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Abstract

Crystallographic groups of solvable Lie groups generalize the crystallographic groups of Euclidean space. The quotient of a solvable Lie group G by the action of a torsion-free crystallographic group of G is an infra-solvmanifold of G. Infra-solvmanifolds are aspherical manifolds that generalize both closed flat manifolds and closed almost flat manifolds (that is, infra-nilmanifolds).

Here we complete the classification of 4-dimensional infra-solvmanifolds by classifying torsion-free crystallographic groups of certain 4-dimensional solvable Lie groups. The classification also includes those crystallographic groups with torsion.

We prove that every 4-dimensional infra-solvmanifold is the boundary of a compact 5-dimensional manifold by constructing an involution on certain 4-dimensional infra-solvmanifolds which is either free, or has 2-dimensional fixed set.

The Ricci signatures (that is, signatures of the Ricci transformation) of 4-dimensional Lie groups have been classified. A Ricci signature can be realized on an infra-solvmanifold M of G if M is a compact isometric quotient of G, where G has left invariant metric with prescribed Ricci signature. We classify which Ricci signatures can be realized on certain 4-dimensional infra-solvmanifolds.

1 Introduction

This dissertation is a study of the crystallographic groups of 4-dimensional solvable Lie groups and their corresponding 4-dimensional infra-solvmanifolds. Crystallographic groups of solvable Lie groups generalize the well known crystallographic groups of $\mathbb{R}^n$. Infra-solvmanifolds are aspherical manifolds that generalize closed flat manifolds. This document is organized as follows: Chapter 1 provides background material and summarizes the main results, Chapter 2 establishes that every 4-dimensional infra-solvmanifold is the boundary of a compact 5-dimensional manifold, Chapter 3 completes the classification of the crystallographic groups of the solvable 4-dimensional geometries, and Chapter 4 explores Ricci signature properties of the 4-dimensional geometries. Finally, we provide some topics for further exploration in Chapter 5.

1.1 Definitions and Notation

1.1.1 (Crystallographic Groups and Infra-Solvmanifolds). For our purposes, we will assume all Lie groups to be connected and simply connected. Let G be a solvable Lie group. The group of *affine diffeomorphisms* of G is:

$$\mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G),$$

which has group operation $(a, A)(b, B) = (aA(b), AB)$. There is an action of $\mathrm{Aff}(G)$ on G. For $(a, A) \in \mathrm{Aff}(G)$ and $g \in G$,

$$(a, A).g = aA(g).$$

Under this action, the subgroup $G \cong \{(g, \mathrm{id}) \mid g \in G\} \subset \mathrm{Aff}(G)$ acts on G as left translations.

Definition 1.1.2. Let K be a maximal compact subgroup of $\mathrm{Aut}(G)$. A discrete subgroup $\mathit{\Pi}$ of $G \rtimes K \subset \mathrm{Aff}(G)$ is called a *crystallographic group* of G when

(1) The quotient $\mathit{\Pi}\backslash G$ is compact.

(2) The translation subgroup $\Gamma := \mathit{\Pi} \cap G$ is of finite index in $\mathit{\Pi}$.

The translation subgroup Γ is normal in $\mathit{\Pi}$, and we refer to the finite group $\Phi := \mathit{\Pi}/\Gamma$ as the *holonomy group* of $\mathit{\Pi}$. Thus, a crystallographic group $\mathit{\Pi}$ fits the diagram of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \mathit{\Pi} & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & G & \longrightarrow & G \rtimes K & \longrightarrow & K & \longrightarrow & 1. \end{array}$$

Condition (2) of Definition 1.1.2 implies that Γ is a discrete subgroup of G and that $\Gamma\backslash G$ is compact. That is, Γ is a *cocompact lattice* of G. For simply connected solvable Lie groups, a result of Mostow [30, Theorem 6.2] implies that $\Gamma\backslash G$ has finite volume if and only if $\Gamma\backslash G$ is compact. So, the terms "lattice" and "cocompact lattice" are equivalent for our purposes. Since Γ acts as left translations on G, it acts freely on G, and we say that the quotient $\Gamma\backslash G$ is a *solvmanifold* when G is solvable, and a *nilmanifold* when G is nilpotent. The quotient $\mathit{\Pi}\backslash G$ is a closed manifold precisely when $\mathit{\Pi}$ acts freely on G.

This is equivalent to $\mathit{\Pi}$ being torsion-free.

Definition 1.1.3. Let $\mathit{\Pi}$ be a torsion-free crystallographic group of G. The quotient $\mathit{\Pi}\backslash G$ is an *infra-solvmanifold* when G is solvable, and an *infra-nilmanifold* when G is nilpotent.

In [22], the various definitions of infra-solvmanifold appearing in the literature are shown to all be equivalent. Here we have adopted Definition 1 in [22].

Note that with the Euclidean metric, $\mathrm{Isom}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{O}(n, \mathbb{R})$, so an infra-solvmanifold for $G = \mathbb{R}^n$ and $K = \mathrm{O}(n, \mathbb{R})$ is simply a closed flat manifold.

Topologically, Condition (2) of Definition 1.1.2 states that an infra-solvmanifold $\mathit{\Pi}\backslash G$ is finitely covered by the solvmanifold $\Gamma\backslash G$ with covering transformation group Φ. This explains the prefix "infra" and is motivated by the classical result that any closed flat n-manifold is finitely covered by a flat torus $\mathbb{Z}^n\backslash\mathbb{R}^n$. Thus, in our context, infra-solvmanifolds generalize closed flat manifolds, and solvmanifolds generalize flat tori.

When G is nilpotent, Auslander proved that Condition (1) of Definition 1.1.2 actually implies Condition (2). When G is abelian, this was due to Bieberbach.

Theorem 1.1.4 (Bieberbach's First Theorem in the nilpotent case [24, Theorem 8.3.2])**.** *Let G be nilpotent and*

$$\mathit{\Pi} \subset G \rtimes K$$

be discrete and such that $\mathit{\Pi}\backslash G$ is compact. Then the translation subgroup $\Gamma = \mathit{\Pi} \cap G$ is of finite index in G, and the quotient $\Gamma\backslash G$ is compact.

When G is solvable, Condition (2) in Definition 1.1.2 is needed, because there are examples of solvable G and discrete subgroups $\Pi \subset G \rtimes K$ with $\Pi \backslash G$ compact, for which $\Pi \cap G$ is *not* of finite index in Π. In other words, Bieberbach's First Theorem does *not* extend to all solvable Lie groups. Dekimpe, Lee, and Raymond provide sufficient criteria in [8] for a solvable Lie group to satisfy Bieberbach's First Theorem.

Crystallographic groups of solvable Lie groups of type (R) satisfy a rigidity property:

Theorem 1.1.5 (Rigidity of crystallographic groups [24, Theorem 8.4.3])**.** *Let G be a solvable Lie group of type (R) and Π, Π' be finite extensions of lattices in G. Then every isomorphism between Π and Π' is conjugation by an element of* $\mathrm{Aff}(G)$.

Consequently, infra-solvmanifolds of solvable Lie groups of type (R) are rigid:

Theorem 1.1.6 (Rigidity of infra-solvmanifolds [24, Theorem 8.4.3])**.** *Homotopy equivalent infra-solvmanifolds of type (R) are affinely diffeomorphic.*

1.1.7 (**The 4-Dimensional Geometries**)**.** The classification of the 4-dimensional geometries appears in [43], analogous to Thurston's eight 3-dimensional geometries [40, 35].

Recall that the 3-dimensional geometry Nil^3 is the group of upper triangular matrices

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix},$$

and the 3-dimensional geometry Sol^3 is the semidirect product $\mathbb{R}^2 \rtimes_{\phi(u)} \mathbb{R}$ where $\phi(u) = \begin{bmatrix} e^{-u} & 0 \\ 0 & e^{u} \end{bmatrix}$.

Notation 1.1.8. Following Hillman's convention [16, 17], the term *4-dimensional solvable geometry* will refer to the type (R), solvable, and unimodular groups $\mathbb{R}^4$, $\mathrm{Nil}^3 \times \mathbb{R}$, Nil^4, $\mathrm{Sol}^3 \times \mathbb{R}$, $\mathrm{Sol}^4_{m,n}$, Sol_0^4, and Sol_1^4.

In dimension 4, except for Sol_1^4, which splits as $\mathrm{Nil}^3 \rtimes \mathbb{R}$, all solvable geometries are of the form $\mathbb{R}^3 \rtimes_{\phi(u)} \mathbb{R}$ for $\phi : \mathbb{R} \to \mathrm{GL}(3, \mathbb{R})$:

$$
\begin{array}{llll}
\mathbb{R}^4: & \phi(u) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \mathrm{Nil}^3 \times \mathbb{R}: & \phi(u) = \begin{bmatrix} 1 & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\mathrm{Nil}^4: & \phi(u) = \begin{bmatrix} 1 & u & \frac{1}{2}u^2 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix} & \mathrm{Sol}^3 \times \mathbb{R}: & \phi(u) = \begin{bmatrix} e^{-u} & 0 & 0 \\ 0 & e^{u} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\mathrm{Sol}_0^4: & \phi(u) = \begin{bmatrix} e^{u} & 0 & 0 \\ 0 & e^{u} & 0 \\ 0 & 0 & e^{-2u} \end{bmatrix} & \mathrm{Sol}^4_{m,n}: & \phi(u) = \begin{bmatrix} e^{\theta u} & 0 & 0 \\ 0 & e^{u} & 0 \\ 0 & 0 & e^{-(1+\theta)u} \end{bmatrix} \\
\mathrm{Sol}_0^{4\prime}: & \phi(u) = \begin{bmatrix} e^{u} & ue^{u} & 0 \\ 0 & e^{u} & 0 \\ 0 & 0 & e^{-2u} \end{bmatrix}. & &
\end{array}
\tag{1.1}
$$

For $\mathrm{Sol}^4_{m,n}$, $\theta > 1$ is such that $\phi(u)$ is conjugate to an element of $\mathrm{SL}(3, \mathbb{Z})$. This guarantees that $\mathrm{Sol}^4_{m,n}$ has a lattice [23]. The characteristic polynomial of $\phi(u)$ is $x^3 - mx^2 + nx - 1$ for $m, n \in \mathbb{Z}$. It is known that $\mathrm{Sol}_0^{4\prime}$ has no compact forms [23] and therefore does not appear in the list of 4-dimensional geometries in [43].

Sol_1^4 can be described as the multiplicative group of matrices

$$\begin{bmatrix} 1 & e^u x & z \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix},$$

which splits as the semidirect product $\mathrm{Nil}^3 \rtimes \mathbb{R}$. It has 1-dimensional center ($u = x = y = 0$ in the above matrix). The quotient of Sol_1^4 by its center is Sol^3; we have the short exact sequence

$$1 \to \mathcal{Z}(\mathrm{Sol}_1^4) \to \mathrm{Sol}_1^4 \to \mathrm{Sol}^3 \to 1.$$

All 4-dimensional solvable geometries are type (R) and unimodular. Therefore, with a left invariant metric on G,

$$\mathrm{Isom}(G) \subseteq G \rtimes K \subset \mathrm{Aff}(G),$$

where K is a maximal compact subgroup of $\mathrm{Aut}(G)$ [13]. In fact, all of the 4-dimensional solvable geometries admit a left invariant metric so that

$$\mathrm{Isom}(G) = G \rtimes K.$$

The discrete cocompact subgroups $\varPi$ of $\mathrm{Isom}(G)$ acting freely on G yield the compact isometric quotients of G, denoted $\varPi\backslash G$.

All 4-dimensional solvable geometries except Sol_0^4 satisfy the generalized First Bieberbach Theorem [8]: if $\varPi \subset \mathrm{Isom}(G) \subseteq G \rtimes K$ ($G \neq \mathrm{Sol}_0^4$) is discrete and cocompact, then the translation subgroup $\Gamma = \varPi \cap G$ is a lattice of G and the quotient $\Phi = \varPi/\Gamma$ is a finite subgroup of $K \subset \mathrm{Aut}(G)$. Therefore, the compact isometric quotients of a 4-dimensional solvable geometry G, excluding Sol_0^4, are in fact infra-solvmanifolds of G. The compact isometric quotients of Sol_0^4 are known to be mapping tori of linear self diffeomorphisms of the 3-torus

T^3. However, they are not infra-solvmanifolds of Sol_0^4. Rather, they can be realized as infra-solvmanifolds of a different solvable 4-dimensional Lie group. However, this Lie group is not of type (R), and it also is not a 4-dimensional solvable geometry. In fact, Sol_0^4 does not even admit a lattice and therefore has no crystallographic groups ([17, Corollary 8.5.1] and [23, Theorem 3.5, Theorem 4.2]).

Conversely, Hillman has shown that every 4-dimensional infra-solvmanifold M is *geometric*. That is, M is diffeomorphic to a compact isometric quotient of one of the solvable 4-dimensional geometries equipped with left invariant metric [16, Theorem 8]. Thus, we have

Theorem 1.1.9 ([16, Theorem 8])**.** (1) *Any 4-dimensional infra-solvmanifold is diffeomorphic to a compact isometric quotient of a 4-dimensional solvable geometry.*

(2) *The compact isometric quotients of G, where G is one of the six 4-dimensional geometries* $\mathbb{R}^4$, $\mathrm{Nil}^3 \times \mathbb{R}$, Nil^4, $\mathrm{Sol}^3 \times \mathbb{R}$, $\mathrm{Sol}^4_{m,n}$, Sol_1^4, *are precisely the infra-solvmanifolds of G.*

1.2 Main Results

It is a remarkable theorem of Hamrick and Royster that for every closed flat n-manifold, M, there is an $(n+1)$-dimensional compact manifold W with $\partial W = M$ [15]. That is, M *bounds.* By the works of Gromov and Ruh, a closed manifold has an almost flat structure if and only if it is diffeomorphic to an infra-nilmanifold $\Pi\backslash G$ [12, 33]. Thus, it seems natural to ask if infra-nilmanifolds bound.

Conjecture 1.2.1 ([10, Conjecture 1])**.** *If M is an n-dimensional infra-nilmanifold, then there exists a compact $(n+1)$-dimensional manifold W with $\partial W = M$.*

Some partial results are known [41]. Hillman has asked if all 4-dimensional infra-solvmanifolds bound [18]. Dimension 4 is the first dimension of interest, as the only 2-dimensional infra-solvmanifolds are the torus and Klein bottle, and both are boundaries. Also, it is well known that all closed 3-dimensional manifolds bound. We shall answer Hillman's question affirmatively:

Theorem 2.4.5. *If M is a 4-dimensional infra-solvmanifold, then M bounds. That is, there is a compact 5-dimensional manifold W with $\partial W = M$.*

Classifying the crystallographic groups of 4-dimensional solvable geometries is important. The crystallographic groups of $\mathbb{R}^4$ have been classified; see for example [1], while Dekimpe has classified those of $\mathrm{Nil}^3 \times \mathbb{R}$ and Nil^4 [7]. The infra-solvmanifolds of $\mathrm{Sol}^4_{m,n}$ have been classified by Hillman [17]. This is equivalent to a classification of the torsion-free crystallographic groups of $\mathrm{Sol}^4_{m,n}$. Partial classifications of the infra-solvmanifolds of $\mathrm{Sol}^3 \times \mathbb{R}$ (torsion-free crystallographic groups of $\mathrm{Sol}^3 \times \mathbb{R}$) are presented in [5, 18]. Here we will complete the classification of the crystallographic groups of ${\mathrm{Sol}_1}^4$, $\mathrm{Sol}^3 \times \mathbb{R}$, and $\mathrm{Sol}^4_{m,n}$. Easily checked criteria for such groups to be torsion-free are also provided.

Theorem 3.3.21. *There are 14 families of* ${\mathrm{Sol}_1}^4$ *crystallographic groups. Let Π be a crystallographic group of* ${\mathrm{Sol}_1}^4$*. Any subgroup of D_4 can be the holonomy group of Π, where D_4 is the dihedral group of 8 elements. The maximal holonomy of a torsion-free* ${\mathrm{Sol}_1}^4$ *crystallographic group, so that $\Pi \backslash G$ is an infra-solvmanifold, is D_4.*

Theorems 3.4.18-3.4.31. *There are 60 families of* $\mathrm{Sol}^3 \times \mathbb{R}$ *crystallographic groups. Let* Π *be a crystallographic group of* $\mathrm{Sol}^3 \times \mathbb{R}$*. Any subgroup of* $D_4 \times \mathbb{Z}_2$ *can be the holonomy group of* Π*. If* Π *is torsion-free, so that* $\Pi \backslash G$ *is an infra-solvmanifold,* Π *must have holonomy* $\{e\}$, $\mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_4$, *or* D_4*. In particular, there is no infra-solvmanifold of* $\mathrm{Sol}^3 \times \mathbb{R}$ *with maximal holonomy* $D_4 \times \mathbb{Z}_2$*.*

Theorem 3.5.13. *There are 7 families of* $\mathrm{Sol}^4_{m,n}$ *crystallographic groups. Let* Π *be a crystallographic group of* $\mathrm{Sol}^4_{m,n}$*. The possible holonomy groups of* Π *are the subgroups of* $\mathbb{Z}_2 \times \mathbb{Z}_2$*. In particular, the maximal compact subgroup of* $\mathrm{Aut}(\mathrm{Sol}^4_{m,n})$, $\mathbb{Z}_2^3$, *cannot be the holonomy of a* $\mathrm{Sol}^4_{m,n}$ *crystallographic group. If* Π *is torsion-free, so that* $\Pi \backslash G$ *is an infra-solvmanifold,* Π *must have holonomy* $\{e\}$ *or* $\mathbb{Z}_2$*.*

We now consider curvature of the 4-dimensional solvable geometries. Let G be a 4-dimensional solvable geometry. With arbitrary left invariant metric on G,

$$\mathrm{Isom}(G) \subseteq G \rtimes K \subset \mathrm{Aff}(G),$$

for a maximal compact subgroup $K \subset \mathrm{Aut}(G)$ [13]. Depending on choice of left invariant metric, a 4-dimensional solvable geometry can have different isometry groups and Ricci signatures. Here are the possible Ricci signatures on some of the 4-dimensional solvable geometries [21]:

G	Possible Ricci signatures
$\mathbb{R}^4$	$(0,0,0,0)$
$\mathrm{Nil}^3 \times \mathbb{R}$	$(0,+,-,-)$
Nil^4	$(0,+,-,-), (+,+,-,-), (+,-,-,-)$
$\mathrm{Sol}^3 \times \mathbb{R}$	$(0,0,0,-), (0,+,-,-), (+,+,-,-), (+,-,-,-)$
$\mathrm{Sol}^4_{m,n}$	$(0,0,0,-), (0,+,-,-), (+,+,-,-), (+,-,-,-)$
Sol_0^4	$(0,0,0,-), (0,+,-,-)$

Table 1.1: Ricci signatures

Suppose we only consider left invariant metrics on G that induce a particular Ricci signature. The compact isometric quotients of G may not account for all infra-solvmanifolds of G, as $\mathrm{Isom}(G)$ may be a *proper* subgroup of $G \rtimes K$, for a maximal compact subgroup K of $\mathrm{Aut}(G)$. Given an infra-solvmanifold M of G, a Ricci signature can be *realized* on M if there is a left invariant metric on G with prescribed Ricci signature such that $M = \Pi\backslash G$, for some $\Pi \subset \mathrm{Isom}(G)$.

Theorems 4.3.5, 4.3.8, 4.3.11. (1) *If M is an infra-nilmanifold of* Nil^4*, then any of the three Ricci signatures* $(0,+,-,-)$*,* $(+,+,-,-)$*,* $(+,-,-,-)$ *can be realized on M.*

(2) *Every infra-solvmanifold of* $\mathrm{Sol}^4_{m,n}$ *is the mapping torus of a linear self diffeomorphism $\mathcal{S}$ of T^3; $\mathcal{S}$ has three distinct real eigenvalues. If all eigenvalues have the same sign, then any of* $(0,0,0,-)$*,* $(0,+,-,-)$*,* $(+,+,-,-)$*,* $(+,-,-,-)$ *can be realized on M. Else, only* $(0,0,0,-)$ *and* $(0,+,-,-)$ *can be realized on M.*

(3) *If M is an infra-solvmanifold of* $\mathrm{Sol}^3 \times \mathbb{R}$ *which has an order 4 element in its holonomy, then only* $(0,0,0,-)$ *can be realized on M.*

2 Cobordism Classes of 4-Dimensional Infra-Solvmanifolds

In this section, we shall prove that every compact 4-dimensional infra-solvmanifold bounds. That is, if M is a 4-dimensional infra-solvmanifold, then there is a compact 5-dimensional manifold W with $\partial W = M$. In other words, the corbordism class of a 4-dimensional infra-solvmanifold in the unoriented Thom cobordism group Ω_4^O is trivial. In light of Hillman's result that 4-dimensional infra-solvmanifolds are geometric (Proposition 1.1.9), it suffices to show that all compact isometric quotients of the solvable 4-dimensional geometries bound. Theorem 2.4.5 is established by exhibiting an involution (if not free) with 2-dimensional fixed set on infra-solvmanifolds with $\mathrm{Nil}^3 \times \mathbb{R}$, Nil^4, $\mathrm{Sol}^3 \times \mathbb{R}$, or Sol_1^4 geometry. We will show that the Stiefel-Whitney number $\omega_1^4(M)$ vanishes by a result of R.E. Stong [38], and from this, it will follow that all Stiefel-Whitney numbers vanish.

2.1 Translational Involutions

We construct an involution on an infra-solvmanifold $M = \mathit{\Pi}\backslash G$ when the center of G, $\mathcal{Z}(G)$, is non-trivial. The involution is induced by left translation. This technique was used to show closed flat n-manifolds bound [15, 11].

Lemma 2.1.1. *Let $M = \mathit{\Pi}\backslash G$ be an infra-solvmanifold with $\mathcal{Z}(G)$ non-trivial. Note $\Gamma \cap \mathcal{Z}(G)$ is a lattice of $\mathcal{Z}(G)$. Let t be a free generator of $\Gamma \cap \mathcal{Z}(G)$ and* *set $s = t^{\frac{1}{2}}$. Translation by s induces an involution on M if and only if $A(s) = s$ modulo $\Gamma \cap \mathcal{Z}(G)$, for all $A \in \Phi$. That is, translation by s commutes with the action of Φ on $\Gamma\backslash G$.*

Proof. Since s commutes with Γ, translation by s defines a free involution on the solvmanifold $\Gamma\backslash G$. To induce an involution on M, translation by s must normalize the action of Φ on $\Gamma\backslash G$. For any $(a, A) \in \Pi$, we have

$$\begin{aligned}(s, \mathrm{id})(a, A)(-s, \mathrm{id}) &= (s \cdot a \cdot As^{-1}, A)\\ &= ((I - A)s \cdot a, A) \text{ (since } s \in \mathcal{Z}(G)).\end{aligned}$$

Therefore, s induces an involution on M when $(I - A)s \in \Gamma \cap \mathcal{Z}(G)$; that is, $A(s) = s$ modulo $\Gamma \cap \mathcal{Z}(G)$, for all $A \in \Phi$. Note that $\mathcal{Z}(G) \cong \mathbb{R}^k$ for some k, and $s \in \mathcal{Z}(G)$. Note that A restricts to a linear map on $\mathcal{Z}(G)$, so that we may write $s \cdot a \cdot As^{-1} = (I - A)s$. □

2.2 Structure of Fixed Set

Let $\hat{M}$ denote the solvmanifold $\Gamma\backslash G$. We have the coverings

$$G \xrightarrow{q} \hat{M} \xrightarrow{p} M.$$

We refer to the involutions induced by translation by s as *translational involutions*. Let $\hat{i}_s : \hat{M} \to \hat{M}$ denote the induced involution on $\hat{M}$, $i_s : M \to M$ denote the induced involution on M, and F denote the fixed set of i_s on M.

Lemma 2.2.1. *The preimage of F in $\hat{M}$ is a finite disjoint union of closed, connected, submanifolds. We can write*

$$p^{-1}(F) = \bigcup_{\eta} E_{\eta},$$

where the union is over all possible injective homomorphisms $\eta : \langle \hat{i}_s \rangle \cong \mathbb{Z}_2 \to \Phi$ *and*

$$E_\eta = \{\hat{x} \in \hat{M} \mid s(\hat{x}) = \eta(\hat{i}_s)(\hat{x})\}.$$

Each E_η *is a finite disjoint union of components of* $p^{-1}(F)$*.*

Proof. The fixed set F of the translational involution must be a finite disjoint union of closed connected submanifolds [6, p.72]. Since p is a finite sheeted covering, $p^{-1}(F)$ also admits the structure of a finite disjoint union of closed connected submanifolds.

If $\hat{x} \in p^{-1}(F)$, then $s(\hat{x}) = (a, A)(\hat{x})$ for some unique $A \in \Phi$ where $(a, A) \in \mathit{\Pi}$. Thus,

$$\hat{x} = s^2(\hat{x}) = (a, A)^2(\hat{x}).$$

Since the deck transformation group acts freely, $(a, A)^2 \in \Gamma$, and thus $A^2 = I$. So $\eta(\hat{i}_s) = A$ defines an injective homomorphism $\eta : \mathbb{Z}_2 \to \Phi$. We warn the reader the action of $\eta(\hat{i}_s) = A \in \Phi$ on $\hat{M}$ is induced not just by the automorphism A, but rather by the affine transformation (a, A). The preimage of F in $\hat{M}$ is indexed by all possible injective homomorphisms $\eta : \mathbb{Z}_2 \to \Phi$. That is,

$$p^{-1}(F) = \bigcup_\eta E_\eta.$$

Note that $E_{\eta_1} = E_{\eta_2}$ when $\eta_1 = \eta_2$ and $E_{\eta_1} \cap E_{\eta_2} = \emptyset$ otherwise.

The actions of s and $\eta(\hat{i}_s)$ commute on $\hat{M}$ by Lemma 2.1.1. By definition, E_η is the fixed set of the involution $\eta(\hat{i}_s)^{-1} \circ \hat{i}_s = \eta(\hat{i}_s) \circ \hat{i}_s$ on $\hat{M}$. So it must be a finite disjoint union of closed connected submanifolds [6, p.72], and therefore must be a finite disjoint union of components of $p^{-1}(F)$. □

When $\exp : \mathfrak{g} \to G$ is a diffeomorphism, for $A \in \mathrm{Aut}(G)$, we have $\mathrm{Fix}(A) = \exp(\mathrm{Fix}(A_*))$, where A_* is the automorphism of $\mathfrak{g}$ induced from A. When G is a 4-dimensional solvable geometry, G is type (E) and exp is a diffeomorphism. So $\mathrm{Fix}(A)$ is always diffeomorphic to $\mathbb{R}^k$. We have the diagram of coverings, where the vertical arrows are inclusions.

$$\begin{array}{ccccc} G & \xrightarrow{q} & \hat{M} & \xrightarrow{p} & M \\ \uparrow & & \uparrow & & \uparrow \\ \bigcup_\eta q^{-1}(E_\eta) & \xrightarrow{q} & \bigcup_\eta E_\eta & \xrightarrow{p} & F. \end{array}$$

Now we analyze $q^{-1}(E_\eta)$.

Lemma 2.2.2. *Assume that* $\exp : \mathfrak{g} \to G$ *is a diffeomorphism. The preimage of* E_η *in* G *under* $q : G \to \hat{M}$ *is a disjoint union of submanifolds of* G*. In fact, if* $\eta(\hat{i}_s) = A \in \Phi$ *with* $(a, A) \in \mathit{\Pi}$*, then any component of* $q^{-1}(E_\eta)$ *is* $\mathrm{Fix}(\gamma s^{-1} a, A)$ *for some* $\gamma \in \Gamma$*. Consequently, the preimage of* E_η *in* G *is*

$$q^{-1}(E_\eta) = \bigcup_{\gamma \in \Gamma} \mathrm{Fix}(\gamma s^{-1} a, A).$$

Further, each $\mathrm{Fix}(\gamma s^{-1} a, A)$ *is a left translate of the connected subgroup* $\mathrm{Fix}(A)$ *of* G *and is diffeomorphic to* $\mathbb{R}^n$*, where* $n = \dim(\mathrm{Fix}(A)) = \dim(\mathrm{Fix}(A_*))$.

Proof. Because E_η is a disjoint union of closed submanifolds and q is a covering, $q^{-1}(E_\eta)$ is a (possibly not connected) submanifold of G without boundary.

Let $A = \eta(\hat{i}_s)$ and let $(a, A) \in \mathit{\Pi}$. An element $\tilde{x} \in G$ projects to $\hat{x} \in E_\eta$ if and only if there exists $\gamma \in \Gamma$ such that $s(\tilde{x}) = \gamma(a, A)(\tilde{x})$, or equivalently,

$$\tilde{x} = (\gamma s^{-1} a, A)(\tilde{x}).$$

That is, $\tilde{x}$ must be in the fixed set of the affine transformation $(\gamma s^{-1} a, A)$.

Consequently, the preimage of E_η in G is

$$q^{-1}(E_\eta) = \bigcup_{\gamma \in \Gamma} \mathrm{Fix}(\gamma s^{-1} a, A).$$

Some sets in the above union may be empty. The fixed set of an affine transformation, if non-empty, is just a translation of the fixed subgroup of its automorphism part; that is, if $x_0 \in \mathrm{Fix}(b, B)$, then

$$\mathrm{Fix}(b, B) = x_0 \mathrm{Fix}(B).$$

To see this, note that if $x_0 y \in x_0 \mathrm{Fix}(B)$ (so $y \in \mathrm{Fix}(B)$), then

$$\begin{aligned}(b, B)(x_0 y) &= bB(x_0)B(y) \\ &= (b, B)(x_0)y \\ &= x_0 y.\end{aligned}$$

For the other inclusion, note that if $z \in \mathrm{Fix}(b, B)$, then $z = x_0 x_0^{-1} z$. We claim that $x_0^{-1} z$ is in $\mathrm{Fix}(B)$:

$$\begin{aligned}B(x_0^{-1} z) &= B(x_0)^{-1} B(z) \\ &= B(x_0)^{-1} b^{-1} b B(z) \\ &= \left[(b, B)(x_0)\right]^{-1} (b, B)(z) \\ &= x_0^{-1} z.\end{aligned}$$

Any two left translates of $\mathrm{Fix}(A)$ are either disjoint or equal. Since exp is a diffeomorphism, any left translate of $\mathrm{Fix}(A)$ is a submanifold of G diffeomorphic to $\mathbb{R}^n$, where $n = \dim(\mathrm{Fix}(A))$. Since Γ is countable, $q^{-1}(E_\eta)$ is expressed as a countable union of submanifolds of G, each of which has dimension $\dim(\mathrm{Fix}(A))$. This forces each component of the submanifold $q^{-1}(E_\eta)$ to have dimension equal to that of $\mathrm{Fix}(A)$.

In fact, we claim a component $\tilde{E}_\eta$ of $q^{-1}(E_\eta)$ is equal to $\mathrm{Fix}(\gamma s^{-1}a, A)$ for some $\gamma \in \Gamma$. The argument above shows that $\tilde{x} \in \tilde{E}_\eta$ belongs to $\mathrm{Fix}(\gamma s^{-1}a, A)$ for some $\gamma \in \Gamma$. Since $\mathrm{Fix}(\gamma s^{-1}a, A)$ is connected,

$$\mathrm{Fix}(\gamma s^{-1}a, A) \subset \tilde{E}_\eta.$$

Also, $\mathrm{Fix}(\gamma s^{-1}a, A)$ is closed in $\tilde{E}_\eta$, since it is closed in G. Note that the inclusion $\mathrm{Fix}(\gamma s^{-1}a, A) \hookrightarrow \tilde{E}_\eta$ is open by invariance of domain, as both manifolds have the same dimension. Consequently, $\mathrm{Fix}(\gamma s^{-1}a, A) = \tilde{E}_\eta$. $\square$

An important consequence of Lemma 2.2.2 is that all components of F lifting to E_η must have the same dimension equal to that of $\mathrm{Fix}(\eta(\hat{i}_s))$, where $\eta(\hat{i}_s) \in \Phi$ is the unique automorphism of G coming from η.

Lemma 2.2.3. *Let $\mathit{\Pi}\backslash G$ be an infra-solvmanifold with translational involution induced by s. Suppose that $A \in \Phi$ has order 2 and $A(s) = s^{-1}$. Let $\eta : \mathbb{Z}_2 \to \Phi$ be the homomorphism $\eta(\hat{i}_s) = A$. Then $E_\eta = \emptyset$.*

Proof. Let $\alpha = (a, A) \in \mathit{\Pi}$ and define $\mathit{\Pi}' = \langle \Gamma, \alpha \rangle$. Note that $\mathit{\Pi}'\backslash G$ is an infra-solvmanifold with $\mathbb{Z}_2$ holonomy and translational involution i'_s induced by s.

We claim that the group generated by $\mathit{\Pi}'$ and s, $\langle \mathit{\Pi}', s \rangle$, is torsion-free. A general element of $\langle \mathit{\Pi}', s \rangle$ with holonomy A is of the form $(s\gamma a, A)$, where $\gamma \in \Gamma$. We have to check that $(s\gamma a, A)^2$ is not the identity, for any $\gamma \in \Gamma$. Now

$$\begin{aligned}(s\gamma a, A)^2 &= (s\gamma a A(s)A(\gamma)A(a), \mathrm{id})\\ &= (\gamma a A(\gamma a), \mathrm{id}), \quad (\text{since } s \in \mathcal{Z}(G))\\ &= (\gamma a, A)^2 \neq (e, \mathrm{id}),\end{aligned}$$

and this cannot be the identity because $(\gamma a, A) \in \mathit{\Pi}$ and $\mathit{\Pi}$ is torsion-free.

Consequently, $\langle \mathit{\Pi}', s\rangle$ is torsion-free and hence $\langle \mathit{\Pi}', s\rangle$ acts freely on G. Therefore s acts as a free involution on the infra-solvmanifold $\mathit{\Pi}'\backslash G$. Note that the preimage of $\mathrm{Fix}(i'_s) \subset \mathit{\Pi}'\backslash G$ in $\Gamma\backslash G$ under the double covering $\Gamma\backslash G \to \mathit{\Pi}'\backslash G$ is precisely E_η. Hence E_η must be empty. □

2.3 Dimension of Fixed Set

Given a 4-dimensional infra-solvmanifold $\mathit{\Pi}\backslash G$ with translational involution i_s induced by translation by s as defined in Lemma 2.1.1, the fixed set F will be a disjoint union of submanifolds. We will need to compute the dimension of $\mathrm{Fix}(i_s)$. By Lemma 2.2.2, a component of F lifts to G as a left translate of $\mathrm{Fix}(\eta(\hat{i}_s))$, where $\eta(\hat{i}_s)$ is an involution in $\mathrm{Aut}(G)$. Every involution in $\mathrm{Aut}(G)$ belongs to a maximal compact subgroup K of $\mathrm{Aut}(G)$. When G is one of the 4-dimensional solvable geometries, $\mathrm{Aut}(G)$ has finitely many components, as $\mathrm{Aut}(G)$ is algebraic. A result of Mostow [29, Theorem 3.1] implies that all maximal compact subgroups of $\mathrm{Aut}(G)$ are conjugate. Therefore, we can fix a maximal compact subgroup K and compute $\dim(\mathrm{Fix}(A))$ for each involution A in K.

Lemma 2.3.1 ([17, p.135])**.** (1) *A maximal compact subgroup of* $\mathrm{Aut}(\mathrm{Nil}^3 \times \mathbb{R})$ *is*

$$\mathrm{O}(2, \mathbb{R}) \times \mathbb{Z}_2.$$

(2) *If* $A \in \mathrm{O}(2, \mathbb{R}) \times \mathbb{Z}_2$ *restricts to the identity on* $\mathbb{R}$ *and has order 2, then*

$$\dim(\mathrm{Fix}(A)) = 2.$$

Proof. A maximal compact subgroup of $\mathrm{Aut}(\mathrm{Nil}^3)$ is $\mathrm{O}(2,\mathbb{R})$ and acts as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & (ax+by) & \frac{1}{2}(abx^2 + 2bcxy + cdy^2 + 2\det(A)z) \\ 0 & 1 & (cx+dy) \\ 0 & 0 & 1 \end{bmatrix}.$$

The induced action of $A \in \mathrm{O}(2,\mathbb{R})$ on $\mathcal{Z}(\mathrm{Nil}^3)$ is multiplication by $\det(A)$. Therefore, $\mathrm{Fix}(A)$ is 1-dimensional on Nil^3 for all $A \in \mathrm{O}(2,\mathbb{R})$. A maximal compact subgroup of $\mathrm{Aut}(\mathrm{Nil}^3 \times \mathbb{R})$ is $\mathrm{O}(2,\mathbb{R}) \times \mathbb{Z}_2$, where $\mathrm{O}(2,\mathbb{R}) \subset \mathrm{Aut}(\mathrm{Nil}^3)$ and $\mathbb{Z}_2$ acts as a reflection on $\mathbb{R}$. Thus, if $A \in \mathrm{O}(2,\mathbb{R}) \times \mathbb{Z}_2$ restricts to the identity on $\mathbb{R}$, then $\dim(\mathrm{Fix}(A)) = 2$. □

Lemma 2.3.2. (1) *A maximal compact subgroup of* $\mathrm{Aut}(\mathrm{Nil}^4)$ *is* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(2) *Let* $A \in \mathrm{Aut}(\mathrm{Nil}^4)$ *have order 2. If* A *restricts to the identity on* $\mathcal{Z}(\mathrm{Nil}^4)$*, then* $\dim(\mathrm{Fix}(A)) = 2$.

Proof. Recall the splitting of Nil^4 as the semidirect product $\mathbb{R}^3 \rtimes \mathbb{R}$. Letting $\mathfrak{g}$ denote the Lie algebra of Nil^4, we have $\mathfrak{g} \cong \mathbb{R}^3 \rtimes \mathbb{R}$, where $\mathbb{R}$ acts by the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

With standard bases $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of $\mathbb{R}^3$ and $\mathbf{e}_4$ of $\mathbb{R}$, $\mathfrak{g}$ has relations

$$[\mathbf{e}_4, \mathbf{e}_2] = \mathbf{e}_1, [\mathbf{e}_4, \mathbf{e}_3] = \mathbf{e}_2.$$

For $A \in \mathrm{Aut}(\mathfrak{g})$, A induces an action on the quotient $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \cong \langle \mathbf{e}_3, \mathbf{e}_4 \rangle$, denote this action by

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We will see that B determines the action of A on $\mathbf{e}_1$ and $\mathbf{e}_2$. The relation $[\mathbf{e}_4, \mathbf{e}_3] = \mathbf{e}_2$, implies $A(\mathbf{e}_2) = \det(B)\mathbf{e}_2$. Compactness forces $\det(B) = \pm 1$.

We also compute

$$\begin{aligned}A(\mathbf{e}_1) &= A([\mathbf{e}_4, \mathbf{e}_2]) = [A(\mathbf{e}_4), A(\mathbf{e}_2)] \\ &= [b\mathbf{e}_3 + d\mathbf{e}_4, \det(B)\mathbf{e}_2] = \det(B)d\mathbf{e}_1.\end{aligned}$$

Again, compactness implies $d = \pm 1$. Since $[\mathbf{e}_3, \mathbf{e}_2]$ vanishes, we have

$$\begin{aligned}0 &= A([\mathbf{e}_3, \mathbf{e}_2]) = [A(\mathbf{e}_3), A(\mathbf{e}_2)] \\ &= [a\mathbf{e}_3 + c\mathbf{e}_4, \det(B)\mathbf{e}_2] = \det(B)c\mathbf{e}_1.\end{aligned}$$

Thus c vanishes and B must be upper triangular of the form

$$B = \begin{bmatrix} \pm 1 & b \\ 0 & \pm 1 \end{bmatrix}.$$

By conjugation we can set $b = 0$. Thus, a maximal compact subgroup of $\mathrm{Aut}(\mathrm{Nil}^4)$ cannot be larger than $\mathbb{Z}_2 \times \mathbb{Z}_2$. Conversely, we see that

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(\pm I_3, 1), (\pm J, -1)\} \subset \mathrm{Aut}(\mathbb{R}^3 \rtimes \mathbb{R}), \tag{2.1}$$

where

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

defines a subgroup of $\mathrm{Aut}(\mathrm{Nil}^4)$. It now follows that a maximal compact subgroup of $\mathrm{Aut}(\mathrm{Nil}^4)$ is $\{(\pm I_3, 1), (\pm J, -1)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Note that $(-I_3, 1)$ and $(-J, -1)$ do not act as the identity on $\mathcal{Z}(\mathrm{Nil}^4)$. The remaining involution $(J, -1)$ restricts to the identity on $\mathcal{Z}(\mathrm{Nil}^4)$ and has 2-dimensional fixed subgroup. □

Lemma 2.3.3. (1) *A maximal compact subgroup of* $\mathrm{Aut}(\mathrm{Sol}_1^4)$ *is* D_4.

(2) *Let* $A \in \mathrm{Aut}(\mathrm{Sol}_1^4)$ *have order 2. If* A *restricts to the identity on* $\mathcal{Z}(\mathrm{Sol}_1^4)$*, then*

$$\dim(\mathrm{Fix}(A)) = 2.$$

Proof. Both $\mathrm{Aut}(\mathrm{Sol}^3)$ and $\mathrm{Aut}(\mathrm{Sol}_1^4)$ have D_4 as their maximal compact subgroup (Propositions 3.1.4 and 3.1.8),

$$D_4 = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle = \mathbb{Z}_4 \rtimes \mathbb{Z}_2.$$

For $A \in D_4$, let $\bar{A}$ be $+1$ if A is diagonal, and -1 if A is off-diagonal. Then A acts on $\mathrm{Sol}^3 = \mathbb{R}^2 \rtimes \mathbb{R}$ as

$$A : \left(\begin{bmatrix} x \\ y \end{bmatrix}, u \right) \longmapsto \left(A \begin{bmatrix} x \\ y \end{bmatrix}, \bar{A}u \right),$$

and on Sol_1^4 as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} 1 & e^u x & z \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & e^{\bar{A}u}(ax+by) & \frac{1}{2}(abx^2 + 2bcxy + cdy^2 + 2\det(A)z) \\ 0 & e^{\bar{A}u} & (cx+dy) \\ 0 & 0 & 1 \end{bmatrix}.$$

For both Sol^3 and Sol_1^4, $\bar{A}$ is the induced action of A on $\mathrm{Sol}^3/\mathbb{R}^2 \cong \mathrm{Sol}_1^4/\mathrm{Nil} \cong \mathbb{R}$. Note that multiplication by $\det(A)$ is the induced action of A on $\mathcal{Z}(\mathrm{Sol}_1^4)$. Thus, the only involution in D_4 restricting to the identity on $\mathcal{Z}(\mathrm{Sol}_1^4)$ is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and evidently it has 2-dimensional fixed subgroup. $\square$

Lemma 2.3.4. (1) *A maximal compact subgroup of* $\mathrm{Aut}(\mathrm{Sol}^3 \times \mathbb{R})$ *is*

$$D_4 \times \mathbb{Z}_2,$$

where $D_4 \subset \mathrm{Aut}(\mathrm{Sol}^3)$ *and* $\mathbb{Z}_2$ *acts as a reflection on* $\mathbb{R}$.

(2) *Let* $A \in D_4 \subset D_4 \times \mathbb{Z}_2 \subset \mathrm{Aut}(\mathrm{Sol}^3 \times \mathbb{R})$ *have order 2.*
If $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ *or* $\pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, *then*

$$\dim(\mathrm{Fix}(A)) = 2.$$

If $A = \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, *then*

$$\dim(\mathrm{Fix}(A)) = 3.$$

Proof. The first statement is clear since a maximal compact subgroup of $\mathrm{Aut}(\mathrm{Sol}^3)$ is D_4. For the second statement, recall that the induced action of A on the quotient $\mathrm{Sol}^3/\mathbb{R}^2 \cong \mathbb{R}$ is $+1$ if A is diagonal and -1 otherwise. □

2.4 Proof of Bounding

The following relations among Stiefel-Whitney classes for 4-manifolds are known.

Lemma 2.4.1 ([37])**.** *For any 4-manifold* M*,*

(1) $\omega_1^2\omega_2 = \omega_1\omega_3 = 0$

(2) $\omega_2^2 = \omega_1^4 + \omega_4$

Therefore, M *is a boundary if and only if the Stiefel-Whitney numbers* $\omega_1^4(M)$ *and* $\omega_4(M)$ *are* 0*.*

A solvmanifold $\Gamma\backslash G$ is parallelizable since one can project a framing of left invariant vector fields from G to $\Gamma\backslash G$. Hence the Euler characteristic $\chi(\Gamma\backslash G)$ vanishes. Since any infra-solvmanifold $\Pi\backslash G$ is finitely covered by a solvmanifold, $\chi(\Pi\backslash G) = 0$. Therefore, the mod 2 Euler characteristic, which is $\omega_4(M)$, vanishes. Hence the only Stiefel Whitney consider is $\omega_1^4(M)$.

Proposition 2.4.2 ([38, Proposition 9.2])**.** *A manifold M^n is unoriented cobordant to a manifold M' with differentiable involution having a fixed set of dimension $n-2$ if and only if $\omega_1^n(M) = 0$.*

We will use this result to show that $\omega_1^4(M) = 0$ for a 4-dimensional infra-solvmanifold, by constructing an involution with 2-dimensional fixed set. It is well known that any closed manifold with free involution bounds. So, henceforth we assume our involution to have non-empty fixed set.

We will also need the following result on the crystallographic groups of $\mathrm{Sol}^3 = \mathbb{R}^2 \rtimes \mathbb{R}$. Note that the nil-radical of Sol^3 is $\mathbb{R}^2$. Let $\Pi \subset \mathrm{Isom}(\mathrm{Sol}^3) = \mathrm{Sol}^3 \rtimes D_4$ be a crystallographic group with lattice Γ and holonomy Φ. Recall the action of D_4 as automorphisms of Sol^3 from Lemma 2.3.3. Let

$$\mathrm{pr}_1 : \mathrm{Sol}^3 \to \mathrm{Sol}^3/\mathbb{R}^2 \cong \mathbb{R}.$$

denote the quotient map. If Γ is a lattice of Sol^3, then Γ meets the nil-radical in a lattice $\Gamma \cap \mathbb{R}^2 \cong \mathbb{Z}^2$ and $\mathrm{pr}_1(\Gamma) \cong \mathbb{Z}$ is a lattice of $\mathbb{R}$ [32, Corollary 8.28].

Proposition 2.4.3 (Lemma 3.2.10)**.** *Let $\Pi \subset \mathrm{Isom}(\mathrm{Sol}^3) = \mathrm{Sol}^3 \rtimes D_4$ be crystallographic and let v denote a generator of $\mathrm{pr}_1(\Gamma) \cong \mathbb{Z}$.*
If $(b, B) \in \Pi$ where $B = \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \subset D_4$, then $\mathrm{pr}_1(b) = v^{\frac{1}{2}}$.

For $\mathrm{Nil}^3 \times \mathbb{R}$ geometry manifolds, we need to study the holonomy representation

$$\rho : \Phi \to \mathrm{Aut}(\mathcal{Z}(\mathrm{Nil}^3 \times \mathbb{R}))$$

closely.

Lemma 2.4.4. *Let $M = \Pi \backslash G$ with $G = \mathrm{Nil}^3 \times \mathbb{R}$. Then there is a set of generators t_1, t_2 for $\Gamma \cap \mathcal{Z}(G) \cong \mathbb{Z}^2$, $t_1 \in \mathcal{Z}(\mathrm{Nil}^3)$, so that with respect to the* *basis* t_1, t_2,

(1) $\rho(\Phi) \subset \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$, or

(2) $\rho(\Phi) \subset \left\langle \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$.

