}{\partial \nu} \\
&+ \int_{(B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}}) \backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})) \bigcap \partial\Omega} v_\epsilon \frac{\partial v_\epsilon}{\partial \nu} \\
=& \int_{\partial B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}}) \bigcap \Omega} v_\epsilon \frac{\partial v_\epsilon}{\partial \nu} - \int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}) \bigcap \Omega} v_\epsilon \frac{\partial v_\epsilon}{\partial \nu} \\
&+ \int_{(B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}}) \backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}))) \bigcap \partial\Omega} (-(1-\epsilon) K_g v_\epsilon + 2\pi(1-\epsilon) v_\epsilon e^{v_\epsilon})
\end{aligned}
$$

Applying Lemma 3.9, we have

$$
\begin{aligned}
&\int_{(B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}}) \backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})) \bigcap \partial\Omega} 2\pi(1-\epsilon) v_\epsilon e^{v_\epsilon} \\
\geq & 2\pi(1-\epsilon) \int_{(B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}}) \backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})) \bigcap \partial\Omega} G e^{v_\epsilon} + o_\epsilon(1) + o_R(1)
\end{aligned}
$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$ and $o_R(1) \to 0$ as $R \to +\infty$.

From (3.8), (3.15) and (3.16) we obtain

$$
\begin{aligned}
&2\pi(1-\epsilon)\int_{(B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}))\bigcap\partial\Omega} Ge^{v_\epsilon} \\
=&\int_{(B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}))\bigcap\partial\Omega} G\frac{\partial v_\epsilon}{\partial\nu}+\int_{(B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}))\bigcap\partial\Omega}(1-\epsilon)K_gG \\
=&\int_{\partial((B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}))\bigcap\Omega)} G\frac{\partial v_\epsilon}{\partial\nu}-\int_{\partial B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega} G\frac{\partial v_\epsilon}{\partial\nu} \\
&+\int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega} G\frac{\partial v_\epsilon}{\partial\nu}+o_\epsilon(1)+o_\delta(1) \\
=&\int_{\partial B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega} v_\epsilon\frac{\partial G}{\partial\nu}-\int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega} v_\epsilon\frac{\partial G}{\partial\nu}+\int_{(B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}))\bigcap\partial\Omega} v_\epsilon\frac{\partial G}{\partial\nu} \\
&-\int_{\partial B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega} G\frac{\partial v_\epsilon}{\partial\nu}+\int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega} G\frac{\partial v_\epsilon}{\partial\nu}+o_\epsilon(1)+o_\delta(1) \\
=&\int_{\partial B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega} v_\epsilon^a\frac{\partial G}{\partial\nu}-\int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega} v_\epsilon\frac{\partial G}{\partial\nu}+\int_{(B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}))\bigcap\partial\Omega} v_\epsilon\frac{\partial G}{\partial\nu} \\
&+\int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega} G\frac{\partial v_\epsilon}{\partial\nu}+o_\epsilon(1)+o_\delta(1) \\
=&\int_{\partial B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega} v_\epsilon^a\frac{\partial G}{\partial\nu}-\int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega} v_\epsilon\frac{\partial G}{\partial\nu}+\int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega} G\frac{\partial v_\epsilon}{\partial\nu} \\
&-\int_{(B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}))\bigcap\partial\Omega} v_\epsilon K_g+o_\epsilon(1)+o_\delta(1)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathbf{I}_2\geq&-(2-\epsilon)\int_{(B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}))\bigcap\partial\Omega} K_gv_\epsilon+\int_{(\partial B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}}))\bigcap\Omega} v_\epsilon\frac{\partial v_\epsilon}{\partial\nu} \\
&-\int_{(\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}))\bigcap\Omega} v_\epsilon\frac{\partial v_\epsilon}{\partial\nu}+\int_{\partial B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega} v_\epsilon^a\frac{\partial G}{\partial\nu} \\
&-\int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega} v_\epsilon\frac{\partial G}{\partial\nu}+\int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega} G\frac{\partial v_\epsilon}{\partial\nu}+o_\epsilon(1)+o_\delta(1)+o_R(1)
\end{aligned}
$$

We now estimate the boundary term in the right hand side of the above inequality. From (3.9), (3.10) and Lemma 3.9, we know that

$$-\int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega}\frac{\partial v_\epsilon}{\partial\nu}(v_\epsilon-G)\geq\pi(-A(\bar{x})-\ln\frac{2\pi}{\sqrt{2}-1})+o_\epsilon(1),$$

and

$$\begin{aligned}&-\int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega}v_\epsilon\frac{\partial G}{\partial\nu}\\&=-\lambda_\epsilon\int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega}\frac{\partial G}{\partial\nu}-(\ln\frac{\sqrt{2}}{R})\int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega}\frac{\partial G}{\partial\nu}+o_\epsilon(1)+o_R(1)\\&=-\lambda_\epsilon\int_{\partial\Omega\backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})}\frac{\partial G}{\partial\nu}-(\ln\frac{\sqrt{2}}{R})\int_{\partial B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})}\frac{\partial G}{\partial\nu}+o_\epsilon(1)+o_R(1)\\&\geq\lambda_\epsilon\int_{\partial\Omega\backslash B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})}K_g+(\ln\frac{\sqrt{2}}{R})(\frac{\pi}{2}+\frac{\pi}{2(\sqrt{2}-1)})+o_\epsilon(1)+o_R(1).\end{aligned}$$

From (3.16) we have

$$v_\epsilon^a\cdot\int_{\partial B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega}\frac{\partial G}{\partial\nu}=v_\epsilon^a\int_{\partial\Omega\backslash B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})}\frac{\partial G}{\partial\nu}=-v_\epsilon^a\int_{\partial\Omega\backslash B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})}K_g,$$

and

$$\begin{aligned}&v_\epsilon^a\cdot\int_{\partial B_\delta(0,\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega}\frac{\partial v_\epsilon}{\partial\nu}\\&=v_\epsilon^a\cdot\int_{(\partial\Omega)\backslash B_\delta(0,\frac{r_\epsilon}{\sqrt{2}})}\frac{\partial v_\epsilon}{\partial\nu}\\&=v_\epsilon^a\int_{(\partial\Omega)\backslash B_\delta(0,\frac{r_\epsilon}{\sqrt{2}})}(-(1-\epsilon)K_g+2\pi(1-\epsilon)e^{v_\epsilon})\\&=-(1-\epsilon)v_\epsilon^a\int_{(\partial\Omega)\backslash B_\delta(0,\frac{r_\epsilon}{\sqrt{2}})}K_g+v_\epsilon^a e^{v_\epsilon^a}\int_{(\partial\Omega)\backslash B_\delta(0,\frac{r_\epsilon}{\sqrt{2}})}2\pi(1-\epsilon)e^{v_\epsilon-v_\epsilon^a}\\&=-(1-\epsilon)v_\epsilon^a\int_{(\partial\Omega)\backslash B_\delta(0,\frac{r_\epsilon}{\sqrt{2}})}K_g+o_\epsilon(1).\end{aligned}$$

From (3.15),

$$
\begin{aligned}
&\int_{(B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\setminus B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}))\bigcap\partial\Omega} K_g v_\epsilon \\
&= v_\epsilon^a \int_{(B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\setminus B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}))\bigcap\partial\Omega} K_g + \int_{(B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\setminus B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}))\bigcap\partial\Omega} K_g(v_\epsilon - v_\epsilon^a) \\
&= v_\epsilon^a \int_{(B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\setminus B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}}))\bigcap\partial\Omega} K_g + o_\delta(1) \\
&= 2\pi v_\epsilon^a - v_\epsilon^a \int_{\partial\Omega\setminus B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})} K_g + v_\epsilon^a \int_{\partial\Omega\bigcap B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})} K_g + o_\delta(1)
\end{aligned}
$$

We conclude

$$
\begin{aligned}
\mathbf{I}_2 \geq \quad & -2\pi(2-\epsilon)v_\epsilon^a + (2-\epsilon)v_\epsilon^a \textstyle\int_{\partial\Omega\bigcap B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})} K_g + \pi(-A(\bar{x}) - \ln\frac{2\pi R}{(\sqrt{2}-1)R}) \\
& +\lambda_\epsilon(2\pi - \textstyle\int_{B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\partial\Omega} K_g) + \textstyle\int_{\partial B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\Omega} G\frac{\partial G}{\partial\nu} \\
& +(\ln\frac{\sqrt{2}}{R})(\frac{\pi}{2} + \frac{\pi}{2(\sqrt{2}-1)}) + o_\epsilon(1) + o_R(1) + o_\delta(1)
\end{aligned}
\tag{3.21}
$$