Proof. We have $\mathcal{Z}(G) = \mathcal{Z}(\mathrm{Nil}^3) \times \mathbb{R}$. Since

$$[G, G] = \mathcal{Z}(\mathrm{Nil}^3) = \mathbb{R},$$

$\mathcal{Z}(\mathrm{Nil}^3)$ is invariant under any automorphism of G. Further, $\Gamma \cap \mathcal{Z}(\mathrm{Nil}^3)$ is a lattice of $\mathcal{Z}(\mathrm{Nil}^3)$ [7, Lemma 1.2.5].

Note that $\rho(\Phi)$ has $\mathcal{Z}(\mathrm{Nil}^3)$ as an invariant subspace. Because $\rho(\Phi)$ can be conjugated (over $\mathrm{GL}(2, \mathbb{R})$) into $\mathrm{O}(\mathcal{Z}(G))$, we can assume that it leaves the orthogonal complement of $\mathcal{Z}(\mathrm{Nil}^3)$ invariant as well. The maximal compact subgroup of $\mathrm{O}(\mathcal{Z}(G))$ leaving $\mathcal{Z}(\mathrm{Nil}^3)$ invariant is

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle.$$

Thus, over $\mathrm{GL}(2, \mathbb{R})$, $\rho(\Phi)$ can be conjugated into this $\mathbb{Z}_2 \times \mathbb{Z}_2$. But over $\mathrm{GL}(2, \mathbb{Z})$, there is one more case.

Let t_1, t_2 be two generators of $\mathit{\Pi} \cap \mathcal{Z}(G) \cong \mathbb{Z}^2$, where t_1 generates $\mathit{\Pi} \cap \mathcal{Z}(\mathrm{Nil}^3)$. It is known that an involution $A \in \mathrm{GL}(2, \mathbb{Z})$ with vanishing trace is $\mathrm{GL}(2, \mathbb{Z})$ conjugate to either $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ [4]. Using this, it is not hard to see that we can keep t_1 the same, but change t_2 to $t_2' = at_1 \pm t_2$ for $a \in \mathbb{Z}$, to put $\rho(\Phi)$ in the desired form. □

We are now ready to prove the main theorem.

Theorem 2.4.5. *All 4-dimensional infra-solvmanifolds are boundaries.*

Proof. The flat 4-dimensional manifolds $M = \Pi\backslash\mathbb{R}^4$ are all boundaries by Hamrick-Royster [15].

When $G = \mathrm{Sol}^4_{m,n}$ or Sol_0^4, any compact form $M = \Pi\backslash G$ is a mapping torus of T^3, and is therefore a T^3 bundle over S^1 ([17, Corollary 8.5.1] and [23, Theorem 3.5, Theorem 4.2]). Because T^3 is orientable, $\omega_1(M)$ is induced from the base of the fibration. That is, let p denote the projection $p : M \to S^1$. Now $\omega_1(M) = p^*(c)$ for some class $c \in H^1(S^1; \mathbb{Z}_2) \cong \mathbb{Z}_2$. To see that ω_1 is induced from a class in the base of the fibration, note that

$$H^1(M; \mathbb{Z}_2) \cong \hom(H_1(M; \mathbb{Z}), \mathbb{Z}_2) \cong \hom(\pi_1(M), \mathbb{Z}_2).$$

It is known that $\omega_1(M)$, under the above isomorphism, is the cohomology class which assigns 1 to an element of $\gamma \in \pi_1(M)$ if the tangent bundle is non-orientable when restricted to γ and 0 if the tangent bundle of M is orientable. Since the fiber of M is orientable, the $\omega_1(M)$ must come from the base. Then:

$$\omega_1(M)^4 = (p^*(c))^4 = p^*(c^4) = p^*(0) = 0.$$

Of course, $\omega_4(M)$ vanishes as well.

Case $G = \mathrm{Nil}^4, \mathrm{Sol}_1^4$:

For $M = \Pi\backslash G$ when G is Nil^4 or Sol_1^4, let $s = t^{\frac{1}{2}}$ where t is a generator of $\Gamma \cap \mathcal{Z}(G)$. We consider the translational involution defined in Lemma 2.1.1. Our explicit computation of maximal compact subgroups of $\mathrm{Aut}(\mathrm{Nil}^4)$ and $\mathrm{Aut}(\mathrm{Sol}_1^4)$ shows that $A(s) = \pm s$ for any holonomy $A \in \Phi$ (recall that s is central). This also follows since $\mathcal{Z}(G)$ is invariant under any automorphism and 1-dimensional. Let $\eta : \mathbb{Z}_2 = \langle s \rangle \to \Phi$ be an injective homomorphism. If $\eta(s)(s) = -s$, then $E_\eta = \emptyset$ by Lemma 2.2.3. If $\eta(s)(s) = s$, then $\eta(s)$

acts as the identity on $\mathcal{Z}(G)$ and $\eta(s)$ has 2-dimensional fixed subgroup on G (Lemmas 2.3.2 and 2.3.3). By Lemma 2.2.2, E_η is 2-dimensional. Therefore, $\mathrm{Fix}(s)$ is 2-dimensional.

Case $G = \mathrm{Sol}^3 \times \mathbb{R}$:

Now consider a $\mathrm{Sol}^3 \times \mathbb{R}$ geometry manifold $\Pi\backslash G$. Let $s = t^{\frac{1}{2}}$ where t is a generator of $\Gamma \cap \mathcal{Z}(G)$. Since $A(s) = \pm s$ for all $A \in \Phi$, s defines an involution on $\Pi\backslash G$. Let $\eta : \mathbb{Z}_2 = \langle s \rangle \to \Phi$ be an injective homomorphism. If $\eta(s)(s) = -s$, then $E_\eta = \emptyset$ by Lemma 2.2.3.

In the $\mathrm{Sol}^3 \times \mathbb{R}$ geometry case, not all involutions in Φ inducing the identity on $\mathcal{Z}(\mathrm{Sol}^3 \times \mathbb{R})$ have 2-dimensional fixed subgroup (Lemma 2.3.4). When $\eta(s) = A$, where A is one of $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ or $\pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, E_η is 2-dimensional, since $\mathrm{Fix}(A)$ is 2-dimensional. But for

$$B = \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$\mathrm{Fix}(B)$ is 3-dimensional. However, we shall see that when $\eta(s) = B$, E_η is empty. Note that the nil-radical of $\mathrm{Sol}^3 \times \mathbb{R}$ is $\mathbb{R}^3$ with quotient $\mathbb{R}$. Let

$$\mathrm{pr} : \mathrm{Sol}^3 \times \mathbb{R} \to \mathbb{R}$$

denote the quotient homomorphism by the nil-radical of $\mathrm{Sol}^3 \times \mathbb{R}$. If we let $\mathrm{pr}_2 : \mathrm{Sol}^3 \times \mathbb{R} \to \mathrm{Sol}^3$ denote the quotient of $\mathrm{Sol}^3 \times \mathbb{R}$ by its center and let $\mathrm{pr}_1 : \mathrm{Sol}^3 \to \mathbb{R}$ denote the quotient of Sol^3 by its nil-radical, then pr factors as $\mathrm{pr}_1 \circ \mathrm{pr}_2$,

$$\mathrm{pr} : \mathrm{Sol}^3 \times \mathbb{R} \xrightarrow{\mathrm{pr}_2 : /\mathcal{Z}(\mathrm{Sol}^3 \times \mathbb{R})} \mathrm{Sol}^3 \xrightarrow{\mathrm{pr}_1 : /\mathbb{R}^2} \mathbb{R}.$$

Now $\mathrm{pr}(\Gamma)$ is a lattice of $\mathbb{R}$. Let v denote a generator of $\mathrm{pr}(\Gamma)$. By Lemma

2.2.2, the preimage of E_η in $\mathrm{Sol}^3 \times \mathbb{R}$ is given by,

$$\bigcup_{\gamma\in\Gamma} \mathrm{Fix}(\gamma s^{-1}b, B) \text{ for } (b, B) \in \Pi.$$

However, all sets $\mathrm{Fix}(\gamma s^{-1}b, B)$ are empty for any $\gamma \in \Gamma$. To see this, suppose $x \in \mathrm{Sol}^3 \times \mathbb{R}$ satisfies

$$\gamma s^{-1}bB(x) = x.$$

We will apply $\mathrm{pr} = \mathrm{pr}_1 \circ \mathrm{pr}_2$ to both sides. Note that $\mathrm{pr}(b) = v^{\frac{1}{2}}$ by Proposition 2.4.3, $\mathrm{pr}(\gamma) = v^n$ for some $n \in \mathbb{Z}$, $\mathrm{pr}(s) = 0$, and

$$\mathrm{pr}(B(x)) = \bar{B}(\mathrm{pr}(x)) = \mathrm{pr}(x)$$

(since B is diagonal, $\bar{B} = +1$). Thus, application of pr yields

$$v^{n+\frac{1}{2}} + \mathrm{pr}(x) = \mathrm{pr}(x),$$

which is a contradiction. This shows that E_η is empty when $\eta(s) = B$.

Therefore, $\mathrm{Fix}(s)$ has no 3-dimensional components and is 2-dimensional in the $\mathrm{Sol}^3 \times \mathbb{R}$ geometry case.

Case $G = \mathrm{Nil}^3 \times \mathbb{R}$:

Finally, consider a $\mathrm{Nil}^3 \times \mathbb{R}$ geometry manifold $\Pi\backslash G$. Now $\mathcal{Z}(\mathrm{Nil}^3 \times \mathbb{R}) = \mathcal{Z}(\mathrm{Nil}^3) \times \mathbb{R}$. We will use either $\mathcal{Z}(\mathrm{Nil}^3)$ or $\mathbb{R}$ to induce an involution on $\Pi\backslash G$ depending on which case of Lemma 2.4.4 occurs.

Suppose we can take t_1, t_2 with $t_1 \in \mathcal{Z}(\mathrm{Nil}^3)$, as a generating set of $\Gamma \cap \mathcal{Z}(G)$ so that $\rho(\Phi)$ is diagonal for this generating set (case (1) of Lemma 2.4.4). Take $s = t_2^{\frac{1}{2}}$ for our involution on M. Lemma 2.2.3 implies that E_η is non-empty only when $\eta(s)(s) = s$. This conditions means that $\eta(s)$ fixes the $\mathbb{R}$ factor in $\mathcal{Z}(\mathrm{Nil}^3 \times \mathbb{R}) = \mathcal{Z}(\mathrm{Nil}^3) \times \mathbb{R}$. But all such $\eta(s)$ fixing $\mathbb{R}$ must have a 2-

dimensional fixed set on $\mathrm{Nil}^3 \times \mathbb{R}$ by Lemma 2.3.1.

Now suppose case (2) of Lemma 2.4.4 occurs. This time, we must take $s = t_1^{\frac{1}{2}}$ for our involution. If $\eta(s)$ acts as a reflection on $\mathcal{Z}(\mathrm{Nil}^3)$, Lemma 2.2.3 implies E_η is empty.

We claim further that E_η is empty when $\eta(s) \in \Phi$ with $\rho(\eta(s)) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. To see this, note that translation by $s = t_1^{\frac{1}{2}}$ is also induced from translation by $s' = t_1^{\frac{1}{2}} t_2^{-1}$. But $\eta(s)(s') = s'^{-1}$. Hence Lemma 2.2.3 implies that E_η is empty in this case.

Thus we conclude that the only non-empty components of $\mathrm{Fix}(s)$ can arise from $\eta(s) \in \Phi$ with $\rho(\eta(s)) = \mathrm{id}$. But Lemma 2.3.1 implies that all such $\eta(s)$ have 2-dimensional fixed set. Therefore, when $G = \mathrm{Nil}^3 \times \mathbb{R}$, in either case of Lemma 2.4.4, we have an involution with fixed set of constant dimension 2.

For any manifold with $\mathrm{Nil}^3 \times \mathbb{R}, \mathrm{Nil}^4, \mathrm{Sol}_1^4$, or $\mathrm{Sol}^3 \times \mathbb{R}$ geometry, we have constructed an involution with 2-dimensional fixed set. By Stong's result (Proposition 2.4.2), $\omega_1^4(M) = 0$. Thus, all Stiefel-Whitney numbers are zero and we have established that all 4-dimensional infra-solvmanifolds bound. $\square$

3 Crystallographic Groups of the 4-Dimensional Solvable Geometries

In the sections below, we compute $\mathrm{Aut}(G)$, and the maximal compact subgroup of $\mathrm{Aut}(G)$, where G is one of Sol^3, $\mathrm{Sol}^3 \times \mathbb{R}$, $\mathrm{Sol}^4_{m,n}$, $\mathrm{Sol}_0^{\,4}$, and $\mathrm{Sol}_1^{\,4}$.

3.1 Computation of $\mathrm{Aut}(G)$

To understand $\mathrm{Aut}(G)$ when G is one of $\mathrm{Sol}^3 = \mathbb{R}^2 \rtimes \mathbb{R}$, $\mathrm{Sol}^4_{m,n} = \mathbb{R}^3 \rtimes \mathbb{R}$, or $\mathrm{Sol}_0^{\,4} = \mathbb{R}^3 \rtimes \mathbb{R}$, first note that Sol^3 has nilradical $\mathbb{R}^2$, and both $\mathrm{Sol}^4_{m,n}$ and $\mathrm{Sol}_0^{\,4}$ have nilradical $\mathbb{R}^3$. The nilradical is invariant under automorphisms, and we will apply the following lemmas.

Lemma 3.1.1. *Let $G = \mathbb{R}^n \rtimes_\phi \mathbb{R}$ and suppose that $\mathbb{R}^n$ is invariant under all automorphisms of G. Then $\alpha \in \mathrm{Aut}(G)$ restricts to an automorphism A of $\mathbb{R}^n$, and hence, induces an automorphism $\bar{A}$ of the quotient $\mathbb{R}$. There is a homomorphism*

$$\begin{array}{ccc} \mathrm{Aut}(\mathbb{R}^n \rtimes \mathbb{R}) & \longrightarrow & \mathrm{Aut}(\mathbb{R}^n) \times \mathrm{Aut}(\mathbb{R}) \\ \alpha & \longrightarrow & (A, \bar{A}) \end{array}$$

Furthermore, A and $\bar{A}$ must satisfy

$$\phi(\bar{A}u) = A\phi(u)A^{-1} \tag{3.1}$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}^n$ be the map (not a homomorphism in general) defined by $\alpha(\mathbf{0}, u) = (f(u), \bar{A}(u)) \in \mathbb{R}^n \rtimes_\phi \mathbb{R}$. Then $\alpha(\mathbf{x}, u) = (A\mathbf{x} + f(u), \bar{A}(u))$. The equality $\alpha\left((\mathbf{x}, u)(\mathbf{y}, v)\right) = \alpha(\mathbf{x}, u)\alpha(\mathbf{y}, v)$ yields

$$f(u) + \phi(\bar{A}u)(A\mathbf{y} + f(v)) = A(\phi(u)(\mathbf{y})) + f(u+v).$$

For this to hold for every $\mathbf{x}, \mathbf{y}, u$, and v, we must have $f(0) = \mathbf{0}$, which yields the desired result. □

Next, we calculate the kernel of the map $\mathrm{Aut}(\mathbb{R}^n \rtimes_\phi \mathbb{R}) \longrightarrow \mathrm{Aut}(\mathbb{R}^n) \times \mathrm{Aut}(\mathbb{R})$.

Lemma 3.1.2. *The kernel of the map* $\mathrm{Aut}(\mathbb{R}^n \rtimes_\phi \mathbb{R}) \longrightarrow \mathrm{Aut}(\mathbb{R}^n) \times \mathrm{Aut}(\mathbb{R})$ *can be identified with the group of crossed homomorphisms* $Z^1(\mathbb{R}, \mathbb{R}^n)$. *The crossed homomorphism* $\eta \in Z^1(\mathbb{R}, \mathbb{R}^n)$ *action is given by*

$$f(\mathbf{x}, u) = (\mathbf{x} + \eta(u), u),$$

and satisfies the cocycle condition

$$\eta(u+v) = \eta(u) + \phi(u)(\eta(v)). \tag{3.2}$$

Proof. Let $f : \mathbb{R}^n \rtimes_\phi \mathbb{R} \to \mathbb{R}^n \rtimes_\phi \mathbb{R}$ be an automorphism which induces the identities on $\mathbb{R}^n$ and $\mathbb{R}$. Then $f(\mathbf{x}, u) = (\mathbf{x} + \eta(u), u)$ for a map $\eta : \mathbb{R} \to \mathbb{R}^n$. We compute

$$\begin{aligned} f(\mathbf{x}, u) f(\mathbf{y}, v) &= (\mathbf{x} + \eta(u) + \phi(u)(\mathbf{y}) + \phi(u)(\eta(v)), u+v), \\ f(\mathbf{x} + \phi(u)(\mathbf{y}), u+v) &= (\mathbf{x} + \phi(u)(\mathbf{y}) + \eta(u+v), u+v), \end{aligned}$$

which yields the desired conclusion. □

Remark 3.1.3. When $H^1(\mathbb{R}, \mathbb{R}^n) = 0$, every crossed homomorphism is principal. This occurs when $\det(I - \phi(u)) \neq 0$. That is, for $\eta \in Z^1(\mathbb{R}, \mathbb{R}^n)$, there exists $\mathbf{x}_0 \in \mathbb{R}^n$ such that $\eta(u) = \delta\mathbf{x}_0 = (I - \phi(u))\mathbf{x}_0$. So when $H^1(\mathbb{R}, \mathbb{R}^n) = 0$, the correspondence $\eta \to \mathbf{x}_0$ gives an isomorphism between the group of crossed homomorphisms and $\mathbb{R}^n$.

Proposition 3.1.4. *We have* $\mathrm{Aut}(\mathrm{Sol}^3) \cong \mathrm{Sol}^3 \rtimes (\mathbb{R}^+ \times D_4)$*, where* D_4 *is the dihedral group with 8 elements. Under this isomorphism,* Sol^3 *acts as inner automorphisms, and* $(\mathbb{R}^+ \times D_4)$ *is identified with the group of matrices* $\mathbb{R}^+ \times D_4 = \left\langle k\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, k\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$*,* $k > 0$*, (*$k = 1$ *yields* D_4*),* $A \in \mathbb{R}^+ \times D_4$ *acts on* Sol^3 *as*

$$A : \left(\begin{bmatrix} x \\ y \end{bmatrix}, u \right) \longmapsto \left(A\begin{bmatrix} x \\ y \end{bmatrix}, \bar{A}u \right).$$

($\bar{A} = +1$ *if* A *is diagonal,* $\bar{A} = -1$ *otherwise.)*

Proof. Any automorphism α restricts to an automorphism of the nilradical $\mathbb{R}^2$ of Sol^3, so Lemma 3.1.1 applies. Since eigenvalues are invariant under conjugation, $\phi(\bar{A}u)$ and $A\phi(u)A^{-1}$ must have the same eigenvalues. Hence $\bar{A} = \pm 1$ is forced. That is, we have

$$\bar{A}(u) = u \quad \text{or} \quad \bar{A}(u) = -u.$$

If $\bar{A}(u) = u$ $\Big(\bar{A}(u) = -u$, respectively$\Big)$, then A must be of the form

$$A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{R}) \left(A = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \in \mathrm{GL}(2, \mathbb{R}), \text{ respectively} \right).$$

Conversely, for any matrix A,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \left(\begin{bmatrix} x \\ y \end{bmatrix}, u \right) \longmapsto \left(A\begin{bmatrix} x \\ y \end{bmatrix}, \bar{A}u \right),$$

where either $a = d = 0$ or $b = c = 0$, defines an automorphism of Sol^3.

Thus the image of $\mathrm{Aut}(\mathrm{Sol}^3) \to \mathrm{GL}(2,\mathbb{R})$ is

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} : a,b,c,d \in \mathbb{R}^* \right\},$$

and this group is isomorphic to $\mathbb{R}^+ \times (\mathbb{R} \rtimes D_4)$, where $\mathbb{R}^+$ is the scalar matrices with positive entries, and

$$\mathbb{R} = \left\{ E^c = \begin{bmatrix} e^{-c} & 0 \\ 0 & e^c \end{bmatrix} : \ c \in \mathbb{R} \right\} \quad \text{and} \quad D_4 = \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \right\}.$$

By Lemma 3.1.2, we obtain a splitting

$$\mathrm{Aut}(\mathrm{Sol}^3) \cong Z^1_\phi(\mathbb{R},\mathbb{R}^2) \rtimes (\mathbb{R}^+ \times (\mathbb{R} \rtimes D_4)).$$

But on Sol^3, every crossed homomorphism is principal, so that $Z^1_\phi(\mathbb{R},\mathbb{R}^2) \cong \mathbb{R}^2$ as described in Remark 3.1.3. Thus we have,

$$\begin{aligned} \mathrm{Aut}(\mathrm{Sol}^3) &\cong \mathbb{R}^2 \rtimes (\mathbb{R}^+ \times (\mathbb{R} \rtimes D_4)) \cong (\mathbb{R}^2 \rtimes \mathbb{R}) \rtimes (\mathbb{R}^+ \times D_4) \\ &\cong \mathrm{Sol}^3 \rtimes (\mathbb{R}^+ \times D_4). \end{aligned}$$

For the last isomorphism, we observe that the action of $\mathbb{R}^2 \rtimes \mathbb{R}$ on Sol^3 is by inner automorphisms. Since Sol^3 has trivial center, $\mathrm{Inn}(\mathrm{Sol}^3) \cong \mathrm{Sol}^3$. □

The computation of $\mathrm{Aut}(\mathrm{Sol}^4_{m,n})$ and $\mathrm{Aut}(\mathrm{Sol}^4_0)$ proceeds in an identical fashion.

Proposition 3.1.5. *We have* $\mathrm{Aut}(\mathrm{Sol}^4_{m,n}) \cong \mathbb{R}^3 \rtimes (\mathbb{R}^*)^3$. *Here* $(\mathbb{R}^*)^3$ *is identified with the group of diagonal matrices in* $\mathrm{GL}(3,\mathbb{R})$ *and acts on the* $\mathbb{R}^3$ *factor linearly in the semi-direct product. For* $(\mathbf{x}_0, A) \in \mathbb{R}^3 \rtimes (\mathbb{R}^*)^3$, $(\mathbf{x}_0, A)$ *acts on* $\mathrm{Sol}^4_{m,n}$ *as follows: for* $(\mathbf{x}, u) \in \mathrm{Sol}^4_{m,n} = \mathbb{R}^3 \rtimes_\phi \mathbb{R}$,

$$(\mathbf{x}, u) \to (A(\mathbf{x}) + (I - \phi(u))\mathbf{x}_0, u).$$

Proof. Any automorphism α restricts to an automorphism of the nilradical $\mathbb{R}^3$ of $\mathrm{Sol}^4_{m,n}$, so Lemma 3.1.1 applies. Since eigenvalues are invariant under conjugation, $\phi(\bar{A}u)$ and $A\phi(u)A^{-1}$ must have the same eigenvalues. Hence $\bar{A} = +1$ is forced. Thus $\phi(u)$ and A commute. Computation shows that A must be diagonal. Thus the image of $\mathrm{Aut}(\mathrm{Sol}^4_{m,n})$ in $\mathrm{Aut}(\mathbb{R}^3)$ consists only of diagonal matrices. Conversely, when A is diagonal, $(\mathbf{x}, u) \to (A\mathbf{x}, u)$ defines an automorphism of $\mathrm{Sol}^4_{m,n}$. So the group of invertible diagonal matrices $(\mathbb{R}^*)^3$ lifts back to $\mathrm{Aut}(\mathrm{Sol}^4_{m,n})$. By Lemma 3.1.2, we obtain a splitting $\mathrm{Aut}(\mathrm{Sol}^4_{m,n}) \cong Z^1_\phi(\mathbb{R}, \mathbb{R}^3) \rtimes (\mathbb{R}^*)^3$. But on $\mathrm{Sol}^4_{m,n}$, every crossed homomorphism is principal. That is, for $\eta \in Z^1_\phi(\mathbb{R}, \mathbb{R}^3)$, $\eta(u) = \delta\mathbf{x}_0 = (I - \phi(u))\mathbf{x}_0$, for some $\mathbf{x}_0 \in \mathbb{R}^3$. $\square$

Note that we could proceed as in Proposition 3.1.4 and express $\mathrm{Aut}(\mathrm{Sol}^4_{m,n})$ in terms of $\mathrm{Inn}(\mathrm{Sol}^4_{m,n}) \cong \mathrm{Sol}^4_{m,n}$, but this will not be necessary.

Proposition 3.1.6. *We have* $\mathrm{Aut}(\mathrm{Sol}_0^{\,4}) \cong \mathbb{R}^3 \rtimes (\mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}^*)$. *Here* $\mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}^*$ *is identified with the group of block diagonal matrices in* $\mathrm{GL}(3,\mathbb{R})$ *given by*

$$\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & b \end{bmatrix},$$

and acts on the $\mathbb{R}^3$ *factor linearly in the semi-direct product. For* $(\mathbf{x}_0, A) \in \mathbb{R}^3 \rtimes (\mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}^*)$, $(\mathbf{x}_0, A)$ *acts on* $\mathrm{Sol}_0^{\,4}$ *as follows: for* $(\mathbf{x}, u) \in \mathrm{Sol}_0^{\,4}$,

$$(\mathbf{x}, u) \to (A(\mathbf{x}) + (I - \phi(u))\mathbf{x}_0, u).$$

Proof. Any automorphism α restricts to an automorphism of the nilradical $\mathbb{R}^3$ of $\mathrm{Sol}_0^{\,4}$, so Lemma 3.1.1 applies. Since eigenvalues are invariant under conjugation, $\phi(\bar{A}u)$ and $A\phi(u)A^{-1}$ must have the same eigenvalues. Hence $\bar{A} = +1$ is forced. Thus $\phi(u)$ and A commute. Computation shows that A must be block diagonal as in the statement of the proposition. Conversely, for $B \in \mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}^*$, $(\mathbf{x}, u) \to (B\mathbf{x}, u)$ defines an automorphism of Sol_0^4. Thus the image of $\mathrm{Aut}(\mathrm{Sol}_0^4)$ in $\mathrm{Aut}(\mathbb{R}^3)$ is $\mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}^*$ and $\mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}^*$ lifts back to $\mathrm{Aut}(\mathrm{Sol}_0^4)$. By Lemma 3.1.2, we obtain a splitting $\mathrm{Aut}(\mathrm{Sol}_0^4) \cong Z^1_\phi(\mathbb{R}, \mathbb{R}^3) \rtimes (\mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}^*)$. But on Sol_0^4, every crossed homomorphism is principal, so that $Z^1_\phi(\mathbb{R}, \mathbb{R}^3) \cong \mathbb{R}^3$. □

Note that both the geometries $\mathrm{Sol}^3 \times \mathbb{R}$ and Sol_1^4 have center $\mathbb{R}$ with quotient isomorphic to Sol^3. So when $G = \mathrm{Sol}^3 \times \mathbb{R}$ or Sol_1^4, we have the short exact sequence

$$1 \to \mathcal{Z}(G) \to G \to \mathrm{Sol}^3 \to 1.$$

When $G = \mathrm{Sol}^3 \times \mathbb{R}$ or Sol_1^4, an automorphism $\hat{\alpha}$ of G, induces an automorphism α of Sol^3, an automorphism $\hat{A}$ of the center, and an automorphism $\bar{A}$ of the quotient $\mathbb{R} = \mathrm{Sol}^3/\mathbb{R}^2$.

$$\begin{array}{ccccc} \mathrm{Aut}(G) & \longrightarrow & \mathrm{Aut}(\mathrm{Sol}^3) \times \mathrm{Aut}(\mathbb{R}) & \longrightarrow & \mathrm{Aut}(\mathbb{R}) \times \mathrm{Aut}(\mathbb{R}^2) \times \mathrm{Aut}(\mathbb{R}) \\ \hat{\alpha} & \longrightarrow & (\alpha, \hat{A}) & \longrightarrow & (\bar{A}, A, \hat{A}) \end{array} \tag{3.3}$$

Recall that A determines $\bar{A}$, according to whether A is diagonal or off-diagonal.

Proposition 3.1.7. *We have* $\mathrm{Aut}(\mathrm{Sol}^3 \times \mathbb{R}) \cong \mathbb{R} \rtimes (\mathrm{Aut}(\mathrm{Sol}^3) \times \mathbb{R}^*)$. *Here* $\mathrm{Aut}(\mathrm{Sol}^3) \times \mathbb{R}^*$ *acts on* $\mathrm{Sol}^3 \times \mathbb{R}$ *in the obvious way as automorphisms on each factor. The* $\mathbb{R}$ *factor is the kernel of the homomorphism* $\mathrm{Aut}(\mathrm{Sol}^3 \times \mathbb{R}) \to \mathrm{Aut}(\mathrm{Sol}^3) \times \mathrm{Aut}(\mathbb{R})$, *it acts as automorphisms as follows: for* $\left(\left(\begin{bmatrix} x \\ y \end{bmatrix}, u\right), t\right) \in \mathrm{Sol}^3 \times \mathbb{R}$ *and* $k \in \mathbb{R}$,

$$\left(\left(\begin{bmatrix} x \\ y \end{bmatrix}, u\right), t\right) \to \left(\left(\begin{bmatrix} x \\ y \end{bmatrix}, u\right), t + ku\right).$$

The action of $\mathrm{Aut}(\mathrm{Sol}^3) \times \mathbb{R}^*$ *on* $\mathbb{R}$ *in* $\mathbb{R} \rtimes (\mathrm{Aut}(\mathrm{Sol}^3) \times \mathbb{R}^*)$ *is given by mul-*

tiplication by $\hat{A}\bar{A}$.

Proof. Clearly, $\text{Aut}(\text{Sol}^3) \times \text{Aut}(\mathbb{R}) \cong \text{Aut}(\text{Sol}^3) \times \mathbb{R}^*$ lifts back to $\text{Aut}(\text{Sol}^3 \times \mathbb{R})$. The kernel of $\text{Aut}(\text{Sol}^3 \times \mathbb{R}) \to \text{Aut}(\text{Sol}^3) \times \text{Aut}(\mathbb{R})$ is the group of crossed homomorphisms $Z^1(\text{Sol}^3, \mathbb{R})$. Since Sol^3 acts trivially on the center $\mathbb{R}$, the crossed homomorphisms become genuine homomorphisms, and $Z^1(\text{Sol}^3, \mathbb{R}) = \hom(\text{Sol}^3, \mathbb{R}) = \hom(\text{Sol}^3/\left[\text{Sol}^3, \text{Sol}^3\right], \mathbb{R}) = \hom(\mathbb{R}, \mathbb{R}) = \mathbb{R}$, acting as translations of the central $\mathbb{R}$-factor of $\text{Sol}^3 \times \mathbb{R}$. We therefore obtain a splitting $\text{Aut}(\text{Sol}^3 \times \mathbb{R}) \cong \mathbb{R} \rtimes (\text{Aut}(\text{Sol}^3) \times \mathbb{R}^*)$. □

Given $\hat{\alpha} \in \text{Aut}(\text{Sol}_1^4)$, A determines $\hat{A}$. That is, $\hat{A} = \det(A)$. Of course, $\bar{A}$ is also determined.

Proposition 3.1.8. *We have* $\text{Aut}(\text{Sol}_1^4) \cong \mathbb{R} \rtimes \text{Aut}(\text{Sol}^3)$. *The group* $\mathbb{R}$ *is the kernel of the homomorphism*

$$\text{Aut}(\text{Sol}_1^4) \to \text{Aut}(\text{Sol}^3).$$

For $k \in \mathbb{R}$, the action on Sol_1^4 *is given by*

$$\begin{bmatrix} 1 & e^u x & z \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & e^u x & z + ku \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix}.$$

The action of $\text{Aut}(\text{Sol}^3)$ *on* $\mathbb{R}$ *in* $\mathbb{R} \rtimes \text{Aut}(\text{Sol}^3)$ *is multiplication by $\hat{A}\bar{A}$.*

Proof. We have seen that the image of $\text{Aut}(\text{Sol}_1^4)$ under

$$\text{Aut}(\text{Sol}_1^4) \to \text{Aut}(\text{Sol}^3) \times \text{Aut}(\mathbb{R}) \to \text{Aut}(\mathbb{R}) \times \text{Aut}(\mathbb{R}^2) \times \text{Aut}(\mathbb{R})$$

is determined by its image in $\text{Aut}(\mathbb{R}^2)$. On the other hand, $\text{Aut}(\text{Sol}^3)$ lifts back to $\text{Aut}(\text{Sol}_1^4)$. Recall the isomorphism $\text{Aut}(\text{Sol}^3) \cong \text{Sol}^3 \rtimes (\mathbb{R}^+ \times D_4)$ given in

Proposition 3.1.4. First, $\mathrm{Sol}^3 \subset \mathrm{Sol}^3 \rtimes (\mathbb{R}^+ \times D_4)$, corresponding to the inner automorphisms of Sol^3 lifts to the inner automorphisms of $\mathrm{Aut}(\mathrm{Sol}_1^4)$. Note that $\mathrm{Inn}(\mathrm{Sol}_1^4) = \mathrm{Inn}(\mathrm{Sol}^3) \cong \mathrm{Sol}^3$.

For the part $\mathbb{R}^+ \times D_4$ of $\mathrm{Aut}(\mathrm{Sol}^3)$, we have that a diagonal or off-diagonal matrix $A \in \mathrm{GL}(2,\mathbb{R})$ can be lifted to an automorphism of Sol_1^4:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} 1 & e^u x & z \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & e^{\bar{A}u}(ax+by) & \frac{1}{2}(abx^2 + 2bcxy + cdy^2 + 2(ad-bc)z) \\ 0 & e^{\bar{A}u} & (cx+dy) \\ 0 & 0 & 1 \end{bmatrix}.$$

This formula is valid only for the cases when either $a = d = 0$ ($\bar{A} = -1$) or $b = c = 0$ ($\bar{A} = +1$).

The kernel of $\mathrm{Aut}(\mathrm{Sol}_1^4) \to \mathrm{Aut}(\mathrm{Sol}^3)$ is the group of crossed homomorphisms $Z^1(\mathrm{Sol}^3, \mathbb{R})$. Since Sol^3 acts trivially on the center $\mathbb{R}$, the crossed homomorphisms become genuine homomorphisms, and $Z^1(\mathrm{Sol}^3,\mathbb{R}) = \hom(\mathrm{Sol}^3,\mathbb{R}) = \hom(\mathbb{R},\mathbb{R}) = \mathbb{R}$. Thus we have a splitting $\mathrm{Aut}(\mathrm{Sol}_1^4) \cong \mathbb{R} \rtimes \mathrm{Aut}(\mathrm{Sol}^3)$. □

In summary,

G	Aut(G)
Sol^3	$\mathrm{Sol}^3 \rtimes (\mathbb{R}^+ \times D_4)$
$\mathrm{Sol}^3 \times \mathbb{R}$	$\mathbb{R} \rtimes (\mathrm{Aut}(\mathrm{Sol}^3) \times \mathbb{R}^*)$
$\mathrm{Sol}^4_{m,n}$	$\mathbb{R}^3 \rtimes (\mathbb{R}^*)^3$
Sol_0^4	$\mathbb{R}^3 \rtimes (\mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}^*)$
Sol_1^4	$\mathbb{R} \rtimes \mathrm{Aut}(\mathrm{Sol}^3)$

Table 3.1: Aut(G) of 4-dimensional solvable geometries

With any left invariant metric on G,

$$\mathrm{Isom}(G) \subset G \rtimes K \subset \mathrm{Aff}(G),$$

where K is a maximal compact subgroup of $\mathrm{Aut}(G)$ [13]. In fact, all of the 4-dimensional solvable geometries admit a left invariant metric so that

$$\mathrm{Isom}(G) = G \rtimes K,$$

where K is any of the maximal compact subgroups listed in the table below. In the table below, we list K for each of the 4-dimensional solvable geometries.

G	$K \subset \mathrm{Isom}(G)$
$\mathrm{Nil}^3 \times \mathbb{R}$	$\mathrm{O}(2,\mathbb{R}) \times \mathbb{Z}_2$
Nil^4	$\mathbb{Z}_2 \times \mathbb{Z}_2$
Sol^3	D_4
$\mathrm{Sol}^3 \times \mathbb{R}$	$D_4 \times \mathbb{Z}_2$
$\mathrm{Sol}^4_{m,n}$	$\mathbb{Z}_2^3$
Sol_0^4	$\mathrm{O}(2,\mathbb{R}) \times \mathbb{Z}_2$
Sol_1^4	D_4

Table 3.2: Maximal compact subgroup of Aut(G)

3.2 Crystallographic Groups of Sol^3

In this section, we shall classify the crystallographic groups of Sol^3. Given a Sol^3 crystallographic group $Q \subset \mathrm{Sol}^3 \rtimes D_4$, $Q \cap \mathrm{Sol}^3$ is a lattice Γ with finite quotient group $\Phi \subseteq D_4$, and we have the short exact sequence

$$1 \to \Gamma \to Q \to \Phi \to 1.$$

The first step will be to classify the lattices of Sol^3 up to isomorphism. For $u \in \mathbb{R}$, let E^u denote the action of $\mathbb{R}$ on $\mathbb{R}^2$ in Sol^3. Our classification, as well as the notation conventions, will be needed in the 4-dimensional case. See [14] for another classification.

3.2.1 (Lattices of Sol^3). In this section, we show the well known fact that there is a one-one correspondence between the isomorphism classes of lattices (discrete, cocompact subgroups) of Sol^3 and the weak-conjugacy classes (see below) of $\mathcal{S} \in \mathrm{SL}(2, \mathbb{Z})$ with trace, $\mathrm{tr}(\mathcal{S}) > 2$.

We say a matrix $H \in \mathrm{GL}(2, \mathbb{Z})$ is *hyperbolic* if it has two distinct real eigenvalues, say λ and $\frac{1}{\lambda}$, with $|\lambda| + |\frac{1}{\lambda}| > 2$. Note that H induces a linear diffeomorphism of T^2 with directions of both expansion and contraction. Let $\mathcal{S} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ with $\mathrm{tr}(\mathcal{S}) > 2$. Such a matrix is a hyperbolic matrix with both eigenvalues $\lambda, \frac{1}{\lambda}$ positive.

Let $\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$ be the extension of $\mathbb{Z}^2$ by $\mathbb{Z}$ with the group operation

$$(\mathbf{x}, m) \cdot (\mathbf{y}, n) = (\mathbf{x} + \mathcal{S}^m \mathbf{y}, m + n).$$

Let

$$\mathbf{e}_1 = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 0 \right), \quad \mathbf{e}_2 = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 0 \right), \quad \mathbf{e}_3 = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 1 \right).$$

Then $\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$ has relations

$$[\mathbf{e}_1, \mathbf{e}_2] = 1, \quad \mathbf{e}_3 \cdot \mathbf{e}_1 \cdot \mathbf{e}_3^{-1} = \mathbf{e}_1^{\sigma_{11}} \cdot \mathbf{e}_2^{\sigma_{21}}, \quad \mathbf{e}_3 \cdot \mathbf{e}_2 \cdot \mathbf{e}_3^{-1} = \mathbf{e}_1^{\sigma_{12}} \cdot \mathbf{e}_2^{\sigma_{22}}. \tag{3.4}$$

Notation 3.2.2. For uniformity of statements, we always take

$$\Delta = \begin{bmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{bmatrix}$$

with $\frac{1}{\lambda} < 1 < \lambda$.

Lemma 3.2.3. *Let $\mathcal{S} \in \mathrm{SL}(2, \mathbb{Z})$ with $\mathrm{tr}(\mathcal{S}) > 2$. Let P and Δ be matrices such that $P\mathcal{S}P^{-1} = \Delta$ and $\det(P) = 1$. Then $Q\mathcal{S}Q^{-1} = \Delta$ (with $\det(Q) = 1$) if and only if Q is of the form $\pm E^d \cdot P$ for $d \in \mathbb{R}$.*

Proof. Recall E^d is a diagonal matrix. If P and Q both diagonalize $\mathcal{S}$, then $(QP^{-1})\Delta(QP^{-1})^{-1} = \Delta$. Since trace$(\mathcal{S}) > 2$, $\mathcal{S}$ has two distinct positive real eigenvalues: $\lambda, 1/\lambda$. For QP^{-1} to leave the two eigenspaces invariant, it should be a diagonal matrix. Since $\det(QP^{-1}) = 1$, $QP^{-1} = \pm E^d$ for some $d \in \mathbb{R}$. Conversely, Suppose $P\mathcal{S}P^{-1} = \Delta$. Then

$$(\pm E^d P)\mathcal{S}(\pm E^d P)^{-1} = E^d(P\mathcal{S}P^{-1})E^{-d} = E^d \Delta E^{-d} = \Delta. \qquad \square$$

With such P and Δ for $\mathcal{S}$, we define a map

$$\begin{aligned} \phi : \mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z} &\longrightarrow \qquad \mathrm{Sol}^3 \\ \left(\begin{bmatrix} x \\ y \end{bmatrix}, u \right) &\longmapsto \left(P \begin{bmatrix} x \\ y \end{bmatrix}, u \ln(\lambda) \right). \end{aligned} \tag{3.5}$$

The relation $P\mathcal{S}P^{-1} = \Delta$ guarantees that ϕ is a homomorphism. Clearly, the image is a lattice of Sol^3. It maps the generators as follows.

$$\begin{aligned} \mathbf{e}_1 &\mapsto \mathbf{t}_1 = P\mathbf{e}_1 \\ \mathbf{e}_2 &\mapsto \mathbf{t}_2 = P\mathbf{e}_2 \\ \mathbf{e}_3 &\mapsto \mathbf{t}_3 = (\mathbf{0}, \ln(\lambda)) \end{aligned} \tag{3.6}$$

We denote image of $\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$ as $\Gamma_{\mathcal{S}}$:

$$\Gamma_{\mathcal{S}} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3 \rangle \subset \mathrm{Sol}^3.$$

Conversely, we show any lattice of Sol^3 is isomorphic to such a $\Gamma_{\mathcal{S}}$, as the following proposition shows. We say $\mathcal{S}, \mathcal{S}' \in \mathrm{SL}(2, \mathbb{Z})$ are *weakly conjugate* if and only if $\mathcal{S}'$ is conjugate, via an element of $\mathrm{GL}(2, \mathbb{Z})$, to $\mathcal{S}$ or $\mathcal{S}^{-1}$.

Notation 3.2.4. We shall refer to a lattice of Sol^3 generated by $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ of the form in assignment (3.6) in Subsection 3.2.1 as a *standard lattice* of Sol^3.

Proposition 3.2.5. *There is a one-one correspondence between the isomorphism classes of* Sol^3*-lattices and the weak-conjugacy classes of* $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(\mathcal{S}) > 2$*. Therefore, any lattice of* Sol^3 *is conjugate to* $\Gamma_{\mathcal{S}}$*, for some* $\mathcal{S}$*, by an inner automorphism of* Sol^3*.*

Proof. Let $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(\mathcal{S}) > 2$. Then the assignment (3.6) in Subsection 3.2.1 yields a lattice $\Gamma_{\mathcal{S}} \subset \mathrm{Sol}^3$.

We show that any lattice Γ is conjugate to $\Gamma_{\mathcal{S}}$ for some $\mathcal{S}$. The intersection of Γ with the nilradical $\mathbb{R}^2$, $\Gamma \cap \mathbb{R}^2$, is a lattice of $\mathbb{R}^2$, [32, Corollary 8.28]. Let $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^2$ be a generating set for $\Gamma \cap \mathbb{R}^2$, and suppose $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}\}$ generates Γ. Then $\mathbf{t}$ is of the form $(\mathbf{a}, \ln(\lambda))$. We can assume that $\lambda > 1$, by taking the inverse of $\mathbf{t}$, if needed. But conjugation by $((E^{\ln(\lambda)} - I)^{-1}\mathbf{a}, 0)$ maps this lattice to

$$\Gamma = \langle (\mathbf{t}_1, 0),\ (\mathbf{t}_2, 0),\ (\mathbf{0}, \ln(\lambda)) \rangle.$$

Define P by $P\mathbf{e}_1 = \mathbf{t}_1$ and $P\mathbf{e}_2 = \mathbf{t}_2$, and set $\mathcal{S} = P^{-1}E^{\ln(\lambda)}P$. We claim that $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$. Since $\mathbf{t}_3\mathbf{t}_1\mathbf{t}_3^{-1} = E^{\ln(\lambda)}\mathbf{t}_1$ must be in Γ, $E^{\ln(\lambda)}\mathbf{t}_1 = \mathbf{t}_1^{\sigma_{11}}\mathbf{t}_2^{\sigma_{21}}$, for some integers σ_{11}, σ_{21}, so that $P^{-1}E^{\ln(\lambda)}P\mathbf{e}_1 = \sigma_{11}\mathbf{e}_1 + \sigma_{21}\mathbf{e}_2$. Similarly, for integers σ_{12}, σ_{22}, $P^{-1}E^{\ln(\lambda)}P\mathbf{e}_2 = \sigma_{12}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2$, which shows that $\mathcal{S}$ is integral.

Notice that, in case $\det(P) < 0$, we can make it positive by taking $-\mathbf{t}_1$ in place of $\mathbf{t}_1$. Furthermore, we can assume that $\det(P) = 1$. For conjugation by $(0, \frac{1}{\sqrt{\det(P)}}I) \in \mathrm{Aff}(\mathrm{Sol}^3)$ maps this lattice to another which has $\det(P) = 1$. Therefore our Γ is, in fact, conjugate to the lattice $\Gamma_{\mathcal{S}}$.

It remains to show the isomorphism class statement. If $\mathcal{S}' = B\mathcal{S}^{\epsilon}B^{-1}$ with $B \in \mathrm{GL}(2,\mathbb{Z})$, where $\epsilon = \pm 1$, then an isomorphism between $\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$ and $\mathbb{Z}^2 \rtimes_{\mathcal{S}'} \mathbb{Z}$ is given by $\mathbf{e}_i \to B\mathbf{e}_i$ for $i = 1, 2$ and $\mathbf{e}_3 \to \epsilon\mathbf{e}_3$.

Conversely, suppose two lattices of Sol^3 are isomorphic. We may assume these lattices are of the form $\Gamma_{\mathcal{S}}$ and $\Gamma_{\mathcal{S}'}$. Our goal is to show that $\mathcal{S}$ and $\mathcal{S}'$ are weakly conjugate. We have an isomorphism

$$\phi : \mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z} \longrightarrow \mathbb{Z}^2 \rtimes_{\mathcal{S}'} \mathbb{Z}.$$

Since $\mathbb{Z}^2$ is the discrete nil-radical (maximal normal nilpotent subgroup) in both groups, ϕ restricts to an isomorphism on the $\mathbb{Z}^2$ factors, say $B \in \mathrm{GL}(2, \mathbb{Z})$. Also, ϕ induces an isomorphism ϵ on the quotients. Then $\epsilon = \pm 1$. Therefore, ϕ is of the form

$$\phi(\mathbf{x}, m) = (B\mathbf{x} + \eta(\mathbf{x}, m), \epsilon m).$$

However, one can quickly see that η is independent of $\mathbf{x}$. So $\eta(\mathbf{x}, m) = \eta(m)$. Then applying ϕ to $(0,1)(\mathbf{x},0)(0,-1) \in \mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$ in two different ways, we get

$$\begin{aligned} \phi((0,1)(\mathbf{x},0)(0,-1)) &= \phi(\mathcal{S}\mathbf{x}, 0) = (B(\mathcal{S}\mathbf{x}), 0) \\ \phi(0,1)\phi(\mathbf{x},0)\phi(0,-1) &= (\eta(1), \epsilon)(B\mathbf{x}, 0)(\eta(1), \epsilon)^{-1} = (\mathcal{S}'^{\epsilon} B\mathbf{x}, 0) \end{aligned}$$

Thus, $B\mathcal{S}\mathbf{x} = \mathcal{S}'^{\epsilon} B\mathbf{x}$ for every $\mathbf{x}$, and we have $\mathcal{S}'^{\epsilon} = B\mathcal{S}B^{-1}$. □

3.2.6 (Classification of Sol^3 Crystallographic Groups). For our classification, we first show that Q can be conjugated in $\mathrm{Aff}(\mathrm{Sol}^3)$ in such a way that the lattice $Q \cap \mathrm{Sol}^3$ is of the form $\Gamma_{\mathcal{S}}$ for some $\mathcal{S}$, see assignment (3.6) in Subsection 3.2.1.

Proposition 3.2.7. *Any crystallographic group Q' of Sol^3 can be conjugated in $\mathrm{Aff}(\mathrm{Sol}^3)$ to $Q \subset \mathrm{Sol}^3 \rtimes D_4$ so that:*

(1) *We have $Q \cap \mathrm{Sol}^3 = \Gamma_{\mathcal{S}}$. That is, the translation subgroup of Q is a standard lattice of Sol^3, generated by $\mathbf{t}_1$, $\mathbf{t}_2$, and $\mathbf{t}_3$ as in* (3.6) *in* Subsection 3.2.1.

(2) *The holonomy group Φ is generated by at most two elements of D_4, and therefore Q is generated by $\langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3 \rangle$, and at most two isometries of the form*

$(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A)$, *for* $A \in D_4$ *and real numbers* a_i.

Proof. Any maximal compact subgroup of $\mathrm{Aut}(\mathrm{Sol}^3)$ is conjugate to $D_4 \subset \mathrm{Aut}(\mathrm{Sol}^3)$. Then we observe that the conjugations performed in the proof of Proposition 3.2.5 leave D_4 invariant. $\square$

For ease of computation, we will assume our Sol^3 crystallographic group is embedded in $\mathrm{Sol}^3 \rtimes D_4$ as in Proposition 3.2.7. Note that $Q \cap \mathbb{R}^2 = \langle \mathbf{t}_1, \mathbf{t}_2 \rangle$ is a lattice of $\mathbb{R}^2$, isomorphic to $\mathbb{Z}^2$. Denote the quotient Q/Γ by $\mathbb{Z}_\Phi$ so that we have the diagram:

$$\begin{array}{ccccccccc} & & 1 & & 1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & \mathbb{Z}^2 & = & \mathbb{Z}^2 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & \Gamma_{\mathcal{S}} & \longrightarrow & Q & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & \downarrow {\scriptstyle /\mathbb{Z}^2} & & \downarrow {\scriptstyle /\mathbb{Z}^2} & & \| & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_\Phi & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array} \tag{3.7}$$

First we will classify all possible extensions $\mathbb{Z}_\Phi$ which can arise as quotients of Sol^3 crystallographic groups.

Lemma 3.2.8. *If* $\bar{A} = -1$, a_3 *can be taken to be* 0. *Otherwise,* $a_3 = 0$ *or* $\frac{1}{2}$ $(\mathrm{mod}\ \mathbb{Z})$.

Proof. For $(\mathbf{t}_3^{a_3}, A) \in \mathbb{Z}_\Phi$, when $\bar{A} = -1$, we can assume that $a_3 = 0$. That is, given $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A) \in Q$, conjugation by $(\mathbf{t}_3^{-\frac{a_3}{2}}, I)$ sets $a_3 = 0$:

$$(\mathbf{t}_3^{-\frac{a_3}{2}}, I)(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A)(\mathbf{t}_3^{\frac{a_3}{2}}, I) = (\mathbf{t}_1^{a_1'}\mathbf{t}_2^{a_2'}\mathbf{t}_3^{0}, A).$$

Note that this conjugation keeps the lattice in standard form.

All elements in D_4 with $\bar{A} = 1$ are involutions, so when $\bar{A} = 1$, we compute in $\mathbb{Z}_\Phi$:

$$(\mathbf{t}_3^{a_3}, A)^2 = (\mathbf{t}_3^{a_3 + \bar{A}(a_3)}, I) = (\mathbf{t}_3^{2a_3}, I).$$

For this to be in $\mathbb{Z} = \langle \mathbf{t}_3 \rangle$, $a_3 = 0$ or $\frac{1}{2} \pmod{\mathbb{Z}}$ is forced. $\square$

Note that we only have to consider subgroups of $D_4 \subset \mathrm{Aut}(\mathrm{Sol}^3)$ up to conjugacy. Using Lemma 3.2.8, we give a complete list of $\mathbb{Z}_\Phi$.

— List of $\mathbb{Z}_\Phi$ — (3.8)

	Φ	$\mathbb{Z}_\Phi$	$a_3, (\ b_3)$
(0)	$\{1\}$	$\mathbb{Z}$	
(1)	$\mathbb{Z}_2 = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$	$\mathbb{Z} \times \mathbb{Z}_2^{(\dagger)}$	$a_3 = 0$
		$\mathbb{Z}$	$a_3 = \frac{1}{2}$
(2)	$\mathbb{Z}_2 = \left\langle \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$	$\mathbb{Z} \times \mathbb{Z}_2$	$a_3 = 0$
		$\mathbb{Z}$	$a_3 = \frac{1}{2}$
(3)	$\mathbb{Z}_2 = \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle$	$\mathbb{Z} \rtimes \mathbb{Z}_2$	$a_3 = 0$
(4)	$\mathbb{Z}_4 = \left\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle$	$\mathbb{Z} \rtimes \mathbb{Z}_4$	$a_3 = 0$
(5)	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \left\langle \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$	$(\mathbb{Z} \times \mathbb{Z}_2) \times \mathbb{Z}_2^{(\dagger)}$	$a_3 = 0,\ b_3 = 0$
		$\mathbb{Z} \times \mathbb{Z}_2,$	$a_3 = 0,\ b_3 = \frac{1}{2}$
		$\mathbb{Z} \times \mathbb{Z}_2^{(\dagger)},$	$a_3 = \frac{1}{2},\ b_3 = 0$
(6)	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$	$(\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$a_3 = 0,\ b_3 = 0$
		$\mathbb{Z} \rtimes \mathbb{Z}_2$	$a_3 = 0,\ b_3 = \frac{1}{2}$
(7)	$\mathbb{Z}_4 \rtimes \mathbb{Z}_2 = \left\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$	$(\mathbb{Z} \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_2^{(\dagger)}$	$a_3 = 0,\ b_3 = 0$
		$(\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	$a_3 = 0,\ b_3 = \frac{1}{2}$

The groups with (†) mark will be shown to not lift to crystallographic groups of Sol^3. That is, there does not exist a Sol^3-crystallographic group whose quotient is such a $\mathbb{Z}_\Phi$. See Corollary 3.2.11.

Recall from diagram (3.7) that Q is an extension

$$1 \to \mathbb{Z}^2 = \langle \mathbf{t}_1, \mathbf{t}_2 \rangle \to Q \to \mathbb{Z}_\Phi \to 1.$$

Having classified all possible $\mathbb{Z}_\Phi$, we now classify all abstract kernels

$$\varphi : \mathbb{Z}_\Phi \to \mathrm{GL}(2, \mathbb{Z}).$$

Of course

$$\varphi(\mathbf{t}_3) = \mathcal{S}$$

is fixed already. For $\bar{\alpha} = (\mathbf{t}_3^{a_3}, A) \in \mathbb{Z}_\Phi$, if $\bar{A} = +1$, $\varphi(\bar{\alpha})$ must commute with $\mathcal{S}$. Otherwise, $\varphi(\bar{\alpha})$ conjugates $\mathcal{S}$ to its inverse.

Recall the embedding

$$\phi : \mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z} \longrightarrow \mathrm{Sol}^3,$$

of assignment (3.6) in Subsection 3.2.1, where $\phi(\mathbf{e}_i) = P\mathbf{e}_i = \mathbf{t}_i$ for $i = 1, 2$ and $\phi(\mathbf{e}_3) = (\mathbf{0}, \ln(\lambda)) = \mathbf{t}_3$. We compute, for $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \in \mathbb{Z}^2$,

$$(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A)(\mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}, I)(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A)^{-1} =$$

$$(\mathbf{t}_3^{a_3} A(\mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2})\mathbf{t}_3^{-a_3}, I).$$

Such conjugation is independent of a_1, a_2 and defines an element of $\mathrm{GL}(2, \mathbb{Z})$. That is,

$$\mathbf{t}_3^{a_3} A(\mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2})\mathbf{t}_3^{-a_3} = \mathbf{t}_1^{m_1}\mathbf{t}_2^{m_2},$$

for $m_1, m_2 \in \mathbb{Z}$. With the definition of P and Δ in assignment (3.6) in Subsection 3.2.1,

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = P^{-1}\Delta^{a_3} A P \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}.$$

We define

$$\widetilde{A} = P^{-1}AP$$
$$\mathcal{S}^{a_3} = P^{-1}\Delta^{a_3}P,$$

so that $S^{a_3}\widetilde{A} = P^{-1}\Delta^{a_3}AP$.

Theorem 3.2.9. *For each $\mathbb{Z}_\Phi$ in the list* (3.8)*, the following is a complete list of homomorphisms $\varphi : \mathbb{Z}_\Phi \to \mathrm{GL}(2,\mathbb{Z})$ with $\varphi(\mathbf{t}_3) = \mathcal{S}$ and*

$$\varphi(\mathbf{t}_3^{a_3}, A) = \mathcal{S}^{a_3}\widetilde{A}$$
$$\varphi(\mathbf{t}_3^{b_3}, B) = \mathcal{S}^{b_3}\widetilde{B},$$

up to conjugation in $\mathrm{GL}(2,\mathbb{Z})$.

(0) Φ *is trivial,*

$\mathbb{Z}_\Phi = \mathbb{Z} = \langle \mathbf{t}_3 \rangle$.

(1) $\Phi = \mathbb{Z}_2$*:* $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = \mathbb{Z} = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A) \rangle$.

- $\varphi(\bar{\alpha}) = -K$ *with* $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$, *and* $\mathcal{S} = nK + I$.

(2a) $\Phi = \mathbb{Z}_2$*:* $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = \mathbb{Z} \times \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A) \rangle$.

- $\varphi(\bar{\alpha}) = A$, *and* $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(\mathcal{S}) > 2$.

(2b) $\Phi = \mathbb{Z}_2$: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = \mathbb{Z} = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A) \rangle$.

- $\varphi(\bar{\alpha}) = -K$ *with* $\det(K) = +1$, $\operatorname{tr}(K) = n > 2$, *and* $\mathcal{S} = nK - I$.

(3) $\Phi = \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

$\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A) \rangle$.

- $\varphi(\bar{\alpha}) = A$, *and* $\mathcal{S} \in \mathrm{SL}(2, \mathbb{Z})$ *with* $\sigma_{12} = -\sigma_{21}$.

(3i) $\Phi = \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

$\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A) \rangle$.

- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, *and* $\mathcal{S} \in \mathrm{SL}(2, \mathbb{Z})$ *with* $\sigma_{11} = \sigma_{22}$.

(4) $\Phi = \mathbb{Z}_4$: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$,

$\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_4 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A) \rangle$.

- $\varphi(\bar{\alpha}) = A$, *and* $\mathcal{S} \in \mathrm{SL}(2, \mathbb{Z})$, *symmetric and* $\operatorname{tr}(\mathcal{S}) > 2$.

(5) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = \mathbb{Z} \times \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A), \bar{\beta} = (\mathbf{t}_3^0, B) \rangle$.

- $\varphi(\bar{\alpha}) = -K$, $\varphi(\bar{\beta}) = B$ (1)+(2a)
- $\mathcal{S} = nK + I$, *where* $K \in \mathrm{GL}(2, \mathbb{Z})$, $\det(K) = -1$, *and* $\operatorname{tr}(K) = n > 0$.

(6a) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = (\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B) \rangle$.

- $\varphi(\bar{\alpha}) = A$, $\varphi(\bar{\beta}) = B$ (3)+(2a)

• $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\sigma_{12} = -\sigma_{21}$ *and* $\mathrm{tr}(\mathcal{S}) > 2$.

(6ai) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = (\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B) \rangle$.

• $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\varphi(\bar{\beta}) = B$ (3i)+(2a)

• $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\sigma_{11} = \sigma_{22}$ *and* $\mathrm{tr}(\mathcal{S}) > 2$.

(6b) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B) \rangle$.

• $\varphi(\bar{\alpha}) = A$, $\varphi(\bar{\beta}) = -K$ (3)+(2b)

• $\mathcal{S} = nK - I$, *where* $K \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(K) = n > 2$; $k_{12} = -k_{21}$.

(6bi) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B) \rangle$.

• $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\varphi(\bar{\beta}) = -K$ (3i)+(2b)

• $\mathcal{S} = nK - I$, *where* $K \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(K) = n > 2$; $k_{11} = k_{22}$.

(7) $\Phi = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = (\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B) \rangle$.

• $\varphi(\bar{\alpha}) = A$, $\varphi(\bar{\beta}) = -K$ *(includes* (6a)*)* (3)+(1)

• $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2,\mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) > 0$; $k_{12} = -k_{21}$.

(7i) $\Phi = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = (\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B) \rangle$.

- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\varphi(\bar{\beta}) = -K$ *(includes* `(6ai)`*)* `(3i)+(1)`
- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2,\mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$; $k_{11} = k_{22}$.

The proof will be deferred to the end of the section.

Lemma 3.2.10. *For* $\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A)\rangle$*, where* $\Phi = \mathbb{Z}_2$*,* $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$*, there is no homomorphism* $\varphi : \mathbb{Z}_\Phi \to \mathrm{GL}(2,\mathbb{Z})$ *with*

$$\varphi(\mathbf{t}_3) = \mathcal{S} \in \mathrm{SL}(2,\mathbb{Z}).$$

Proof. By applying the Rational Zero Theorem to the characteristic polynomial of $\mathcal{S}$, we see that λ must be irrational. The matrix

$$\mathcal{S}\varphi(\bar{\alpha}) = P^{-1}\Delta P P^{-1} A P = P^{-1}\Delta A P$$

must be integral. The sum of its eigenvalues, $\frac{1}{\lambda} - \lambda$, must be an integer. On the other hand, since $\frac{1}{\lambda} + \lambda$ is an integer, we infer that 2λ is an integer, so that λ is rational, which is a contradiction. $\square$.

Corollary 3.2.11. *In the list of* $\mathbb{Z}_\Phi$ (3.8)*, the 4 cases with* (†) *cannot be the quotient of a* Sol^3*-crystallographic group.*

Proposition 3.2.12 (Procedure of finding $\mathcal{S}$ and its diagonalizing matrix P)**.** *With a group* $\mathbb{Z}_\Phi$ *in the list* (3.8)*, qualifying* P *and* $\mathcal{S}$ *can be found as follows:*

(1) *If* $a_3 = 0$, $\varphi(\bar{\alpha}) = \mathcal{S}^0 \widetilde{A} = \widetilde{A} = P^{-1}AP$ *should satisfy the following.*

ⓐ *If* $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$*, then* $\widetilde{A} = A$ *(no condition on* $\mathcal{S}$*).*

ⓑ *If* $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$*, then* $\widetilde{A} = A$*, and* $\sigma_{12} = \sigma_{21}$*.*

ⓒ *If* $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, *then* A *can be* $\mathrm{SL}(2,\mathbb{Z})$*-conjugate to either* A *itself or* $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. *For* $\widetilde{A} = A$, $\sigma_{12} = -\sigma_{21}$; *for* $\widetilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\sigma_{11} = \sigma_{22}$.

(2) *If* $a_3 = \frac{1}{2}$, *then* $\bar{A} = +1$. *There are two cases.*

ⓐ *If* $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, *the condition* $\mathcal{S}^{a_3} \circ P^{-1}AP \in \mathrm{GL}(2,\mathbb{Z})$ *forces* $\mathcal{S} = nK - I$ *with* $\det(K) = +1$, $\mathrm{tr}(K) = n > 2$. *Then* $\varphi(\bar{\alpha}) = \mathcal{S}^{a_3} \circ P^{-1}AP = -K$.

ⓑ *If* $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, *the condition* $\mathcal{S}^{a_3} \circ P^{-1}AP \in \mathrm{GL}(2,\mathbb{Z})$ *forces* $\mathcal{S} = nK + I$ *with* $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$. *Then* $\varphi(\bar{\alpha}) = \mathcal{S}^{a_3} \circ P^{-1}AP = -K$.

(3) *If* Φ *has two generators, negotiate so that both conditions are met.*

That is, $\varphi : \mathbb{Z}_\Phi \to \mathrm{GL}(2,\mathbb{Z})$ *given by*

$$\varphi(\mathbf{t}_3) = \mathcal{S}$$
$$\varphi(\alpha) = \mathcal{S}^{a_3} \circ P^{-1}AP$$
$$\varphi(\beta) = \mathcal{S}^{b_3} \circ P^{-1}BP$$

must be a homomorphism.

Once we found $\mathcal{S}$ and $\varphi(\bar{\alpha}) = \mathcal{S}^{a_3} \circ P^{-1}AP$, we form $\alpha = (\mathbf{t}_3^{a_3}, A)$. Then the unique extension $\langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha \rangle$ satisfies the following:

$$\mathbf{t}_3(\mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2})\mathbf{t}_3^{-1} = \mathbf{t}_1^{m_1}\mathbf{t}_2^{m_2}, \text{ where } \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \mathcal{S}\begin{bmatrix} n_1 \\ n_2 \end{bmatrix},$$
$$\alpha(\mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2})\alpha^{-1} = \mathbf{t}_1^{w_1}\mathbf{t}_2^{w_2}, \text{ where } \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \varphi(\bar{\alpha})\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}.$$

The proof requires a series of lemmas.

When $a_3 = 0$, $\varphi(\bar{\alpha}) = \mathcal{S}^0\widetilde{A} = \widetilde{A}$. There are many matrices $\widetilde{A}$, depending on P. We show how to cut down the number of possibilities for $\widetilde{A}$. Essentially, we will be selecting a new set of generators for the lattice $\Gamma_{\mathcal{S}}$ to obtain a simpler formula for $\varphi(\bar{\alpha}) = \widetilde{A}$. The following is known (see e.g., [4, p.182]).

Lemma 3.2.13. *Let* $A \in \mathrm{GL}(2,\mathbb{Z})$, $\mathrm{tr}(A) = 0$. *Then* A *is* $\mathrm{SL}(2,\mathbb{Z})$*-conjugate to one and only one of the following matrices:*

$$\pm\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The last two matrices are $\mathrm{SL}(2,\mathbb{Z})$*-conjugate to their own negatives.*

However, the first two (order 4 elements) are only conjugate by a matrix of negative determinant. Thus, they are not $\mathrm{SL}(2,\mathbb{R})$-conjugate. A very important fact for us is:

Corollary 3.2.14. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ *are* $\mathrm{GL}(2,\mathbb{R})$*-conjugate but not* $\mathrm{GL}(2,\mathbb{Z})$*-conjugate.*

For the lemma below, recall that our convention is $P\mathcal{S}P^{-1} = \begin{bmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{bmatrix}$ with $\lambda > 1$.

Lemma 3.2.15. *Let* $P \in \mathrm{SL}(2,\mathbb{R})$, $\mathcal{S} = (\sigma_{ij}) \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{tr}(\mathcal{S}) > 2$. *Then*

(a) $P^{-1}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *if and only if* $P = \pm\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$.
$\mathcal{S}$ *is diagonalized by such a* P *if and only if* $\sigma_{12} = \sigma_{21}$.

(b1) $P^{-1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *if and only if* $P = \pm\begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$.
$\mathcal{S}$ *is diagonalized by such a* P *if and only if* $\sigma_{12} = -\sigma_{21}$.

(b2) $P^{-1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ *if and only if* $P = \pm\frac{1}{\sqrt{2}}\begin{bmatrix} t & -\frac{1}{t} \\ t & \frac{1}{t} \end{bmatrix}$*,* $t \neq 0$*.*
$\mathcal{S}$ *is diagonalized by such a* P *if and only if* $\sigma_{11} = \sigma_{22}$*.*

Proof (of Proposition 3.2.12). For cases (1)ⓑ and (1)ⓒ, let $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$, P be such that

$$P\mathcal{S}P^{-1} = \Delta$$
$$P^{-1}AP = B \in \mathrm{GL}(2,\mathbb{Z}).$$

For a $U \in \mathrm{SL}(2,\mathbb{Z})$, set $\mathcal{S}' = U\mathcal{S}U^{-1}$ and $P' = PU^{-1}$. We can embed $\mathbb{Z}^2 \rtimes_{\mathcal{S}'} \mathbb{Z}$ into Sol^3 using P' so that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Sol}^3 & = & \mathrm{Sol}^3 \\ \uparrow & & \uparrow \\ \mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z} & \longrightarrow & \mathbb{Z}^2 \rtimes_{U\mathcal{S}U^{-1}} \mathbb{Z} \end{array} \qquad \begin{array}{ccc} (Px, v\ln(\lambda)) & = & (PU^{-1}(Ux), v\ln(\lambda)) \\ \uparrow & & \uparrow \\ (x,v) & \longrightarrow & (Ux, v) \end{array}$$

Therefore the image of both $\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$ and $\mathbb{Z}^2 \rtimes_{\mathcal{S}'} \mathbb{Z}$ in Sol^3 is $\Gamma_{\mathcal{S}}$; we have not changed the lattice. The effect is just a change of basis. For this new embedding of our lattice, note that

$$\widetilde{A}' = P'^{-1}AP' = UP^{-1}APU^{-1} = U\widetilde{A}U^{-1}.$$

In case (1)ⓑ, we use this fact and Lemma 3.2.13 to find a U such that $U\widetilde{A}U^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; whereas in case (1)ⓒ we can find a U such that $U\widetilde{A}U^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $U\widetilde{A}U^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(2) Now we look at the case $a_3 = \frac{1}{2}$.

(2)ⓐ Let $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

$$\mathcal{S}^{\frac{1}{2}}\widetilde{A} = (P^{-1}\Delta^{\frac{1}{2}}P)(P^{-1}AP) = P^{-1}\begin{bmatrix} -\frac{1}{\sqrt{\lambda}} & 0 \\ 0 & -\sqrt{\lambda} \end{bmatrix}P$$
$$= -\tfrac{1}{\sqrt{T+2}}\begin{bmatrix} \sigma_{11}+1 & \sigma_{12} \\ \sigma_{21} & \sigma_{22}+1 \end{bmatrix}.$$

Note that this is one of the square roots of $\mathcal{S}$. That is,

$$\left(-\tfrac{1}{\sqrt{T+2}}\begin{bmatrix} \sigma_{11}+1 & \sigma_{12} \\ \sigma_{21} & \sigma_{22}+1 \end{bmatrix}\right)^2 = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}.$$

The condition $\mathcal{S}^{\frac{1}{2}}\widetilde{A} \in \mathrm{GL}(2,\mathbb{Z})$ requires that the trace of $\mathcal{S}$ be of the form n^2-2 for some integer $n>0$. Since $\sqrt{T+2}=n$, $\mathcal{S}$ must be of the form

$$\mathcal{S} = nK - I, \quad \text{where } K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}.$$

Then $\mathrm{tr}(\mathcal{S}) = n^2-2$ and $\det(\mathcal{S}) = 1$ imposes

$$k_{11}+k_{22} = n \neq 0,$$
$$k_{11}k_{22} - k_{12}k_{21} = 1.$$

With such an $\mathcal{S}$,

$$\varphi(\bar{\alpha}) = \mathcal{S}^{\frac{1}{2}}\widetilde{A} = -\tfrac{1}{\sqrt{T+2}}(\mathcal{S}+I) = -\frac{1}{n}(nK) = -K \text{ (assuming } \mathrm{tr}(K) = n > 2).$$

(2)ⓑ Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

$$\mathcal{S}^{\frac{1}{2}}\widetilde{A} = (P^{-1}\Delta^{\frac{1}{2}}P)(P^{-1}AP) = P^{-1}\begin{bmatrix} \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & -\sqrt{\lambda} \end{bmatrix}P$$
$$= -\tfrac{1}{\sqrt{T-2}}\begin{bmatrix} \sigma_{11}-1 & \sigma_{12} \\ \sigma_{21} & \sigma_{22}-1 \end{bmatrix}.$$

Note that this is one of the square roots of $\mathcal{S}$. That is,

$$\left(-\tfrac{1}{\sqrt{T-2}}\begin{bmatrix} \sigma_{11}-1 & \sigma_{12} \\ \sigma_{21} & \sigma_{22}-1 \end{bmatrix}\right)^2 = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}.$$

The condition $\mathcal{S}^{\frac{1}{2}}\widetilde{A} \in \mathrm{GL}(2,\mathbb{Z})$ requires that the trace of $\mathcal{S}$ be of the form n^2+2 for some integer $n > 0$. Since $\sqrt{T-2} = n$, $\mathcal{S}$ must be of the form

$$\mathcal{S} = nK + I, \quad \text{where } K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}.$$

Then $\mathrm{tr}(\mathcal{S}) = n^2+2$ and $\det(\mathcal{S}) = 1$ imposes

$$k_{11} + k_{22} = n \neq 0,$$
$$k_{11}k_{22} - k_{12}k_{21} = -1.$$

With such an $\mathcal{S}$,

$$\varphi(\bar{\alpha}) = \mathcal{S}^{\frac{1}{2}}\widetilde{A} = -\tfrac{1}{\sqrt{T-2}}(\mathcal{S} - I) = -\frac{1}{n}(nK) = -K \text{ (assuming } \mathrm{tr}(K) = n > 0).$$

This finishes the proof of Proposition 3.2.12, and hence of Theorem 3.2.9. For Theorem 3.2.9, when there is more than one generator, apply criteria from both cases (1) and (2) from Proposition 3.2.12. □

Theorem 3.2.16 below will be needed in our classification of both the crystallographic groups of Sol^3 and $\mathrm{Sol}^4_{m,n}$, and we state it in suitable generality.

Theorem 3.2.16. *For each homomorphism $\varphi : \mathbb{Z}_\Phi \to \mathrm{GL}(k, \mathbb{Z})$, where $\mathbb{Z}_\Phi$ is an extension of $\mathbb{Z} = \langle \mathbf{t}_{k+1} \rangle$ by a finite group Φ, with $\varphi(\mathbf{t}_3) = \mathcal{S}$ in* Theorem 3.2.9 *($k = 2$, Sol^3 case), or $\varphi(\mathbf{t}_4) = \mathcal{S}$ in* Section 3.5 *($k = 3$, $\mathrm{Sol}^4_{m,n}$ case), we have an isomorphism $H^2_\varphi(\mathbb{Z}_\Phi, \mathbb{Z}^k) \cong H^1(\Phi; \mathrm{Coker}(I-\mathcal{S}))$ where $\mathrm{Coker}(I-\mathcal{S}) \cong (I-\mathcal{S})^{-1}\mathbb{Z}^k/\mathbb{Z}^k \subset T^k$ is a finite abelian group. Thus the set of all equivalence classes of extensions Q,*

$$1 \longrightarrow \mathbb{Z}^k \longrightarrow Q \longrightarrow \mathbb{Z}_\Phi \longrightarrow 1,$$

is in one-one correspondence with $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$.

Proof. Since $\det(I - \mathcal{S}) \neq 0$, $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ is finite, as $\mathrm{Coker}(I - \mathcal{S})$ is finite.

First we verify that $\varphi(\mathbb{Z}_\Phi) \subset \mathrm{GL}(k, \mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}^k)$ leaves the group $(I - \mathcal{S})^{-1}\mathbb{Z}^k \subset \mathbb{R}^k$ containing $\mathbb{Z}^k$ invariant. Suppose $\mathbf{a} \in \mathbb{R}^k$ such that $(I - \mathcal{S})\mathbf{a} = \mathbf{z} \in \mathbb{Z}^k$. Then,

$$\begin{aligned}(I-\mathcal{S})\big(\varphi(\mathbf{t}_{k+1})\mathbf{a}\big) &= (I-\mathcal{S})\big(\mathcal{S}\mathbf{a}\big)\\ &= \mathcal{S}\big((I-\mathcal{S})(\mathbf{a})\big)\\ &= \mathcal{S}(\mathbf{z}) \in \mathbb{Z}^k.\end{aligned}$$

Now for $\varphi(\bar{\alpha})$, if $\bar{A} = +1$, then

$$\begin{aligned}(I-\mathcal{S})\big(\varphi(\bar{\alpha})\mathbf{a}\big) &= \varphi(\bar{\alpha})(I-\mathcal{S}^{\bar{A}})\mathbf{a}\\ &= \varphi(\bar{\alpha})(I-\mathcal{S})\mathbf{a}\\ &= \varphi(\bar{\alpha})\mathbf{z} \in \mathbb{Z}^k;\end{aligned}$$

and if $\bar{A} = -1$, then

$$\begin{aligned}(I-\mathcal{S})\big(\varphi(\bar{\alpha})\mathbf{a}\big) &= \varphi(\bar{\alpha})(I-\mathcal{S}^{\bar{A}})\mathbf{a}\\ &= \varphi(\bar{\alpha})(-\mathcal{S}^{-1})(I-\mathcal{S})\mathbf{a}\\ &= \varphi(\bar{\alpha})(-\mathcal{S}^{-1})\mathbf{z} \in \mathbb{Z}^k.\end{aligned}$$

These show that, if $\mathbf{a} \in (I-\mathcal{S})^{-1}\mathbb{Z}^k$, then so are $\varphi(\mathbf{t}_{k+1})\mathbf{a}$ and $\varphi(\bar{\alpha})\mathbf{a}$. Consequently, $(I-\mathcal{S})^{-1}\mathbb{Z}^k$ is $\varphi(\mathbb{Z}_\Phi)$-invariant. Since $\mathbf{a} - \varphi(\mathbf{t}_{k+1})\mathbf{a} = (I-\mathcal{S})\mathbf{a} \in \mathbb{Z}^k$, $\mathbf{t}_{k+1}$ acts trivially on $\mathrm{Coker}(I-\mathcal{S})$. We obtain an induced action of $\mathbb{Z}_\Phi/\langle \mathbf{t}_{k+1}\rangle \cong \Phi$ on $\mathrm{Coker}(I-\mathcal{S})$, and so $H^1(\Phi; \mathrm{Coker}(I-\mathcal{S}))$ is defined.

Suppose we have a class in $H^2(\mathbb{Z}_\Phi; \mathbb{Z}^k)$, which defines an extension Q. Since $\mathbb{Z}^k \subset \mathbb{R}^k$ has the unique automorphism extension property, there exists a push-out $\widetilde{Q}$ [24, (5.3.4)] fitting the commuting diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & Q & \longrightarrow & \mathbb{Z}_\Phi & \longrightarrow & 1\\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbb{R}^k & \longrightarrow & \widetilde{Q} & \longrightarrow & \mathbb{Z}_\Phi & \longrightarrow & 1.\end{array}$$

We claim that $\widetilde{Q}$ is the split extension $\mathbb{R}^k \rtimes \mathbb{Z}_\Phi$. Note that $\mathbb{Z}_\Phi$ contains $\mathbb{Z} = \langle \mathbf{t}_{k+1}\rangle$ as a finite index subgroup, and $H^2(\mathbb{Z}; \mathbb{R}^k)$ vanishes. Then $H^2(\mathbb{Z}_\Phi; \mathbb{R}^k)$ is annihilated by the index of $\mathbb{Z} = \langle \mathbf{t}_{k+1}\rangle$ in $\mathbb{Z}_\Phi$ [2, Proposition 10.1]. Therefore, $H^2(\mathbb{Z}_\Phi; \mathbb{R}^k)$ vanishes. Thus we can always assume that such a Q is embedded in $\mathbb{R}^k \rtimes \mathbb{Z}_\Phi$. Since $\mathbb{Z} \subset \mathbb{Z}_\Phi$ lifts back to $\Gamma_\mathcal{S}$, it lifts back to $\widetilde{Q}$ so that $\widetilde{Q}$ contains $(\mathbf{0}, \mathbf{t}_{k+1}) \in \mathbb{R}^k \rtimes \mathbb{Z}_\Phi$. For any element $\mathbf{t}_{k+1}^n \bar{\alpha} \in \mathbb{Z}_\Phi$, pick a preimage $\alpha = (a, \bar{\alpha}) \in \mathbb{R}^k \rtimes \mathbb{Z}_\Phi$, taking care that $a = \mathbf{0}$ if $\bar{\alpha} = \mathrm{id}$. Clearly, $\mathbf{t}_{k+1}^n \bar{\alpha} \mapsto a$ defines a map $\eta : \mathbb{Z}_\Phi \to \mathbb{R}^k/\mathbb{Z}^k = T^k$. Evidently, η maps into $\mathrm{Coker}(I-\mathcal{S}) \subset T^k$, which in turn, yields

$$\eta : \Phi \to \mathrm{Coker}(I-\mathcal{S}).$$

We claim that η is a crossed homomorphism. Let $\bar{\alpha}, \bar{\beta} \in \Phi$, and let $\eta(\bar{\alpha}) = \mathbf{a}, \eta(\bar{\beta}) = \mathbf{b}$. Then $\eta(\mathbf{t}_{k+1}^m \bar{\alpha}) = \mathbf{a}, \eta(\mathbf{t}_{k+1}^n \bar{\beta}) = \mathbf{b}$, and we have

$$\begin{aligned}(\mathbf{a}, \mathbf{t}_{k+1}^m \bar{\alpha})(\mathbf{b}, \mathbf{t}_{k+1}^n \bar{\beta}) &= (\mathbf{a} + \varphi(\mathbf{t}_{k+1}^m \bar{\alpha})(\mathbf{b}), \mathbf{t}_{k+1}^m \bar{\alpha} \mathbf{t}_{k+1}^n \bar{\beta}) \\ &= (\mathbf{a} + \varphi(\mathbf{t}_{k+1}^m)(\varphi(\bar{\alpha})(\mathbf{b})), \mathbf{t}_{k+1}^m (\bar{\alpha} \mathbf{t}_{k+1}^n \bar{\alpha}^{-1}) \bar{\alpha} \bar{\beta}).\end{aligned}$$

Since $\bar{\alpha} \mathbf{t}_{k+1}^n \bar{\alpha}^{-1} = \mathbf{t}_{k+1}^l$ for some l,

$$\begin{aligned}\eta(\bar{\alpha}\bar{\beta}) &= \mathbf{a} + \varphi(\mathbf{t}_{k+1}^m)(\varphi(\bar{\alpha})(\mathbf{b})) \\ &= \eta(\bar{\alpha}) + \mathcal{S}^m(\varphi(\bar{\alpha})(\eta(\bar{\beta}))) \\ &= \eta(\bar{\alpha}) + \varphi(\bar{\alpha})(\eta(\bar{\beta})),\end{aligned}$$

where the last equality holds because $\phi(\bar{\alpha})(\eta(\bar{\beta})) \in \mathrm{Coker}(I - \mathcal{S})$, and the action of $\mathcal{S}$ on $\mathrm{Coker}(I - \mathcal{S})$ is trivial (if $\mathbf{a} \in \mathrm{Coker}(I - \mathcal{S})$, then $(I - \mathcal{S})\mathbf{a} \in \mathbb{Z}^k$, and hence $\mathbf{a} = \mathcal{S}\mathbf{a}$ modulo $\mathbb{Z}^k$). Thus η is a crossed homomorphism. Conversely, such a crossed homomorphism η clearly gives rise to an extension Q. Thus, we obtain a surjective map

$$Z^1(\Phi; \mathrm{Coker}(I - S)) \to H^2(\mathbb{Z}_\Phi; \mathbb{Z}^k),$$

which we claim is a homomorphism. To see this, given

$$\eta : \Phi \to \mathrm{Coker}(I - \mathcal{S}),$$

we find a 2-cocycle $f : \mathbb{Z}_\Phi \times \mathbb{Z}_\Phi \to \mathbb{Z}^k$ representing the extension Q corresponding to η. Fix a lift $\widetilde{\eta} : \Phi \to (I - S)^{-1}(\mathbb{Z}^k)$ (not a homomorphism in general) of η. Then we can write any element of Q as

$$(\mathbf{n} + \widetilde{\eta}(\bar{\alpha}), \mathbf{t}_{k+1}^m \bar{\alpha}),$$

where $\mathbf{n} \in \mathbb{Z}^k$ and $m \in \mathbb{Z}$. Now, for

$$(\mathbf{n}_1 + \widetilde{\eta}(\bar{\alpha}), \mathbf{t}_{k+1}^{m_1}\bar{\alpha}) \text{ and } (\mathbf{n}_2 + \widetilde{\eta}(\bar{\beta}), \mathbf{t}_{k+1}^{m_2}\bar{\beta}) \in Q,$$

$$\begin{aligned}
&(\mathbf{n}_1 + \widetilde{\eta}(\bar{\alpha}), \mathbf{t}_{k+1}^{m_1}\bar{\alpha})(\mathbf{n}_2 + \widetilde{\eta}(\bar{\beta}), \mathbf{t}_{k+1}^{m_2}\bar{\beta}) = \\
&(\mathbf{n}_1 + \mathcal{S}^m\varphi(\bar{\alpha})(\mathbf{n}_2) + \widetilde{\eta}(\bar{\alpha}) + \mathcal{S}^m\varphi(\bar{\alpha})(\widetilde{\eta}(\bar{\beta})), \mathbf{t}_{k+1}^{m_1}\bar{\alpha}\mathbf{t}_{k+1}^{m_2}\bar{\beta}).
\end{aligned}$$

Therefore, Q is represented by the 2-cocycle $f : \mathbb{Z}_\Phi \times \mathbb{Z}_\Phi \to \mathbb{Z}^k$ defined by

$$f(\mathbf{t}_{k+1}^{m_1}\bar{\alpha}, \mathbf{t}_{k+1}^{m_2}\bar{\beta}) = \widetilde{\eta}(\bar{\alpha}) + \mathcal{S}^m\varphi(\bar{\alpha})(\widetilde{\eta}(\bar{\beta})) - \widetilde{\eta}(\bar{\alpha}\bar{\beta}).$$

It is now clear that addition of crossed homomorphisms in $Z^1(\Phi; \mathrm{Coker}(I-S))$ corresponds to addition of 2-cocycles in $Z^2(\mathbb{Z}_\Phi; \mathbb{Z}^k)$.

Now we shall prove that Q splits if and only if the corresponding η is a coboundary, i.e. $\eta \in B^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$. Note that this will imply that $Z^1(\Phi; \mathrm{Coker}(I - S)) \to H^2(\mathbb{Z}_\Phi; \mathbb{Z}^k)$ induces an isomorphism

$$H^1(\Phi; \mathrm{Coker}(I - S)) \cong H^2(\mathbb{Z}_\Phi; \mathbb{Z}^k).$$

A splitting $\mathbb{Z}_\Phi \to Q$ induces a homomorphism

$$s : \mathbb{Z}_\Phi \to \widetilde{Q}.$$

Suppose $s(\mathbf{t}_{k+1}) = (z, \mathbf{t}_{k+1})$ with $z \in \mathbb{Z}^k$. (Even in this case, our definition of η shows that, we will pick $(\mathbf{0}, \mathbf{t}_{k+1})$ as our preimage of $\mathbf{t}_{k+1}$ so that $\eta(\mathbf{t}_{k+1}) = \mathbf{0}$, and $\eta(\bar{\alpha}) = \mathbf{a}$ if $s(\bar{\alpha}) = (\mathbf{a}, \bar{\alpha})$ for others). Let $y = (\mathcal{S} - I)^{-1}z$. Then

$$\begin{aligned}(y, I)(z, \mathbf{t}_{k+1})(-y, I) &= (y + z - \phi(\mathbf{t}_{k+1})(y), \mathbf{t}_{k+1}) \\ &= (z + (I - \mathcal{S})(y), \mathbf{t}_{k+1}) \\ &= (\mathbf{0}, \mathbf{t}_{k+1}),\end{aligned}$$

and

$$\begin{aligned}(y, I)(\mathbf{a}, \bar{\alpha})(-y, I) &= (y + \mathbf{a} - \phi(\bar{\alpha})y, \bar{\alpha}) \\ &= (\mathbf{a} + (I - \phi(\bar{\alpha}))y, \bar{\alpha}),\end{aligned}$$

by letting $\mathbf{a} + (I - \phi(\bar{\alpha}))y = \mathbf{v}$,

$$= (\mathbf{v}, \bar{\alpha}).$$

Now,

$$\begin{aligned}(\mathbf{v}, \bar{\alpha})(\mathbf{0}, \mathbf{t}_{k+1})(\mathbf{v}, \bar{\alpha})^{-1} &= (\mathbf{v} - (\bar{\alpha}\mathbf{t}_{k+1}\bar{\alpha}^{-1})\mathbf{v}, \bar{\alpha}\mathbf{t}_{k+1}\bar{\alpha}^{-1}) \\ &= (\mathbf{v} - \mathbf{t}_{k+1}^{\bar{A}}\mathbf{v}, \mathbf{t}_{k+1}^{\bar{A}}) \\ &= ((I - \mathcal{S}^{\bar{A}})\mathbf{v}, \mathbf{t}_{k+1}^{\bar{A}}).\end{aligned}$$

Since $\mathbb{Z}$ is normal in $\mathbb{Z}_\Phi$, for s to be a homomorphism, we must have $(I - \mathcal{S}^{\bar{A}})\mathbf{v} = \mathbf{0} \bmod \mathbb{Z}$. This happens if and only if $\mathbf{v} = \mathbf{0} \bmod \mathbb{Z}$, which holds if and only if

$$\eta(\bar{\alpha}) = \mathbf{a} = (\phi(\bar{\alpha}) - I)(-y) = (\delta y)(\bar{\alpha}),$$

so that η is a coboundary. □

Remark 3.2.17 (Cocycles and Coboundaries)**.** For each subgroup Φ of D_4, we describe both $Z^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ and $B^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$. For $\Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we need to check that the commutator of $(\mathbf{a},\bar{\alpha})$ and $(\mathbf{b},\bar{\beta})$ is in $\mathbb{Z}^2$. For $\mathbb{Z}_4$, there is no cocycle condition to check (since $I+\varphi(\bar{\alpha})+\varphi(\bar{\alpha})^2+\varphi(\bar{\alpha})^3=0$). Likewise for $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$, there is no cocycle condition for the order 4 element.

(1) $\Phi = \mathbb{Z}_2 = \langle\bar{\alpha}\rangle$. Then

$$Z^1(\Phi;\mathrm{Coker}(I-\mathcal{S})) = \{\mathbf{a}\in \mathrm{Coker}(I-\mathcal{S}):\ (I+\varphi(\bar{\alpha}))\mathbf{a}=\mathbf{0}\},$$
$$B^1(\Phi;\mathrm{Coker}(I-\mathcal{S})) = \{\mathbf{a}=(I-\varphi(\bar{\alpha}))\mathbf{v}:\ \mathbf{v}\in \mathrm{Coker}(I-\mathcal{S})\}.$$

(2) $\Phi = \mathbb{Z}_4 = \langle\bar{\alpha}\rangle$. Then

$$Z^1(\Phi;\mathrm{Coker}(I-\mathcal{S})) = \{\mathbf{a}\in \mathrm{Coker}(I-\mathcal{S})\},$$
$$B^1(\Phi;\mathrm{Coker}(I-\mathcal{S})) = \{\mathbf{a}=(I-\varphi(\bar{\alpha}))\mathbf{v}:\ \mathbf{v}\in \mathrm{Coker}(I-\mathcal{S})\}.$$

(3) $\Phi = \mathbb{Z}_2\times\mathbb{Z}_2 = \langle\bar{\alpha},\bar{\beta}\rangle$. Then

$$Z^1(\Phi;\mathrm{Coker}(I-\mathcal{S})) = \{(\mathbf{a},\mathbf{b})\in \mathrm{Coker}(I-\mathcal{S}) : (I+\varphi(\bar{\alpha}))\mathbf{a}=\mathbf{0},$$
$$(I+\varphi(\bar{\beta}))\mathbf{b}=\mathbf{0}, (I-\varphi(\bar{\alpha}))\mathbf{b}=(I-\varphi(\bar{\beta}))\mathbf{a}\},$$

$$B^1(\Phi;\mathrm{Coker}(I-\mathcal{S})) = \{(\mathbf{a},\mathbf{b})=((I-\varphi(\bar{\alpha}))\mathbf{v},(I-\varphi(\bar{\beta}))\mathbf{v}) :$$
$$\mathbf{v}\in \mathrm{Coker}(I-\mathcal{S})\}.$$

(4) $\Phi = \mathbb{Z}_4\rtimes\mathbb{Z}_2 = \langle\bar{\alpha},\bar{\beta}|\ \bar{\alpha}^2,\bar{\beta}^2,(\bar{\beta}\bar{\alpha})^4\rangle$. Then

$$Z^1(\Phi;\mathrm{Coker}(I-\mathcal{S})) = \{(\mathbf{a},\mathbf{b})\in \mathrm{Coker}(I-\mathcal{S}) : (I+\varphi(\bar{\alpha}))\mathbf{a}=\mathbf{0},$$
$$(I+\varphi(\bar{\beta}))\mathbf{b}=\mathbf{0}\},$$

$$B^1(\Phi;\mathrm{Coker}(I-\mathcal{S})) = \{(\mathbf{a},\mathbf{b})=((I-\varphi(\bar{\alpha}))\mathbf{v},(I-\varphi(\bar{\beta}))\mathbf{v}) :$$
$$\mathbf{v}\in \mathrm{Coker}(I-\mathcal{S})\}.$$

Suppose we found an extension Q; that is, $\eta \in H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ with $\eta(\bar{\alpha}) = \mathbf{a}$. Then

$$Q = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A)\rangle$$

has the following presentation:

$$\begin{aligned}
&\mathbf{t}_3(\mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2})\mathbf{t}_3^{-1} = \mathbf{t}_1^{m_1}\mathbf{t}_2^{m_2}, \text{ where } \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \mathcal{S}\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, \\
&\alpha(\mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2})\alpha^{-1} = \mathbf{t}_1^{m_1}\mathbf{t}_2^{m_2}, \text{ where } \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \varphi(\bar{\alpha})\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, \\
&\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{w_1}\mathbf{t}_2^{w_2}\mathbf{t}_3^{\bar{A}}, \text{ where } \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \left(I - \mathcal{S}^{\bar{A}}\right)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \qquad (3.9) \\
&\alpha^2 = \mathbf{t}_1^{v_1}\mathbf{t}_2^{v_2}\mathbf{t}_3^{(1+\bar{A})a_3}, \text{ where } \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (I + \varphi(\bar{\alpha}))\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \text{ if } A^2 = I, \\
&\alpha^4 = \mathrm{id}, \text{ if } \mathrm{ord}(A) = 4.
\end{aligned}$$

Remark 3.2.18. An element of $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ determines a unique (up to isomorphism) extension of $\mathbb{Z}^2$ by $\mathbb{Z}_\Phi$, which we can also view as

$$1 \to \Gamma_{\mathcal{S}} \to Q \to \Phi \to 1.$$

This also follows from group cohomology. For an element of $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ fixes the exponents a_1, a_2, a_3 in equation (3.9), and defines an abstract kernel $\Phi \to \mathrm{Out}(\Gamma_{\mathcal{S}})$. As $\Gamma_{\mathcal{S}}$ has trivial center, $H^2(\Phi; \mathcal{Z}(\Gamma_{\mathcal{S}}))$ is trivial.