We have, from (3.19)-(3.21), that

$$
\begin{aligned}
I_\epsilon(v_\epsilon) \geq & -\frac{\pi}{8}A(\bar{x}) - \frac{1}{8}\ln\frac{2\pi}{\sqrt{2}-1} + \frac{\pi\ln\sqrt{2}}{4} + \lambda(\frac{\pi}{4} - \frac{1}{8}\int_{B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\partial\Omega} K_g) \\
& - \frac{\pi}{4}\epsilon v_\epsilon^a + \frac{1-\epsilon}{4}\int_{B_{r_\epsilon}(0,-\frac{r_\epsilon}{\sqrt{2}})\bigcap\partial\Omega} K_g v_\epsilon^a - \frac{1}{8}\int_{(\partial\Omega)\setminus B_\delta(0,-\frac{r_\epsilon}{\sqrt{2}})} GK_g \\
& + \frac{1-\epsilon}{4}\int_{\partial\Omega} K_g(v_\epsilon - v_\epsilon^a) + \frac{c}{8} + o_\epsilon(1) + o_R(1) + o_\delta(1).
\end{aligned}
$$

From (3.14), we know that $v_\epsilon^a \to -\infty$. Also, since $\lambda_\epsilon \cdot vol(B_{r_\epsilon}(0, -\frac{r_\epsilon}{\sqrt{2}})) = o_\epsilon(1)$, we know from Lemma 3.9, that $v_\epsilon^a \cdot vol(B_{r_\epsilon}(0, -\frac{r_\epsilon}{\sqrt{2}})) = o_\epsilon(1)$. We finally have

$$
\lim_{\epsilon\to 0} E_\epsilon \geq -\frac{\pi}{8}A(\bar{x}) - \frac{1}{8}\ln\frac{2\pi}{\sqrt{2}-1} + \frac{\pi\ln\sqrt{2}}{4} + C.
$$

for some constant C. Thus Theorem 1.2 is proved.

Chapter 4

Conclusion and Future Work

In this research, we study the inequalities related to Moser-Trudinger-Onofri inequality. We first prove an analog Hardy inequality with sharp constant involving exponential weight function in chapter 2. When $n = 2$, this inequality allows us to give a direct and simple proof of Onofri inequality on S^2. In chapter 3, we start with Lebedev-Milin inequality and derive the sharp local trace inequality

$$\ln \int_{-1}^{1} e^u \frac{dy_1}{2\pi} \leq \frac{1}{4\pi} \int_D |\nabla u|^2 dy + C \quad \forall u \in H^1(D),$$

where $D = \{(y_1, y_2) \in \mathbb{R}^2_+ : y_1^2 + (y_2+1)^2 < 2\}$. This inequality and partition of unity are used to establish the following sharp inequality on general bounded domain $\Omega \in \mathbb{R}^2$,

$$\ln \int_{\partial\Omega} e^u dS \leq (\frac{1}{4\pi} + \epsilon) \int_\Omega |\nabla u|^2 dy + C_1 \int_\Omega u^2 \ dy + C_2 \qquad \forall u \in H^1(\Omega).$$

This inequality is followed by the $\epsilon-$level sharp trace inequality on simply connected domain $\Omega \in \mathbb{R}^2$,

$$\ln \int_{\partial\Omega} e^u dS \leq (\frac{1}{4\pi} + \epsilon) \int_\Omega |\nabla u|^2 dy + C_3 \quad \forall u \in H^1(\Omega), \quad \text{with} \int_{\partial\Omega} K_g \cdot u dS_g = 0.$$

We then define a functional and study the behavior of the minimizing sequence as the parameter ϵ goes to zero. It leads to the sharp trace inequality on two-dimensional simply connected domain,

$$\ln \int_{\partial\Omega} e^u dS \leq \frac{1}{4\pi} \int_{\Omega} |\nabla u|^2 dy + \frac{1}{2\pi} \int_{\partial\Omega} K_g \cdot u dS + C(\Omega) \quad \forall u \in H^1(\Omega).$$

It is quite natural to ask whether one can adapt this trace inequality for two-dimensional compact manifold with boundary. I plan to continue my research on extending such trace inequality to two-dimensional compact manifold with boundary.

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