Corollary 3.2.19. *Let* $Q = \langle \Gamma_{\mathcal{S}}, (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A)\rangle$ *be an extension in* Theorem 3.2.16 *with lattice* $\Gamma_{\mathcal{S}} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\rangle$*. Suppose* $\varphi(\bar{\alpha}) = -K$ *and* $\mathcal{S} = nK \pm I$ *(in* Theorem 3.2.9*,* A *has order 2,* $\bar{A} = 1$*, and* $a_3 = \frac{1}{2}$*). Then* $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) = 0$*. Therefore, there exists* $\mathbf{t}_1^{v_1}\mathbf{t}_2^{v_2}$ *which conjugates* $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{\frac{1}{2}}, A)$ *to* $(\mathbf{t}_3^{\frac{1}{2}}, A)$ *and leaves* $\Gamma_{\mathcal{S}}$ *invariant.*

Proof. We have $\det(I - \varphi(\bar{\alpha})) = 1 + \det(\varphi(\bar{\alpha})) - \operatorname{tr}(\varphi(\bar{\alpha}))$. If $\det(\varphi(\bar{\alpha})) = -1$, this quantity is nonzero since $\operatorname{tr}(\varphi(\bar{\alpha})) \neq 0$. Else, if $\det(\varphi(\bar{\alpha})) = 1$, we had that $|\operatorname{tr}(\varphi(\bar{\alpha}))| > 2$. Consequently, $I - \varphi(\bar{\alpha})$ is non-singular and we may take $\mathbf{v} = (I - \varphi(\bar{\alpha}))^{-1}\mathbf{a}$. Then our $\mathbf{v}$ conjugates $(\mathbf{t}_3^{\frac{1}{2}}, A)$ to $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{\frac{1}{2}}, A)$. It remains to show $\mathbf{v} \in (I - \mathcal{S})^{-1}\mathbb{Z}^2$. First observe that $\varphi(\bar{\alpha})$ is a square root of $\mathcal{S}$.

$$\begin{aligned}\varphi(\bar{\alpha})^2 &= P^{-1}\sqrt{\Delta}A\sqrt{\Delta}AP = P^{-1}\Delta A^2 P \text{ (since } \bar{A} = +1, A \text{ is diagonal)}\\ &= P^{-1}\Delta P = \mathcal{S}.\end{aligned}$$

Now

$$(I - \mathcal{S})\mathbf{v} = (I + \varphi(\bar{\alpha}))(I - \varphi(\bar{\alpha}))\mathbf{v} = (I + \varphi(\bar{\alpha}))\mathbf{a} \in \mathbb{Z}^2. \qquad \square$$

Lemma 3.2.20. *When* $\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A) \in Q$ *satisfies* $a_3 = \frac{1}{2}$ *and* $\bar{A} = 1$, $\gamma\alpha$ *is not torsion for any* $\gamma \in \Gamma_{\mathcal{S}}$.

Proof. Note that A is necessarily of order 2. Let $\operatorname{pr} : \mathrm{Sol}^3 \to \mathbb{R}$ denote the quotient homomorphism of Sol^3 by its nil-radical $\mathbb{R}^2$. Write $\gamma \in \Gamma_{\mathcal{S}}$ as $\mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_3^{n_3}$. Application of pr to $(\gamma\alpha)^2$ yields

$$\operatorname{pr}(\gamma\alpha)^2 = 2n_3 + 1,$$

from which we infer $\gamma\alpha$ is not torsion. $\square$

Theorem 3.2.21 (Classification of Sol^3-geometry)**.** *The following is a complete classification crystallographic groups of* Sol^3*. In each case, we give* Φ, $\mathbb{Z}_\Phi$*, the abstract kernel* $\varphi : \mathbb{Z}_\Phi \to \mathrm{GL}(2, \mathbb{Z})$ *(which gives the action of* $\mathbb{Z}_\Phi$ *on* $\langle \mathbf{t}_1, \mathbf{t}_2 \rangle$*), equations defining* $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ *from* $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$*, and conditions for* Q *to be torsion-free. Note that* $\varphi(\mathbf{t}_3) = \mathcal{S}$ *by definition.*

For each abstract kernel $\mathbb{Z}_\Phi \to \mathrm{GL}(2, \mathbb{Z})$ give in Theorem 3.2.9, we give

(1) The holonomy Φ, the conditions on $\mathcal{S}$, and the abstract kernel

$$\phi : \mathbb{Z}_\Phi \to \mathrm{GL}(2,\mathbb{Z})$$

(this fixes the exponents on $\mathbf{t}_3$).

(2) A computation of $H^1(\Phi; \mathrm{Coker}(I-\mathcal{S}))$ (this fixes the exponents on $\mathbf{t}_1, \mathbf{t}_2$).

(3) When the group is torsion-free.

(0) $\Phi = $ trivial, $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(\mathcal{S}) > 2$; $Q = \Gamma_\mathcal{S}$.

- Torsion-free

(1) $\Phi = \mathbb{Z}_2$: $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbb{Z}_\Phi = \mathbb{Z} = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A)\rangle$.

- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2,\mathbb{Z})$ with $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$.
- $\varphi(\bar{\alpha}) = -K$
- $H^1(\Phi; \mathrm{Coker}(I-\mathcal{S}))$ is trivial so that $\mathbf{a} = \mathbf{0}$.
- $Q = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha = (\mathbf{t}_3^{\frac{1}{2}}, A)\rangle$.
- Torsion-free

(2a) $\Phi = \mathbb{Z}_2$: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbb{Z}_\Phi = \mathbb{Z} \times \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A)\rangle$.

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(\mathcal{S}) > 2$.
- $\varphi(\bar{\alpha}) = A$
- $H^1(\Phi; \mathrm{Coker}(I-\mathcal{S})) = \frac{\mathrm{Coker}(I-\mathcal{S})}{2\mathrm{Coker}(I-\mathcal{S})} \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2$. That is, all $\mathbf{a}$ with $(I-\mathcal{S})\mathbf{a} \equiv \mathbf{0}$, taken modulo $2\mathbf{a}$.
- $Q = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A)\rangle$.
- Always has torsion

(2b) $\Phi = \mathbb{Z}_2$: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbb{Z}_\Phi = \mathbb{Z} = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A)\rangle$.

- $\mathcal{S} = nK - I$, $K \in \mathrm{GL}(2,\mathbb{Z})$ with $\det(K) = +1$, $\mathrm{tr}(K) = n > 2$.
- $\varphi(\bar{\alpha}) = -K$
- $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ is trivial so that $\mathbf{a} = \mathbf{0}$.
- $Q = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha = (\mathbf{t}_3^{\frac{1}{2}}, A)\rangle$.
- Torsion-free

(3) $\Phi = \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A)\rangle$.

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$, $\mathrm{tr}(\mathcal{S}) > 2$, with $\sigma_{12} = -\sigma_{21}$.
- $\varphi(\bar{\alpha}) = A$
- $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) \subseteq \mathbb{Z}_2$ can be described as all $\mathbf{a}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$, $a_2 \equiv -a_1$ taken modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ where $(I - \mathcal{S})\mathbf{v} \equiv \mathbf{0}$.
- $Q = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A)\rangle$.
- Always has torsion

(3i) $\Phi = \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A)\rangle$.

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$, $\mathrm{tr}(\mathcal{S}) > 2$, with $\sigma_{11} = \sigma_{22}$.
- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2$ can be described as all $\mathbf{a}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$, $2a_1 \equiv 0$ (so $a_1 = 0$ or $\frac{1}{2}$) modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$ where $(I - \mathcal{S})\mathbf{v} \equiv \mathbf{0}$.
- $Q = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A)\rangle$.
- Torsion-free if and only if $a_1 = \frac{1}{2}$ and $a_2 \not\equiv \frac{(\sigma_{11}+1)(2n+1)}{2\sigma_{12}}$ for any $n \in \mathbb{Z}$.

(4) $\Phi = \mathbb{Z}_4$: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_4 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A) \rangle$.

- $\mathcal{S} \in \mathrm{SL}(2, \mathbb{Z})$, symmetric, and $\mathrm{tr}(\mathcal{S}) > 2$.
- $\varphi(\bar{\alpha}) = A$
- $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) \cong \frac{\mathrm{Coker}(I-\mathcal{S})}{\mathrm{Im}(I-A)} \subseteq \mathbb{Z}_2$. That is, all $\mathbf{a}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$, taken modulo $(I - A)\mathbf{a}$.
- $Q = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A) \rangle$.
- Always has torsion

(5) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = \mathbb{Z} \times \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A), \bar{\beta} = (\mathbf{t}_3^0, B) \rangle$.

- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2, \mathbb{Z})$ with $\det(K) = -1$ and $\mathrm{tr}(K) = n > 0$.
- $\varphi(\bar{\alpha}) = -K$, $\varphi(\bar{\beta}) = B$ (1)+(2a)
- $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) \cong \frac{\mathrm{Coker}(I+K)}{2\mathrm{Coker}(I+K)} \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2$. That is, all $\mathbf{b}$ with $(I + K)\mathbf{b} \equiv \mathbf{0}$, taken modulo $2\mathbf{b}$.
- $Q = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha = (\mathbf{t}_3^{\frac{1}{2}}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}, B) \rangle$
- Always has torsion

(6a) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = (\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B) \rangle$.

- $\mathcal{S} \in \mathrm{SL}(2, \mathbb{Z})$ with $\sigma_{12} = -\sigma_{21}$ and $\mathrm{tr}(\mathcal{S}) > 2$.
- $\varphi(\bar{\alpha}) = A$, $\varphi(\bar{\beta}) = B$ (3)+(2a)
- $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ can be described as all $\mathbf{a}$, $\mathbf{b}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$ and

$(I-\mathcal{S})\mathbf{b} \equiv \mathbf{0}$, $a_2 \equiv -a_1$, $b_1 - b_2 - 2a_1 \equiv 0$, $\mathbf{a}$ and $\mathbf{b}$ modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $2\mathbf{v}$ for $\mathbf{v} \in \mathrm{Coker}(I-\mathcal{S})$, respectively.

- $Q = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}, B)\rangle$.

- Always has torsion

(6ai) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = (\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B)\rangle$.

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\sigma_{11} = \sigma_{22}$ and $\mathrm{tr}(\mathcal{S}) > 2$.

- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\varphi(\bar{\beta}) = B$ (3i)+(2a)

- $H^1(\Phi; \mathrm{Coker}(I-\mathcal{S}))$ can be described as all $\mathbf{a}$, $\mathbf{b}$ with $(I-\mathcal{S})\mathbf{a} \equiv \mathbf{0}$ and $(I-\mathcal{S})\mathbf{b} \equiv \mathbf{0}$, $2a_1 \equiv 0$, $2b_2 - 2a_2 \equiv 0$, $\mathbf{a}$ and $\mathbf{b}$ taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$ and $2\mathbf{v}$ for $\mathbf{v} \in \mathrm{Coker}(I-\mathcal{S})$, respectively.

- $Q = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}, B)\rangle$.

- Always has torsion

(6b) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B)\rangle$.

- $\mathcal{S} = nK - I$, $K \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(K) = n > 2$; and $k_{12} = -k_{21}$.

- $\varphi(\bar{\alpha}) = A$, $\varphi(\bar{\beta}) = -K$ (3)+(2b)

- $H^1(\Phi; \mathrm{Coker}(I-\mathcal{S})) \cong H^1(\langle A\rangle; \mathrm{Coker}(I+K)) \subsetneq \mathbb{Z}_2$, that is, all $\mathbf{a}$ with $(I+K)\mathbf{a} \equiv \mathbf{0}$ and $a_2 \equiv -a_1$ modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, where $(I+K)\mathbf{v} \equiv \mathbf{0}$.

- $Q = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}, B)\rangle$.

- Always has torsion

(6bi) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B) \rangle$.

- $\mathcal{S} = nK - I$, $K \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(K) = n > 2$; $k_{11} = k_{22}$.
- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\varphi(\bar{\beta}) = -K$ (3i)+(2b)
- $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) \cong H^1(\langle A \rangle; \mathrm{Coker}(I + K)) \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2$, that is, all $\mathbf{a}$ with $(I + K)\mathbf{a} \equiv \mathbf{0}$ and $2a_1 \equiv 0$ taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$, where $(I + K)\mathbf{v} \equiv \mathbf{0}$.
- $Q = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}, B) \rangle$.
- Torsion-free if and only if $a_1 = \frac{1}{2}$ and $a_2 \not\equiv \frac{(k_{11}-1)(2n+1)}{2k_{12}}$ for any $n \in \mathbb{Z}$.

(7) $\Phi = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = (\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B) \rangle$.

- $\mathcal{S} = nK + I, K \in \mathrm{GL}(2,\mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) > 0$; $k_{12} = -k_{21}$.
- $\varphi(\bar{\alpha}) = A$, $\varphi(\bar{\beta}) = -K$ (includes (6a)) (3)+(1)
- $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ is all $\mathbf{a}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$ and $a_2 \equiv -a_1$, taken modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, where $(I + K)\mathbf{v} \equiv \mathbf{0}$.
- $Q = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}, B) \rangle$.
- Always has torsion

(7i) $\Phi = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

$\mathbb{Z}_\Phi = (\mathbb{Z} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B) \rangle$

- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2,\mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) > 0$; $k_{11} = k_{22}$.

- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\varphi(\bar{\beta}) = -K$ (includes `(6a`i`)`) `(3`i`)+(1)`
- $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ is all $\mathbf{a}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$ and $2a_1 \equiv 0$, taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$, where $(I + K)\mathbf{v} \equiv \mathbf{0}$.
- $Q = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}, B)\rangle$.
- Always has torsion

See equation (3.9) for a description of the relations. Of course, when the holonomy has two generators α, β, we also have:

$$[\alpha, \beta] = \mathbf{t} \text{ for } \mathbf{t} \in \Gamma_{\mathcal{S}}, \quad \text{if } \Phi = \mathbb{Z}_2 \times \mathbb{Z}_2;$$

$$(\alpha\beta)^4 = \mathbf{t} \text{ for } \mathbf{t} \in \Gamma_{\mathcal{S}} \quad \text{if } \Phi = \mathbb{Z}_4 \rtimes \mathbb{Z}_2.$$

Proof. Theorem 3.2.9 classifies all the $\mathbb{Z}_\Phi$ and abstract kernels $\phi : \mathbb{Z}_\Phi \to \mathrm{GL}(2, \mathbb{Z})$ up to conjugation in $\mathrm{GL}(2, \mathbb{Z})$ (which is essentially up to a change of generating set for the lattice). This fixes a_3, b_3. Theorem 3.2.16 allows us to find a_1, a_2, b_1, b_2 for each abstract kernel φ. This is all we need for a Sol^3 crystallographic group Q. The proof of torsion-free criteria for cases `(3i)`, `(6bi)` is postponed until Corollary 3.2.24.

`(0)` See Theorem 3.3.2.

`(1)` By Proposition 3.2.12 (2)ⓑ, $\mathcal{S} = nK + I$, where $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$. Then $\varphi(\bar{\alpha}) = \mathcal{S}^{a_3} \circ P^{-1}AP = -K$. Corollary 3.2.19 shows $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ is trivial, and thus we can take $a_1 = a_2 = 0$. By Lemma 3.2.20, Q is torsion-free.

(2a) Let $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. There is no condition on $\mathcal{S}$, as $\varphi(\bar{\alpha}) = \mathcal{S}^0 \widetilde{A} = A \in \mathrm{GL}(2,\mathbb{Z})$ automatically. Now $\mathbf{a}$ must satisfy $(I - \mathcal{S})\mathbf{a} \in \mathbb{Z}^2$ modulo $(I - \varphi(\bar{\alpha}))\mathbf{a} = 2\mathbf{a}$, since $(I + \varphi(\bar{\alpha}))\mathbf{a} = \mathbf{0} \in \mathbb{Z}^2$ already. Note that all elements of $H^1(\Phi; \mathrm{Coker}(I - S))$ are of order 2, and is generated by at most 2 elements. Therefore, $H^1(\Phi; \mathrm{Coker}(I - S))$ is at most $\mathbb{Z}_2 \times \mathbb{Z}_2$. The square of α is always the identity. Therefore, Q has torsion.

(2b) Let

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \ a_3 = \tfrac{1}{2}.$$

By Proposition 3.2.12 (2)ⓑ, $\mathcal{S} = nK - I$ with $\det(K) = +1$, $\mathrm{tr}(K) = n > 2$. Then $\varphi(\bar{\alpha}) = \mathcal{S}^{a_3} \circ P^{-1}AP = -K$. By Corollary 3.2.19, we can take $a_1 = a_2 = 0$ so that $\alpha = (\mathbf{t}_3^{\frac{1}{2}}, A)$. By Lemma 3.2.20, Q is torsion-free.

(3) Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \widetilde{A} = A.$$

The conditions on $\mathcal{S}$ follow from 3.2.12 (1)ⓒ. Let $\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A)$. Then $\mathbf{a}$ must satisfy $(I - \mathcal{S})\mathbf{a} \in \mathbb{Z}^2$,

$$(I+\varphi(\bar{\alpha}))\mathbf{a} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{a} \equiv \mathbf{0} \text{ modulo } (I-\varphi(\bar{\alpha}))\mathbf{v} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{v}, \text{ for } (I-\mathcal{S})\mathbf{v} \in \mathbb{Z}^2,$$

computing, we obtain the claimed conditions. Applying the coboundary operator to the cocycles yields:

$$(I - \varphi(\bar{\alpha})) \begin{bmatrix} a_1 \\ -a_1 \end{bmatrix} = \begin{bmatrix} 2a_1 \\ -2a_1 \end{bmatrix}.$$

This computation implies that $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ is at most $\mathbb{Z}_2$.

To show that Q always has torsion, note that $a_2 \equiv -a_1$, and

$$\begin{aligned}\alpha^2 &= (\mathbf{t}_1^{a_1}\mathbf{t}_2^{-a_1}, A)^2 = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{-a_1} \cdot A(\mathbf{t}_1^{a_1}\mathbf{t}_2^{-a_1}), I)\\ &= (\mathbf{t}_1^{a_1}\mathbf{t}_2^{-a_1} \cdot \mathbf{t}_2^{a_1}\mathbf{t}_1^{-a_1}, I)\\ &= (e, I).\end{aligned}$$

(3i) Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \widetilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \neq A.$$

Conditions on $\mathcal{S}$ follow from 3.2.12 (1)ⓒ. Then $\mathbf{a}$ must satisfy $(I-\mathcal{S})\mathbf{a} \in \mathbb{Z}^2$,

$$(I+\varphi(\bar{\alpha}))\mathbf{a} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{a} \equiv \mathbf{0} \text{ modulo } (I-\varphi(\bar{\alpha}))\mathbf{v} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{v}, \text{ for } (I-\mathcal{S})\mathbf{v} \in \mathbb{Z}^2.$$

This shows that $a_1 = 0$ or $\frac{1}{2}$. Note that $H^1(\Phi; \mathrm{Coker}(I-\mathcal{S}))$ can be at most $\mathbb{Z}_2 \times \mathbb{Z}_2$. The torsion criteria follow from Corollary 3.2.24.

(4) Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{and } \widetilde{A} = A.$$

The conditions on $\mathcal{S}$ follow from 3.2.12 (1)ⓑ. By Remark 3.2.17, we just need $\mathbf{a} \in \mathrm{Coker}(I-\mathcal{S})$, taken modulo $\mathrm{Im}(I-\phi(\bar{\alpha}))$. Note that $\det(I-\phi(\bar{\alpha})) = 2$, which implies that $H^1(\Phi; \mathrm{Coker}(I-S))$ is at most $\mathbb{Z}_2$. As $\alpha^4 = e$, Q always has torsion.

(5) Let $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle A, B \rangle$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and } a_3 = \tfrac{1}{2}.$$

This is a combination of `(1)+(2a)`. From case `(1)`, $\mathcal{S} = I + nK \in \mathrm{GL}(2, \mathbb{Z})$ with $\det(K) = -1$. By Corollary 3.2.19, we can take $a_1 = a_2 = 0$. We need $\mathbf{b}$ to satisfy $(I - \mathcal{S})\mathbf{b} \equiv \mathbf{0}$. However, the cocycle conditions for $\mathbb{Z}_2 \times \mathbb{Z}_2$ in Remark 3.2.17 show that we must have $(I - \phi(\bar{\alpha}))\mathbf{b} = (I + K)\mathbf{b} \equiv \mathbf{0}$. In fact, since $(I - \mathcal{S}) = (I - K)(I + K)$, this condition implies $(I - \mathcal{S})\mathbf{b} \equiv \mathbf{0}$. Since we have already fixed $a_1 = a_2 = 0$, for the coboundary in Theorem 3.2.16, we take $\mathbf{b}$ modulo $(I - \phi(\bar{\beta}))\mathbf{v} = 2\mathbf{v}$ only when $\mathbf{v}$ satisfies $(I - \phi(\bar{\alpha}))\mathbf{v} = (I + K)\mathbf{v} \equiv \mathbf{0}$. Hence

$$H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) \cong \frac{\mathrm{Coker}(I + K)}{2\mathrm{Coker}(I + K)} \subseteq \mathbb{Z}_2.$$

As $\beta^2 = e$, Q is never torsion-free.

`(6a)` $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and $\widetilde{A} = A$.

This is a combination of `(3)+(2a)`. We have $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$ and $(I - \mathcal{S})\mathbf{b} \equiv \mathbf{0}$. Also, $\mathbf{a}$ and $\mathbf{b}$ also must satisfy the cocycle conditions in Remark 3.2.17. Note that $(I + \phi(\bar{\alpha}))\mathbf{a} \equiv \mathbf{0}$ forces $a_2 \equiv -a_1$, whereas $(I - \varphi(\bar{\alpha}))\mathbf{b} - (I - \varphi(\bar{\beta}))\mathbf{a} \equiv \mathbf{0}$ forces $b_1 - b_2 - 2a_1 \equiv 0$, $-b_1 + b_2 - 2a_2 \equiv 0$. Since $a_2 \equiv -a_1$, the second equation is redundant. We take $\mathbf{a}$ and $\mathbf{b}$ modulo $(I - \varphi(\bar{\alpha}))\mathbf{v}$ and $(I - \varphi(\bar{\beta}))\mathbf{v}$, respectively, where $(I - \mathcal{S})\mathbf{v} \equiv \mathbf{0}$. As Q contains a subgroup of type `(3)`, Q has torsion.

`(6a`i`)` $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and $\widetilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \neq A$.

Similar to case `(6a)`, this is a combination of `(3`i`)+(2a)`. From the cocycle condition $(I + \varphi(\bar{\alpha}))\mathbf{a} \equiv \mathbf{0}$, $a_1 = 0$ or $\frac{1}{2}$. The description of $H^1(\Phi, \mathrm{Coker}(I - \mathcal{S}))$ follows just like in case `(6a)`. As $\beta^2 = e$, Q has torsion.

`(6b)` $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\widetilde{A} = A$, and $b_3 = \frac{1}{2}$.

This is a combination of (3)+(2b). To have $\widetilde{A} = A$, Proposition 3.2.12 (1)ⓒ, (2)ⓐ force $\sigma_{12} = -\sigma_{21}$ and $\mathcal{S} = nK - I$, where $K \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(K) = n > 2$, which taken together yield the desired conditions on K. By Corollary 3.2.19, we can take $b_1 = b_2 = 0$. Hence,

$$\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A),$$
$$\beta = (\mathbf{t}_3^{\frac{1}{2}}, B).$$

The cocycle conditions in Remark 3.2.17 force $(I + \varphi(\bar{\alpha}))\mathbf{a} \equiv \mathbf{0}$ as well as $(I - \varphi(\bar{\beta}))\mathbf{a} = (I + K)\mathbf{a} \equiv \mathbf{0}$. This shows $\mathbf{a} \in \mathrm{Coker}(I + K)$. Since $b_1, b_2 = 0$ is fixed, take $\mathbf{a}$ modulo $(I - \varphi(\bar{\alpha}))\mathbf{v}$ only when $(I - \varphi(\bar{\beta}))\mathbf{v} = (I + K)\mathbf{v} \equiv \mathbf{0}$. Hence

$$H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) \cong H^1(\langle A\rangle; \mathrm{Coker}(I + K)) \subseteq \mathbb{Z}_2.$$

There is always torsion in Q, as it contains a subgroup of type (3).

(6bi) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\widetilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $b_3 = \frac{1}{2}$.

This is a combination of (3i)+(2b); Similar to case (6b), we apply Proposition 3.2.12 (1)ⓒ, (2)ⓐ to obtain $\mathcal{S} = nK - I$, where $K \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(K) = n > 2$, $k_{11} = k_{22}$. By Corollary 3.2.19, we can take $b_1 = b_2 = 0$. Hence,

$$\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A),$$
$$\beta = (\mathbf{t}_3^{\frac{1}{2}}, B).$$

The argument to establish

$$H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) \cong H^1(\langle A\rangle; \mathrm{Coker}(I + K)) \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2$$

is identical to that of (6b). In this case, we use $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ rather than $\varphi(\bar{\alpha}) = A$ and obtain the desired conditions on a_1, a_2. The torsion criteria follow from Corollary 3.2.24.

(7) $\Phi = \mathbb{Z}_4 \rtimes \mathbb{Z}_2 : A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \widetilde{A} = A,$ and $b_3 = \frac{1}{2}$.
We apply Proposition 3.2.12 (1)ⓒ, (2)ⓑ to obtain the conditions $\mathcal{S} = nK + I$, where $K \in \mathrm{GL}(2, \mathbb{Z})$ with $\det(K) = -1$ and $\operatorname{tr}(K) > 0$; and $k_{12} = -k_{21}$. This is a combination (3)+(1) which includes (6a). By Corollary 3.2.19, we can take $b_1 = b_2 = 0$. Therefore,

$$\alpha = (\mathbf{t}_1^{a_1} \mathbf{t}_2^{a_2}, A),\ \beta = (\mathbf{t}_3^{\frac{1}{2}}, B).$$

For $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$, the only cocycle condition that $\mathbf{a}$ must satisfy is $(I + \varphi(\bar{\alpha}))\mathbf{a} \equiv \mathbf{0}$, which forces $a_2 \equiv -a_1$ (Remark 3.2.17). However, we have fixed $b_1 = b_2 = 0$. Therefore, when computing the coboundaries, we can take $\mathbf{a}$ modulo $(I - \varphi(\bar{\alpha}))\mathbf{v}$ only for $(I - \mathcal{S})\mathbf{v}$ where $\mathbf{v}$ also satisfies $(I - \varphi(\bar{\beta}))\mathbf{v} = (I + K)\mathbf{v} \equiv \mathbf{0}$. Note that $(I + K)\mathbf{v} \equiv \mathbf{0}$ actually implies $(I - \mathcal{S})\mathbf{v} \equiv \mathbf{0}$ since $(I - \mathcal{S}) = (I - K)(I + K)$. There is always torsion in Q, as it contains a subgroup of type (4).

(7i) $\Phi = \mathbb{Z}_4 \rtimes \mathbb{Z}_2 : A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \widetilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, b_3 = \frac{1}{2}$.
We apply Proposition 3.2.12 (1)ⓒ, (2)ⓑ to obtain $\mathcal{S} = nK + I$, where $K \in \mathrm{GL}(2, \mathbb{Z})$ with $\det(K) = -1$ and $\operatorname{tr}(K) = n > 0$; and $k_{11} = k_{22}$.
This is a combination of (3i)+(1), which includes (6ai). By Corollary 3.2.19, we can take $b_1 = b_2 = 0$. Therefore,

$$\alpha = (\mathbf{t}_1^{a_1} \mathbf{t}_2^{a_2}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}, B).$$

The description for $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ follows as in case `(7)`, using $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ rather than $\varphi(\bar{\alpha}) = A$. There is always torsion in Q, as it contains a subgroup of type `(4)`.

This finishes the proof of Theorem 3.2.21. □

3.2.22 (**Detecting Torsion in** Sol^3 **Crystallographic Groups).** A crystallographic group Q of Sol^3 is torsion-free precisely if it acts freely on Sol^3. Using this fact, we will give proofs of the torsion criteria in Theorem 3.2.21.

Proposition 3.2.23 (Compare [35, Theorem 5.3])**.** *Every* Sol^3*-geometry manifold* $Q\backslash\mathrm{Sol}^3$ *has the structure of*

- *The mapping torus of a hyperbolic linear self-diffeomorphism of* T^2 *(*Q *is of type* `(0)`*,* `(1)`*, or* `(2b)`*).*
- *The union of two twisted I-bundles over Klein bottles (*Q *is of type* `(3`i`)` *or* `(6b`i`)`*).*

When Q *is of type* `(3`i`)`*,* $Q\backslash\mathrm{Sol}^3$ *can be described as* $T^2 \times I$ *with* $T^2 \times \{0\}$ *identified to itself by the affine involution* $\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)$*, and* $T^2 \times \{1\}$ *identified to itself by the affine involution* $\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} \sigma_{11} & -\sigma_{12} \\ \sigma_{21} & -\sigma_{11} \end{bmatrix}\right)$*.*

If $-\varphi(\bar{\alpha}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ *is used instead of* $\varphi(\bar{\alpha})$*, then* $Q\backslash\mathrm{Sol}^3$ *can be described as* $T^2 \times I$ *with* $T^2 \times \{0\}$ *identified to itself by the affine involution* $\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right)$*, and* $T^2 \times \{1\}$ *identified to itself by the affine involution* $\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} -\sigma_{11} & \sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{bmatrix}\right)$*.*

When Q is of type (6bi)*, $Q\backslash\mathrm{Sol}^3$ can be described as $T^2 \times I$ with $T^2 \times \{0\}$ identified to itself by the affine involution* $\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)$*, and $T^2 \times \{1\}$ identified to itself by the affine involution* $\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} -k_{11} & k_{12} \\ -k_{21} & k_{11} \end{bmatrix}\right)$*.*

Proof. If $Q = \Gamma_{\mathcal{S}}$ is a Sol^3 crystallographic group of type (0), then $Q \cong \mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$. The action of $\Gamma_{\mathcal{S}}$ on Sol^3 is equivalent to an action of $\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$ on $\mathbb{R}^2 \rtimes_{\mathcal{S}} \mathbb{R}$. A fundamental domain for this action is given by the unit cube I^3 and evidently $Q\backslash\mathrm{Sol}^3$ is given by $T^2 \times I$ with $T^2 \times \{0\}$ identified to $T^2 \times \{1\}$ via $\mathcal{S}$, which we view as a hyperbolic self-diffeomorphism of T^2. Note that $\mathbb{R}^2 \to \mathbb{R}^2 \rtimes_{\mathcal{S}} \mathbb{R} \to \mathbb{R}$ induces the fiber bundle with infinite cyclic structure group generated by $\mathcal{S}$

$$T^2 \to \Gamma_{\mathcal{S}}\backslash\mathrm{Sol}^3 \to S^1.$$

If Q is of type (1), then $Q\backslash\mathrm{Sol}^3$ is the quotient of $\Gamma_{\mathcal{S}}\backslash\mathrm{Sol}^3$ by the involution defined by $\alpha = \left(\mathbf{t}_3^{\frac{1}{2}}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)$. On the base of $T^2 \to \Gamma_{\mathcal{S}}\backslash\mathrm{Sol}^3 \to S^1$, α acts as a translation. Thus a fundamental domain for the action of α is given by $T^2 \times \left[0, \frac{1}{2}\right]$. Note that α identifies $T^2 \times \{0\}$ with $T^2 \times \{\frac{1}{2}\}$ via $\varphi(\bar{\alpha}) = -K$, which is a square root of $\mathcal{S}$. This $\varphi(\bar{\alpha})$ is a hyperbolic matrix with one eigenvalue positive and the other negative, and $Q\backslash\mathrm{Sol}^3$ is the mapping torus of $\varphi(\bar{\alpha})$. The argument for type (2b) is identical, in this case we obtain the mapping torus of a hyperbolic matrix with both eigenvalues negative.

Now suppose Q is of type (3i). Then Q is the quotient of $\Gamma_{\mathcal{S}}\backslash\mathrm{Sol}^3$ by the involution defined by $\alpha = \left(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$. Here α acts as a reflection on the base S^1. As in the preceding case, a fundamental domain for this action is given by $T^2 \times \left[0, \frac{1}{2}\right]$. Now α identifies $T^2 \times \{0\}$ to itself and $T^2 \times \{\frac{1}{2}\}$ to itself. We seek criteria for α to act freely on these boundary tori so that they are identified to Klein bottles. Then cutting $Q\backslash\mathrm{Sol}^3$ along $T^2 \times \{\frac{1}{4}\}$ yields two twisted I-bundles over Klein bottles. Indeed, $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A).\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2} = \mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}A(\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2})$ shows that α acts on T^2 as the affine transformation $(\mathbf{a}, \varphi(\bar{\alpha}))$. For an arbitrary element $\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2}\mathbf{t}_3^{\frac{1}{2}} \in T^2 \times \{\frac{1}{2}\}$,

$$\begin{aligned}
\mathbf{t}_3(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A).\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2}\mathbf{t}_3^{\frac{1}{2}} &= \mathbf{t}_3\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}A(\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2})A(\mathbf{t}_3^{\frac{1}{2}}) \\
&= \mathbf{t}_3\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}A(\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2})\mathbf{t}_3^{-\frac{1}{2}} \\
&= \left(\mathbf{t}_3\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{-1}\right)\left(\mathbf{t}_3A(\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2})\mathbf{t}_3^{-1}\right)\mathbf{t}_3^{\frac{1}{2}} \in T^2 \times \{\frac{1}{2}\}.
\end{aligned}$$

Since conjugation by $\mathbf{t}_3$ is the action of $\mathcal{S}$, we see that α acts on T^2 as the affine transformation $(\mathcal{S}\mathbf{a}, \mathcal{S}\varphi(\bar{\alpha}))$. But since $\mathbf{a} \in \mathrm{Coker}(I - \mathcal{S})$, this simplifies to $(\mathbf{a}, \mathcal{S}\varphi(\bar{\alpha}))$. The condition that $\sigma_{11} = \sigma_{22}$ ensures that $\mathcal{S}\varphi(\bar{\alpha})$ has order 2. Note that cutting $Q\backslash\mathrm{Sol}^3$ along $T^2 \times \{\frac{1}{4}\}$ results in two components, each a twisted I-bundle over a Klein bottle.

The argument in case `(6b`i`)` is nearly identical. In this case, note that Q contains a group of type `(2b)`, say Q', as an index 2 subgroup. Therefore, $Q\backslash\mathrm{Sol}^3$ is the quotient of $Q'\backslash\mathrm{Sol}^3$ by $\alpha = \left(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$. But because $Q'\backslash\mathrm{Sol}^3$ admits the structure of a T^2 bundle over S^1, the construction for `(3`i`)` applies. A fundamental domain for the action of α on $Q'\backslash\mathrm{Sol}^3$ is given by $T^2 \times \{\frac{1}{4}\}$. As in case `(3`$i$`)`, α acts on $T^2 \times \{0\}$ affinely as $(\mathbf{a}, \varphi(\bar{\alpha}))$. For an arbitrary element $\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2}\mathbf{t}_3^{\frac{1}{4}} \in T^2 \times \{\frac{1}{4}\}$, we have

$$\begin{aligned}
&(\mathbf{t}_3^{\frac{1}{2}}, B)(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A).\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2}\mathbf{t}_3^{\frac{1}{4}} \\
&\qquad= \mathbf{t}_3^{\frac{1}{2}}B(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2})BA(\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2})BA(\mathbf{t}_3^{\frac{1}{4}}) \\
&\qquad= \mathbf{t}_3^{\frac{1}{2}}B(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2})\mathbf{t}_3^{-\frac{1}{2}}\mathbf{t}_3^{\frac{1}{2}}BA(\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2})\mathbf{t}_3^{-\frac{1}{4}} \\
&\qquad= \left(\mathbf{t}_3^{\frac{1}{2}}B(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2})\mathbf{t}_3^{-\frac{1}{2}}\right)\left(\mathbf{t}_3^{\frac{1}{2}}BA(\mathbf{t}_1^{x_1}\mathbf{t}_2^{x_2})\mathbf{t}_3^{-\frac{1}{2}}\right)\mathbf{t}_3^{\frac{1}{4}} \in T^2 \times \{\frac{1}{4}\}.
\end{aligned}$$

Now conjugation by $(\mathbf{t}_3^{\frac{1}{2}}, B)$ is the action of $\varphi(\bar{\beta}) = -K$ on T^2. Hence α acts affinely on $T^2 \times \{\frac{1}{4}\}$ as $(\varphi(\bar{\beta})\mathbf{a}, \varphi(\bar{\beta})\varphi(\bar{\alpha}))$. But we had the condition $(I - \varphi(\bar{\beta}))\mathbf{a} = (I + K)\mathbf{a} \in \mathbb{Z}^2$, so this simplifies to $(\mathbf{a}, \varphi(\bar{\beta})\varphi(\bar{\alpha}))$. As in case (3*i*), we obtain a union of two twisted I-bundles over Klein bottles. □

Corollary 3.2.24 (Torsion in Sol^3 Crystallographic Groups)**.** *Consider the crystallographic groups of* Sol^3 *in* Theorem 3.2.21*.*

A group of type (3*i*) *is torsion-free if and only if*

$$a_1 = \frac{1}{2} \quad \text{and } a_2 \not\equiv \frac{(\sigma_{11}+1)(2n+1)}{2\sigma_{12}} \text{ for any } n \in \mathbb{Z}. \tag{3.10}$$

A group of type (6b*i*) *is torsion-free if and only if*

$$a_1 = \frac{1}{2} \quad \text{and } a_2 \not\equiv \frac{(k_{11}-1)(2n+1)}{2k_{12}} \text{ for any } n \in \mathbb{Z}. \tag{3.11}$$

Proof. By Proposition 3.2.23, for type (3*i*) groups, we need the affine involutions $\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)$ and $\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} \sigma_{11} & -\sigma_{12} \\ \sigma_{21} & -\sigma_{11} \end{bmatrix}\right)$ to act freely on T^2, which occurs precisely when a_1 and a_2 satisfy (3.10).

For type (6*i*) groups, we need the affine involutions $\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)$ and $\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} -k_{11} & k_{12} \\ -k_{21} & k_{11} \end{bmatrix}\right)$ to act freely on T^2, which occurs precisely when a_1 and a_2 satisfy (3.11) □

3.3 Crystallographic Groups of Sol_1^4

Here we provide a classification of the crystallographic groups of Sol_1^4. Given a crystallographic group $\Pi \subset \mathrm{Sol}_1^4 \rtimes D_4$, $\widetilde{\Gamma} := \Pi \cap \mathrm{Sol}_1^4$ is a lattice of Sol_1^4

with finite quotient group $\Phi \subseteq D_4$. We have the short exact sequence

$$1 \to \widetilde{\Gamma} \to \Pi \to \Phi \to 1.$$

The first step will be to classify the lattices $\widetilde{\Gamma}$ of Sol_1^4.

3.3.1 (Lattices of Sol_1^4**).** Let G denote either $\mathrm{Sol}_1^4 = \mathrm{Nil} \rtimes \mathbb{R}$ or $\mathrm{Sol}^3 \times \mathbb{R}$. Given a lattice $\widetilde{\Gamma}_{\mathcal{S}}$ of G, $\widetilde{\Gamma} \cap \mathcal{Z}(G) \cong \mathbb{Z}$ is a lattice of $\mathcal{Z}(G) \cong \mathbb{R}$ and the projection map

$$G \to G/\mathcal{Z}(G) \cong \mathrm{Sol}^3$$

maps $\widetilde{\Gamma}_{\mathcal{S}}$ to a lattice of Sol^3, which is isomorphic to $\Gamma_{\mathcal{S}}$, for some $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{trace}(\mathcal{S}) > 2$. Thus, $\widetilde{\Gamma}_{\mathcal{S}}$ is the central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\Gamma}_{\mathcal{S}} \longrightarrow \Gamma_{\mathcal{S}} \longrightarrow 1.$$

Such central extensions of $\mathbb{Z}$ by $\Gamma_{\mathcal{S}}$ are classified by the second cohomology group $H^2(\Gamma_{\mathcal{S}};\mathbb{Z})$.

Theorem 3.3.2. *Let* $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ *with* $\mathrm{trace}(\mathcal{S}) > 2$. *There is a one-one correspondence between the equivalence classes of all central extensions*

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma_{\mathcal{S}} \longrightarrow 1$$

and the set $\mathbb{Z} \oplus \mathit{Coker}(\mathcal{S} - I)$.

Proof. Recall $\Gamma_{\mathcal{S}} = \mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$.

$$\begin{aligned} H^2(\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z};\mathbb{Z}) &= \mathrm{Free}\left(H_2(\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z};\mathbb{Z})\right) \oplus \mathrm{Torsion}\left(H_1(\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z};\mathbb{Z})\right) \\ &= \mathbb{Z} \oplus (\mathbb{Z}^2/(\mathcal{S}-I)\mathbb{Z}^2) \\ &= \mathbb{Z} \oplus \mathrm{Coker}(\mathcal{S}-I). \end{aligned}$$

The first term $\mathbb{Z}$ comes from the central direction. The second term $\mathrm{Coker}(\mathcal{S}-I)$ is finite, because $\det(\mathcal{S}-I) = 2 - \mathrm{tr}(\mathcal{S}) \neq 0$. For $\{q,(m_1,m_2)\} \in \mathbb{Z} \oplus \mathrm{Coker}(\mathcal{S}-I)$, we shall denote the corresponding $\widetilde{\Gamma}$ by $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$. □

Here we shall show that $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ with $q \neq 0$ embeds as a lattice in Sol_1^4. An $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(\mathcal{S}) > 2$ produces P and Δ. Then we had the map (3.5) in Subsection 3.2.1:

$$\begin{aligned} \phi : \mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z} \longrightarrow & \qquad \mathrm{Sol}^3 \\ \left(\begin{bmatrix} x \\ y \end{bmatrix}, u \right) \longmapsto & \left(P \begin{bmatrix} x \\ y \end{bmatrix}, u \ln(\lambda) \right). \end{aligned} \tag{3.12}$$

We combine this with $\mathrm{Sol}^3 \to \mathrm{Sol}_1^4$

$$\left(\begin{bmatrix} x \\ y \end{bmatrix}, u \right) \longmapsto \begin{bmatrix} 1 & e^u x & * \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix}.$$

The only ambiguities are the central slots. So let

$$\begin{aligned} \mathbf{e}_1 = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 0 \right) \longmapsto (P\mathbf{e}_1, 0) \longmapsto \mathbf{t}_1 = & \begin{bmatrix} 1 & p_{11} & c_1 \\ 0 & 1 & p_{21} \\ 0 & 0 & 1 \end{bmatrix}, \\ \mathbf{e}_2 = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 0 \right) \longmapsto (P\mathbf{e}_2, 0) \longmapsto \mathbf{t}_2 = & \begin{bmatrix} 1 & p_{12} & c_2 \\ 0 & 1 & p_{22} \\ 0 & 0 & 1 \end{bmatrix}, \\ \mathbf{e}_3 = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 1 \right) \longmapsto (0, \ln(\lambda)) \longmapsto \mathbf{t}_3 = & \begin{bmatrix} 1 & 0 & c_3 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ \mathbf{t}_4 = & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{3.13}$$

where c_i's are to be determined. Then $[\mathbf{t}_1, \mathbf{t}_2] = \mathbf{t}_4$ (regardless the values of c_i's).

Lemma 3.3.3. *For any integers q, m_1, m_2, there exist unique c_1, c_2 for which $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4\}$ forms a group $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ with the presentation*

$$\begin{aligned}\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}} \mid [\mathbf{t}_1, \mathbf{t}_2] = \mathbf{t}_4,\ \mathbf{t}_4 \textit{ is central,}& \\ \mathbf{t}_3\mathbf{t}_1\mathbf{t}_3^{-1} = \mathbf{t}_1^{\sigma_{11}}\mathbf{t}_2^{\sigma_{21}}\mathbf{t}_4^{\frac{m_1}{q}},& \\ \mathbf{t}_3\mathbf{t}_2\mathbf{t}_3^{-1} = \mathbf{t}_1^{\sigma_{12}}\mathbf{t}_2^{\sigma_{22}}\mathbf{t}_4^{\frac{m_2}{q}}\rangle.&\end{aligned}$$

Consequently, $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ is solvable and contains $\Gamma_q = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_4^{\frac{1}{q}} \rangle$ as its discrete nil-radical, where Γ_q is a lattice of Nil.

Proof. We only need to verify the last two equalities. But they become a system of equations on c_i's

$$\begin{aligned}(1-\sigma_{11})c_1 - \sigma_{21}c_2 &= \frac{m_1}{q} - \frac{\sigma_{21}(\sigma_{12}+1-\sigma_{11}+\sigma_{11}\sqrt{T^2-4})}{2\sqrt{T^2-4}}, \\ -\sigma_{12}c_1 + (1-\sigma_{22})c_2 &= \frac{m_2}{q} + \frac{\sigma_{12}(\sigma_{21}+1-\sigma_{22}-\sigma_{22}\sqrt{T^2-4})}{2\sqrt{T^2-4}},\end{aligned} \tag{3.14}$$

where $T = \sigma_{11}+\sigma_{22}$. Since $I-\mathcal{S}$ is non-singular, there exists a unique solution for c_1, c_2. □

Remark 3.3.4. (1) There are 1-parameter family of P's diagonalizing $\mathcal{S}$ with $\det(P) = 1$. In the above lemma, the c_i's are independent of choice of P because equation (3.14) has coefficients only from the matrix $\mathcal{S}$.

(2) Note that a lattice $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ of Sol_1^4 projects to the lattice $\Gamma_{\mathcal{S}}$ of Sol^3.

(3) The equation (3.14) also shows the cohomology classification. Suppose $\{c_1, c_2\}$ and $\{c_1', c_2'\}$ are solutions for the equations with $\{m_1, m_2\}$ and $\{m_1', m_2'\}$, respectively. Then $(c_1'-c_1, c_2'-c_2) \in \left(\frac{1}{q}\mathbb{Z}\right)^2$ if and only if $(m_1' -$

$m_1, m_2' - m_2) \in \mathrm{Coker}(\mathcal{S}^T - I) \cong \mathrm{Coker}(\mathcal{S} - I)$. This happens if and only if $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)} = \widetilde{\Gamma}_{(\mathcal{S};q,m_1',m_2')}$.

(4) Notice that c_3 does not show up in the presentation of the lattice $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$. This implies that c_3 can be changed without affecting the isomorphism type of the lattice.

Notation 3.3.5 (Standard lattice)**.** The lattice generated by

$$\mathbf{t}_1 = \begin{bmatrix} 1 & p_{11} & c_1 \\ 0 & 1 & p_{21} \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix} 1 & p_{12} & c_2 \\ 0 & 1 & p_{22} \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{t}_3 = \begin{bmatrix} 1 & 0 & c_3 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{t}_4^{\frac{1}{q}} = \begin{bmatrix} 1 & 0 & \frac{1}{q} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $c_3 = 0$, is called a *standard lattice* of Sol_1^4.

Therefore, any lattice of Sol_1^4 is isomorphic to a standard lattice. However, a non-standard lattice (i.e., $c_3 \neq 0$) will be needed when we consider finite extensions of $\widetilde{\Gamma}_{\mathcal{S}}$ (in the holonomy $\mathbb{Z}_4$ case).
The following lemmas on lattices of Sol_1^4 will be needed in the next section.

Lemma 3.3.6. *Let* $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ *be a lattice of* Sol_1^4*, embedded as in assignment (3.13). Let* $r_1, r_2 \in \mathbb{Q}$*. Then*

$$\mathbf{t}_1^{r_1}\mathbf{t}_2^{r_2} = \mathbf{t}_2^{r_2}\mathbf{t}_1^{r_1}\mathbf{t}_4^{r_1 r_2}.$$

Proof. We compute that

$$[\mathbf{t}_1^{r_1}, \mathbf{t}_2^{r_2}] = \mathbf{t}_4^{r_1 r_2 \det(P)} = \mathbf{t}_4^{r_1 r_2}.$$ □

Lemma 3.3.7. *Let* $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ *be a lattice of* Sol_1^4*, embedded as in assignment (3.13). Let* $a_1, a_2 \in \mathbb{Q}$*. Then, for* $\bar{A} = \pm 1$*,*

$$\mathbf{t}_3^{\bar{A}}\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{-\bar{A}} = \mathbf{t}_1^{l_1}\mathbf{t}_2^{l_2}\mathbf{t}_4^{v}, \ \textit{where} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \mathcal{S}^{\bar{A}} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \ \textit{and } v \in \mathbb{Q}.$$

Proof. The definition of $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ shows that $\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \mathcal{S}^{\bar{A}} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. It remains to show that v rational. Because a_1 and a_2 are rational, there is a positive integer n so that $na_1, na_2 \in \mathbb{Z}$. By Lemma 3.3.6,

$$\begin{aligned} (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2})^n &= \mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2} \cdots \mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2} (n \text{ times}) \\ &= \mathbf{t}_1^{na_1}\mathbf{t}_2^{na_2}\mathbf{t}_4^{u'}, \text{ for } u' \in \mathbb{Q}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{t}_3^{\bar{A}}(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2})^n\mathbf{t}_3^{-\bar{A}} &= \mathbf{t}_3^{\bar{A}}\mathbf{t}_1^{na_1}\mathbf{t}_2^{na_2}\mathbf{t}_4^{u'}\mathbf{t}_3^{-\bar{A}} \\ &= \mathbf{t}_3^{\bar{A}}\mathbf{t}_1^{na_1}\mathbf{t}_2^{na_2}\mathbf{t}_3^{-\bar{A}}\mathbf{t}_4^{u'} \\ &= \mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_4^{u} \text{ for some } n_1, n_2 \in \mathbb{Z}, \text{ and } u \in \mathbb{Q}, \end{aligned}$$

where the last equality follows from that na_1 and na_2 are integers, together with the relations in Lemma 3.3.3.

On the other hand, we have that

$$\begin{aligned} \mathbf{t}_3^{\bar{A}}(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2})^n\mathbf{t}_3^{-\bar{A}} &= \mathbf{t}_3^{\bar{A}}\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{-\bar{A}} \cdots \mathbf{t}_3^{\bar{A}}\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{-\bar{A}} (n \text{ times}) \\ &= \mathbf{t}_1^{l_1}\mathbf{t}_2^{l_2}\mathbf{t}_4^{v} \cdots \mathbf{t}_1^{l_1}\mathbf{t}_2^{l_2}\mathbf{t}_4^{v} (n \text{ times}) \\ &= \mathbf{t}_1^{nl_1}\mathbf{t}_2^{nl_2}\mathbf{t}_4^{nv+w} \text{ for some } w \in \mathbb{Q}, \end{aligned}$$

where the last equality follows from Lemma 3.3.6.

Consequently, we have

$$\mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_4^{u} = \mathbf{t}_1^{nl_1}\mathbf{t}_2^{nl_2}\mathbf{t}_4^{nv+w}.$$

This forces $n_1 = nl_1$ and $n_2 = nl_2$. Therefore,

$$nv + w = u.$$

Since $n \in \mathbb{Z}$, $u, w \in \mathbb{Q}$, it follows that $v \in \mathbb{Q}$. □

Example 3.3.8. For $\mathcal{S} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$. We have $\lambda = \frac{1}{2}\left(5 + \sqrt{21}\right)$. Take

$$P = \begin{bmatrix} -\frac{5d}{\sqrt{21}} & \left(\frac{1}{2} - \frac{1}{2\sqrt{21}}\right) d \\ -\frac{1}{d} & \frac{-1-\sqrt{21}}{10d} \end{bmatrix},$$

($d \neq 0$ arbitrary) and $q = 1$ to get

$$c_1 = \tfrac{1}{42}\left(5(7+\sqrt{21}) + 28m_1 - 70m_2\right)$$
$$c_2 = \tfrac{1}{42}\left(-(49+\sqrt{21}) - 14m_1 + 14m_2\right).$$

Therefore, $\widetilde{\Gamma}_{(\mathcal{S};1,m_1,m_2)}$ is represented as follows:

$$\mathbf{t}_1 = \begin{bmatrix} 1 & -\frac{5d}{\sqrt{21}} & \frac{1}{42}\left(28m_1 - 70m_2 + 5\sqrt{21} + 35\right) \\ 0 & 1 & -\frac{1}{d} \\ 0 & 0 & 1 \end{bmatrix},$$
$$\mathbf{t}_2 = \begin{bmatrix} 1 & -\frac{1}{42}\left(-21 + \sqrt{21}\right) d & \frac{1}{42}\left(-14m_1 + 14m_2 - \sqrt{21} - 49\right) \\ 0 & 1 & \frac{-1-\sqrt{21}}{10d} \\ 0 & 0 & 1 \end{bmatrix},$$
$$\mathbf{t}_3 = \begin{bmatrix} 1 & 0 & c_3 \\ 0 & \frac{1}{2}\left(5+\sqrt{21}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{t}_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

These have relations

$$\mathbf{t}_3\mathbf{t}_1\mathbf{t}_3^{-1} = \mathbf{t}_1^2\mathbf{t}_2^5\mathbf{t}_4^{m_1},\ \mathbf{t}_3\mathbf{t}_2\mathbf{t}_3^{-1} = \mathbf{t}_1\mathbf{t}_2^3\mathbf{t}_4^{m_2}.$$

Notice that

$$\mathcal{S} - I = \begin{bmatrix} 1 & 1 \\ 5 & 2 \end{bmatrix},$$

and $\mathrm{Coker}(\mathcal{S} - I) = \mathbb{Z}_3$. Therefore, depending on (m_1, m_2), we have 3 non-equivalent $\widetilde{\Gamma}_{(\mathcal{S};1,m_1,m_2)}$'s. Of course, one can throw in a finer generator $\mathbf{t}_4^{\frac{1}{q}}$, in place of $\mathbf{t}_4$.

3.3.9 (Classification of Sol_1^4 Crystallographic Groups). Let $\Pi \subset \mathrm{Sol}_1^4 \rtimes K$ be a crystallographic group of Sol_1^4, where K is a maximal compact subgroup of $\mathrm{Aut}(\mathrm{Sol}_1^4)$. As all maximal compact subgroups of Sol_1^4 are conjugate, we can assume that K is the maximal compact subgroup

$$D_4 = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$$

of $\mathrm{Aut}(\mathrm{Sol}_1^4)$, whose action on Sol_1^4 is described in Proposition 3.1.8. Furthermore, we can conjugate Π in $\mathrm{Aff}(\mathrm{Sol}_1^4)$ so that the lattice of Π is some $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$, where $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ is embedded in Sol_1^4 as in assignment (3.13) in Subsection 3.3.1.

Proposition 3.3.10. (1) *Any crystallographic group Π of Sol_1^4 can be conjugated into $\mathrm{Sol}_1^4 \rtimes D_4$ in such a way that*

$$\Pi \cap \mathrm{Sol}_1^4 = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}} \rangle,$$

where

$$\mathbf{t}_1 = \begin{bmatrix} 1 & p_{11} & c_1 \\ 0 & 1 & p_{21} \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{t}_2 = \begin{bmatrix} 1 & p_{12} & c_2 \\ 0 & 1 & p_{22} \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{t}_3 = \begin{bmatrix} 1 & 0 & c_3 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{t}_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(2) *The holonomy group Φ is generated by at most two elements of D_4, and thus Π is generated by $\langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}} \rangle$ and at most two isometries of the form $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A)$, for $A \in D_4$ and real numbers a_i.*

Proof. Let $\widetilde{\Gamma} = \Pi \cap \mathrm{Sol}_1^4$. This lattice must meet the center of Sol_1^4 in a lattice: $\widetilde{\Gamma} \cap \mathcal{Z}(\mathrm{Sol}_1^4)$ is a lattice of $\mathcal{Z}(\mathrm{Sol}_1^4)$, say generated by $\mathbf{t}_4^{\frac{1}{q}}$. Also $\widetilde{\Gamma} \cap \mathrm{Nil}$ is a lattice of the nilradical Nil, so we can find generators $\langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_4^{\frac{1}{q}} \rangle$ of this lattice as given in the statement. The remaining one generator for the lattice $\widetilde{\Gamma}$ must project down to a generator of the quotient $\mathbb{Z}$. It must be of the form

$$\mathbf{t}_3'' = \begin{bmatrix} 1 & a & c_3 \\ 0 & \lambda & b \\ 0 & 0 & 1 \end{bmatrix}$$

Conjugation by $\begin{bmatrix} 1 & \frac{a}{1-\lambda} & 0 \\ 0 & \lambda & -\frac{b\lambda}{1-\lambda} \\ 0 & 0 & 1 \end{bmatrix}$ maps $\mathbf{t}_3''$ to the form of $\mathbf{t}_3$. Recall that $\widetilde{\Gamma}/\mathcal{Z}(\widetilde{\Gamma})$ gives rise to $\mathcal{S}$, and we can find $P = (p_{ij})$ diagonalizing $\mathcal{S}$. Assuming $\det(P) = 1$ (this can always be achieved, as in the case of Sol^3), we have $[\mathbf{t}_1, \mathbf{t}_2] = \mathbf{t}_4$. Therefore, any lattice is conjugate to a lattice $\langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}} \rangle$. $\square$

However, we will see that we can always take $c_3 = 0$, except possibly in the case when the holonomy Φ is $\mathbb{Z}_4$. Because lattices of Sol_1^4 project to lattices of Sol^3, the projections $\mathrm{Sol}_1^4 \to \mathrm{Sol}^3$ and $\mathrm{Aut}(\mathrm{Sol}_1^4) \to \mathrm{Aut}(\mathrm{Sol}^3)$ induce a projection $\mathrm{Sol}_1^4 \rtimes D_4 \to \mathrm{Sol}^3 \rtimes D_4$ which carries a Sol_1^4 crystallographic group to a Sol^3 crystallographic group. Furthermore, when $\mathit{\Pi}$ is embedded in $\mathrm{Sol}_1^4 \rtimes D_4$ as in Proposition 3.3.10, the lattice $\widetilde{\Gamma}_{\mathcal{S}} = \widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ projects to a standard lattice $\Gamma_{\mathcal{S}}$ of Sol^3. That is, we have the following commuting diagram:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \frac{1}{q}\mathbb{Z} & = & \frac{1}{q}\mathbb{Z} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \widetilde{\Gamma}_{\mathcal{S}} & \longrightarrow & \mathit{\Pi} & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Gamma_{\mathcal{S}} & \longrightarrow & Q & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

Here is an outline of our procedure for obtaining the Sol_1^4 crystallographic groups.

(1) For each Sol^3 crystallographic group Q from Theorem 3.2.21, classify lifts of $\Gamma_{\mathcal{S}}$ to lattices of $\widetilde{\Gamma}_{\mathcal{S}}$ of Sol_1^4 (Theorem 3.3.2).

(2) Show that the abstract kernel $\Phi \to \mathrm{Out}(\Gamma_{\mathcal{S}})$ will induce an abstract kernel $\Phi \to \mathrm{Out}(\widetilde{\Gamma}_{\mathcal{S}})$, after possibly making the central generator of $\widetilde{\Gamma}_{\mathcal{S}}$ finer (i.e., taking a larger value of q).

(3) Once an abstract kernel $\Phi \to \mathrm{Out}(\widetilde{\Gamma}_{\mathcal{S}})$ is fixed, show that after possibly taking a larger value of q, the obstruction in $H^3(\Phi;\mathbb{Z})$ to the existence of $\mathit{\Pi}$ vanishes. Then the equivalence classes of extensions

$$1 \to \widetilde{\Gamma}_{\mathcal{S}} \to \mathit{\Pi} \to \Phi \to 1$$

are classified by $H^2(\Phi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = H^2(\Phi;\mathbb{Z})$. The details follow.

Our goal is finding all crystallographic groups $\mathit{\Pi}$ of Sol_1^4 which project down to Q. In general, it is *not* true that there exists Π fitting the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} & & \mathbb{Z} & = & \mathbb{Z} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & \widetilde{\Gamma}_{\mathcal{S}} & \longrightarrow & \Pi & \longrightarrow & \Phi & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Gamma_{\mathcal{S}} & \longrightarrow & Q & \longrightarrow & \Phi & \longrightarrow & 1 \end{array}$$

Even though $\widetilde{\Gamma}_{\mathcal{S}}$ always exists, for Π to exist, sometimes the kernel $\mathbb{Z}$ needs to be "inflated". It turns out that, after appropriate inflation, an extension Π always exists. We will need the following lemma.

Proposition 3.3.11. *Let Q be a* Sol^3 *crystallographic group with standard lattice $\Gamma_{\mathcal{S}}$. Suppose in* Proposition 3.3.10, *we have fixed the c_i, as well as set $q = 1$, thus fixing the lattice $\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4 \rangle$. For any generator $A \in \Phi$, let*

$$\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A) = (a, A).$$

The powers a_1, a_2, and a_3 are determined by the abstract kernel $\Phi \to \mathrm{Out}(\Gamma_{\mathcal{S}})$ (Remark 3.2.18). *Consider the effect that conjugation by α has on $\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)}$. Note that conjugation by α is independent of a_4. We have the relations:*

$$\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_1^{m_1}\mathbf{t}_2^{m_2}\mathbf{t}_4^{v_1}, \text{ where } \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \varphi(\bar{\alpha})\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_4^{v_2}, \text{ where } \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \varphi(\bar{\alpha})\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{w_1}\mathbf{t}_2^{w_2}\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{v_3}, \text{ where } \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \left(I - \mathcal{S}^{\bar{A}}\right)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix},$$

$$\alpha\mathbf{t}_4\alpha^{-1} = \mathbf{t}_4^{\hat{A}},$$

where v_1 and v_2 are rational numbers. We can adjust c_3 so that v_3 is rational, as well.

Proof. First we claim that the image of $\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)}$ under conjugation by α,

$$\mu(\alpha)(\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)}) = \alpha\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)}\alpha^{-1},$$

is a lattice of Sol_1^4 lifting the standard lattice $\Gamma_{\mathcal{S}}$ of Sol^3.

As conjugation by α defines an automorphism of Sol_1^4, $\mu(\alpha)(\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)})$ is again a subgroup of Sol_1^4. Since conjugation by α is an affine diffeomorphism of Sol_1^4, $\mu(\alpha)(\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)})$ is also discrete. Because $\mu(\alpha)(\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)})$ lifts the standard lattice $\Gamma_{\mathcal{S}}$ of Sol^3, and maps the generator $\mathbf{t}_4$ of

$$\mathcal{Z}(\mathrm{Sol}_1^4) \cap \widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)}$$

to $\mathbf{t}_4^{\hat{A}}$, it follows that $\mu(\alpha)(\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)})$ is cocompact. Therefore $\mu(\alpha)(\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)})$ is a lattice of Sol_1^4 lifting $\Gamma_{\mathcal{S}}$.

All such lifts are given in Lemma 3.3.3. From equation (3.14) in Subsection 3.3.1, we see that for any two solutions c_1, c_2 and c_1', c_2', both $c_1' - c_1$ and $c_2' - c_2$ must be rational. Thus v_1 and v_2 are rational numbers.

For

$$\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{w_1}\mathbf{t}_2^{w_2}\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{v_3}, \text{ where } \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \left(I - \mathcal{S}^{\bar{A}}\right)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \tag{3.15}$$

we need only consider two cases by Theorem 3.2.21. First, consider the case when $a_3 = \frac{1}{2}$. Then A must be diagonal, so that $\bar{A} = +1$:

$$\alpha = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{a_4}, A)$$
$$\bar{A} = +1.$$

In the computation below, let $\hat{\mathbf{t}}_3$ denote $\mathbf{t}_3$ with the $(1,3)$-slot set to be zero, so that $\mathbf{t}_3 = \hat{\mathbf{t}}_3\mathbf{t}_4^{c_3}$.

$$\begin{aligned}
\alpha\mathbf{t}_3\alpha^{-1} &= \mathbf{t}_3^{\frac{1}{2}} A(\mathbf{t}_3)\mathbf{t}_3^{-\frac{1}{2}} \\
&= \mathbf{t}_3^{\frac{1}{2}} A(\hat{\mathbf{t}}_3\mathbf{t}_4^{c_3})\mathbf{t}_3^{-\frac{1}{2}} \\
&= \mathbf{t}_3^{\frac{1}{2}} \hat{\mathbf{t}}_3\mathbf{t}_4^{\hat{A}c_3}\mathbf{t}_3^{-\frac{1}{2}} \\
&= \mathbf{t}_3^{\frac{1}{2}} \hat{\mathbf{t}}_3\mathbf{t}_4^{c_3}\mathbf{t}_4^{-c_3}\mathbf{t}_4^{\hat{A}c_3}\mathbf{t}_3^{-\frac{1}{2}} \\
&= \mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_3\mathbf{t}_3^{-\frac{1}{2}}\mathbf{t}_4^{(\hat{A}-1)c_3} = \mathbf{t}_3\mathbf{t}_4^{(\hat{A}-1)c_3}.
\end{aligned}$$

Since $\hat{A} = \pm 1$, there is always a choice of c_3 which makes $(\hat{A}-1)c_3 \in \mathbb{Q}$. Now consider the case when $a_3 = 0$, $\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_4^{a_4}, A)$. We compute:

$$\begin{aligned}
\alpha\mathbf{t}_3\alpha^{-1} &= \mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}A(\mathbf{t}_3)\mathbf{t}_2^{-a_2}\mathbf{t}_1^{-a_1} \\
&= \mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}A(\hat{\mathbf{t}}_3\mathbf{t}_4^{c_3})\mathbf{t}_2^{-a_2}\mathbf{t}_1^{-a_1} \\
&= \mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\hat{\mathbf{t}}_3^{\bar{A}}\mathbf{t}_4^{\hat{A}c_3}\mathbf{t}_2^{-a_2}\mathbf{t}_1^{-a_1} \\
&= \mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\hat{\mathbf{t}}_3^{\bar{A}}\mathbf{t}_4^{\bar{A}c_3}\mathbf{t}_4^{-\bar{A}c_3}\mathbf{t}_4^{\hat{A}c_3}\mathbf{t}_2^{-a_2}\mathbf{t}_1^{-a_1} \\
&= \mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{(\hat{A}-\bar{A})c_3}\mathbf{t}_2^{-a_2}\mathbf{t}_1^{-a_1} \\
&= \left(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{\bar{A}}\mathbf{t}_2^{-a_2}\mathbf{t}_1^{-a_1}\mathbf{t}_3^{-\bar{A}}\right)\mathbf{t}_3{}^{\bar{A}}\mathbf{t}_4^{(\hat{A}-\bar{A})c_3}.
\end{aligned}$$

Now by Lemmas 3.3.6 and 3.3.7, and using that a_1, a_2 are rational, we have

$$\begin{aligned}
\left(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{\bar{A}}\mathbf{t}_2^{-a_2}\mathbf{t}_1^{-a_1}\mathbf{t}_3^{-\bar{A}}\right)\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{(\hat{A}-\bar{A})c_3} &= \left(\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_4^{u}\right)\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{(\hat{A}-\bar{A})c_3} \\
&= \mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{u+(\hat{A}-\bar{A})c_3},
\end{aligned}$$

for a rational number u. Equating this with equation (3.15), we obtain

$$\mathbf{t}_1^{w_1}\mathbf{t}_2^{w_2}\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{v_3} = \mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{u+(\hat{A}-\bar{A})c_3}.$$

Now $w_1 = b_1$ and $w_2 = b_2$ is forced. Therefore,

$$v_3 = u + (\hat{A} - \bar{A})c_3.$$

Because $\hat{A} = \pm 1$, $\bar{A} = \pm 1$, and u is rational, c_3 can always be chosen so that v_3 is rational. $\square$

Proposition 3.3.12. *Let Q be a crystallographic group of* Sol^3*, from* Theorem 3.2.21. *Then there exists c_i and q so that the abstract kernel $\Phi \to \mathrm{Out}(\Gamma_{\mathcal{S}})$ induces $\Phi \to \mathrm{Out}(\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)})$, where $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}} \rangle$ is a lattice of* Sol_1^4 *lifting $\Gamma_{\mathcal{S}}$.*

Proof. The abstract kernel of $\Phi \to \mathrm{Out}(\Gamma_{\mathcal{S}})$ is given by, for $A \in \Phi$,

$$\mu(\alpha) : \Gamma_{\mathcal{S}} \to \Gamma_{\mathcal{S}}, \text{ where } \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A) \in Q.$$

For any integer $q > 0$, we can form the lattice of Sol_1^4

$$\langle \widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)}, \mathbf{t}_4^{\frac{1}{q}} \rangle = \widetilde{\Gamma}_{(\mathcal{S};q,qn_1,qn_2)}.$$

That is, we add a finer generator of the central direction to $\widetilde{\Gamma}_{(\mathcal{S};1,n_1,n_2)}$.

Now, for each generator of $A \in \Phi$, the v_i in Proposition 3.3.11 are rational. Therefore, for q large enough, $\widetilde{\Gamma}_{(\mathcal{S};q,qn_1,qn_2)}$ is invariant under conjugation by $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A)$, for each $A \in \Phi$. As this conjugation is independent of lift of $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A)$ to $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A)$, with $m_1 = qn_1$ and $m_2 = qn_2$, we obtain an abstract kernel $\Phi \to \mathrm{Out}(\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)})$. $\square$

Proposition 3.3.13. *Let $Q \hookrightarrow \mathrm{Sol}^3 \rtimes D_4$ be a crystallographic group for* Sol^3 *containing $\Gamma_{\mathcal{S}}$. Assume that the abstract kernel $\Phi \to \mathrm{Out}(\Gamma_{\mathcal{S}})$ induces $\Phi \to \mathrm{Out}(\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)})$. Then for some $p > 0$, there exists Π which fits the* *following commuting diagram*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \frac{1}{pq}\mathbb{Z} & = & \frac{1}{pq}\mathbb{Z} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \widetilde{\Gamma}_{(\mathcal{S};pq,pm_1,pm_2)} & \longrightarrow & \mathit{\Pi} & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Gamma_{\mathcal{S}} & \longrightarrow & Q & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Proof. Since the center of $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ is $\frac{1}{q}\mathbb{Z}$ and Φ is finite, $H^3(\Phi;\frac{1}{q}\mathbb{Z})$ is finite. This means the obstruction class to the existence of the extension vanishes if we use $\frac{1}{pq}\mathbb{Z}$ for the coefficients, for some $p > 0$. That is, it vanishes inside $H^3(\Phi;\frac{1}{pq}\mathbb{Z})$. Thus, with such pq, the center of $\widetilde{\Gamma}_{(\mathcal{S};pq,pm_1,pm_2)}$ is $\frac{1}{pq}\mathbb{Z}$, and an extension $\mathit{\Pi}$ exists. □

So we can assume that after appropriate inflation, there exists an extension $\mathit{\Pi}$ with lattice $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$, for some $q > 0$. The Seifert Construction will show that such an abstract extension actually embeds in $\mathrm{Sol}_1^4 \rtimes D_4$ as a crystallographic group.

Theorem 3.3.14. *Let* $\widetilde{\Gamma} = \widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ *be a lattice of* Sol_1^4, *and*

$$1 \longrightarrow \widetilde{\Gamma} \longrightarrow \mathit{\Pi} \longrightarrow \Phi \longrightarrow 1$$

be an extension of $\widetilde{\Gamma}$ *by a finite group* Φ. *Then there exists a homomorphism*

$$\theta : \mathit{\Pi} \to \mathrm{Sol}_1^4 \rtimes D_4 \subset \mathrm{Sol}_1^4 \rtimes \mathrm{Aut}(\mathrm{Sol}_1^4)$$

carrying $\widetilde{\Gamma}$ *onto a standard lattice. This* θ *is injective if and only if* $\Phi \subset D_4$.

Such θ is unique up to conjugation by an element of $\mathrm{Sol}_1^4 \rtimes \mathrm{Aut}(\mathrm{Sol}_1^4)$.

Proof. Sol_1^4 is completely solvable, and we can apply the Seifert construction [24, Theorem 7.3.2] with $G = \mathrm{Sol}_1^4$ and $W = \{\text{point}\}$. Since Φ is finite, the homomorphism $\varphi : \mathit{\Pi} \to \mathrm{Out}(\widetilde{\Gamma}) \to \mathrm{Out}(\mathrm{Sol}_1^4)$ has finite image in $\mathrm{Out}(\mathrm{Sol}_1^4)$, and it lifts back to a finite subgroup C of $\mathrm{Aut}(\mathrm{Sol}_1^4)$. But this C can be conjugated into $D_4 \subset \mathrm{Aut}(\mathrm{Sol}_1^4)$, a maximal compact subgroup. Consequently, we have a commuting diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{\Gamma} & \longrightarrow & \mathit{\Pi} & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Sol}_1^4 & \longrightarrow & \mathrm{Sol}_1^4 \rtimes D_4 & \longrightarrow & D_4 & \longrightarrow & 1
\end{array}$$

The essence of the argument is showing that the cohomology set $H^2(\Phi; \mathrm{Sol}_1^4)$ is trivial for any finite group Φ. The uniqueness is a result of [24, Corollary 7.7.4]. It also comes from $H^1(\Phi; \mathrm{Sol}_1^4) = 0$. □

After inflation, the Seifert Construction produces a crystallographic group of Sol_1^4. Often we can assume that $c_3 = 0$, that is, $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$ is a standard lattice of Sol_1^4. Recall that $\mathrm{Aut}(\mathrm{Sol}_1^4) = \mathbb{R} \rtimes \mathrm{Aut}(\mathrm{Sol}^3)$ (Proposition 3.1.8), where $\hat{k} \in \mathbb{R}$ acts by

$$\begin{bmatrix} 1 & e^u x & z \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & e^u x & z + ku \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix}.$$

We have the following:

Theorem 3.3.15. *For all holonomy groups, except $\mathbb{Z}_4$, a crystallographic group $\mathit{\Pi}$ of Sol_1^4 embeds into $\mathrm{Sol}_1^4 \rtimes D_4$ in such a way that $\mathit{\Pi} \cap \mathrm{Sol}_1^4$ is a standard lattice ($c_3 = 0$).*

Proof. Let e denote the identity element of Sol_1^4. For the statement concerning c_3, conjugation by $(e, \hat{k})$ with $k = -\frac{c_3}{\lambda}$ sets $c_3 = 0$ in $\mathbf{t}_3$. However, this conjugation moves D_4 to $\hat{k} D_4 \hat{k}^{-1}$.

Suppose every $A \in \Phi$ satisfies $\bar{A}\hat{A} = +1$. Since such A commute with $\hat{k}$, conjugation by $(e, \hat{k})$ leaves the holonomy group Φ inside D_4 while moving $\mathbf{t}_3$ so that $c_3 = 0$. This applies to, from the list of Theorem 3.2.9, all the groups lifting Sol^3 crystallographic groups of type `(2a)`, `(2b)`, `(3)`, `(3i)`, `(6a)`, `(6ai)`, `(6b)`, and `(6bi)`.

Suppose Φ contains $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then Corollary 3.2.19 and Lemma 3.3.16 show that a generator α of Π projecting to $A \in \Phi$ can be conjugated to $\alpha = (\mathbf{t}_3^{\frac{1}{2}}, A)$ (so that $a_1 = a_2 = a_4 = 0$). Then, we shall show that $\mathbf{t}_3 = \hat{\mathbf{t}}_3 \mathbf{t}_4^{c_3}$ can be replaced by $\hat{\mathbf{t}}_3$ (where $\hat{\mathbf{t}}_3$ is $\mathbf{t}_3$ with $c_3 = 0$).

$$
\begin{aligned}
\alpha^2 &= (\mathbf{t}_3^{\frac{1}{2}}, A)^2 = ((\hat{\mathbf{t}}_3 \mathbf{t}_4^{c_3})^{\frac{1}{2}}, A)^2 = (\hat{\mathbf{t}}_3 \mathbf{t}_4^{c_3})^{\frac{1}{2}} A ((\hat{\mathbf{t}}_3 \mathbf{t}_4^{c_3})^{\frac{1}{2}}) \\
&= \hat{\mathbf{t}}_3^{\frac{1}{2}} \mathbf{t}_4^{\frac{c_3}{2}} \cdot \hat{\mathbf{t}}_3^{\frac{1}{2}} \mathbf{t}_4^{-\frac{c_3}{2}} = \hat{\mathbf{t}}_3.
\end{aligned}
$$

Thus $\hat{\mathbf{t}}_3 = \alpha^2 \in \Pi$, and we can take $\hat{\mathbf{t}}_3$ instead of $\mathbf{t}_3$ as a generator for the same group (which is apparently redundant since α is in the set already). From the list in Theorem 3.2.9, the groups `(1)`, `(5)`, `(7)` and `(7i)` contain such an A in the holonomy.

The only case that is not covered by these two cases is when $\Phi = \mathbb{Z}_4$ (type `(4)` in the list), which is discussed below in our main classification (Theorem 3.3.21). □

Lemma 3.3.16. *If* $\det(A) = -1$, *by conjugation,* a_4 *can be made 0.*

Proof. Suppose $\det(A) = -1$. Conjugation by $\mathbf{t}_4^{-\frac{a_4}{2}}$ fixes the lattice $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$, and moves $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A)$ to $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A)$. □

Proposition 3.3.17 (Fixing a_4, b_4). *Consider the commuting diagram in* Proposition 3.3.13*. Given Q and integers q, m_1, m_2, we had $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$. The only thing that remains for the construction of Π is fixing a_4, b_4. As is well known, all the extensions Π in the short exact sequence*

$$1 \to \widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)} \to \Pi \to \Phi \to 1$$

are classified by $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)})) = H^2(\Phi; \mathbb{Z})$. When $\Phi = \langle A \rangle$,

$$H^2(\mathbb{Z}_p; \mathbb{Z}) = \begin{cases} 0, & \text{if } \hat{A} = -1; \\ \mathbb{Z}_p, & \text{if } \hat{A} = 1, \end{cases}$$

see [25, Theorem 7.1, p.122] .

In actual calculation, this becomes an equation

$$\alpha^p = \mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_3^{n_3}\mathbf{t}_4^{k_4}$$

for integers n_i and $k_4 = \frac{i}{q}$, $i = 0, 1, \cdots, p-1$. □

Remark 3.3.18. When $\Phi = \langle A, B \rangle$ is not cyclic, $\hat{A} = \hat{B} = +1$ never happens, so we can set one of a_4, b_4 to zero. Thus, $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ is cyclic for all Φ.

3.3.19 (Detecting Torsion in Sol_1^4 Crystallographic Groups). Given a lattice $\widetilde{\Gamma}_{\mathcal{S}}$ of Sol_1^4 (which projects to a lattice $\Gamma_{\mathcal{S}}$ of Sol^3), the short exact sequence

$$1 \to \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}) \to \widetilde{\Gamma}_{\mathcal{S}} \to \Gamma_{\mathcal{S}} \to 1$$

induces an S^1-bundle over the solvmanifold $\Gamma_{\mathcal{S}}\backslash\mathrm{Sol}^3$,

$$S^1 \to \widetilde{\Gamma}_{\mathcal{S}}\backslash(\mathrm{Sol}_1^4) \to \Gamma_{\mathcal{S}}\backslash\mathrm{Sol}^3.$$

The following lemma will be useful for determining when a Sol_1^4 crystallographic group has torsion.

Lemma 3.3.20. *Let* $\widetilde{\Gamma}_{\mathcal{S}}$ *be a lattice of* Sol_1^4*, projecting to a standard lattice* $\Gamma_{\mathcal{S}}$ *of* Sol^3*, and suppose that for* $\alpha \in \mathrm{Sol}_1^4 \rtimes D_4$*, the group* $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha \rangle$ *is crystallographic. Let* $\bar{\alpha}$ *denote the projection of* α *to* $\mathrm{Sol}^3 \rtimes D_4$*. When the automorphism part of* α *acts as a reflection on the center of* Sol_1^4*,* Π *is torsion-free if and only if* $\langle \Gamma_{\mathcal{S}}, \bar{\alpha} \rangle$ *is torsion-free.*

Proof. Obviously, if $\langle \Gamma_{\mathcal{S}}, \bar{\alpha} \rangle$ is torsion-free, then Π must be torsion-free. For the converse, suppose that $\langle \Gamma_{\mathcal{S}}, \bar{\alpha} \rangle$ has torsion. In this case, the action of $\bar{\alpha}$ on the solvmanifold $\Gamma_{\mathcal{S}} \backslash \mathrm{Sol}^3$ must fix a point. Observe that the action of α on the solvmanifold $\widetilde{\Gamma}_{\mathcal{S}} \backslash \mathrm{Sol}_1^4$ is S^1 fiber preserving. Therefore, a circle fiber is left invariant under the action of α. Since α acts as reflection on the fiber, α must fix a point. Therefore, Π has torsion. $\square$

Theorem 3.3.21 (Classification of Sol_1^4-geometry)**.** *The following is a complete list of crystallographic groups of* Sol_1^4*, viewed as lifts of* Sol^3 *crystallographic groups, see* Theorem 3.2.21.

We assume that in each case q, c_1, c_2, and c_3 are fixed so as to induce an abstract kernel $\Phi \to \mathrm{Out}(\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)})$ with vanishing obstruction in $H^3(\Phi; \widetilde{\Gamma}_{\mathcal{S}})$. Recall that this was possible by Proposition 3.3.13 and Theorem 3.3.14. In all cases, except, $\Phi = \mathbb{Z}_4$, we can take $c_3 = 0$ (Theorem 3.3.15). In the $\mathbb{Z}_4$ holonomy case, we have two different (up to isomorphism) choices for c_3. Whenever the holonomy group contains an automorphism which is off-diagonal, the orbifold is non-orientable. To avoid repetition, we do not give the classification of the exponents on $\mathbf{t}_1, \mathbf{t}_2$ by $H^2_\varphi(\mathbb{Z}_\Phi; \mathbb{Z}^2) \cong H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$, as these exponents are fixed by Q (see Theorem 3.2.21).

(0) $\Phi = \text{trivial}$

- $c_3 = 0$
- $\mathit{\Pi} = \widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$.
- Torsion-free

(1) $\Phi = \mathbb{Z}_2$: $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

- $c_3 = 0$
- $\mathit{\Pi} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_3^{\frac{1}{2}}, A) \rangle$.
- $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ is trivial.
- Torsion-free

(2a) $\Phi = \mathbb{Z}_2$: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

- $c_3 = 0$
- $\mathit{\Pi} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_1^{a_1} \mathbf{t}_2^{a_2} \mathbf{t}_4^{a_4}, A) \rangle$.
- $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$. There are two choices for a_4, the solutions of $\alpha^2 = \mathbf{t}_4^{\frac{i}{q}}$ $(i = 0, 1)$.
- Torsion-free for $i = 1$

(2b) $\Phi = \mathbb{Z}_2$: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

- $c_3 = 0$
- $\mathit{\Pi} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_3^{\frac{1}{2}} \mathbf{t}_4^{a_4}, A) \rangle$.
- $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2q}$.
- Torsion-free

(3) $\Phi = \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- $c_3 = 0$
- $\mathit{\Pi} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A)\rangle$.
- $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ is trivial.
- Always has torsion

(3i) $\Phi = \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- $c_3 = 0$
- $\mathit{\Pi} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A)\rangle$.
- $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ is trivial.
- $\mathit{\Pi}$ is torsion-free if and only if $a_1 = \frac{1}{2}$ and $a_2 \not\equiv \frac{(\sigma_{11}+1)(2n+1)}{2\sigma_{12}}$ for any $n \in \mathbb{Z}$.

(4) $\Phi = \mathbb{Z}_4$: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

- There are two choices for c_3 in $\mathbf{t}_3$. They are solutions of $d = 0$ or $d = \frac{1}{q}$ for c_3, where $\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{(1-\sigma_{22})a_1+\sigma_{12}a_2}\mathbf{t}_2^{\sigma_{21}a_1+(1-\sigma_{11})a_2}\mathbf{t}_3^{-1}\mathbf{t}_4^{d}$. Each corresponds to a different abstract kernel $\Phi \to \mathrm{Out}(\widetilde{\Gamma}_{\mathcal{S}})$.
- $\mathit{\Pi} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_4^{a_4}, A)\rangle$.
- $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_4$. There are 4 choices for a_4 which make $\alpha^4 = \mathbf{t}_4^{\frac{i}{q}}$, $(i = 0, 1, 2, 3)$
- $\mathit{\Pi}$ is torsion-free only when $i = 1, 3$.

(5) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

- $c_3 = 0$

- $\mathit{\Pi} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_3^{\frac{1}{2}}, A)\beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_4^{b_4}, B)\rangle$.

- $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$. There are two choices for b_4, the solutions of $\beta^2 = \mathbf{t}_4^{\frac{i}{q}}, (i = 0, 1)$.

- $\mathit{\Pi}$ is torsion-free only when $i = 1$.

(6a) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

- $c_3 = 0$

- $\mathit{\Pi} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_4^{b_4}, B)\rangle$.

- $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$. There are two choices for b_4, the solutions of $\beta^2 = \mathbf{t}_4^{\frac{i}{q}}, (i = 0, 1)$.

- Always has torsion

(6ai) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

- $c_3 = 0$

- $\mathit{\Pi} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_4^{b_4}, B)\rangle$.

- $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$. There are two choices for b_4, the solutions of $\beta^2 = \mathbf{t}_4^{\frac{i}{q}}, (i = 0, 1)$.

- $\mathit{\Pi}$ is torsion-free if and only if $i = 1$, $a_1 = \frac{1}{2}$, $a_2 = b_2 + \frac{1}{2}$, $b_1 \not\equiv \frac{\sigma_{12}(2n+1)}{2(\sigma_{11}-1)} + \frac{1}{2}$, $b_2 \not\equiv \frac{(\sigma_{11}+1)(2m+1)}{2\sigma_{12}} + \frac{1}{2}$ for any $m, n \in \mathbb{Z}$.

(6b) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

- $c_3 = 0$

- $\mathit{\Pi} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B)\rangle$.

- $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$. There are two choices for b_4, the solutions of $\beta^2 = \mathbf{t}_3\mathbf{t}_4^{\frac{i}{q}}, (i = 0, 1)$.

• Always has torsion

(6bi) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

• $c_3 = 0$

• $\Pi = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B)\rangle$.

• $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$. There are two choices for b_4, the solutions of $\beta^2 = \mathbf{t}_3\mathbf{t}_4^{\frac{i}{q}}, (i = 0, 1)$.

• Π is torsion-free if and only if $a_1 = \frac{1}{2}$ and $a_2 \not\equiv \frac{(k_{11}-1)(2n+1)}{2k_{12}}$ for any $n \in \mathbb{Z}$.

(7) $\Phi = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

• $c_3 = 0$

• $\Pi = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B)\rangle$.

• $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_4$. There are 4 choices for b_4, the solutions of $(\beta\alpha)^4 = \mathbf{t}_4^{\frac{j}{q}}, (j = 0, 1, 2, 3)$.

• Always has torsion

(7i) $\Phi = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

• $c_3 = 0$

• $\Pi = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4^{\frac{1}{q}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B)\rangle$.

• $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_4$. There are 4 choices for b_4, the solutions of $(\beta\alpha)^4 = \mathbf{t}_4^{\frac{j}{q}}, (j = 0, 1, 2, 3)$.

• Π is torsion-free if and only if $j = 1, 3$ and $a_1 = \frac{1}{2}$ and $a_2 = -\frac{k_{21}+1}{2k_{11}} + \frac{i}{k_{11}}$ for $i = 0, \ldots, k_{11} - 1$.

Proof. In cases (2a), (2b) and (4), $\Phi = \mathbb{Z}_p$, $p = 2$ or 4. Let $\mathbf{t}_4^{\frac{1}{q}}$ be a free generator of $\widetilde{\Gamma}_{\mathcal{S}} \cap \mathcal{Z}(\mathrm{Sol}_1^4)$. Since $\det(A) = +1$, α^p has $\mathbf{t}_4$ component $\mathbf{t}_4{}^{p \cdot a_4 + \ell}$, where ℓ is independent of a_4. The equation

$$p \cdot a_4 + \ell = 0$$

has solutions

$$a_4, a_4 + \tfrac{1}{p \cdot q}, \cdots, a_4 + \tfrac{p-1}{p \cdot q},$$

each corresponding to a different class in $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$. In fact, the number ℓ is always a rational number, and hence so is a_4 (or b_4). The remaining cases when $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$ or D_4 are similar. We set one of exponents on $\mathbf{t}_4 = 0$ by Lemma 3.3.16, and apply the above technique to find the remaining exponent on $\mathbf{t}_4$.

(0) See Theorem 3.3.2.

(1) Since $\hat{A} = -1$, Lemma 3.3.16 implies a_4 can be conjugated to zero. So, $\mathit{\Pi} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \alpha = (\mathbf{t}_3^{\frac{1}{2}}, A) \rangle$. As Q is torsion-free, so is $\mathit{\Pi}$.

(2a) There are two choices for a_4, the solutions of $\alpha^2 = \mathbf{t}_4^{\frac{i}{q}}, (i = 0, 1)$. Recall that Q is never torsion-free, and thus $\mathit{\Pi}$ is torsion-free only when $\alpha^2 = \mathbf{t}_4^{\frac{1}{q}}$. That is, when $i = 1$, the class representing $\mathit{\Pi}$ in $H^2(\mathbb{Z}_2; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ corresponds to the non-split (torsion-free) extension

$$1 \to \widetilde{\Gamma}_{\mathcal{S}} \to \mathit{\Pi} \to \mathbb{Z}_2 \to 1.$$

(2b) By the Sol^3 case, we can take $a_1 = a_2 = 0$ so that $\alpha = (\mathbf{t}_3^{\frac{1}{2}} \mathbf{t}_4^{a_4}, A)$. Then $\alpha^2 = \mathbf{t}_3 \mathbf{t}_4^{2a_4}$. Therefore, $a_4 = 0$ or $\frac{1}{2q}$. As Q is torsion-free, so is $\mathit{\Pi}$.

(3) By Lemma 3.3.16, we may assume $a_4 = 0$, equivalently, $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$

vanishes. Let $\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A)$. As Q always has torsion, Lemma 3.3.20 applies to show that Π has torsion. Alternatively, explicit computation, like in the Sol^3 case, also shows that Π has torsion. For $a_2 \equiv -a_1$, and one can compute that

$$\alpha^2 = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{-a_1}, A)^2 = (e, I).$$

`(3`i`)` By Lemma 3.3.16, we may assume $a_4 = 0$, that is $H^2(\Phi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ vanishes. Therefore, $\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A)$. Lemma 3.3.20 applies to show that Π is torsion-free precisely when Q is torsion-free, and hence precisely when the criteria from Theorem 3.2.21 `(3i)` are satisfied.

`(4)` This is the only case where a non-standard lattice is present, that is $c_3 \neq 0$. We have the relations

$$\begin{aligned}
[\mathbf{t}_1\mathbf{t}_2] &= \mathbf{t}_4, \quad \mathbf{t}_4 \text{ central},\\
\mathbf{t}_3\mathbf{t}_1\mathbf{t}_3^{-1} &= \mathbf{t}_1^{\sigma_{11}}\mathbf{t}_2^{\sigma_{21}}\mathbf{t}_4^{m_1},\\
\mathbf{t}_3\mathbf{t}_2\mathbf{t}_3^{-1} &= \mathbf{t}_1^{\sigma_{12}}\mathbf{t}_2^{\sigma_{22}}\mathbf{t}_4^{m_2},\\
\alpha\mathbf{t}_1\alpha^{-1} &= \mathbf{t}_2^{-1}\mathbf{t}_4^{v_1}\\
\alpha\mathbf{t}_2\alpha^{-1} &= \mathbf{t}_1\mathbf{t}_4^{v_2}\\
\alpha\mathbf{t}_3\alpha^{-1} &= \mathbf{t}_1^{(1-\sigma_{22})a_1+\sigma_{12}a_2}\mathbf{t}_2^{\sigma_{21}a_1+(1-\sigma_{11})a_2}\mathbf{t}_3^{-1}\mathbf{t}_4^{u_4+2c_3},\\
\alpha^4 &= \mathbf{t}_4^{4a_4-(a_1-a_2)^2+v_4}, \text{ where}
\end{aligned}$$

$$
\begin{aligned}
(\sigma_{11}+\sigma_{22}-2)v_1 &= (-\sigma_{21}+\sigma_{22}-1)m_1+(\sigma_{11}-\sigma_{21}-1)m_2\\
&\quad+\tfrac{1}{2}(\sigma_{11}-\sigma_{11}\sigma_{21}+\sigma_{22}-(\sigma_{11}^2+\sigma_{21}-2\sigma_{11}\sigma_{21})\sigma_{22}-\sigma_{11}\sigma_{22}^2)\\
&\quad-(\sigma_{11}+\sigma_{22}-2)\cdot a_1\\
(\sigma_{11}+\sigma_{22}-2)v_2 &= (1-\sigma_{21}-\sigma_{22})m_1+(\sigma_{11}+\sigma_{21}-1)m_2\\
&\quad-\tfrac{1}{2}(\sigma_{11}-\sigma_{22})(-1-\sigma_{21}+\sigma_{11}\sigma_{22})-(\sigma_{11}+\sigma_{22}-2)\cdot a_2\\
2\sigma_{12}(\sigma_{11}+\sigma_{22}-2)u_4 &= \big((\quad\sigma_{22}m_1-\sigma_{12}m_2)+\tfrac{1}{2}(\quad\sigma_{11}-\sigma_{12})\sigma_{12}\sigma_{22}\big)\cdot a_1\\
&\quad+\big((-\sigma_{12}m_1+\sigma_{11}m_2)+\tfrac{1}{2}(-\sigma_{12}+\sigma_{22})\sigma_{11}\sigma_{12}\big)\cdot a_2\\
&\quad+\tfrac{1}{2}\sigma_{12}\sigma_{22}\cdot a_1^2+(\sigma_{22}-\sigma_{11}\sigma_{22})\cdot a_1a_2+\tfrac{1}{2}(\sigma_{11}-2)\sigma_{12}\cdot a_2^2,\\
(\sigma_{11}+\sigma_{22}-2)v_4 &= 2a_1(\sigma_{22}+\sigma_{11}(\sigma_{21}\sigma_{22}-\sigma_{21}-\sigma_{22}^2)+2(\sigma_{22}-1)m_1-2\sigma_{21}m_2)\\
&\quad+2a_2(\sigma_{11}-\sigma_{22}(\sigma_{11}\sigma_{21}-\sigma_{21}+\sigma_{11}^2)-2\sigma_{21}m_1+2(\sigma_{11}-1)m_2)
\end{aligned}
$$

Recall that m_1, m_2 and all σ_{ij}'s are integers. Since

$$
\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{\sigma_{11}+\sigma_{22}-2}\begin{bmatrix} \sigma_{11}-1 & \sigma_{12} \\ \sigma_{21} & \sigma_{22-1} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad u_1, u_2 \in \mathbb{Z},
$$

v_1, v_2, v_4 and u_4 are all *rational numbers*. Note that c_3 and a_4 appear only once in $\alpha\mathbf{t}_3\alpha^{-1}$ and α^4, respectively. The above equalities show that both c_3 and a_4 must also be rational numbers. With a pre-determined q in $\varPi\cap\mathcal{Z}(\mathrm{Sol}_1^4) = \langle \mathbf{t}_4^{\frac{1}{q}}\rangle$, suppose $c_3 = -\frac{u_4}{2}$ (a solution for $u_4+2c_3=0$). Then $c_3' = c_3+\frac{1}{2q}$ is another solution so that

$$
\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{(1-\sigma_{22})a_1+\sigma_{12}a_2}\mathbf{t}_2^{\sigma_{21}a_1+(1-\sigma_{11})a_2}\mathbf{t}_3^{-1}\mathbf{t}_4^{d},
$$

where $d=0$ or $\frac{1}{q}$. Unless c_3 is a multiple of $\frac{1}{q}$, the corresponding lattice is non-standard. In case $c_3 = (\frac{1}{q})^p$, then $\mathbf{t}_3 = \hat{\mathbf{t}}_3\cdot(\mathbf{t}_4^{\frac{1}{q}})^p$, where $\hat{\mathbf{t}}_3$ has no $\mathbf{t}_4$-component. Since $(\mathbf{t}_4^{\frac{1}{q}})^p\in\varPi$, we can replace the generator $\mathbf{t}_3$ by $\hat{\mathbf{t}}_3$ so that the lattice becomes standard.

For a_4, we have

$$\alpha^4 = \mathbf{t}_4^{4a_4-(a_1-a_2)^2+v_4},$$

Then there are 4 choices for a_4, $a_4 = \frac{(a_1-a_2)^2-v_4+i}{4q}, (i = 0,1,2,3)$. These are the solutions of

$$\alpha^4 = \mathbf{t}_4^{\frac{i}{q}}, \quad (i = 0,1,2,3).$$

For $i = 0, 2$, Π has torsion. To see this when $i = 2$, note that

$$(\mathbf{t}_4^{-\frac{1}{q}}\alpha^2)^2 = \mathbf{t}_4^{-\frac{2}{q}}\mathbf{t}_4^{\frac{2}{q}} = e.$$

For $i = 1, 3$, Π is torsion-free.

(5) By Corollary 3.2.19, we can take $a_1 = a_2 = 0$. Since $\det(A) = -1$, we may assume $a_4 = 0$ by Lemma 3.3.16. There are two choices for b_4, the solutions of $\beta^2 = \mathbf{t}_4^{\frac{i}{q}}, (i = 0,1)$. Thus $H^2(\Phi, \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$. Like case (2a), Π is torsion-free only when $\beta^2 = \mathbf{t}_4^{\frac{1}{q}}$.

(6a) There are two choices for b_4, the solutions of $\beta^2 = \mathbf{t}_4^{\frac{i}{q}}, (i = 0,1)$. That is, $H^2(\Phi, \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$. As Π contains a group of type (3a), it is never torsion-free. (6a*i*) There are two choices for b_4, the solutions of $\beta^2 = \mathbf{t}_4^{\frac{i}{q}}, (i = 0,1)$.

That is, $H^2(\Phi, \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$. For Π to be torsion-free, the subgroups $\langle\widetilde{\Gamma}_{\mathcal{S}}, \alpha\rangle$, $\langle\widetilde{\Gamma}_{\mathcal{S}}, \beta\rangle$, and $\langle\widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta\rangle$, where

$$\alpha\beta = \left(\mathbf{t}_1^{a_1+b_1}\mathbf{t}_2^{a_2-b_2}\mathbf{t}_4^{b_4'}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\right)$$

must all be torsion-free. The group $\langle\widetilde{\Gamma}_{\mathcal{S}}, \beta\rangle$ is torsion-free precisely when b_4 satisfies $\beta^2 = \mathbf{t}_4^{\frac{1}{q}}$.

We need criteria for $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A)$ and $(\mathbf{t}_1^{a_1+b_1}\mathbf{t}_2^{a_2-b_2}, AB)$ to be of infinite order. By Lemma 3.3.20, these they are of infinite order precisely when they are infinite order, projected to Sol^3. By computing when the appropriate affine maps in Proposition 3.2.23 are fixed point free, we obtain the conditions $a_1 = \frac{1}{2}$, $a_2 = b_2 + \frac{1}{2}$, $b_1 \not\equiv \frac{\sigma_{12}(2n+1)}{2(\sigma_{11}-1)} + \frac{1}{2}$ $b_2 \not\equiv \frac{(\sigma_{11}+1)(2m+1)}{2\sigma_{12}} + \frac{1}{2}$ for any $m, n \in \mathbb{Z}$.

(6b) Note that, $a_4 = 0$ by Lemma 3.3.16, and there are two choices for b_4, the solutions of $\beta^2 = \mathbf{t}_3\mathbf{t}_4^{\frac{i}{q}}, (i = 0, 1)$. Hence $H^2(\Phi, \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$. As Π contains a subgroup of type (3), which has torsion, it must have torsion as well.

(6bi) Note $a_4 = 0$ by Lemma 3.3.16, and there are two choices for b_4, the solutions of $\beta^2 = \mathbf{t}_3\mathbf{t}_4^{\frac{i}{q}}, (i = 0, 1)$. Thus $H^2(\Phi, \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$. Recall from Theorem 3.2.21 that Q (and hence Π, by Lemma 3.3.20) is torsion-free precisely when the affine maps

$$\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)$$
$$\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} -k_{11} & k_{12} \\ -k_{21} & k_{11} \end{bmatrix}\right)$$

are fixed point free on T^2, which occurs precisely when $a_1 = \frac{1}{2}$ and $a_2 \not\equiv \frac{(k_{11}-1)(2n+1)}{2k_{12}}$ for any $n \in \mathbb{Z}$.

(7) We may take $a_4 = 0$ without loss of generality by Lemma 3.3.16. $(\beta\alpha)^4 = \mathbf{t}_4^{4b_4+\ell}$ shows that there are 4 choices for b_4, the solutions of $(\beta\alpha)^4 = \mathbf{t}_4{}^{\frac{j}{q}}, (j = 0, 1, 2, 3)$. Hence $H^2(D_4; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_4$. As Π contains a subgroup of type (3), it has torsion.

(7i) Like case (7), we take $a_4 = 0$ and there are 4 choices for b_4, the solutions of $(\beta\alpha)^4 = \mathbf{t}_4{}^{\frac{j}{q}}, (j = 0, 1, 2, 3)$, so that $H^2(D_4; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_4$. For Π to be torsion-free, $\langle\widetilde{\Gamma}_{\mathcal{S}}, \beta\alpha\rangle$ is necessarily torsion-free. This forces b_4 to satisfy $(\beta\alpha)^4 = \mathbf{t}_4^{\frac{j}{q}}, (j = 1, 3)$. Note that β and the holonomy element corresponding to $-B$, $\alpha\beta\alpha$, are of infinite order when projected to Sol^3, and hence also on Sol_1^4. Thus the only remaining holonomy elements to consider are α and

$$\beta\alpha\beta = \left(\mathbf{t}_1^{-k_{11}a_1 - k_{12}a_2}\mathbf{t}_2^{-k_{21}a_1 - k_{11}a_2}\mathbf{t}_4^{2b_4+v}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\right).$$

We need criteria for $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A)$ and $(\mathbf{t}_1^{-k_{11}a_1-k_{12}a_2}\mathbf{t}_2^{-k_{21}a_1-k_{11}a_2}A, BAB)$ to be infinite order, when projected to Sol^3.

By Proposition 3.2.23, we just need to ensure that the appropriate affine maps are fixed point free on T^2, and this occurs precisely when

$$\begin{aligned} &a_1 = \tfrac{1}{2}, \quad a_2 \not\equiv \tfrac{(\sigma_{11}+1)(2n+1)}{2\sigma_{12}}, \\ &\tfrac{-k_{21}}{2} - k_{11}a_2 = \tfrac{1}{2}, \\ &\tfrac{-k_{11}}{2} - k_{12}a_2 \not\equiv \tfrac{\sigma_{12}(2n+1)}{2(\sigma_{11}-1)}. \end{aligned}$$

With $a_1 = \frac{1}{2}$ and $a_2 = -\frac{k_{21}+1}{2k_{11}} + \frac{i}{k_{11}}$ for $i = 0, \ldots, k_{11} - 1$, using that $\det(K) = -1$ and $K^2 = \mathcal{S}$, one can compute that the remaining criteria are satisfied. In fact, we compute

$$\begin{aligned} \frac{(\sigma_{11}+1)(2n+1)}{2\sigma_{12}} &= \frac{(k_{11}^2 + k_{12}k_{21} + 1)(2n+1)}{4k_{11}k_{12}} \\ &= \frac{2k_{12}k_{21}(2n+1)}{4k_{11}k_{12}} = \frac{k_{21}(2n+1)}{2k_{11}}. \end{aligned}$$

Now, for $m \in \mathbb{Z}$,

$$\begin{aligned} a_2 = -\frac{k_{21}+1}{2k_{11}} + \frac{i}{k_{11}} &= \frac{k_{21}(2n+1)}{2k_{11}} + m \implies \\ \frac{-k_{21} - 1 + 2i}{2k_{11}} &= \frac{k_{21}(2n+1) + 2mk_{11}}{2k_{11}} \implies \\ \frac{-1 + 2i}{2k_{11}} &= \frac{2k_{21}(n+1) + 2mk_{11}}{2k_{11}}, \end{aligned}$$

a contradiction for any integers i, n, m, as the numerators are of different parity.

For the remaining criterion, we compute

$$\begin{aligned}\frac{-k_{11}}{2} - k_{12}a_2 &= \frac{-k_{11}}{2} - k_{12}\left(-\frac{k_{21}+1}{2k_{11}} + \frac{i}{k_{11}}\right)\\ &= \frac{-k_{11}^2 + k_{12}k_{21} + k_{12} - 2k_{12}i}{2k_{11}}\\ &= \frac{1 + k_{12} - 2k_{12}i}{2k_{11}}.\end{aligned}$$

Now, for $m \in \mathbb{Z}$,

$$\begin{aligned}\frac{1+k_{12}-2k_{12}i}{2k_{11}} &= \frac{\sigma_{12}(2n+1)}{2(\sigma_{11}-1)} + m \implies\\ \frac{1+k_{12}-2k_{12}i}{2k_{11}} &= \frac{2k_{11}k_{12}(2n+1)}{2(k_{11}^2+k_{12}k_{21}-1)} + m \implies\\ \frac{1+k_{12}-2k_{12}i}{2k_{11}} &= \frac{2k_{11}k_{12}(2n+1)}{4k_{11}^2} + m = \frac{k_{12}(2n+1)+2mk_{11}}{2k_{11}} \implies\\ \frac{1-2k_{12}i}{2k_{11}} &= \frac{2(nk_{12}+mk_{11})}{2k_{11}},\end{aligned}$$

a contradiction for any integers i, n, m, as the numerators are of different parity. Therefore, with $a_1 = \frac{1}{2}$ and $a_2 = -\frac{k_{21}+1}{2k_{11}} + \frac{i}{k_{11}}$ for $i = 0, \ldots, k_{11}-1$, and $(\beta\alpha)^4 = \mathbf{t}_4^{\frac{j}{q}}, (j = 1, 3)$, $\mathit{\Pi}$ is torsion-free.

This completes the proof of Theorem 3.3.21. □

We have shown that every Sol_1^4 crystallographic group $\mathit{\Pi}$ can be conjugated so that the holonomy Φ is a subgroup of our fixed $D_4 \subset \mathrm{Aut}(\mathrm{Sol}_1^4)$. We take $c_3 = 0$ (standard lattice) for all holonomy groups except $\mathbb{Z}_4$. In this case, we have two values for c_3. This corresponds to two distinct abstract kernels, which give rise to different isomorphism classes. Alternatively, in the $\mathbb{Z}_4$ case, one may assume $c_3 = 0$. Then, however, Φ is not a subgroup of D_4. Rather,

$\Phi \subset \hat{k} D_4 \hat{k}^{-1}$, where $\hat{k} \in \mathbb{R}$ acts by

$$\begin{bmatrix} 1 & e^u x & z \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & e^u x & z + ku \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark 3.3.22. Let Π and Π' be two crystallographic groups of Sol_1^4, with projections to Sol^3 crystallographic groups, Q and Q' , respectively. That is,

$$\Pi / (\Pi \cap \mathcal{Z}(\mathrm{Sol}_1^4)) \cong Q,$$
$$\Pi' / (\Pi' \cap \mathcal{Z}(\mathrm{Sol}_1^4)) \cong Q'.$$

Our classification is up to equivalence of extensions, a finer notion of equivalence than isomorphism. However, we observe that Q and Q' being isomorphic is necessary for Π and Π' to be isomorphic. This is easy to see from the rigidity property of these crystallographic groups (Theorem 1.1.5). Any isomorphism between Π and Π' is conjugation by an element of $\mathrm{Aff}(\mathrm{Sol}_1^4) = \mathrm{Sol}_1^4 \rtimes \mathrm{Aut}(\mathrm{Sol}_1^4)$. Such a conjugation must carry $\Pi \cap \mathcal{Z}(\mathrm{Sol}_1^4)$ to $\Pi' \cap \mathcal{Z}(\mathrm{Sol}_1^4)$, and hence must induce an isomorphism between Q and Q'.

3.3.23 (**Examples**)**.** First we show that $\mathrm{Sol}_1^4 \rtimes D_4$ embeds into the affine group $\mathrm{Aff}(4) \hookrightarrow \mathrm{GL}(5, \mathbb{R})$. The following correspondence is an injective homomorphism of Lie groups, $\mathrm{Sol}_1^4 \hookrightarrow \mathrm{Aff}(4)$,

$$\begin{bmatrix} 1 & e^u x & z \\ 0 & e^u & y \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{1}{2} e^{-u} y & \frac{e^u x}{2} & 0 & z - \frac{xy}{2} \\ 0 & e^{-u} & 0 & 0 & x \\ 0 & 0 & e^u & 0 & y \\ 0 & 0 & 0 & 1 & u \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (3.16)$$

Moreover, the automorphisms

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \quad \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \in \mathrm{Aut}(\mathrm{Sol}_1^{\,4})$$

can also be embedded as

$$\begin{bmatrix} ad & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -bc & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

respectively, where a, b, c, d are ± 1. Note that, if we remove the first row and the first column from Aff(4), we get a representation of Sol^3 into Aff(3).

If we write the element $(\mathbf{a}, A) \in \mathrm{Sol}_1^{\,4} \rtimes D_4$ as the product $\mathbf{a} \cdot A$, then the group operation of $\mathrm{Sol}_1^{\,4} \rtimes D_4$ is compatible with the matrix product in this affine group. The action of A on $\mathbf{a}$ is by conjugation. That is,

$$\begin{aligned} (\mathbf{a} \cdot A)(\mathbf{b} \cdot B) &= \mathbf{a} A \mathbf{b} B \\ &= \mathbf{a}(A \mathbf{b} A^{-1}) \cdot AB \\ &= (\mathbf{a}, A) \cdot (\mathbf{b}, B). \end{aligned}$$

Example 3.3.24 ((4) Non-standard lattice)**.** This is an example where c_3 is non-zero (Theorem 3.3.21). Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_4^{a_4}, A)$. Let $\mathcal{S} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$. Then $\lambda = 3 + 2\sqrt{2}$, and with

$$P = \begin{bmatrix} -\frac{1}{2}\sqrt{2+\sqrt{2}} & \frac{\sqrt{2-\sqrt{2}}}{2} \\ -\frac{1}{\sqrt{2\left(2+\sqrt{2}\right)}} & -\frac{1}{2}\sqrt{2+\sqrt{2}} \end{bmatrix},$$

$\mathit{\Pi} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \alpha \rangle$, where $\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_4^{a_4}, A) \in \mathrm{Sol}_1^{\,4} \rtimes \mathrm{Aut}(\mathrm{Sol}_1^{\,4})$, has a representation into Aff(4):

$$\mathbf{t}_1 = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2}(2m_1 - m_2 - 3) \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{t}_2 = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2}(-m_1 - 1) \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{t}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & c_3 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & \ln\left(3 + 2\sqrt{2}\right) \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{t}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$(a, A) = \begin{bmatrix} 1 & -\frac{a_1}{2} & -\frac{a_2}{2} & 0 & \frac{1}{2}(2a_4 - a_2(m_1 + 1) + a_1(a_2 + 2m_1 - m_2 - 3)) \\ 0 & 0 & 1 & 0 & a_1 \\ 0 & -1 & 0 & 0 & a_2 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Π has a presentation

$$
\begin{aligned}
&[\mathbf{t}_1, \mathbf{t}_2] = \mathbf{t}_4, \quad \text{and } \mathbf{t}_4 \text{ is central}\\
&\mathbf{t}_3\mathbf{t}_1\mathbf{t}_3^{-1} = \mathbf{t}_1\mathbf{t}_2^2\mathbf{t}_4^{m_1},\ \mathbf{t}_3\mathbf{t}_2\mathbf{t}_3^{-1} = \mathbf{t}_1^2\mathbf{t}_2^5\mathbf{t}_4^{m_2},\\
&\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_2^{-1}\mathbf{t}_4^{\frac{1}{2}(-4-2a_1+m_1-m_2)},\ \alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_1\mathbf{t}_4^{\frac{1}{2}(2-2a_2-3m_1+m_2)},\\
&\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{-4a_1+2a_2}\mathbf{t}_2^{2a_1}\mathbf{t}_3^{-1}\mathbf{t}_4^{5a_1^2+2c_3+a_1(-5+5m_1-2m_2)+a_2(3-a_2-2m_1+m_2)},\\
&\alpha\mathbf{t}_4\alpha^{-1} = \mathbf{t}_4,\ \alpha^4 = \mathbf{t}_4^{-a_1^2+4a_4-a_2(2+a_2+2m_1)+2a_1(-3+a_2+2m_1-m_2)}.
\end{aligned}
$$

From $(I-\mathcal{S})^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$, we have $\operatorname{Coker}(I-\mathcal{S}) = \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by

$$
\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}.
$$

The coboundary is

$$
\operatorname{Im}(I - \varphi(\bar{\alpha})) = \operatorname{Im} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\}.
$$

Thus we have only two cases

$$
\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.
$$

For simplicity, we shall assume $m_1 = m_2 = 0$.
With $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\Pi = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \alpha \rangle$, $\alpha = (\mathbf{t}_4^{a_4}, A) \in \mathrm{Sol}_1^4 \rtimes \mathrm{Aut}(\mathrm{Sol}_1^4)$, has

a presentation

$$\begin{aligned}
&[\mathbf{t}_1, \mathbf{t}_2] = \mathbf{t}_4, \quad \text{and } \mathbf{t}_4 \text{ is central}\\
&\mathbf{t}_3\mathbf{t}_1\mathbf{t}_3^{-1} = \mathbf{t}_1\mathbf{t}_2^2,\ \mathbf{t}_3\mathbf{t}_2\mathbf{t}_3^{-1} = \mathbf{t}_1^2\mathbf{t}_2^5,\\
&\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_2^{-1}\mathbf{t}_4^{-2},\ \alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_1\mathbf{t}_4,\\
&\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_3^{-1}\mathbf{t}_4^{2c_3},\ \alpha\mathbf{t}_4\alpha^{-1} = \mathbf{t}_4,\\
&\alpha^4 = \mathbf{t}_4^{4a_4}.
\end{aligned}$$

The minimum q for $\widetilde{\Gamma}_{\mathcal{S}}$ is $q = 1$, and we have choices $a_4 = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and $c_3 = 0, \frac{1}{2}$ (any combination of a_4 and c_3), with the same center. So, there are 8 distinct groups. Half of them (with $c_3 = 0$) have standard lattices, and the rest (with $c_3 = \frac{1}{2}$) have non-standard lattices.

With $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$, $\mathit{\Pi} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \alpha \rangle$, where $\alpha = (\mathbf{t}_1^{\frac{1}{2}}\mathbf{t}_4^{a_4}, A) \in \mathrm{Sol}_1^4 \rtimes \mathrm{Aut}(\mathrm{Sol}_1^4)$, has a presentation

$$\begin{aligned}
&[\mathbf{t}_1, \mathbf{t}_2] = \mathbf{t}_4, \quad \text{and } \mathbf{t}_4 \text{ is central}\\
&\mathbf{t}_3\mathbf{t}_1\mathbf{t}_3^{-1} = \mathbf{t}_1\mathbf{t}_2^2,\ \mathbf{t}_3\mathbf{t}_2\mathbf{t}_3^{-1} = \mathbf{t}_1^2\mathbf{t}_2^5,\\
&\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_2^{-1}\mathbf{t}_4^{-\frac{5}{2}},\ \alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_1\mathbf{t}_4,\\
&\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{-2}\mathbf{t}_2\mathbf{t}_3^{-1}\mathbf{t}_4^{-\frac{5}{4}+2c_3}, \alpha\mathbf{t}_4\alpha^{-1} = \mathbf{t}_4,\\
&\alpha^4 = \mathbf{t}_4^{-\frac{13}{4}+4a_4}.
\end{aligned}$$

The minimum q for $\widetilde{\Gamma}_{\mathcal{S}}$ is $q = 2$ (which comes out of $\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_2^{-1}\mathbf{t}_4^{-\frac{5}{2}}$), and we have choices $a_4 = \frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}$ and $c_3 = \frac{1}{8}$ or $\frac{3}{8}$ (any combination of a_4 and c_3), with the same center. So, there are 8 distinct groups. All these groups have non-standard lattices, because no c_3 is an integer multiple of $\frac{1}{q}$ for $q = 2$.

Example 3.3.25 ((6bi)). Let $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle A, B \rangle$, where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ with } \widetilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; \widetilde{B} = B.$$

Then $\mathcal{S}$ must be of the form $nK - I$. Let us take

$$K = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}. \text{ Then, } \mathcal{S} = \begin{bmatrix} 17 & 12 \\ 24 & 17 \end{bmatrix}.$$

Note that $k_{11} = k_{22}$ and $\operatorname{tr}(K) = n = 6 > 2$. Therefore, $\mathcal{S}$ satisfies the conditions for case (6bi) and

$$\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A),$$
$$\beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B).$$

Then $\lambda = 17 + 12\sqrt{2}$, and with

$$P = \begin{bmatrix} -\frac{1}{\sqrt[4]{2}} & \frac{1}{2^{3/4}} \\ -\frac{1}{\sqrt[4]{2}} & -\frac{1}{2^{3/4}} \end{bmatrix},$$

the equation in Lemma 3.3.3 yields

$$c_1 = \tfrac{1}{4}(102 + \sqrt{2} + 2m_1 - 3m_2)$$
$$c_2 = \tfrac{1}{8}(-204 - \sqrt{2} - 3m_1 + 4m_2).$$

Coker$(I + K)$ is $\mathbb{Z}_4 \times \mathbb{Z}_2$, and we have

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{i}{4} \\ \frac{j}{2} \end{bmatrix}, \quad i = 0, 1, 2, 3; \; j = 0, 1.$$

From $(I + \varphi(\bar{\alpha}))\mathbf{a} = \mathbf{0}$, we have

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{i}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{j}{2} \end{bmatrix}, \quad i, j = 0, 1. \tag{3.17}$$

For any $\mathbf{v} \in \mathrm{Coker}(I+K)$,

$$(\delta \mathbf{v})(A) = (I - \widetilde{A})\mathbf{v} = \begin{bmatrix} 0 \\ 2v_2 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, $H^1(\langle A \rangle; \mathrm{Coker}(I+K)) = \mathbb{Z}_2 \times \mathbb{Z}_2$ (quotient of $\mathbb{Z}_4 \times \mathbb{Z}_2$) is generated by $\mathbf{a}$ in (3.17) above. Our group Π has presentation

$$\begin{aligned}
&[\mathbf{t}_1, \mathbf{t}_2] = \mathbf{t}_4,\ [\mathbf{t}_i, \mathbf{t}_4] = 1,\ \text{for } i = 1, 2, 3,\\
&\mathbf{t}_3\mathbf{t}_1\mathbf{t}_3^{-1} = \mathbf{t}_1^{17}\mathbf{t}_2^{24}\mathbf{t}_4^{m_1},\ \mathbf{t}_3\mathbf{t}_2\mathbf{t}_3^{-1} = \mathbf{t}_1^{12}\mathbf{t}_2^{17}\mathbf{t}_4^{m_2},\\
&\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_1\mathbf{t}_4^{-51-a_2-m_1+\frac{3}{2}m_2},\ \alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_2^{-1}\mathbf{t}_4^{-a_1},\\
&\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{-16a_1+12a_2}\mathbf{t}_2^{8(3a_1-2a_2)}\mathbf{t}_3^{-1}\\
&\qquad\qquad \cdot \mathbf{t}_4^{204a_1^2+a_1(-272a_2+17(60+m_1)-24m_2)+a_2(-714+90a_2-12m_1+17m_2)},\\
&\alpha\mathbf{t}_4\alpha^{-1} = \mathbf{t}_4^{-1},\\
&\beta\mathbf{t}_1\beta^{-1} = \mathbf{t}_1^{-3}\mathbf{t}_2^{-4}\mathbf{t}_4^{-6+\frac{1}{2}m_1-m_2},\ \beta\mathbf{t}_2\beta^{-1} = \mathbf{t}_1^{-2}\mathbf{t}_2^{-3}\mathbf{t}_4^{\frac{1}{2}(-108-m_1+m_2)},\\
&\beta\mathbf{t}_3\beta^{-1} = \mathbf{t}_3,\ \beta\mathbf{t}_4\beta^{-1} = \mathbf{t}_4,\\
&\alpha^2 = \mathbf{t}_1^{2a_1}\mathbf{t}_4^{-a_1(51+a_2+m_1)+\frac{3}{2}a_1m_2}\\
&\beta^2 = \mathbf{t}_3\mathbf{t}_4^{2b_4},\\
&[\alpha, \beta] = \mathbf{t}_1^{4a_1-2a_2}\mathbf{t}_2^{-4(a_1-a_2)}\mathbf{t}_3^{-1}\mathbf{t}_4^{6a_1^2-2b_4+\frac{1}{2}a_2(306+10a_2+5m_1-7m_2)+a_1(-204-12a_2-\frac{7}{2}m_1+5m_2)}.
\end{aligned}$$

Of the 8 groups, seven groups have torsion, and one is torsion-free.

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}: \quad \alpha^2 = \mathrm{id},$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}: \quad \text{Both } \mathit{\Pi} \text{ and } Q \text{ are torsion-free},$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}: \quad (\mathbf{t}_2\mathbf{t}_3\alpha)^2 = \mathrm{id},$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}: \quad (\mathbf{t}_1^{-1}\mathbf{t}_3\alpha)^2 = \mathrm{id}.$$

Note that $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$ is the only case that satisfies $a_1 = \frac{1}{2}$ and $a_2 \not\equiv \frac{(k_{11}-1)(2m+1)}{2k_{12}} = \frac{2m+1}{2} \equiv \frac{1}{2}$, as stated in Theorem 3.3.21.

Example 3.3.26 ((7i))**.** This example has maximal holonomy group D_4, but it does not contain all the possible holonomy actions. For example, groups of type (6b) or (6bi) are not subgroups of this group. Let $\Phi = \mathbb{Z}_4 \rtimes \mathbb{Z}_2 = \langle A, B \rangle$, where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and

$$\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A),$$

$$\beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B).$$

Our $\mathcal{S}$ is of the form

$$\mathcal{S} = nK + I,$$

where

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

with $\det(K) = -1$ and $\operatorname{tr}(K) = n \neq 0$. Now for $\varphi(\bar{\alpha})$, we have two choices. We can take $\widetilde{A} = A$ or $\widetilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Since the former yields always a torsion group, we take the latter case. Therefore we consider case (7i), $k_{11} = k_{22}$.

As an example, we take

$$K = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad \text{therefore } n = k_{11} + k_{22} = 2, \quad \mathcal{S} = nK + I = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}.$$

Then $\lambda = 3 + 2\sqrt{2}$, and with

$$P = \begin{bmatrix} -\frac{1}{\sqrt[4]{2^3}} & \frac{1}{\sqrt[4]{2}} \\ -\frac{1}{\sqrt[4]{2^3}} & -\frac{1}{\sqrt[4]{2}} \end{bmatrix},$$

the equation in Lemma 3.3.3 yields

$$c_1 = \frac{1}{8}(-12 + \sqrt{2} + 4m_1 - 4m_2)$$
$$c_2 = \frac{1}{4}(-\sqrt{2} - 4m_1 + 2m_2).$$

Recall we can take $c_3 = 0$ by Theorem 3.3.21. Our crystallographic group

$\mathit{\Pi} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4, \alpha, \beta \rangle$ has a representation into Aff(4):

$$\mathbf{t}_1 = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2}(m_1 - m_2 - 3) \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{t}_2 = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2}(m_2 - 2m_1) \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{t}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & \ln\left(3 + 2\sqrt{2}\right) \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{(because } c_3 = 0\text{)},$$

$$\mathbf{t}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$(a, A) = \begin{bmatrix} -1 & -\frac{a_2}{2} & -\frac{a_1}{2} & 0 & \frac{1}{2}(a_1(a_2 + m_1 - m_2 - 3) + a_2(m_2 - 2m_1)) \\ 0 & 1 & 0 & 0 & a_1 \\ 0 & 0 & -1 & 0 & a_2 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$(b, B) = \begin{bmatrix} -1 & 0 & 0 & 0 & b_4 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2}\log\left(3 + 2\sqrt{2}\right) \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have

$$\operatorname{Coker}(I-\mathcal{S}) = (\mathbb{Z}_2)^2 = \left\{ \tfrac{1}{2}\begin{bmatrix}0\\0\end{bmatrix}, \tfrac{1}{2}\begin{bmatrix}0\\1\end{bmatrix}, \tfrac{1}{2}\begin{bmatrix}1\\0\end{bmatrix}, \tfrac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} \right\}.$$

Now

$$\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \varphi(\bar{\beta}) = -K,$$

yields

$$I+\varphi(\bar{\alpha}) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix},\ I+\varphi(\bar{\beta}) = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}.$$

Then $(I+\varphi(\bar{\alpha}))\mathbf{a} \equiv \mathbf{0}$ yields $2a_1 \equiv 0$, which is not a new condition. We have 4 choices for $\mathbf{a}$,

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix}0\\0\end{bmatrix}, \tfrac{1}{2}\begin{bmatrix}0\\1\end{bmatrix}, \tfrac{1}{2}\begin{bmatrix}1\\0\end{bmatrix}, \tfrac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}.$$

The coboundary $\operatorname{Im}(I-\varphi(\bar{\alpha}))$ yields the trivial group, and hence there are 4 non-equivalent groups Π. The group Π has a presentation

$$[\mathbf{t}_1,\mathbf{t}_2] = \mathbf{t}_4,\ [\mathbf{t}_i,\mathbf{t}_4] = 1\ (i=1,2,3),$$

$$\mathbf{t}_3\mathbf{t}_1\mathbf{t}_3^{-1} = \mathbf{t}_1^3\mathbf{t}_2^2\mathbf{t}_4^{m_1},\ \mathbf{t}_3\mathbf{t}_2\mathbf{t}_3^{-1} = \mathbf{t}_1^4\mathbf{t}_2^3\mathbf{t}_4^{m_2},$$

$$\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_1\mathbf{t}_4^{3-a_2-m_1+m_2},\ \alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_2^{-1}\mathbf{t}_4^{-a_1},$$

$$\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{-2a_1+4a_2}\mathbf{t}_2^{2(a_1-a_2)}\mathbf{t}_3^{-1}\mathbf{t}_4^{3a_1^2-a_1(3+6a_2-3m_1+2m_2)+a_2(6+2a_2-4m_1+3m_2)},$$

$$\alpha\mathbf{t}_4\alpha^{-1} = \mathbf{t}_4^{-1},$$

$$\beta\mathbf{t}_1\beta^{-1} = \mathbf{t}_1^{-1}\mathbf{t}_2^{-1}\mathbf{t}_4^{\frac{1}{2}(-1-2m_1+m_2)},\ \beta\mathbf{t}_2\beta^{-1} = \mathbf{t}_1^{-2}\mathbf{t}_2^{-1}\mathbf{t}_4^{-4+m_1-m_2},$$

$$\beta\mathbf{t}_3\beta^{-1} = \mathbf{t}_3,\ \beta\mathbf{t}_4\beta^{-1} = \mathbf{t}_4^{-1},$$

$$\alpha^2 = \mathbf{t}_1^{2a_1}\mathbf{t}_4^{-a_1(-3+a_2+m_1-m_2)},$$

$$\beta^2 = \mathbf{t}_3,$$

$$(\alpha\beta)^4 = \mathbf{t}_4^{-4b_4+a_1^2+4a_1a_2+2a_2^2-2a_1(3-m_1+m_2)-2a_2(2m_1-m_2)}.$$

Out of these 4 groups (by the choice of a_i's), only one group can be torsion-free, and 3 groups have torsion.

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} : \quad \alpha^2 = \mathrm{id}.$$
$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} : \qquad \left(\mathbf{t}_2^{-1}(\alpha\beta)^2\alpha\right)^2 = \mathrm{id}$$
$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} : \qquad a_2 = -\frac{k_{21}+1}{2k_{11}} \equiv 0$$

When $m_i = 0$, $a_1 = \frac{1}{2}$, $a_2 = 0$, $q = 4$ (minimum), b_4 takes values $\frac{j}{16}$, $0 \leq j \leq 4$. When $b_4 = 0$ or $\frac{2}{16}$, $\mathit{\Pi}$ is torsion-free. When $b_4 = \frac{1}{16}$ or $\frac{3}{16}$, $\mathit{\Pi}$ has torsion.

3.4 Crystallographic Groups of $\mathrm{Sol}^3 \times \mathbb{R}$

Here we provide a classification of the crystallographic groups of $\mathrm{Sol}^3 \times \mathbb{R}$. Given a crystallographic group $\mathit{\Pi} \subset (\mathrm{Sol}^3 \times \mathbb{R}) \rtimes (D_4 \times \mathbb{Z}_2)$, $\widetilde{\Gamma} = \mathit{\Pi} \cap (\mathrm{Sol}^3 \times \mathbb{R})$ is a lattice of $\mathrm{Sol}^3 \times \mathbb{R}$ with finite quotient group $\Psi \subseteq D_4 \times \mathbb{Z}_2$. We have the short exact sequence

$$1 \to \widetilde{\Gamma} \to \mathit{\Pi} \to \Psi \to 1.$$

3.4.1 (Lattices of $\mathrm{Sol}^3 \times \mathbb{R}$). In the cohomology classification of lattices in Theorem 3.3.2, the $\mathrm{Sol}^3 \times \mathbb{R}$ case corresponds precisely to when $q = 0$. We will see that $\widetilde{\Gamma}_{(\mathcal{S};0,m_1,m_2)}$ embeds as a lattice of $\mathrm{Sol}^3 \times \mathbb{R}$.

An $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(\mathcal{S}) > 2$ produces P and Δ. Using P and Δ, we embed $\mathbb{Z}^2 \rtimes \mathbb{Z}$ into Sol^3 as in assignment (3.5) in Subsection 3.2.1, which we now lift to $\mathrm{Sol}^3 \times \mathbb{R}$. Recall the definition of $\mathrm{Sol}^3 \times \mathbb{R}$ as $\mathbb{R}^3 \rtimes_{\phi(u)} \mathbb{R}$, where

$$\phi(u) = \begin{bmatrix} e^{-u} & 0 & 0 \\ 0 & e^{u} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The only ambiguities are the central slots. So let

$$\begin{aligned}
\mathbf{e}_1 = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 0 \right) \longmapsto (P\mathbf{e}_1, 0) \longmapsto \mathbf{t}_1 = \left(\begin{bmatrix} p_{11} \\ p_{21} \\ c_1 \end{bmatrix}, 0 \right), \\
\mathbf{e}_2 = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 0 \right) \longmapsto (P\mathbf{e}_2, 0) \longmapsto \mathbf{t}_2 = \left(\begin{bmatrix} p_{12} \\ p_{22} \\ c_2 \end{bmatrix}, 0 \right), \\
\mathbf{e}_3 = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 1 \right) \longmapsto (0, \ln(\lambda)) \longmapsto \mathbf{t}_3 = \left(\begin{bmatrix} 0 \\ 0 \\ c_3 \end{bmatrix}, \ln(\lambda) \right), \\
\mathbf{t}_4 = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, 0 \right),
\end{aligned} \tag{3.18}$$

where c_i's will be determined later. Then $[\mathbf{t}_1, \mathbf{t}_2] = e$ (regardless the values of c_i's).

3.4.2 (Warning). In this notation, the central factor of $\mathrm{Sol}^3 \times \mathbb{R}$ goes to the 3rd slot of $\mathbb{R}^3 \subset \mathbb{R}^3 \rtimes \mathbb{R}$. That is,

$$\mathrm{Sol}^3 \times \mathbb{R} \ni (e, s) \mapsto \left(\begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix}, 0 \right) \in \mathbb{R}^3 \rtimes \mathbb{R}.$$

Lemma 3.4.3. *For any integers m_1, m_2, there exist unique c_1, c_2 for which*

$\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4\}$ *forms a group* $\widetilde{\Gamma}_{(\mathcal{S};0,m_1,m_2)}$ *with the presentation*

$$\begin{aligned}\widetilde{\Gamma}_{(\mathcal{S};0,m_1,m_2)} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4 |\ [\mathbf{t}_1, \mathbf{t}_2] = \mathbf{t}_4^0 = e,\ \mathbf{t}_4 \textit{ is central,}\\ \mathbf{t}_3\mathbf{t}_1\mathbf{t}_3^{-1} = \mathbf{t}_1^{\sigma_{11}}\mathbf{t}_2^{\sigma_{21}}\mathbf{t}_4^{m_1},\\ \mathbf{t}_3\mathbf{t}_2\mathbf{t}_3^{-1} = \mathbf{t}_1^{\sigma_{12}}\mathbf{t}_2^{\sigma_{22}}\mathbf{t}_4^{m_2}\rangle.\end{aligned}$$

Consequently, $\widetilde{\Gamma}_{(\mathcal{S};0,m_1,m_2)}$ *is solvable and contains* $\mathbb{Z}^3 = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_4\rangle$ *as its discrete nil-radical, a lattice of* $\mathbb{R}^3$ *in* $\mathbb{R}^3 \subset \mathbb{R}^3 \rtimes \mathbb{R}$.

Proof. We only need to verify the last two equalities. But they become a system of equations on c_i's

$$\begin{aligned}(1-\sigma_{11})c_1 - \sigma_{21}c_2 = m_1\\ -\sigma_{12}c_1 + (1-\sigma_{22})c_2 = m_2\end{aligned} \tag{3.19}$$

Since $I - \mathcal{S}$ is non-singular, there exists a unique solution for c_1, c_2. □

Remark 3.4.4. (1) There are 1-parameter family of P's diagonalizing $\mathcal{S}$ with $\det(P) = 1$. In the above lemma, the c_i's are independent of choice of P because equation (3.19) has coefficients only from the matrix $\mathcal{S}$..

(2) The equation (3.19) also shows the cohomology classification. Suppose $\{c_1, c_2\}$ and $\{c_1', c_2'\}$ are solutions for the equations with $\{m_1, m_2\}$ and $\{m_1', m_2'\}$, respectively. Then $(c_1' - c_1, c_2' - c_2) \in \mathbb{Z}^2$ if and only if $(m_1' - m_1, m_2' - m_2) \in \mathrm{Coker}(\mathcal{S}^T - I) \cong \mathrm{Coker}(\mathcal{S} - I)$. This happens if and only if $\widetilde{\Gamma}_{(\mathcal{S};0,m_1,m_2)} = \widetilde{\Gamma}_{(\mathcal{S};0,m_1',m_2')}$.

(3) Notice that c_3 does not show up in the presentation of the lattice $\widetilde{\Gamma}_{(\mathcal{S};0,m_1,m_2)}$. This implies that c_3 can be changed without affecting the isomorphism type of the lattice.

Notation 3.4.5 (Standard lattice)**.** The lattice generated by

$$\mathbf{t}_1 = \left(\begin{bmatrix} p_{11} \\ p_{21} \\ c_1 \end{bmatrix}, 0 \right), \mathbf{t}_2 = \left(\begin{bmatrix} p_{12} \\ p_{22} \\ c_2 \end{bmatrix}, 0 \right), \mathbf{t}_3 = \left(\begin{bmatrix} 0 \\ 0 \\ c_3 \end{bmatrix}, \ln(\lambda) \right), \mathbf{t}_4 = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, 0 \right)$$

with $c_3 = 0$, is called a *standard lattice* of $\mathrm{Sol}^3 \times \mathbb{R}$.

Any lattice of $\mathrm{Sol}^3 \times \mathbb{R}$ is isomorphic to a standard lattice.

3.4.6 (**Classification of** $\mathrm{Sol}^3 \times \mathbb{R}$ **Crystallographic Groups).** Given a crystallographic group $\Pi \subset \mathrm{Isom}(\mathrm{Sol}^3 \times \mathbb{R}) = (\mathrm{Sol}^3 \times \mathbb{R}) \rtimes (D_4 \times \mathbb{Z}_2)$, we can conjugate Π in $\mathrm{Aff}(\mathrm{Sol}^3 \times \mathbb{R})$ so that $\widetilde{\Gamma}_{(\mathcal{S};0,m_1,m_2)}$ satisfies criteria of Notation 3.4.5, *except possibly* for the condition that $c_3 = 0$. Indeed, like the case of Sol_1^4, we will see that $c_3 = 0$ cannot always be achieved when we have fixed our maximal compact subgroup of $\mathrm{Aut}(\mathrm{Sol}^3 \times \mathbb{R})$ to be $D_4 \times \mathbb{Z}_2$. Note that $\widetilde{\Gamma}_{(\mathcal{S};0,m_1,m_2)}$ projects to a standard lattice of Sol^3.

Notation 3.4.7. We will let τ denote the automorphism of $\mathrm{Sol}^3 \times \mathbb{R}$ given by a reflection in the central direction. In the description of $\mathrm{Sol}^3 \times \mathbb{R}$ as $\mathbb{R}^3 \rtimes \mathbb{R}$, $(\mathbf{x}, t) \in \mathbb{R}^3 \rtimes \mathbb{R}$ is mapped to $(T\mathbf{x}, t) \in \mathbb{R}^3 \rtimes \mathbb{R}$ under τ, where

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Proposition 3.4.8. (1) *Any crystallographic group* Π *of* $\mathrm{Sol}^3 \times \mathbb{R}$ *can be conjugated into* $(\mathrm{Sol}^3 \times \mathbb{R}) \rtimes (D_4 \times \mathbb{Z}_2)$ *in such way that*

$$\widetilde{\Gamma}_{\mathcal{S}} = \Pi \cap (\mathrm{Sol}^3 \times \mathbb{R}) = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4 \rangle$$

where

$$\mathbf{t}_1 = \left(\begin{bmatrix} p_{11} \\ p_{21} \\ c_1 \end{bmatrix}, 0 \right), \ \mathbf{t}_2 = \left(\begin{bmatrix} p_{12} \\ p_{22} \\ c_2 \end{bmatrix}, 0 \right), \ \mathbf{t}_3 = \left(\begin{bmatrix} 0 \\ 0 \\ c_3 \end{bmatrix}, \ln(\lambda) \right), \ \mathbf{t}_4 = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, 0 \right).$$

(2) *The holonomy group* Ψ *is generated by at most three elements of* $D_4 \times \mathbb{Z}_2$*, and therefore* Π *is generated by* $\langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4 \rangle$ *and at most three isometries of the form* $(\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{b_3}\mathbf{t}_4^{b_4}, B)$ *for* $B \in D_4 \times \mathbb{Z}_2$ *and real numbers* b_i*.*

Proof. Let $\rho : \mathrm{Sol}^3 \times \mathbb{R} \to \mathrm{Sol}^3$ denote projection. Given a lattice $\widetilde{\Gamma}_{\mathcal{S}}$ of $\mathrm{Sol}^3 \times \mathbb{R}$, $\widetilde{\Gamma}_{\mathcal{S}}$ must meet the center of $\mathrm{Sol}^3 \times \mathbb{R}$ (the $\mathbb{R}$ factor of $\mathrm{Sol}^3 \times \mathbb{R}$) in a lattice $\widetilde{\Gamma}_{\mathcal{S}} \cap \mathcal{Z}(\mathrm{Sol}^3 \times \mathbb{R}) \cong \mathbb{Z}$. Let $\mathbf{t}_4$ denote a generator of $\widetilde{\Gamma}_{\mathcal{S}} \cap \mathbb{R}$. Then $\mathbf{t}_4$ must be of the form

$$\mathbf{t}_4 = \left(\begin{bmatrix} 0 \\ 0 \\ \ell \end{bmatrix}, 0 \right).$$

Let $\hat{\ell}$ denote the automorphism of $\mathrm{Sol}^3 \times \mathbb{R}$ defined by multiplication by ℓ on the center:

$$\hat{\ell} : \left(\begin{bmatrix} x \\ y \\ t \end{bmatrix}, u \right) \longrightarrow \left(\begin{bmatrix} x \\ y \\ \ell t \end{bmatrix}, u \right).$$

After conjugating $\widetilde{\Gamma}_{\mathcal{S}}$ by $(e, \hat{\ell})^{-1}$ in $\mathrm{Aff}(\mathrm{Sol}^3 \times \mathbb{R})$, $\mathbf{t}_4$ has the desired form. For the remaining generators, note that $\rho(\widetilde{\Gamma}_{\mathcal{S}})$ must project to a lattice of Sol^3. The conjugations by elements of $\mathrm{Aff}(\mathrm{Sol}^3)$ in Proposition 3.2.5 also lift to conjugations by elements of $\mathrm{Aff}(\mathrm{Sol}^3 \times \mathbb{R})$ (see Proposition 3.1.7), so that $\mathbf{t}_1, \mathbf{t}_2$, and $\mathbf{t}_3$ have the desired form. Therefore, we can assume that $\rho(\widetilde{\Gamma}_{\mathcal{S}})$ is a standard Sol^3 lattice $\Gamma_{\mathcal{S}}$. Finally, observe that the conjugations in $\mathrm{Aff}(\mathrm{Sol}^3 \times \mathbb{R})$ do not change the maximal compact subgroup $D_4 \times \mathbb{Z}_2$. □

Similar to the Sol_1^4 case, the quotient of $\mathit{\Pi}$ by $\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}) = \langle \mathbf{t}_4 \rangle$,

$$Q = \mathit{\Pi}/\langle \mathbf{t}_4 \rangle = \mathit{\Pi}/\mathbb{Z},$$

is a finite extension of the Sol^3 lattice $\Gamma_{\mathcal{S}}$ by Ψ. In fact, we have the following commuting diagram.

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{Z} & = & \mathbb{Z} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \widetilde{\Gamma}_{\mathcal{S}} & \longrightarrow & \mathit{\Pi} & \longrightarrow & \Psi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Gamma_{\mathcal{S}} & \longrightarrow & Q & \longrightarrow & \Psi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Recall that we have let τ denote the generator of the $\mathbb{Z}_2$ factor in $D_4 \times \mathbb{Z}_2$. The inclusion $\Psi \hookrightarrow D_4 \times \mathbb{Z}_2$ lifts an inclusion of $\Phi \hookrightarrow D_4$, where Φ is one of the holonomy groups in List (3.8) in Subsection 3.2.6. That is, there is a $\Phi \subseteq D_4$ for which the diagram commutes:

$$\begin{array}{ccc}
\Psi & \longrightarrow & D_4 \times \mathbb{Z}_2 \\
\downarrow & & \downarrow {\scriptstyle /\mathbb{Z}_2 = \langle \tau \rangle} \\
\Phi & \longrightarrow & D_4
\end{array}$$

Crystallographic groups of $\mathrm{Sol}^3 \times \mathbb{R}$ do not in general project to crystallographic groups of Sol^3, as τ projects to the trivial automorphism of Sol^3. Using the notation of our Sol^3 crystallographic group classification, Q fits the short exact sequence

$$1 \to \mathbb{Z}^2 = \langle \mathbf{t}_1, \mathbf{t}_2 \rangle \to Q \to \mathbb{Z}_\Psi \to 1,$$

to which Theorem 3.2.16 applies. The group $\mathbb{Z}_\Psi$ fits the short exact sequence

$$1 \to \mathbb{Z} = \langle \mathbf{t}_3 \rangle \to \mathbb{Z}_\Psi \to \Psi \to 1.$$

Similar to the Sol^3 case, $\mathbb{Z}_\Psi$ is a subgroup of $\frac{1}{2}\mathbb{Z} \rtimes (D_4 \times \mathbb{Z}_2)$, where $\frac{1}{2}\mathbb{Z} = \langle \mathbf{t}_3^{\frac{1}{2}} \rangle$. That is, any element of $\mathbb{Z}_\Psi$ can be expressed as $(\mathbf{t}_3^{a_3}, A)$, where $a_3 = 0$ or $\frac{1}{2}$, and $A \in (D_4 \times \mathbb{Z}_2)$. We need to classify all such $\mathbb{Z}_\Psi$ and abstract kernels $\varphi : \mathbb{Z}_\Psi \to \mathrm{GL}(2, \mathbb{Z})$. The inclusion $\mathbb{Z}_\Psi \hookrightarrow \frac{1}{2}\mathbb{Z} \rtimes (D_4 \times \mathbb{Z}_2)$ lifts an inclusion of $\mathbb{Z}_\Phi \hookrightarrow \frac{1}{2}\mathbb{Z} \rtimes D_4$, where $\mathbb{Z}_\Phi$ is a group of one of the types from list (3.8) in Subsection 3.2.6. That is, the following diagram commutes (note that $\mathbb{Z}_2 = \langle \tau \rangle$ is central in $\frac{1}{2}\mathbb{Z} \rtimes (D_4 \times \mathbb{Z}_2)$):

$$\begin{array}{ccc} \mathbb{Z}_\Psi & \longrightarrow & \frac{1}{2}\mathbb{Z} \rtimes (D_4 \times \mathbb{Z}_2) \\ \downarrow & & \downarrow {\scriptstyle /\mathbb{Z}_2 = \langle \tau \rangle} \\ \mathbb{Z}_\Phi & \longrightarrow & \frac{1}{2}\mathbb{Z} \rtimes D_4 \end{array}$$

There are many more such groups in the $\mathrm{Sol}^3 \times \mathbb{R}$ case than in the Sol^3 case. For example, notice that $\mathbb{Z}_\Psi$ may contain an element of the form $\bar{\gamma} = (\mathbf{t}_3^0, \tau)$ with $\varphi(\bar{\gamma}) = I$. Such a $\mathbb{Z}_\Psi$ did not arise in the Sol^3 case.

Here is our methodology for classifying the $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic groups. Essentially, we classify all Q, then examine how to lift them to $\mathrm{Sol}^3 \times \mathbb{R}$.

(1) Classify the possible abstract kernels $\varphi : \mathbb{Z}_\Psi \to \mathrm{GL}(2, \mathbb{Z})$ with $\varphi(\mathbf{t}_3) = \mathcal{S}$. This fixes the holonomy group and the exponents on $\mathbf{t}_3$.

(2) Apply Theorem 3.2.16 to obtain all Q. This fixes the exponents of $\mathbf{t}_1, \mathbf{t}_2$, as elements of $H^2(\mathbb{Z}_\Psi; \mathbb{Z}^2) \cong H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$.

(3) Classify lifts of $\Gamma_\mathcal{S}$ to lattices of $\widetilde{\Gamma}_\mathcal{S}$ of $\mathrm{Sol}^3 \times \mathbb{R}$ (Theorem 3.3.2).

(4) Determine which $\mathrm{Sol}^3 \times \mathbb{R}$ lattices $\widetilde{\Gamma}_\mathcal{S}$ admit an abstract kernel $\Psi \to \mathrm{Out}(\widetilde{\Gamma}_\mathcal{S})$, induced by conjugation by lifts of elements of Q.

(5) Once an abstract kernel $\Psi \to \mathrm{Out}(\widetilde{\Gamma}_{\mathcal{S}})$ is fixed, the equivalence classes of extensions

$$1 \to \widetilde{\Gamma}_{\mathcal{S}} \to \Pi \to \Psi \to 1$$

are classified by $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = H^2(\Psi; \mathbb{Z})$, fixing the exponents on $\mathbf{t}_4$.

Lemma 3.4.9 (cf. Lemma 3.3.16)**.** *If* $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A\tau) \in \Pi$ *($A \in D_4$, τ is reflection on center of* $\mathrm{Sol}^3 \times \mathbb{R}$*), then by conjugation, a_4 can be made 0.*

Proof. Conjugation by $\mathbf{t}_4^{-\frac{a_4}{2}}$ fixes the lattice $\widetilde{\Gamma}_{(\mathcal{S};0,m_1,m_2)}$, and moves $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A)$ to $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}, A)$. □

Lemma 3.4.10 (cf. Lemma 3.2.10)**.** *Let*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

If $\mathbb{Z}_\Psi$ contains an element of the form $\bar{\alpha} = (\mathbf{t}_3^0, A)$ or $(\mathbf{t}_3^0, A\tau)$, then $\mathbb{Z}_\Psi$ cannot arise as the quotient of a $\mathrm{Sol}^3 \times \mathbb{R}$ *crystallographic group.*

Proof. The matrix

$$\varphi(\mathbf{t}_3)\varphi(\bar{\alpha}) = \mathcal{S}\varphi(\bar{\alpha}) = P^{-1}\Delta P P^{-1} A P = P^{-1}\Delta A P$$

must be integral. The sum of its eigenvalues, $\frac{1}{\lambda} - \lambda$, must be an integer. On the other hand, since $\frac{1}{\lambda} + \lambda$ is an integer, we infer that 2λ is an integer, so that λ is rational, which is a contradiction. A similar argument works for $-A$ as well. □

3.4.11 (**Existence of** $\Psi \to \mathrm{Out}(\widetilde{\Gamma}_{\mathcal{S}})$)**.** Given a Sol^3 crystallographic group Q with abstract kernel $\Psi \to \Gamma_{\mathcal{S}}$, we showed the *existence* of a Sol_1^4 lattice $\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)}$, lifting $\Gamma_{\mathcal{S}}$, together with induced abstract kernel $\Psi \to \mathrm{Out}(\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)})$ (Proposition 3.3.13).

Given $\beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{b_3}\mathbf{t}_4^{b_4}, B) \in (\mathrm{Sol}^3 \times \mathbb{R}) \rtimes (D_4 \times \mathbb{Z}_2)$, here we provide precise conditions for a $\mathrm{Sol}^3\times\mathbb{R}$ lattice $\widetilde{\Gamma}_{\mathcal{S}} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4\rangle$ to be invariant under conjugation by β, so that we obtain an abstract kernel $\Psi \to \mathrm{Out}(\widetilde{\Gamma}_{(\mathcal{S};q,m_1,m_2)})$. The computations below are independent of b_4 and so we set $b_4 = 0$. The conditions on c_i in the following tables are of course modulo the integers.

$\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}, A)$	$\mu(\alpha)$ on $\mathbf{t}_1, \mathbf{t}_2$	Conditions on c_i
$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\varphi(\bar{\alpha}) = A$	$\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_1^{-1}\mathbf{t}_4^{2c_1}$ $\alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_2^{-1}\mathbf{t}_4^{2c_2}$	$c_1 = 0$ or $\frac{1}{2}$ $c_2 = 0$ or $\frac{1}{2}$
$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\varphi(\bar{\alpha}) = A$	$\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_2^{-1}\mathbf{t}_4^{c_1+c_2}$ $\alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_1\mathbf{t}_4^{-c_1+c_2}$	$(c_1, c_2) = (0, 0)$ $(c_1, c_2) = (\frac{1}{2}, \frac{1}{2})$
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\varphi(\bar{\alpha}) = A$	$\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_2\mathbf{t}_4^{c_1-c_2}$ $\alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_1\mathbf{t}_4^{-c_1+c_2}$	$c_1 - c_2 = 0$
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_1$ $\alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_2^{-1}\mathbf{t}_4^{2c_2}$	$c_2 = 0$ or $\frac{1}{2}$

Table 3.3: Conjugation on $\mathbf{t}_1$ and $\mathbf{t}_2$ when holonomy action is trivial on center and $a_3 = 0$

$\alpha = (\mathbf{t}_3^{\frac{1}{2}}, A)$	$\mu(\alpha)$ on $\mathbf{t}_1, \mathbf{t}_2$	Conditions on c_i
$\varphi(\bar{\alpha}) = -K$	$\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_1^{-k_{11}}\mathbf{t}_2^{-k_{21}}\mathbf{t}_4^{c_1+k_{11}c_1+k_{21}c_2}$ $\alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_1^{-k_{12}}\mathbf{t}_2^{-k_{22}}\mathbf{t}_4^{c_2+k_{12}c_1+k_{22}c_2}$	$(c_1, c_2) \in \mathrm{Coker}(I+K^T)$

Table 3.4: Conjugation on $\mathbf{t}_1$ and $\mathbf{t}_2$ when holonomy action is trivial on center and $a_3 = \frac{1}{2}$

$\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^0, A\tau)$	$\mu(\alpha)$ on $\mathbf{t}_1, \mathbf{t}_2$	Conditions on c_i
$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\alpha}) = A$	$\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_1^{-1}$ $\alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_2^{-1}$	None
$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \varphi(\bar{\alpha}) = A$	$\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_2^{-1}\mathbf{t}_4^{-c_1+c_2}$ $\alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_1\mathbf{t}_4^{-c_1-c_2}$	$(c_1, c_2) = (0, 0)$ $(c_1, c_2) = (\frac{1}{2}, \frac{1}{2})$
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \varphi(\bar{\alpha}) = A$	$\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_2\mathbf{t}_4^{-c_1-c_2}$ $\alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_1\mathbf{t}_4^{-c_1-c_2}$	$c_1 + c_2 = 0$
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_1\mathbf{t}_4^{-2c_1}$ $\alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_2^{-1}$	$c_1 = 0$ or $\frac{1}{2}$
$A = I$	$\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_1\mathbf{t}_4^{-2c_1}$ $\alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_2\mathbf{t}_4^{-2c_2}$	$c_1 = 0$ or $\frac{1}{2}$ $c_2 = 0$ or $\frac{1}{2}$

Table 3.5: Conjugation on $\mathbf{t}_1$ and $\mathbf{t}_2$ when holonomy action is reflection on center and $a_3 = 0$

$\alpha = (\mathbf{t}_3^{\frac{1}{2}}, A\tau)$	$\mu(\alpha)$ on $\mathbf{t}_1, \mathbf{t}_2$	Conditions on c_i
$\varphi(\bar{\alpha}) = -K$	$\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_1^{-k_{11}}\mathbf{t}_2^{-k_{21}}\mathbf{t}_4^{-c_1+k_{11}c_1+k_{21}c_2}$ $\alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_1^{-k_{12}}\mathbf{t}_2^{-k_{22}}\mathbf{t}_4^{-c_2+k_{12}c_1+k_{22}c_2}$	$(c_1, c_2) \in \mathrm{Coker}(K^T - I)$
$\varphi(\bar{\alpha}) = K$	$\alpha\mathbf{t}_1\alpha^{-1} = \mathbf{t}_1^{k_{11}}\mathbf{t}_2^{k_{21}}\mathbf{t}_4^{-c_1-k_{11}c_1-k_{21}c_2}$ $\alpha\mathbf{t}_2\alpha^{-1} = \mathbf{t}_1^{k_{12}}\mathbf{t}_2^{k_{22}}\mathbf{t}_4^{-c_2-k_{12}c_1-k_{22}c_2}$	$(c_1, c_2) \in \mathrm{Coker}(I + K^T)$

Table 3.6: Conjugation on $\mathbf{t}_1$ and $\mathbf{t}_2$ when holonomy action is reflection on center and $a_3 = \frac{1}{2}$

We now study the holonomy action on the $\mathbf{t}_3$ generator of a $\mathrm{Sol}^3 \times \mathbb{R}$ lattice; that is, we study $\alpha\mathbf{t}_3\alpha^{-1}$. From the Sol^3 relations, equation (3.9) in Subsection

3.2.6:

$$\alpha \mathbf{t}_3 \alpha^{-1} = \mathbf{t}_1^{w_1}\mathbf{t}_2^{w_2}\mathbf{t}_3^{\bar{A}}\mathbf{t}_4^{w_3}, \text{ where } \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \left(I - \mathcal{S}^{\bar{A}}\right)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

Of course, w_1 and w_2 are integers; we need w_3 to be an integer.

$\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}, A)$	$\mu(\alpha)$ on $\mathbf{t}_3$	Conditions on c_i
$\bar{A} = +1$	$\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{w_1}\mathbf{t}_2^{w_2}\mathbf{t}_3\mathbf{t}_4^{-c_1w_1-c_2w_2}$	$c_1w_1 + c_2w_2 = 0$
$\bar{A} = -1$	$\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{w_1}\mathbf{t}_2^{w_2}\mathbf{t}_3^{-1}\mathbf{t}_4^{2c_3-c_1w_1-c_2w_2}$	$c_3 = \frac{c_1w_1+c_2w_2}{2}$ or $\frac{c_1w_1+c_2w_2}{2} + \frac{1}{2}$

Table 3.7: Conjugation on $\mathbf{t}_3$ when holonomy action is trivial on center and $a_3 = 0$

$\alpha = (\mathbf{t}_3^{\frac{1}{2}}, A)$	$\mu(\alpha)$ on $\mathbf{t}_3$	Conditions on c_i
$\varphi(\bar{\alpha}) = -K, (\bar{A} = +1)$	$\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_3$	None

Table 3.8: Conjugation on $\mathbf{t}_3$ when holonomy action is trivial on center and $a_3 = \frac{1}{2}$

$\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}, A\tau)$	$\mu(\alpha)$ on $\mathbf{t}_3$	Conditions on c_i
$\bar{A} = +1$	$\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{w_1}\mathbf{t}_2^{w_2}\mathbf{t}_3\mathbf{t}_4^{-2c_3-c_1w_1-c_2w_2}$	$c_3 = -\frac{c_1w_1+c_2w_2}{2}$ or $-\frac{c_1w_1+c_2w_2}{2} + \frac{1}{2}$
$\bar{A} = -1$	$\alpha\mathbf{t}_3\alpha^{-1} = \mathbf{t}_1^{w_1}\mathbf{t}_2^{w_2}\mathbf{t}_3^{-1}\mathbf{t}_4^{-c_1w_1-c_2w_2}$	$c_1w_1 + c_2w_2 = 0$

Table 3.9: Conjugation on $\mathbf{t}_3$ when holonomy action is reflection on center and $a_3 = 0$

$\alpha = (\mathbf{t}_3^{\frac{1}{2}}, A\tau)$	$\mu(\alpha)$ on $\mathbf{t}_3$	Conditions on c_i
$\varphi(\bar{\alpha}) = \pm K, (\bar{A} = +1)$	$\alpha \mathbf{t}_3 \alpha^{-1} = \mathbf{t}_3 \mathbf{t}_4^{-2c_3}$	$c_3 = 0$ or $\frac{1}{2}$

Table 3.10: Conjugation on $\mathbf{t}_3$ when holonomy action is reflection on center and $a_3 = \frac{1}{2}$

Remark 3.4.12. When $c_1 = c_2 = c_3 = 0$, so that $\widetilde{\Gamma}_{\mathcal{S}} = \Gamma_{\mathcal{S}} \times \mathbb{Z}$, the above criteria on the c_i are immediately satisfied.

Lemma 3.4.13 (cf. Theorem 3.3.15)**.** *Let $\Pi \subset (\mathrm{Sol}^3 \times \mathbb{R}) \rtimes (D_4 \times \mathbb{Z}_2)$ be a* $\mathrm{Sol}^3 \times \mathbb{R}$ *crystallographic group, with lattice embedded as in* Proposition 3.4.8*. If all elements of Π are of the form $(\mathbf{x}, A) \in \Pi$ with $\bar{A} = 1$, or $(\mathbf{x}, A\tau) \in \Pi$ with $\bar{A} = -1$, then we can assume that $c_3 = 0$ (This is analogous to the* Sol_1^4 *case when $\bar{A}\hat{A} = +1$ for all holonomy elements A).*

Proof. Let $\hat{k}$ denote the automorphism of $\mathrm{Sol}^3 \times \mathbb{R}$ defined as:

$$\hat{k} : \left(\begin{bmatrix} x \\ y \\ t \end{bmatrix}, u \right) \longrightarrow \left(\begin{bmatrix} x \\ y \\ t + ku \end{bmatrix}, u \right).$$

Setting $k = \frac{c_3}{\ln(\lambda)}$, we see that conjugation by $(e, \hat{k})^{-1} \in \mathrm{Aff}(\mathrm{Sol}^3 \times \mathbb{R})$ sets $c_3 = 0$, while fixing the maximal compact subgroup $D_4 \times \mathbb{Z}_2$. □

3.4.14 (**Computation of Cyclic Powers).** Before our main classification result, we provide computations of the cyclic powers of all possible α. This will be useful for determining the exponents on $\mathbf{t}_4$, as well as checking whether or not a crystallographic group has torsion.

$\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A)$	$\alpha^n \qquad (n = \mathrm{ord}(A))$
$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\alpha}) = A$	$\alpha^2 = \mathbf{t}_4^{2(a_4+a_1c_1+a_2c_2)}$
$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \varphi(\bar{\alpha}) = A$	$\alpha^4 = \mathbf{t}_4^{4(a_4+a_1c_1+a_2c_2)}$
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \varphi(\bar{\alpha}) = A$	$\alpha^2 = \mathbf{t}_1^{a_1+a_2}\mathbf{t}_2^{a_1+a_2}\mathbf{t}_4^{2a_4+a_1c_1+a_2c_2-a_2c_1-a_1c_2}$
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\alpha^2 = \mathbf{t}_1^{2a_1}\mathbf{t}_4^{2(a_4+a_2c_2)}$

Table 3.11: Cyclic powers when holonomy acts trivially on center and $a_3 = 0$

$\alpha = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{a_4}, A)$	$\alpha^n \qquad (n = \mathrm{ord}(A))$
$\varphi(\bar{\alpha}) = -K, (\bar{A} = +1)$	$\alpha^2 = \mathbf{t}_3\mathbf{t}_4^{2a_4}$

Table 3.12: Cyclic powers when holonomy acts trivially on center and $a_3 = \frac{1}{2}$

$\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A\tau)$	$\alpha^n \qquad (n = \mathrm{ord}(A))$
$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\alpha}) = A$	$\alpha^2 = e$
$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \varphi(\bar{\alpha}) = A$	$\alpha^4 = e$
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \varphi(\bar{\alpha}) = A$	$\alpha^2 = \mathbf{t}_1^{a_1+a_2}\mathbf{t}_2^{a_1+a_2}\mathbf{t}_4^{-(a_1c_1+a_2c_2+a_2c_1+a_1c_2)}$
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\alpha^2 = \mathbf{t}_1^{2a_1}\mathbf{t}_4^{-2a_1c_1}$
$A = I,$	$\alpha^2 = \mathbf{t}_1^{2a_1}\mathbf{t}_2^{2a_2}\mathbf{t}_4^{-2(a_1c_1+a_2c_2)}$

Table 3.13: Cyclic powers when holonomy acts as reflection on center and $a_3 = 0$

$\alpha = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{a_4}, A\tau)$	$\alpha^n \qquad (n = \mathrm{ord}(A))$
$\varphi(\bar{\alpha}) = \pm K, (\bar{A} = +1)$	$\alpha^2 = \mathbf{t}_3\mathbf{t}_4^{-c_3}$

Table 3.14: Cyclic powers when holonomy acts as reflection on center and $a_3 = \frac{1}{2}$

3.4.15 (Detecting Torsion in $\mathrm{Sol}^3 \times \mathbb{R}$ Crystallographic Groups). Given a lattice $\widetilde{\Gamma}_{\mathcal{S}}$ of $\mathrm{Sol}^3 \times \mathbb{R}$ (which projects to a lattice $\Gamma_{\mathcal{S}}$ of Sol^3), the short exact sequence

$$1 \to \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}) \to \widetilde{\Gamma}_{\mathcal{S}} \to \Gamma_{\mathcal{S}} \to 1$$

induces an S^1-bundle over the solvmanifold $\Gamma_{\mathcal{S}}\backslash\mathrm{Sol}^3$,

$$S^1 \to \widetilde{\Gamma}_{\mathcal{S}}\backslash(\mathrm{Sol}^3 \times \mathbb{R}) \to \Gamma_{\mathcal{S}}\backslash\mathrm{Sol}^3.$$

Lemma 3.4.16 (cf. Lemma 3.3.20)**.** *Let $\widetilde{\Gamma}_{\mathcal{S}}$ be a lattice of $\mathrm{Sol}^3 \times \mathbb{R}$, projecting to a standard lattice $\Gamma_{\mathcal{S}}$ of Sol^3, and suppose that for $\alpha \in (\mathrm{Sol}^3 \times \mathbb{R}) \rtimes (D_4 \times \mathbb{Z}_2)$, the group $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha \rangle$ is crystallographic. Let $\bar{\alpha}$ denote the projection of α to $\mathrm{Sol}^3 \rtimes D_4$. When the automorphism part of α acts as a reflection on the center of $\mathrm{Sol}^3 \times \mathbb{R}$, Π is torsion-free if and only if $\langle \Gamma_{\mathcal{S}}, \bar{\alpha} \rangle$ is torsion-free.*

Proof. Obviously, if $\langle \Gamma_{\mathcal{S}}, \bar{\alpha} \rangle$ is torsion-free, then Π must be torsion-free. For the converse, suppose that $\langle \Gamma_{\mathcal{S}}, \bar{\alpha} \rangle$ has torsion. In this case, the action of $\bar{\alpha}$ on the solvmanifold $\Gamma_{\mathcal{S}}\backslash\mathrm{Sol}^3$ must fix a point. Observe that the action of α on the solvmanifold $\widetilde{\Gamma}_{\mathcal{S}}\backslash(\mathrm{Sol}^3 \times \mathbb{R})$ is S^1 fiber preserving. Therefore, a circle fiber is left invariant under the action of α. Since α acts as reflection on the fiber, α must fix a point. Therefore, Π has torsion. □

Lemma 3.4.17 (cf. Lemma 3.2.20)**.** *When $\beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{b_3}\mathbf{t}_4^{b_4}, B) \in \Pi$ satisfies $b_3 = \frac{1}{2}$ and $\bar{B} = 1$, $\gamma\beta$ is not torsion for any $\gamma \in \widetilde{\Gamma}_{\mathcal{S}}$.*

Proof. Note that B is necessarily of order 2. Let $\mathrm{pr} : \mathrm{Sol}^3 \times \mathbb{R} \to \mathbb{R}$ denote the quotient homomorphism of $\mathrm{Sol}^3 \times \mathbb{R}$ by its nil-radical $\mathbb{R}^3$. Write $\gamma \in \widetilde{\Gamma}_{\mathcal{S}}$ as $\mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_3^{n_3}\mathbf{t}_4^{n_4}$. Application of pr to $(\gamma\beta)^2$ yields

$$\mathrm{pr}(\gamma\beta)^2 = 2n_3 + 1,$$

from which we infer $\gamma\beta$ is not torsion. □

We are now ready the classify the $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic groups. We will enumerate the $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic groups associated to each abstract kernel $\varphi : \mathbb{Z}_\Psi \to \mathrm{GL}(2, \mathbb{Z})$. The $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic groups will be ordered according to the abstract kernel in Theorem 3.2.9 that φ lifts. We start our classification with the trivial case (`(0)` Family). In all cases, we give:

(1) The holonomy Ψ, the conditions on $\mathcal{S}$, and the abstract kernel

$$\varphi : \mathbb{Z}_\Psi \to \mathrm{GL}(2, \mathbb{Z})$$

(this fixes the exponents on $\mathbf{t}_3$).

(2) A computation of $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ (this fixes the exponents on $\mathbf{t}_1, \mathbf{t}_2$). As a matter of notation, we set

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \text{ and } \mathbf{r} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix},$$

the exponents on $\mathbf{t}_1, \mathbf{t}_2$ corresponding to holonomy elements A or $A\tau$, B or $B\tau$, and τ, respectively.

(3) Conditions on c_1, c_2, c_3 (this fixes the lattice $\widetilde{\Gamma}_{\mathcal{S}}$, and induces an abstract kernel $\Psi \to \mathrm{Out}(\widetilde{\Gamma}_{\mathcal{S}})$).

(4) Exponents on $\mathbf{t}_4$, which are classified by $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$.

(5) When the group is torsion-free.

Theorem 3.4.18 ((0) Family)**.** *The crystallographic groups of* $\mathrm{Sol}^3 \times \mathbb{R}$*, lifting an abstract kernel* $\mathbb{Z}_\Phi \to \mathrm{GL}(2,\mathbb{Z})$ *of case* (0) *of* Theorem 3.2.9 *are:*

(0a) $\Psi = \{e\}$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3 \rangle$

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(\mathcal{S}) > 2$
- $c_3 = 0$
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}} \rangle$
- Torsion-free

(0b) $\Psi = \mathbb{Z}_2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\gamma} = (\mathbf{t}_3^0, \tau) \rangle$

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(\mathcal{S}) > 2$
- $\varphi(\bar{\gamma}) = I$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is all $\mathbf{r}$ with $(I - \mathcal{S})\mathbf{r} \equiv \mathbf{0}$ and $2\mathbf{r} \equiv \mathbf{0}$
- $c_1 = 0$ or $\frac{1}{2}$, and $c_2 = 0$ or $\frac{1}{2}$
- Two choices for c_3 (up to isomorphism), see Table 3.9
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \gamma = (\mathbf{t}_1^{t_1}\mathbf{t}_2^{t_2}\mathbf{t}_3^0, \tau) \rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ is trivial.
- Torsion-free when $\mathbf{r} \neq \mathbf{0}$.

(0c) $\Psi = \mathbb{Z}_2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\gamma} = (\mathbf{t}_3^{\frac{1}{2}}, \tau) \rangle$

$\mathcal{S} = nK - I$, where $K \in \mathrm{GL}(2,\mathbb{Z})$ with $\det(K) = +1$, $\mathrm{tr}(K) = n > 2$.

- $\varphi(\bar{\gamma}) = K$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is trivial, so that we can take $\mathbf{r} = \mathbf{0}$.
- $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$

- $c_3 = 0$
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \gamma = (\mathbf{t}_3^{\frac{1}{2}}, \tau) \rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ is trivial.
- Torsion-free

Proof. The arguments in Theorem 3.2.9 establish the three different possibilities for $\mathbb{Z}_\Psi$. In particular, `(2b)` from Theorem 3.2.9 gives the conditions on $\mathcal{S}$ in case `(0c)`. The difference is that $\varphi(\bar{\gamma})$ is given by K, not $-K$, as in Theorem 3.2.9 `(2b)`.

In case `(0b)`, the cocycle condition for $\mathbf{r} \in \mathrm{Coker}(I - \mathcal{S})$ is that $(I + \varphi(\bar{\gamma}))\mathbf{r} = 2\mathbf{r} \equiv \mathbf{0}$, whereas the coboudary vanishes. This shows our description of $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$. In case `(0c)`, $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ vanishes by Corollary 3.2.19.

In all cases, the conditions on c_1, c_2 follow from the Tables 3.5 and 3.6. We can take $c_3 = 0$ in case `(0a)` by Lemma 3.4.13, and $c_3 = 0$ is forced in case `(0c)` by Table 3.14. Table 3.9 shows that we have two choices for c_3 in case `(0b)`.

Evidently, groups of type `(0b)` are torsion-free when $\mathbf{r} \neq \mathbf{0}$. The groups of type `(0c)` are torsion-free Lemma 3.4.17. In all three cases, we take $a_4 = 0$ by Lemma 3.4.9, and $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ vanishes. □

Theorem 3.4.19 (`(1)` Family)**.** *The crystallographic groups of* $\mathrm{Sol}^3 \times \mathbb{R}$*, lifting an abstract kernel* $\mathbb{Z}_\Phi \to \mathrm{GL}(2, \mathbb{Z})$ *of case* `(1)` *of* Theorem 3.2.9*,*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \textit{ are}$$

(1a) $\Psi = \mathbb{Z}_2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A)\rangle$

- $\mathcal{S} = nK + I$, where $K \in \mathrm{GL}(2, \mathbb{Z})$ with $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$.
- $\varphi(\bar{\alpha}) = -K$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is trivial, so that we can take $\mathbf{a} = \mathbf{0}$.
- $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$
- $c_3 = 0$
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_3^{\frac{1}{2}} \mathbf{t}_4^{a_4}, A)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$
- Torsion-free

(1b) $\Psi = \mathbb{Z}_2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A\tau)\rangle$

- $\mathcal{S} = nK + I$, where $K \in \mathrm{GL}(2, \mathbb{Z})$ with $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$.
- $\varphi(\bar{\alpha}) = -K$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is trivial, so that we can take $\mathbf{a} = \mathbf{0}$.
- $(c_1, c_2) \in \mathrm{Coker}(K^T - I)$
- $c_3 = 0$
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_3^{\frac{1}{2}}, A\tau)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ is trivial.
- Torsion-free

(1c) $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A), \bar{\gamma} = (\mathbf{t}_3^0, \tau)\rangle$

- $\mathcal{S} = nK + I$, where $K \in \mathrm{GL}(2, \mathbb{Z})$ with $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$.
- $\varphi(\bar{\alpha}) = -K, \varphi(\bar{\gamma}) = I$

- $H^1(\Psi; \text{Coker}(I - \mathcal{S}))$ is all $\mathbf{r}$ with $(I + K)\mathbf{r} \equiv \mathbf{0}$ and $2\mathbf{r} \equiv \mathbf{0}$
- $(c_1, c_2) \in \text{Coker}(I + K^T)$ and $c_1 = 0$ or $\frac{1}{2}$, $c_2 = 0$ or $\frac{1}{2}$
- One choice for c_3, Table 3.9 gives two different values for c_3, but only one satisfies $(\alpha\gamma)^2 \in \widetilde{\Gamma}_{\mathcal{S}}$.
- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{a_4}, A), \gamma = (\mathbf{t}_1^{t_1}\mathbf{t}_2^{t_2}\mathbf{t}_3^{0}, \tau) \rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$
- Torsion-free when $\mathbf{r} \neq \mathbf{0}$.

Proof. First, observe that

$$\begin{aligned}\mathbb{Z}_\Psi &= \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A), \bar{\gamma} = (\mathbf{t}_3^{\frac{1}{2}}, \tau) \rangle \\ &= \langle \mathbf{t}_3, (\mathbf{t}_3^{0}, A\tau), (\mathbf{t}_3^{\frac{1}{2}}, \tau) \rangle\end{aligned}$$

cannot be lifted to a $\text{Sol}^3 \times \mathbb{R}$ crystallographic group by Lemma 3.4.10, and we therefore disregard this case.

In cases `(1a)` and `(1b)`, $H^1(\Psi; \text{Coker}(I - \mathcal{S}))$ vanishes by Corollary 3.2.19. In case `(1c)`, Corollary 3.2.19 implies we may take $\mathbf{a} = \mathbf{0}$. Then the cocycle conditions imply that $\mathbf{r}$ must satisfy

$$(I + K)\mathbf{r} \equiv \mathbf{0}$$

and

$$(I + \varphi(\bar{\gamma}))\mathbf{r} = 2\mathbf{r} \equiv \mathbf{0}.$$

In all cases, the conditions on c_1, c_2 follow from the Tables 3.5, 3.4, and 3.6. We can take $c_3 = 0$ in case `(1a)` by Lemma 3.4.13, and $c_3 = 0$ is forced in case `(1b)` by Table 3.14. Table 3.9 shows that we have two choices for c_3

in case (1c). However,

$$(\alpha\gamma)^2 = \mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_3\mathbf{t}_4^{-c_3+v},$$

for integers n_i, and a rational number v. Only one choice for c_3 (modulo the integers) will ensure $-c_3 + v$ is integral.

The two choices for a_4 in cases (1a) and (1c) arise from Table 3.12:

$$\alpha^2 = \mathbf{t}_3\mathbf{t}_4^{2a_4}. \qquad \square$$

Theorem 3.4.20 ((2a) Family)**.** *The crystallographic groups of* Sol$^3 \times \mathbb{R}$*, lifting an abstract kernel* $\mathbb{Z}_\Phi \to \mathrm{GL}(2,\mathbb{Z})$ *of case* (2a) *of* Theorem 3.2.9*,*

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \textit{ are}$$

(2aa) $\Psi = \mathbb{Z}_2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A)\rangle$

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(\mathcal{S}) > 2$
- $\varphi(\bar{\alpha}) = A$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S})) = \frac{\mathrm{Coker}(I-\mathcal{S})}{2\mathrm{Coker}(I-\mathcal{S})}$. That is, all $\mathbf{a}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$, taken modulo $2\mathbf{a}$.
- $c_1 = 0$ or $\frac{1}{2}$, $c_2 = 0$ or $\frac{1}{2}$
- $c_3 = 0$, must satisfy conditions in Table 3.7
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^0\mathbf{t}_4^{a_4}, A)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = -(a_1c_1 + a_2c_2) + \frac{i}{2}, i = 0, 1$
- Torsion-free when $i = 1$

(2ab) $\Psi = \mathbb{Z}_2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau)\rangle$

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(\mathcal{S}) > 2$

- $\varphi(\bar{\alpha}) = A$

- $H^1(\Psi; \mathrm{Coker}(I-\mathcal{S})) = \frac{\mathrm{Coker}(I-\mathcal{S})}{2\mathrm{Coker}(I-\mathcal{S})}$. That is, all $\mathbf{a}$ with $(I-\mathcal{S})\mathbf{a} \equiv \mathbf{0}$, taken modulo $2\mathbf{a}$.

- Two choices for c_3 (up to isomorphism), see Table 3.9

- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}, A\tau)\rangle$

- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ is trivial.

- Always has torsion

(2ac) $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\gamma} = (\mathbf{t}_3^0, \tau)\rangle$

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(\mathcal{S}) > 2$

- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\gamma}) = I$

- $H^1(\Psi; \mathrm{Coker}(I-\mathcal{S}))$ is all $\mathbf{a}, \mathbf{r}$ with $(I-\mathcal{S})\mathbf{a} \equiv (I-\mathcal{S})\mathbf{r} \equiv \mathbf{0}$ satisfying $2\mathbf{r} \equiv \mathbf{0}$

- $c_1 = 0$ or $\frac{1}{2}$, $c_2 = 0$ or $\frac{1}{2}$

- Two choices for c_3 (up to isomorphism), see Tables 3.7 and 3.9

- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \gamma = (\mathbf{t}_1^{t_1}\mathbf{t}_2^{t_2}\mathbf{t}_3^{0}, \tau)\rangle$

- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = -(a_1c_1 + a_2c_2) + \frac{i}{2}(i = 0, 1)$

- Always has torsion

(2ad) $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\gamma} = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle$

- $\mathcal{S} = nK - I$, where $K \in \mathrm{GL}(2,\mathbb{Z})$ with $\det(K) = +1$, $\mathrm{tr}(K) = n > 2$.

- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\gamma}) = K$

• $H^1(\Psi;\mathrm{Coker}(I-\mathcal{S})) = \frac{\mathrm{Coker}(I-K)}{2\mathrm{Coker}(I-K)}$. That is, all $\mathbf{a}$ with $(I-K)\mathbf{a} \equiv \mathbf{0}$, taken modulo $2\mathbf{a}$.

• $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$ and $c_1 = 0$ or $\frac{1}{2}$, $c_2 = 0$ or $\frac{1}{2}$

• $c_3 = 0$, must satisfy conditions in Table 3.7

• $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \gamma = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle$

• $H^2(\Psi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = -(a_1c_1 + a_2c_2) + \frac{i}{2}(i = 0, 1)$

•Torsion-free when $i = 1$

(2ae) $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau), \bar{\gamma} = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle$

• $\mathcal{S} = nK - I$, where $K \in \mathrm{GL}(2,\mathbb{Z})$ with $\det(K) = +1$, $\mathrm{tr}(K) = n > 2$.

• $\varphi(\bar{\alpha}) = A, \varphi(\bar{\gamma}) = K$

• $H^1(\Psi;\mathrm{Coker}(I-\mathcal{S})) = \frac{\mathrm{Coker}(I-K)}{2\mathrm{Coker}(I-K)}$. That is, all $\mathbf{a}$ with $(I-K)\mathbf{a} \equiv \mathbf{0}$, taken modulo $2\mathbf{a}$.

• $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$

• $c_3 = 0$, must satisfy conditions in Table 3.9

• $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A\tau), \gamma = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle$

• $H^2(\Psi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, two choices for a_4. We have $(\alpha\gamma)^2 = \mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_3\mathbf{t}_4^{2a_4+v}$ for integers n_1, and some rational number v, and therefore two solutions for a_4.

• Always has torsion

Proof. The computation for $H^1(\Psi;\mathrm{Coker}(I-\mathcal{S}))$ for cases (2aa) and (2ab) follows immediately from that of case (2a) in Theorem 3.2.21. That for cases

`(2ac)`, `(2ad)`, and `(2ae)` follows from computing the appropriate cocycle conditions in Remark 3.2.17, together with application of Corollary 3.2.19. In particular, in cases `(2ad)` and `(2ae)`, we take $\mathbf{r} = \mathbf{0}$ by Corollary 3.2.19; the commutator cocycle condition in cases `(2ad)` and `(2ae)` then reduces to

$$(I - \phi(\bar{\gamma}))\mathbf{a} = (I - \phi(\bar{\alpha}))\mathbf{r}$$
$$(I - K)\mathbf{a} \equiv \mathbf{0}.$$

In all cases, the conditions on c_1, c_2 follow from the Tables 3.3, 3.5, and 3.6. We can take $c_3 = 0$ in case `(2aa)` by Lemma 3.4.13. We have two choices for c_3 in cases `(2ab)` and `(2ac)`. For cases `(2ad)` and `(2ae)`, notice that from Table 3.14, $\gamma^2 = \mathbf{t}_3\mathbf{t}_4^{-c_3}$, so that $c_3 = 0$ is forced.

The two choices for a_4 in cases `(2aa)`, `(2ac)`, and `(2ad)` arise from Table 3.12:

$$\alpha^2 = \mathbf{t}_4^{2(a_4+a_1c_1+a_2c_2)}.$$

The torsion criteria are clear, given that Sol^3 groups of type `(2a)` in Theorem 3.2.21 always have torsion. Notice that γ in case `(2ad)` cannot contribute torsion by Lemma 3.4.17. Lemma 3.4.16 implies that `(2ab)` always has torsion, and that `(2ac)` also has torsion as it contains case `(2ab)`. Case `(2ae)` is always torsion by Lemma 3.4.16. □

Theorem 3.4.21 (`(2b)` Family)**.** *The crystallographic groups of* $\mathrm{Sol}^3 \times \mathbb{R}$*, lifting an abstract kernel* $\mathbb{Z}_\Phi \to \mathrm{GL}(2,\mathbb{Z})$ *of case* `(2b)` *of* Theorem 3.2.9*,*

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ are}$$

(2ba) $\Psi = \mathbb{Z}_2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A)\rangle$

- $\mathcal{S} = nK - I$, where $K \in \mathrm{GL}(2,\mathbb{Z})$ with $\det(K) = +1$, $\mathrm{tr}(K) = n > 2$.
- $\varphi(\bar{\alpha}) = -K$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is trivial so that $\mathbf{a} = \mathbf{0}$.
- $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$
- $c_3 = 0$
- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{a_4}, A)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = 0,\ \frac{1}{2}$
- Torsion-free

(2bb) $\Psi = \mathbb{Z}_2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A\tau)\rangle$

- $\mathcal{S} = nK - I$, where $K \in \mathrm{GL}(2,\mathbb{Z})$ with $\det(K) = +1$, $\mathrm{tr}(K) = n > 2$.
- $\varphi(\bar{\alpha}) = -K$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is trivial so that $\mathbf{a} = \mathbf{0}$.
- $(c_1, c_2) \in \mathrm{Coker}(K^T - I)$
- $c_3 = 0$
- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_3^{\frac{1}{2}}, A\tau)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ is trivial.
- Torsion-free

(2bc) $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A), \bar{\gamma} = (\mathbf{t}_3^{0}, \tau)\rangle$

- $\mathcal{S} = nK - I$, where $K \in \mathrm{GL}(2,\mathbb{Z})$ with $\det(K) = +1$, $\mathrm{tr}(K) = n > 2$.
- $\varphi(\bar{\alpha}) = -K, \varphi(\bar{\gamma}) = I$

- $H^1(\Psi;\text{Coker}(I-\mathcal{S}))$ all $\mathbf{r}$ with $(I+K)\mathbf{r}\equiv\mathbf{0}$, $2\mathbf{r}\equiv\mathbf{0}$.

- $(c_1,c_2)\in\text{Coker}(I+K^T)$ and $c_1=0$ or $\frac{1}{2}$, $c_2=0$ or $\frac{1}{2}$

- One choice for c_3, Table 3.9 gives two different values for c_3, but only one satisfies $(\alpha\gamma)^2\in\widetilde{\Gamma}_{\mathcal{S}}$.

- $\mathit{\Pi}=\langle\widetilde{\Gamma}_{\mathcal{S}},\alpha=(\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{a_4},A),\gamma=(\mathbf{t}_1^{t_1}\mathbf{t}_2^{t_2}\mathbf{t}_3^{0},\tau)\rangle$

- $H^2(\Psi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))=\mathbb{Z}_2$, $a_4=0,\ \frac{1}{2}$

- Torsion-free when $\mathbf{r}\neq\mathbf{0}$.

Proof. First, note that

$$\begin{aligned}\mathbb{Z}_\Psi&=\langle\mathbf{t}_3,\bar{\alpha}=(\mathbf{t}_3^{\frac{1}{2}},A),\bar{\gamma}=(\mathbf{t}_3^{\frac{1}{2}},\tau)\rangle\\&=\langle\mathbf{t}_3,\bar{\alpha}=(\mathbf{t}_3^{0},A\tau),\bar{\gamma}=(\mathbf{t}_3^{\frac{1}{2}},\tau)\rangle\end{aligned}$$

is a lift of type `(2a)` from the Sol^3 classification, and it corresponds to type `(2ae)` in Theorem 3.4.20.

The computation for $H^1(\Psi;\text{Coker}(I-\mathcal{S}))$ for cases `(2ba)` and `(2bb)` follows immediately from that of case `(2b)` in Theorem 3.2.21. That for case `(2bc)` follows from computing the appropriate cocycle conditions in Remark 3.2.17, together with application of Corollary 3.2.19.

In all cases, the conditions on c_1,c_2 follow from the Tables 3.5, 3.4, and 3.6. We can take $c_3=0$ in case `(2ba)` by Lemma 3.4.13. From Table 3.14, $c_3=0$ is forced in case `(2bb)`. Table 3.9 shows that we have two choices for c_3 in case `(2bc)`. However,

$$(\alpha\gamma)^2=\mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_3\mathbf{t}_4^{-c_3+v},$$

for integers n_i, and a rational number v. Only one choice for c_3 (modulo the integers) will ensure $-c_3 + v$ is integral.

For the torsion criteria, notice that Sol^3 groups of type `(2b)` were always torsion-free. □

Theorem 3.4.22 (`(3)` Family)**.** *The crystallographic groups of* $\mathrm{Sol}^3 \times \mathbb{R}$*, lifting an abstract kernel* $\mathbb{Z}_\Phi \to \mathrm{GL}(2, \mathbb{Z})$ *of case* `(3)` *of* Theorem 3.2.9,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \textit{are}$$

`(3a)` $\Psi = \mathbb{Z}_2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A) \rangle$

- $\mathcal{S} \in \mathrm{SL}(2, \mathbb{Z})$ with $\sigma_{12} = -\sigma_{21}$ and $\mathrm{tr}(\mathcal{S}) > 2$.
- $\varphi(\bar{\alpha}) = A$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ can be described as all $\mathbf{a}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$, $a_2 \equiv -a_1$ taken modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ where $(I - \mathcal{S})\mathbf{v} \equiv \mathbf{0}$.
- $c_1 - c_2 = 0$
- Two choices for c_3 (up to isomorphism), see Table 3.7
- $\Pi = \langle \widetilde{\Gamma}_\mathcal{S}, \alpha = (\mathbf{t}_1^{a_1} \mathbf{t}_2^{a_2} \mathbf{t}_3^0 \mathbf{t}_4^{a_4}, A) \rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_\mathcal{S})) = \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$
- Torsion-free when $a_4 = \frac{1}{2}$

`(3b)` $\Psi = \mathbb{Z}_2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau) \rangle$

- $\mathcal{S} \in \mathrm{SL}(2, \mathbb{Z})$ with $\sigma_{12} = -\sigma_{21}$ and $\mathrm{tr}(\mathcal{S}) > 2$.
- $\varphi(\bar{\alpha}) = A$

- $H^1(\Psi;\mathrm{Coker}(I-\mathcal{S}))$ can be described as all $\mathbf{a}$ with $(I-\mathcal{S})\mathbf{a}\equiv\mathbf{0}$, $a_2\equiv -a_1$ taken modulo $(v_1-v_2)\begin{bmatrix}1\\-1\end{bmatrix}$ where $(I-\mathcal{S})\mathbf{v}\equiv\mathbf{0}$.
- $c_1+c_2=0$
- $c_3=0$, must satisfy conditions in Table 3.9
- $\Pi=\langle\widetilde{\Gamma}_{\mathcal{S}},\alpha=(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0},A\tau)\rangle$
- $H^2(\Psi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ is trivial.
- Always has torsion

(3c) $\Psi=\mathbb{Z}_2^2$, $\mathbb{Z}_\Psi=\langle\mathbf{t}_3,\bar{\alpha}=(\mathbf{t}_3^0,A),\bar{\gamma}=(\mathbf{t}_3^0,\tau)\rangle$

- $\mathcal{S}\in\mathrm{SL}(2,\mathbb{Z})$ with $\sigma_{12}=-\sigma_{21}$ and $\mathrm{tr}(\mathcal{S})>2$.
- $\varphi(\bar{\alpha})=A,\varphi(\bar{\gamma})=I$
- $H^1(\Psi;\mathrm{Coker}(I-\mathcal{S}))$ can be described as all $\mathbf{a}$, $\mathbf{r}$ with $(I-\mathcal{S})\mathbf{a}\equiv(I-\mathcal{S})\mathbf{r}\equiv\mathbf{0}$, $\mathbf{r}=(0,0)$ or $(\frac{1}{2},\frac{1}{2})$,
and $a_2\equiv -a_1$ taken modulo $(v_1-v_2)\begin{bmatrix}1\\-1\end{bmatrix}$ where $(I-\mathcal{S})\mathbf{v}\equiv\mathbf{0}$.
- $(c_1,c_2)=(0,0)$ or $(c_1,c_2)=(\frac{1}{2},\frac{1}{2})$
- Two choices for c_3 (up to isomorphism), see Tables 3.7 and 3.9
- $\Pi=\langle\widetilde{\Gamma}_{\mathcal{S}},\alpha=(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4},A),\gamma=(\mathbf{t}_1^{t_1}\mathbf{t}_2^{t_2}\mathbf{t}_3^{0},\tau)\rangle$
- $H^2(\Psi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))=\mathbb{Z}_2$, $a_4=0$ or $\frac{1}{2}$
- Always has torsion

(3d) $\Psi=\mathbb{Z}_2^2$, $\mathbb{Z}_\Psi=\langle\mathbf{t}_3,\bar{\alpha}=(\mathbf{t}_3^0,A),\bar{\gamma}=(\mathbf{t}_3^{\frac{1}{2}},\tau)\rangle$

- $\mathcal{S}=nK-I$, where $K\in\mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(K)=n>2$ and $k_{12}=-k_{21}$.
- $\varphi(\bar{\alpha})=A,\varphi(\bar{\gamma})=K$

- $H^1(\Psi; \text{Coker}(I - \mathcal{S}))$ can be described as all $\mathbf{a}$ with $(I - K)\mathbf{a} \equiv \mathbf{0}$, $a_2 \equiv -a_1$ taken modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ where $(I - K)\mathbf{v} \equiv \mathbf{0}$.
- $(c_1, c_2) \in \text{Coker}(I + K^T)$, $c_1 - c_2 = 0$
- $c_3 = 0$, must satisfy conditions in Table 3.7
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \gamma = (\mathbf{t}_3^{\frac{1}{2}}, \tau) \rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$
- Always has torsion

Proof. The computation for $H^1(\Psi; \text{Coker}(I - \mathcal{S}))$ for cases `(3a)` and `(3b)` follows immediately from that of case `(3)` in Theorem 3.2.21. That for cases `(3c)` and `(3d)` follows from computing the appropriate cocycle conditions in Remark 3.2.17, together with application of Corollary 3.2.19. Let us examine `(3c)` in greater detail. Notice that the cyclic power cocycle condition forces $2\mathbf{r} \equiv 0$. Furthermore, consider the commutator cocycle condition in case `(3c)`:

$$(I - \varphi(\alpha))\mathbf{r} = (I - \varphi(\gamma))\mathbf{a}$$
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{r} = \mathbf{0}$$

This forces $\mathbf{r}$ to be either $(0, 0)$ or $(\frac{1}{2}, \frac{1}{2})$.

In all cases, the conditions on c_1, c_2 follow from the Tables 3.3, 3.5, and 3.6. We can take $c_3 = 0$ in case `(3b)` by Lemma 3.4.13. From Table 3.14, $c_3 = 0$ is forced in case `(3d)`. For cases `(3a)` and `(3c)`, we have two choices for c_3.

For the choices we have for a_4 in cases `(3a)`, `(3c)`, and `(3d)`, note that

$c_1 - c_2 = 0$ and by Table 3.11,

$$\alpha^2 = \mathbf{t}_1^{a_1+a_2}\mathbf{t}_2^{a_1+a_2}\mathbf{t}_4^{2a_4+a_1c_1+a_2c_2-a_2c_1-a_1c_2} = \mathbf{t}_4^{2a_4+a_1(c_1-c_2)-a_2(c_1-c_2)} = \mathbf{t}_4^{2a_4}.$$

For the torsion criteria, notice that Sol^3 groups of type `(3)` always have torsion. Thus Lemma 3.4.16 applies to show that α is always a torsion element in case `(3b)`, $\alpha\gamma$ is a torsion element in case `(3c)`, and $\alpha\gamma$ is a torsion element in case `(3d)`. Only in case `(3a)` can we resolve torsion in the central direction of $\mathrm{Sol}^3 \times \mathbb{R}$.

□

Theorem 3.4.23 (`(3`i`)` Family)**.** *The crystallographic groups of* $\mathrm{Sol}^3 \times \mathbb{R}$*, lifting an abstract kernel* $\mathbb{Z}_\Phi \to \mathrm{GL}(2,\mathbb{Z})$ *of case* `(3`i`)` *of* Theorem 3.2.9*,*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \textit{ are}$$

`(3`i`a)` $\Psi = \mathbb{Z}_2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A)\rangle$

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\sigma_{11} = \sigma_{22}$ and $\mathrm{tr}(\mathcal{S}) > 2$.
- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ can be described as all $\mathbf{a}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$, $2a_1 \equiv 0$ (so $a_1 = 0$ or $\frac{1}{2}$) modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$ where $(I - \mathcal{S})\mathbf{v} \equiv \mathbf{0}$.
- $c_2 = 0$ or $\frac{1}{2}$
- Two choices for c_3 (up to isomorphism), see Table 3.7
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = -a_2c_2 + \frac{i}{2}(i = 0, 1)$

- When $i = 0$, torsion-free when $a_1 = \frac{1}{2}$ and $a_2 \not\equiv \frac{(\sigma_{11}+1)(2n+1)}{2\sigma_{12}}$ for any $n \in \mathbb{Z}$
- When $i = 1$, always torsion-free

(3*i*b) $\Psi = \mathbb{Z}_2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau)\rangle$

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\sigma_{11} = \sigma_{22}$ and $\mathrm{tr}(\mathcal{S}) > 2$.
- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ can be described as all $\mathbf{a}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$, $2a_1 \equiv 0$ (so $a_1 = 0$ or $\frac{1}{2}$) modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$ where $(I - \mathcal{S})\mathbf{v} \equiv \mathbf{0}$.
- $c_1 = 0$ or $\frac{1}{2}$
- $c_3 = 0$, must satisfy conditions in Table 3.9
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_\mathcal{S}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^0, A\tau)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_\mathcal{S}))$ is trivial.
- Torsion-free when $a_1 = \frac{1}{2}$ and $a_2 \not\equiv \frac{(\sigma_{11}+1)(2n+1)}{2\sigma_{12}}$ for any $n \in \mathbb{Z}$.

(3*i*c) $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\gamma} = (\mathbf{t}_3^0, \tau)\rangle$

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\sigma_{11} = \sigma_{22}$ and $\mathrm{tr}(\mathcal{S}) > 2$.
- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\gamma}) = I$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ can be described as all $\mathbf{a}$, $\mathbf{r}$ with $(I - \mathcal{S})\mathbf{a} \equiv (I - \mathcal{S})\mathbf{r} \equiv \mathbf{0}$, $2\mathbf{r} \equiv 0$, $2a_1 \equiv 0$ with $\mathbf{a}$ taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$ where $(I - \mathcal{S})\mathbf{v} \equiv \mathbf{0}$.
- $c_1 = 0$ or $\frac{1}{2}$, and $c_2 = 0$ or $\frac{1}{2}$
- Two choices for c_3 (up to isomorphism), see Tables 3.7 and 3.9

- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \gamma = (\mathbf{t}_1^{t_1}\mathbf{t}_2^{t_2}\mathbf{t}_3^{0}, \tau)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = -a_2c_2 + \frac{i}{2}(i = 0, 1)$
- Torsion-free when $\mathbf{r} \neq \mathbf{0}$, $t_1 + a_1 = \frac{1}{2}$, $t_2 + a_2 \not\equiv \frac{(\sigma_{11}+1)(2n+1)}{2\sigma_{12}}$ for any $n \in \mathbb{Z}$, and...
  - when $i = 0$, in addition $a_1 = \frac{1}{2}$ and $a_2 \not\equiv \frac{(\sigma_{11}+1)(2n+1)}{2\sigma_{12}}$ for any $n \in \mathbb{Z}$
  - when $i = 1$, no additional condition

(3*i*d) $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\gamma} = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle$

- $\mathcal{S} = nK - I$, where $K \in \mathrm{SL}(2, \mathbb{Z})$ with $\mathrm{tr}(K) = n > 2$ and $k_{11} = k_{22}$.
- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\gamma}) = K$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ can be described as all $\mathbf{a}$ with $(I - K)\mathbf{a} \equiv \mathbf{0}$, $2a_1 \equiv 0$ modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$ where $(I - K)\mathbf{v} \equiv \mathbf{0}$.
- $c_2 = 0$ or $\frac{1}{2}$, and $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$
- $c_3 = 0$, must satisfy conditions in Table 3.7
- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \gamma = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = -a_2c_2 + \frac{i}{2}(i = 0, 1)$
- Torsion-free ...
  - when $i = 0$, $a_1 = \frac{1}{2}$ and $a_2 \not\equiv \frac{(k_{11}+1)(2n+1)}{2k_{12}}$ for any $n \in \mathbb{Z}$.
  - when $i = 1$, the $\langle \widetilde{\Gamma}_{\mathcal{S}}, (\alpha\gamma)\rangle$ must be torsion-free, projected to Sol^3.

Proof. The computation for $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ for cases (3*i*a) and (3*i*b) follows immediately from that of case (3*i*) in Theorem 3.2.21. That for cases (3*i*c) and (3*i*d) follows from computing the appropriate cocycle conditions in

Remark 3.2.17, together with application of Corollary 3.2.19. Let us examine (3*i*c) in greater detail. Notice that the cyclic power cocycle condition forces $2\mathbf{r} \equiv 0$. We do not obtain an additional condition on $\mathbf{r}$ from commutator cocycle condition in case (3*i*c):

$$(I - \varphi(\alpha))\mathbf{r} = (I - \varphi(\gamma))\mathbf{a}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{r} = \mathbf{0}.$$

In all cases, the conditions on c_1, c_2 follow from the Tables 3.3, 3.5, and 3.6. We can take $c_3 = 0$ in case (3*i*b) by Lemma 3.4.13. From Table 3.14, $c_3 = 0$ is forced in case (3*i*d). For cases (3*i*a) and (3*i*c), we have two choices for c_3.

For the choices we have for a_4 in cases (3*i*a), (3*i*c), and (3*i*d), note that

$$\alpha^2 = \mathbf{t}_1^{2a_1} \mathbf{t}_4^{2(a_4 + a_2 c_2)}.$$

For case (3*i*a), when $i = 1$, Π is always torsion-free. But when $i = 0$, the projection of Π to a Sol^3 crystallographic group of type (3*i*) must be torsion-free. The torsion criteria now follow from Theorem 3.2.21, case (3*i*).

For case (3*i*b), by Lemma 3.4.16, the projection of Π to a Sol^3 crystallographic group of type (3*i*) must be torsion-free, and hence the desired criteria.

For case (3*i*c), we first require that $\mathbf{r} \neq \mathbf{0}$ for γ to be of infinite order. In addition,

$$\gamma\alpha = (\mathbf{t}_1^{a_1 + t_1} \mathbf{t}_2^{a_2 + t_2} \mathbf{t}_4^{a_4}, A\tau)$$

must be infinite order when projected to Sol^3 by Lemma 3.4.16. Thus we obtain the stated conditions on $a_1 + t_1$ and $a_2 + t_2$.

Now, when $i = 1$, α is never torsion. But when $i = 0$, the projection of α to Sol^3 must be of infinite order, and we obtain additional conditions on a_1 and a_2.

For case $(\mathtt{3}i\mathtt{d})$, note that when $i = 0$, both

$$\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A)$$
$$\alpha\gamma = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{\frac{1}{2}}, A\tau)$$

must be torsion-free, when projected to Sol^3. This occurs precisely when the Sol^3 crystallographic group of type $(\mathtt{3}i)$,

$$Q' = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3^{\frac{1}{2}}, \alpha \rangle$$

is torsion-free. Note that the lattice of Q' is $\langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3^{\frac{1}{2}} \rangle = \mathbb{Z}^2 \rtimes_K \mathbb{Z}$. The desired criteria on a_1 and a_2 now follow by applying the result of Theorem 3.2.21, case $(\mathtt{3}i)$ with K in place of $\mathcal{S}$.

When $i = 1$, α and γ are always of infinite order. But Lemma 3.4.16 implies that

$$\alpha\gamma = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{\frac{1}{2}}, A\tau)$$

must be of infinite order, when projected to Sol^3. This occurs precisely when the Sol^3 crystallographic group of type (3i),

$$Q'' = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \alpha\gamma \rangle$$

is torsion-free. We can conjugate $\alpha\gamma$ by $\mathbf{t}_3^{-\frac{1}{4}}$ to obtain

$$\mathbf{t}_3^{-\frac{1}{4}}(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{\frac{1}{2}}, A\tau)\mathbf{t}_3^{\frac{1}{4}} = (\mathbf{t}_1^{a_1'}\mathbf{t}_2^{a_2'}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A\tau)$$

Now one could apply Theorem 3.2.21, case (3i) to obtain criteria on a_1', a_2'. □

Theorem 3.4.24 ((4) Family)**.** *The crystallographic groups of* $\mathrm{Sol}^3 \times \mathbb{R}$*, lifting an abstract kernel* $\mathbb{Z}_\Phi \to \mathrm{GL}(2,\mathbb{Z})$ *of case* (4) *of* Theorem 3.2.9*,*

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \textit{are}$$

(4a) $\Psi = \mathbb{Z}_4$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A) \rangle$

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$, symmetric and $\mathrm{tr}(\mathcal{S}) > 2$.
- $\varphi(\bar{\alpha}) = A$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S})) \cong \frac{\mathrm{Coker}(I-\mathcal{S})}{\mathrm{Im}(I-A)}$.
- $(c_1, c_2) = (0,0)$ or $(c_1, c_2) = (\frac{1}{2}, \frac{1}{2})$
- Two choices for c_3 (up to isomorphism), see Table 3.7
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A) \rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_4$, $a_4 = -(a_1c_1 + a_2c_2) + \frac{i}{4}(i = 0,1,2,3)$
- Torsion-free for $i = 1, 3$

(4b) $\Psi = \mathbb{Z}_4$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau) \rangle$

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$, symmetric and $\mathrm{tr}(\mathcal{S}) > 2$.
- $\varphi(\bar{\alpha}) = A$

- $H^1(\Psi; \mathrm{Coker}(I-\mathcal{S})) \cong \frac{\mathrm{Coker}(I-\mathcal{S})}{\mathrm{Im}(I-A)}$.

- $(c_1, c_2) = (0, 0)$ or $(c_1, c_2) = (\frac{1}{2}, \frac{1}{2})$

- $c_3 = 0$, must satisfy conditions in Table 3.9

- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}, A\tau) \rangle$

- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ is trivial.

- Always has torsion

`(4c)` $\Psi = \mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\gamma} = (\mathbf{t}_3^0, \tau) \rangle$

- $\mathcal{S} \in \mathrm{SL}(2, \mathbb{Z})$, symmetric and $\mathrm{tr}(\mathcal{S}) > 2$.

- $\varphi(\bar{\alpha}) = A$

- $H^1(\Psi; \mathrm{Coker}(I-\mathcal{S}))$ is all $\mathbf{a}$, $\mathbf{r}$ with $(I-\mathcal{S})\mathbf{a} \equiv (I-\mathcal{S})\mathbf{r} \equiv \mathbf{0}$, $\mathbf{r} = (0,0)$ or $(\frac{1}{2}, \frac{1}{2})$, and $\mathbf{a}$ taken modulo $(I-A)\mathbf{v}$, for $(I-\mathcal{S})\mathbf{v} \equiv \mathbf{0}$.

- $(c_1, c_2) = (0, 0)$ or $(c_1, c_2) = (\frac{1}{2}, \frac{1}{2})$

- Two choices for c_3 (up to isomorphism), see Tables 3.7 and 3.9

- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \gamma = (\mathbf{t}_1^{t_1}\mathbf{t}_2^{t_2}\mathbf{t}_3^{0}, \tau) \rangle$

- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_4$, $a_4 = -(a_1c_1 + a_2c_2) + \frac{i}{4}(i = 0, 1, 2, 3)$

- Always has torsion

`(4d)` $\Psi = \mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\gamma} = (\mathbf{t}_3^{\frac{1}{2}}, \tau) \rangle$

- $\mathcal{S} = nK - I$, where $K \in \mathrm{SL}(2, \mathbb{Z})$, symmetric with $\mathrm{tr}(K) = n > 2$.

- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\gamma}) = K$

- $H^1(\Psi; \mathrm{Coker}(I-S)) \cong \frac{\mathrm{Coker}(I-K)}{\mathrm{Im}(I-A)}$.

- $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$, $(c_1, c_2) = (0, 0)$ or $(c_1, c_2) = (\frac{1}{2}, \frac{1}{2})$

- $c_3 = 0$, must satisfy conditions in Table 3.7
- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \gamma = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_4$, $a_4 = -(a_1c_1 + a_2c_2) + \frac{i}{4}(i = 0, 1, 2, 3)$
- Always has torsion

Proof. The computation for $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ for cases `(4a)` and `(4b)` follows from that of case `(4)` in Theorem 3.2.21. That for cases `(4c)` and `(4d)` follows from computing the appropriate cocycle conditions in Remark 3.2.17, together with application of Corollary 3.2.19. For case `(4c)`, notice that the cyclic power cocycle condition for γ forces $2\mathbf{r} \equiv 0$. We obtain an additional condition on $\mathbf{r}$ from the commutator cocycle condition in case `(4c)`:

$$(I - \varphi(\alpha))\mathbf{r} = (I - \varphi(\gamma))\mathbf{a}$$
$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{r} = \mathbf{0},$$

which forces $(t_1, t_2) = (0, 0)$ or $(t_1, t_2) = (\frac{1}{2}, \frac{1}{2})$.

In all cases, the conditions on c_1, c_2 follow from the Tables 3.3, 3.5, and 3.6. We can take $c_3 = 0$ in case `(4b)` by Lemma 3.4.13. From Table 3.14, $c_3 = 0$ is forced in case `(4d)`. For cases `(4a)` and `(4c)`, we have two choices for c_3.

For the choices we have for a_4 in cases `(4a)`, `(4c)`, and `(4d)`, note that from Table 3.11

$$\alpha^4 = \mathbf{t}_4^{4(a_4 + a_1c_1 + a_2c_2)}.$$

Recall that Sol^3 groups of type `(4)` always have torsion. Lemma 3.4.16 shows that groups of type `(4b)`, `(4c)`, and `(4d)` all have torsion. Only in case

(4a) can we resolve torsion in the central direction, just like the case of Sol_1^4 crystallographic groups of type (4) (Theorem 3.3.21). $\square$

Theorem 3.4.25 ((5) Family)**.** *The crystallographic groups of* $\mathrm{Sol}^3 \times \mathbb{R}$*, lifting an abstract kernel* $\mathbb{Z}_\Phi \to \mathrm{GL}(2, \mathbb{Z})$ *of case* (5) *of* Theorem 3.2.9,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ are}$$

(5a) $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A), \bar{\beta} = (\mathbf{t}_3^0, B) \rangle$

- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2, \mathbb{Z})$, $\det(K) = -1$, and $\mathrm{tr}(K) = n > 0$.
- $\varphi(\bar{\alpha}) = -K, \varphi(\bar{\beta}) = B$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S})) \cong \frac{\mathrm{Coker}(I+K)}{2\mathrm{Coker}(I+K)}$
- $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$, $c_1 = 0$ or $\frac{1}{2}$, $c_2 = 0$ or $\frac{1}{2}$
- $c_3 = 0$, must satisfy conditions in Table 3.7
- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_3^{\frac{1}{2}} \mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_1^{b_1} \mathbf{t}_2^{b_2} \mathbf{t}_3^0 \mathbf{t}_4^{b_4}, B) \rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$, $b_4 = -(a_1 c_1 + a_2 c_2) + \frac{i}{2} (i = 0, 1)$
- Torsion-free when $i = 1$

(5b) $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A\tau), \bar{\beta} = (\mathbf{t}_3^0, B) \rangle$

- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2, \mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$.
- $\varphi(\bar{\alpha}) = -K, \varphi(\bar{\beta}) = B$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S})) \cong \frac{\mathrm{Coker}(I+K)}{2\mathrm{Coker}(I+K)}$
- $(c_1, c_2) \in \mathrm{Coker}(K^T - I)$, $c_1 = 0$ or $\frac{1}{2}$, $c_2 = 0$ or $\frac{1}{2}$
- $c_3 = 0$, must satisfy conditions in Table 3.7
- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_3^{\frac{1}{2}}, A\tau), \beta = (\mathbf{t}_1^{b_1} \mathbf{t}_2^{b_2} \mathbf{t}_3^0 \mathbf{t}_4^{b_4}, B) \rangle$

- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $b_4 = -(a_1c_1 + a_2c_2) + \frac{i}{2}(i = 0, 1)$
- Torsion-free when $i = 1$

(5c) $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A), \bar{\beta} = (\mathbf{t}_3^0, B\tau)\rangle$

- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2, \mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$.
- $\varphi(\bar{\alpha}) = -K, \varphi(\bar{\beta}) = B$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S})) \cong \frac{\mathrm{Coker}(I+K)}{2\mathrm{Coker}(I+K)}$
- $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$
- Unique choice for c_3 (up to isomorphism), found by solving $(\alpha\beta)^2 \in \widetilde{\Gamma}_{\mathcal{S}}$, this c_3 must satisfy the conditions in Table 3.9
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^0, B\tau)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$
- Always has torsion

(5d) $\Psi = \mathbb{Z}_2^3$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A), \bar{\beta} = (\mathbf{t}_3^0, B), \bar{\gamma} = (\mathbf{t}_3^0, \tau)\rangle$

- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2, \mathbb{Z})$, $\det(K) = -1$, and $\mathrm{tr}(K) = n > 0$.
- $\varphi(\bar{\alpha}) = -K, \varphi(\bar{\beta}) = B, \varphi(\bar{\gamma}) = I$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is all $\mathbf{b}$, $\mathbf{r}$ with $(I + K)\mathbf{b} \equiv (I + K)\mathbf{r} \equiv \mathbf{0}$, $2\mathbf{r} \equiv \mathbf{0}$, and $\mathbf{b}$ taken modulo $2\mathbf{b}$.
- $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$, $c_1 = 0$ or $\frac{1}{2}$, $c_2 = 0$ or $\frac{1}{2}$
- Unique choice for c_3 (up to isomorphism), found by solving $(\alpha\gamma)^2 \in \widetilde{\Gamma}_{\mathcal{S}}$, this c_3 must satisfy the conditions in Tables 3.7 and 3.9
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^0\mathbf{t}_4^{b_4}, B), \gamma = (\mathbf{t}_1^{t_1}\mathbf{t}_2^{t_2}\mathbf{t}_3^0, \tau)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$, $b_4 = -(a_1c_1 + a_2c_2) + \frac{i}{2}(i = 0, 1)$

- Always has torsion

Proof. First, observe that

$$\begin{aligned}\mathbb{Z}_\Psi &= \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A), \bar{\beta} = (\mathbf{t}_3^0, B), \bar{\gamma} = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle \\ &= \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau), \bar{\beta} = (\mathbf{t}_3^0, B), \bar{\gamma} = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle\end{aligned}$$

cannot be lifted to a $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic group by Lemma 3.4.10, and we therefore disregard this case.

Also, note that

$$\begin{aligned}\mathbb{Z}_\Psi &= \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, A\tau), \bar{\beta} = (\mathbf{t}_3^0, B\tau)\rangle \\ &= \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^{\frac{1}{2}}, -A), \bar{\beta} = (\mathbf{t}_3^0, B\tau)\rangle\end{aligned}$$

is conjugate to a $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic group of type `(5c)`, as A and $-A$ are conjugate in $D_4 \times \mathbb{Z}_2$.

The statements regarding $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ for cases `(5a)`, `(5b)`, and `(5c)` follow from that of case `(5)` in Theorem 3.2.21. The statement for Case `(5d)` follows from the cocycle conditions in Remark 3.2.17, together with application of Corollary 3.2.19.

The conditions on the c_i, as well as the choices for the exponents on $\mathbf{t}_4$, follow from our proceeding classification of $\mathrm{Sol}^3 \times \mathbb{R}$ groups with cyclic holonomy.

For the torsion criteria, note that in cases `(5a)` and `(5b)`, both $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\rangle$ and $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta\rangle$ are torsion-free by Lemma 3.4.17. In these cases we must take $i = 1$ to get that $\langle \widetilde{\Gamma}_{\mathcal{S}}, \beta\rangle$ (and hence Π) is torsion-free. Groups of type `(5c)` have torsion by Lemma 3.4.16. As a group of type `(5d)` contains a group of type `(5c)`, it likewise has torsion. □

Theorem 3.4.26 (`(6a)` Family)**.** *The crystallographic groups of* Sol$^3 \times \mathbb{R}$*, lifting an abstract kernel* $\mathbb{Z}_\Phi \to \mathrm{GL}(2,\mathbb{Z})$ *of case* `(6a)` *of* Theorem 3.2.9*,*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ are}$$

`(6aa)` $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B)\rangle$

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\sigma_{12} = -\sigma_{21}$ and $\mathrm{tr}(\mathcal{S}) > 2$.
- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\beta}) = B$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is $\mathbf{a}$, $\mathbf{b}$ with $(I-\mathcal{S})\mathbf{a} \equiv (I-\mathcal{S})\mathbf{b} \equiv \mathbf{0}$, $a_2 \equiv -a_1$, $b_1 - b_2 - 2a_1 \equiv 0$, $\mathbf{a}$ and $\mathbf{b}$ modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $2\mathbf{v}$ for $\mathbf{v} \in \mathrm{Coker}(I - \mathcal{S})$, respectively.
- $(c_1, c_2) = (0,0)$ or $(\frac{1}{2}, \frac{1}{2})$
- Two choices for c_3 (up to isomorphism), see Table 3.7
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_\mathcal{S}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{0}\mathbf{t}_4^{b_4}, B)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_\mathcal{S})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$, $b_4 = -(a_1c_1 + a_2c_2) + \frac{i}{2}(i = 0,1)$
- Always has torsion

`(6ab)` $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau), \bar{\beta} = (\mathbf{t}_3^0, B)\rangle$

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\sigma_{12} = -\sigma_{21}$ and $\mathrm{tr}(\mathcal{S}) > 2$.
- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\beta}) = B$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is $\mathbf{a}$, $\mathbf{b}$ with $(I-\mathcal{S})\mathbf{a} \equiv (I-\mathcal{S})\mathbf{b} \equiv \mathbf{0}$, $a_2 \equiv -a_1$, $b_1 - b_2 - 2a_1 \equiv 0$, $\mathbf{a}$ and $\mathbf{b}$ modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $2\mathbf{v}$ for $\mathbf{v} \in \mathrm{Coker}(I - \mathcal{S})$, respectively.

- $(c_1, c_2) = (0,0)$ or $(\frac{1}{2}, \frac{1}{2})$

- $c_3 = 0$, must satisfy conditions in Table 3.7

- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}, A\tau), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{0}\mathbf{t}_4^{b_4}, B)\rangle$

- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $b_4 = -(a_1c_1 + a_2c_2) + \frac{i}{2}(i = 0, 1)$

- Always has torsion

`(6ac)` $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B\tau)\rangle$

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\sigma_{12} = -\sigma_{21}$ and $\mathrm{tr}(\mathcal{S}) > 2$.

- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\beta}) = B$

- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is $\mathbf{a}$, $\mathbf{b}$ with $(I - \mathcal{S})\mathbf{a} \equiv (I - \mathcal{S})\mathbf{b} \equiv \mathbf{0}$, $a_2 \equiv -a_1$, $b_1 - b_2 - 2a_1 \equiv 0$, $\mathbf{a}$ and $\mathbf{b}$ modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $2\mathbf{v}$ for $\mathbf{v} \in \mathrm{Coker}(I - \mathcal{S})$, respectively.

- $c_1 - c_2 = 0$

- Two choices for c_3 (up to isomorphism), see Tables 3.7 and 3.9

- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{0}, B\tau)\rangle$

- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$

- Always has torsion

`(6ad)` $\Psi = \mathbb{Z}_2^3$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B), \bar{\gamma} = (\mathbf{t}_3^0, \tau)\rangle$

- $\mathcal{S} \in \mathrm{SL}(2,\mathbb{Z})$ with $\sigma_{12} = -\sigma_{21}$ and $\mathrm{tr}(\mathcal{S}) > 2$.

- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\beta}) = B, \varphi(\bar{\gamma}) = I$

- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is $\mathbf{a}$, $\mathbf{b}$, $\mathbf{r}$ with $(I - \mathcal{S})\mathbf{a} \equiv (I - \mathcal{S})\mathbf{b} \equiv (I - \mathcal{S})\mathbf{r} \equiv \mathbf{0}$, $\mathbf{r} = (0,0)$ or $(\frac{1}{2}, \frac{1}{2})$, $a_2 \equiv -a_1$, $b_1 - b_2 - 2a_1 \equiv 0$, $\mathbf{a}$ and $\mathbf{b}$ modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $2\mathbf{v}$ for $\mathbf{v} \in \mathrm{Coker}(I - \mathcal{S})$, respectively.

- $(c_1, c_2) = (0, 0)$ or $(\frac{1}{2}, \frac{1}{2})$

- Two choices for c_3 (up to isomorphism), see Tables 3.7 and 3.9

- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{0}\mathbf{t}_4^{b_4}, B), \gamma = (\mathbf{t}_1^{t_1}\mathbf{t}_2^{t_2}\mathbf{t}_3^{0}, \tau) \rangle$

- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$, $b_4 = -(a_1c_1 + a_2c_2) + \frac{i}{2} (i = 0, 1)$

- Always has torsion

(6ae) $\Psi = \mathbb{Z}_2^3$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B), \bar{\gamma} = (\mathbf{t}_3^{\frac{1}{2}}, \tau) \rangle$

- $\mathcal{S} = nK - I$, where $K \in \mathrm{SL}(2, \mathbb{Z})$ with $\mathrm{tr}(K) = n > 2$ and $k_{12} = -k_{21}$.

- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\beta}) = B, \varphi(\bar{\gamma}) = K$

- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is $\mathbf{a}$, $\mathbf{b}$ with $(I - K)\mathbf{a} \equiv (I - K)\mathbf{b} \equiv \mathbf{0}$, $a_2 \equiv -a_1$, $b_1 - b_2 - 2a_1 \equiv 0$, $\mathbf{a}$ and $\mathbf{b}$ modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $2\mathbf{v}$ for $\mathbf{v} \in \mathrm{Coker}(I - K)$, respectively.

- $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$, and $(c_1, c_2) = (0, 0)$ or $(\frac{1}{2}, \frac{1}{2})$

- $c_3 = 0$, must satisfy conditions in Table 3.7

- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{0}\mathbf{t}_4^{b_4}, B), \gamma = (\mathbf{t}_3^{\frac{1}{2}}, \tau) \rangle$

- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$, $b_4 = -(a_1c_1 + a_2c_2) + \frac{i}{2} (i = 0, 1)$

- Always has torsion

(6af) $\Psi = \mathbb{Z}_2^3$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B\tau), \bar{\gamma} = (\mathbf{t}_3^{\frac{1}{2}}, \tau) \rangle$

- $\mathcal{S} = nK - I$, where $K \in \mathrm{SL}(2, \mathbb{Z})$ with $\mathrm{tr}(K) = n > 2$ and $k_{12} = -k_{21}$.

- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\beta}) = B, \varphi(\bar{\gamma}) = K$

- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is $\mathbf{a}$, $\mathbf{b}$ with $(I - K)\mathbf{a} \equiv (I - K)\mathbf{b} \equiv \mathbf{0}$, $a_2 \equiv -a_1$, $b_1 - b_2 - 2a_1 \equiv 0$, $\mathbf{a}$ and $\mathbf{b}$ modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $2\mathbf{v}$ for $\mathbf{v} \in \mathrm{Coker}(I - K)$, respectively.

- $c_1 - c_2 = 0$, $(c_1, c_2) \in \text{Coker}(I + K^T)$
- $c_3 = 0$, must satisfy conditions in Table 3.7 and Table 3.9
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{0}\mathbf{t}_4^{b_4}, B\tau), \gamma = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$, two choices for b_4. We have $(\beta\gamma)^2 = \mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_3\mathbf{t}_4^{2b_4+v}$ for integers n_1, and some rational number v, and therefore two solutions for b_4.
- Always has torsion

Proof. First, observe that if a $\text{Sol}^3 \times \mathbb{R}$ crystallographic group $\mathit{\Pi}$ has

$$\begin{aligned}\mathbb{Z}_\Psi &= \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau), \bar{\beta} = (\mathbf{t}_3^0, B\tau)\rangle \\ &= \langle \mathbf{t}_3, (\mathbf{t}_3^0, -A), (\mathbf{t}_3^0, B\tau)\rangle,\end{aligned}$$

then it is conjugate to a $\text{Sol}^3 \times \mathbb{R}$ crystallographic group of type `(6ac)`, as $-A$ and A are conjugate in $D_4 \times \mathbb{Z}_2$ (and $B\tau$ remains fixed under this conjugation).

The statements regarding $H^1(\Psi; \text{Coker}(I - \mathcal{S}))$ for cases `(6aa)`, `(6ab)`, and `(6ac)` follow from that of case `(6a)` in Theorem 3.2.21. The statements for cases `(6ad)`, `(6ae)`, and `(6af)` follow from the cocycle conditions in Remark 3.2.17, together with application of Corollary 3.2.19.

The conditions on the c_i, as well as the choices for the exponents on $\mathbf{t}_4$, follow from our proceeding classification of $\text{Sol}^3 \times \mathbb{R}$ groups with cyclic holonomy. In particular, note that in case `(6aa)`, from the cyclic cases, we obtain two conditions on (c_1, c_2)

$$\begin{aligned}&c_1 - c_2 = 0 \\ &c_1 = 0 \text{ or } \frac{1}{2}, c_2 = 0 \text{ or } \frac{1}{2},\end{aligned}$$

and these two conditions force $(c_1, c_2) = (0, 0)$ or $(\frac{1}{2}, \frac{1}{2})$.

All $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic groups of this type have torsion, just like their corresponding Sol^3 crystallographic groups. In case `(6aa)`, we need the groups

$$\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha \rangle$$
$$\langle \widetilde{\Gamma}_{\mathcal{S}}, \beta \rangle$$
$$\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta \rangle$$

to all be torsion-free. All these groups are torsion, when projected to Sol^3 crystallographic groups, and so we must resolve the torsion using the $\mathbb{R}$ direction of $\mathrm{Sol}^3 \times \mathbb{R}$. We have:

$$\begin{aligned}
a_4 = 0, i = 0 &\implies \text{ all have torsion} \\
a_4 = \frac{1}{2}, i = 0 &\implies \langle \widetilde{\Gamma}_{\mathcal{S}}, \beta \rangle \text{ has torsion} \\
a_4 = 0, i = 1 &\implies \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha \rangle \text{ has torsion} \\
a_4 = \frac{1}{2}, i = 1 &\implies \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta \rangle \text{ has torsion}
\end{aligned}$$

Now groups of type `(6ad)` and `(6ae)` have torsion, as they contain a group of type `(6aa)` as a subgroup. To see that groups of type `(6ab)`, `(6ac)`, and `(6af)` have torsion, we may appeal to Lemma 3.4.16. □

Theorem 3.4.27 ((6ai) Family)**.** *The crystallographic groups of* $\mathrm{Sol}^3 \times \mathbb{R}$ *lifting an abstract kernel* $\mathbb{Z}_\Phi \to \mathrm{GL}(2, \mathbb{Z})$ *of case* (6ai) *of* Theorem 3.2.9,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ are}$$

(6aia) $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B) \rangle$

- $\mathcal{S} \in \mathrm{SL}(2, \mathbb{Z})$ with $\sigma_{11} = \sigma_{22}$ and $\mathrm{tr}(\mathcal{S}) > 2$.

- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\beta}) = B$

- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is all $\mathbf{a}$, $\mathbf{b}$ with $(I - \mathcal{S})\mathbf{a} \equiv (I - \mathcal{S})\mathbf{b} \equiv \mathbf{0}$, $2a_1 \equiv 0$, $2b_2 - 2a_2 \equiv \mathbf{0}$, $\mathbf{a}$ and $\mathbf{b}$ taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$ and $2\mathbf{v}$ for $\mathbf{v} \in \mathrm{Coker}(I - \mathcal{S})$, respectively.

- $c_1 = 0$ or $\frac{1}{2}$, $c_2 = 0$ or $\frac{1}{2}$

- Two choices for c_3 (up to isomorphism), see Table 3.7

- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{0}\mathbf{t}_4^{b_4}, B)\rangle$

- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = -a_2c_2 + \frac{i}{2}(i = 0, 1)$, $b_4 = -(a_1c_1 + a_2c_2) + \frac{j}{2}(j = 0, 1)$

- Torsion-free when $j = 1$, and...

  - when $i = 0$, $a_1 = \frac{1}{2}$ and $a_2 \not\equiv \frac{(\sigma_{11}+1)(2n+1)}{2\sigma_{12}}$ for any $n \in \mathbb{Z}$

  - when $i = 1$, $a_1 + b_1 \neq \frac{\sigma_{12}(2n+1)}{2(\sigma_{11}-1)}$ for any $n \in \mathbb{Z}$ and $a_2 - b_2 = \frac{1}{2}$

(6aib) $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau), \bar{\beta} = (\mathbf{t}_3^0, B)\rangle$

- $\mathcal{S} \in \mathrm{SL}(2, \mathbb{Z})$ with $\sigma_{11} = \sigma_{22}$ and $\mathrm{tr}(\mathcal{S}) > 2$.

- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\beta}) = B$

- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is all $\mathbf{a}$, $\mathbf{b}$ with $(I - \mathcal{S})\mathbf{a} \equiv (I - \mathcal{S})\mathbf{b} \equiv \mathbf{0}$, $2a_1 \equiv 0$, $2b_2 - 2a_2 \equiv \mathbf{0}$, $\mathbf{a}$ and $\mathbf{b}$ taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$ and $2\mathbf{v}$ for $\mathbf{v} \in \mathrm{Coker}(I - \mathcal{S})$, respectively.

- $c_1 = 0$ or $\frac{1}{2}$, $c_2 = 0$ or $\frac{1}{2}$

- $c_3 = 0$, must satisfy conditions in Table 3.7

- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}, A\tau), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{0}\mathbf{t}_4^{b_4}, B)\rangle$

- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $b_4 = -(a_1c_1 + a_2c_2) + \frac{j}{2}(j = 0, 1)$

- Torsion-free when $j = 1$, $a_1 = \frac{1}{2}$, $a_2 = b_2 + \frac{1}{2}$, $b_1 \not\equiv \frac{\sigma_{12}(2n+1)}{2(\sigma_{11}-1)} + \frac{1}{2}$, $b_2 \not\equiv \frac{(\sigma_{11}+1)(2m+1)}{2\sigma_{12}} + \frac{1}{2}$ for any $m, n \in \mathbb{Z}$.

(6a*i*c) $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B\tau)\rangle$

- $\mathcal{S} \in \mathrm{SL}(2, \mathbb{Z})$ with $\sigma_{11} = \sigma_{22}$ and $\mathrm{tr}(\mathcal{S}) > 2$.
- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\beta}) = B$
- $H^1(\Psi; \mathrm{Coker}(I-\mathcal{S}))$ is all $\mathbf{a}$, $\mathbf{b}$ with $(I-\mathcal{S})\mathbf{a} \equiv (I-\mathcal{S})\mathbf{b} \equiv \mathbf{0}$, $2a_1 \equiv 0$, $2b_2 - 2a_2 \equiv \mathbf{0}$, $\mathbf{a}$ and $\mathbf{b}$ taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$ and $2\mathbf{v}$ for $\mathbf{v} \in \mathrm{Coker}(I-\mathcal{S})$, respectively.

- $c_2 = 0$ or $\frac{1}{2}$

- Two choices for c_3 (up to isomorphism), see Tables 3.7 and 3.9

- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^0\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^0, B\tau)\rangle$

- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = -a_2c_2 + \frac{i}{2}(i = 0, 1)$

- Always has torsion

(6a*i*d) $\Psi = \mathbb{Z}_2^3$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B), \bar{\gamma} = (\mathbf{t}_3^0, \tau)\rangle$

- $\mathcal{S} \in \mathrm{SL}(2, \mathbb{Z})$ with $\sigma_{11} = \sigma_{22}$ and $\mathrm{tr}(\mathcal{S}) > 2$.
- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\beta}) = B, \varphi(\bar{\gamma}) = I$
- $H^1(\Psi; \mathrm{Coker}(I-\mathcal{S}))$ is all $\mathbf{a}$, $\mathbf{b}$, $\mathbf{r}$ with $(I-\mathcal{S})\mathbf{a} \equiv (I-\mathcal{S})\mathbf{b} \equiv (I-\mathcal{S})\mathbf{r} \equiv \mathbf{0}$, $2a_1 \equiv 0$, $2b_2 - 2a_2 \equiv \mathbf{0}$, $2\mathbf{r} = \mathbf{0}$, $\mathbf{a}$ and $\mathbf{b}$ taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$ and $2\mathbf{v}$ for $\mathbf{v} \in \mathrm{Coker}(I-\mathcal{S})$, respectively.

- $c_1 = 0$ or $\frac{1}{2}$, $c_2 = 0$ or $\frac{1}{2}$

• Two choices for c_3 (up to isomorphism), see Tables 3.7 and 3.9

• $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{0}\mathbf{t}_4^{b_4}, B), \gamma = (\mathbf{t}_1^{t_1}\mathbf{t}_2^{t_2}\mathbf{t}_3^{0}, \tau)\rangle$

• $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = -a_2c_2 + \frac{i}{2}(i = 0, 1)$, $b_4 = -(a_1c_1 + a_2c_2) + \frac{j}{2}(j = 0, 1)$

• Always has torsion

`(6aie)` $\Psi = \mathbb{Z}_2^3$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B), \bar{\gamma} = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle$

• $\mathcal{S} = nK - I$, where $K \in \mathrm{SL}(2, \mathbb{Z})$ with $\mathrm{tr}(K) = n > 2$ and $k_{11} = k_{22}$.

• $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\beta}) = B, \varphi(\bar{\gamma}) = K$

• $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is all $\mathbf{a}$, $\mathbf{b}$ with $(I-K)\mathbf{a} \equiv (I-K)\mathbf{b} \equiv \mathbf{0}$, $2a_1 \equiv 0$, $2b_2 - 2a_2 \equiv \mathbf{0}$, $\mathbf{a}$ and $\mathbf{b}$ taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$ and $2\mathbf{v}$ for $\mathbf{v} \in \mathrm{Coker}(I-K)$, respectively.

• $c_1 = 0$ or $\frac{1}{2}$, $c_2 = 0$ or $\frac{1}{2}$, and $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$

• $c_3 = 0$, must satisfy conditions in Table 3.7

• $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{0}\mathbf{t}_4^{b_4}, B), \gamma = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle$

• $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = -a_2c_2 + \frac{i}{2}(i = 0, 1)$, $b_4 = -(a_1c_1 + a_2c_2) + \frac{j}{2}(j = 0, 1)$

• Torsion-free when $j = 1$, and when the torsion-free criteria for $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha, \beta\rangle$ `(6aia)`, and $\langle \widetilde{\Gamma}_{\mathcal{S}}, \gamma\alpha, \beta\rangle$ `(6aib)` are satisfied.

`(6aif)` $\Psi = \mathbb{Z}_2^3$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B\tau), \bar{\gamma} = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle$

• $\mathcal{S} = nK - I$, where $K \in \mathrm{SL}(2, \mathbb{Z})$ with $\mathrm{tr}(K) = n > 2$ and $k_{11} = k_{22}$.

• $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\beta}) = B, \varphi(\bar{\gamma}) = K$

- $H^1(\Psi; \mathrm{Coker}(I-\mathcal{S}))$ can be described as all $\mathbf{a}$, $\mathbf{b}$ with $(I-K)\mathbf{a} \equiv (I-K)\mathbf{b} \equiv \mathbf{0}$, $2a_1 \equiv 0$, $2b_2 - 2a_2 \equiv \mathbf{0}$, $\mathbf{a}$ and $\mathbf{b}$ taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$ and $2\mathbf{v}$ for $\mathbf{v} \in \mathrm{Coker}(I-K)$, respectively.

- $c_2 = 0$ or $\frac{1}{2}$, $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$

- $c_3 = 0$, must satisfy conditions in Table 3.7 and Table 3.9

- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{0}\mathbf{t}_4^{b_4}, B\tau), \gamma = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle$

- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = -a_2c_2 + \frac{i}{2}(i = 0, 1)$, two choices for b_4. We have $(\beta\gamma)^2 = \mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_3\mathbf{t}_4^{2b_4+v}$ for integers n_1, and some rational number v, and therefore two solutions for b_4.

- Always has torsion

Proof. Observe that if a $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic group $\mathit{\Pi}$ has

$$\begin{aligned} \mathbb{Z}_\Psi &= \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau), \bar{\beta} = (\mathbf{t}_3^0, B\tau)\rangle \\ &= \langle \mathbf{t}_3, (\mathbf{t}_3^0, -A), (\mathbf{t}_3^0, B\tau)\rangle, \end{aligned}$$

then it is conjugate to a $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic group of type (6aic), similar to the situation in Theorem 3.4.26.

See Theorem 3.2.21, case (6ai), for $H^1(\Psi; \mathrm{Coker}(I-\mathcal{S}))$ in cases (6aia), (6aib), and (6aic). For cases (6aid), (6aie), and (6aif) use the cocycle conditions in Remark 3.2.17, together with Corollary 3.2.19. The conditions on the c_i follow from previous cases. The computation of $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ is similar to that in Theorem 3.4.26, and so we obtain the stated exponents on $\mathbf{t}_4$.

The groups of type (6aic) (and hence also those of type (6aid) and (6aif)) always have torsion by Lemma 3.4.16.

In case (6aia), we must take $j=1$ to ensure that $\langle\widetilde{\Gamma}_{\mathcal{S}},\beta\rangle$ is torsion-free. When $i=0$, $\langle\widetilde{\Gamma}_{\mathcal{S}},\alpha\beta\rangle$ is torsion-free. When $i=1$, $\langle\widetilde{\Gamma}_{\mathcal{S}},\alpha\rangle$ is torsion-free. Thus,

$$i=0 \implies \langle\widetilde{\Gamma}_{\mathcal{S}},\alpha\rangle \text{ must be torsion-free projected to } \mathrm{Sol}^3$$
$$i=1, \implies \langle\widetilde{\Gamma}_{\mathcal{S}},\alpha\beta\rangle \text{ must be torsion-free projected to } \mathrm{Sol}^3$$

Both the groups $\langle\widetilde{\Gamma}_{\mathcal{S}},\alpha\rangle$ and $\langle\widetilde{\Gamma}_{\mathcal{S}},\alpha\beta\rangle$ are Sol^3 crystallographic groups of type (3i). The projection of $\alpha\beta$ to Sol^3 is

$$(\mathbf{t}_1^{a_1+b_1}\mathbf{t}_2^{a_2-b_2}, AB).$$

By computing when the appropriate affine maps in Proposition 3.2.23 are fixed point free, we obtain the desired conditions. See also Theorem 3.3.21, case (6ai).

In case (6aib), we must take $j=1$. Lemma 3.4.16 then implies that both the groups $\langle\widetilde{\Gamma}_{\mathcal{S}},\alpha\rangle$ and $\langle\widetilde{\Gamma}_{\mathcal{S}},\alpha\beta\rangle$ must both be torsion-free when projected to Sol^3. The proof for this case now proceeds as in Theorem 3.3.21, case (6ai).

For case (6aie), we must take $j=1$, which is is necessary for the group $\langle\widetilde{\Gamma}_{\mathcal{S}},\gamma,\beta\rangle$ to be torsion-free. Then, Π will be torsion-free precisely when the groups $\langle\widetilde{\Gamma}_{\mathcal{S}},\alpha,\beta\rangle$ and $\langle\widetilde{\Gamma}_{\mathcal{S}},\gamma\alpha,\beta\rangle$ are torsion-free, as the union of these three $\mathbb{Z}_2^2$ holonomy $\mathrm{Sol}^3\times\mathbb{R}$ crystallographic groups exhausts Π. □

Theorem 3.4.28 ((6b) Family)**.** *The crystallographic groups of* $\mathrm{Sol}^3\times\mathbb{R}$ *lifting an abstract kernel* $\mathbb{Z}_\Phi\to\mathrm{GL}(2,\mathbb{Z})$ *of case* (6b) *of* Theorem 3.2.9*,*

$$A=\begin{bmatrix}0&1\\1&0\end{bmatrix}, B=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}, \text{ are}$$

(6ba) $\Psi=\mathbb{Z}_2^2$, $\mathbb{Z}_\Psi=\langle\mathbf{t}_3,\bar{\alpha}=(\mathbf{t}_3^0,A),\bar{\beta}=(\mathbf{t}_3^{\frac{1}{2}},B)\rangle$

- $\mathcal{S}=nK-I$, where $K\in\mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(K)=n>2$ and $k_{12}=-k_{21}$.

- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\beta}) = -K$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S})) \cong H^1(\langle A \rangle; \mathrm{Coker}(I + K))$, that is, all $\mathbf{a}$ with $(I + K)\mathbf{a} \equiv \mathbf{0}$ and $a_2 \equiv -a_1$ modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, for $(I + K)\mathbf{v} \equiv \mathbf{0}$.
- $c_1 - c_2 = 0$, $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$
- Two choices for c_3 (up to isomorphism), see Table 3.7
- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B) \rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$, $b_4 = 0$, $\frac{1}{2}$
- Torsion-free when $a_4 = \frac{1}{2}$, $b_4 = 0$

(6bb) $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B) \rangle$

- $\mathcal{S} = nK - I$, where $K \in \mathrm{SL}(2, \mathbb{Z})$ with $\mathrm{tr}(K) = n > 2$ and $k_{12} = -k_{21}$.
- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\beta}) = -K$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S})) \cong H^1(\langle A \rangle; \mathrm{Coker}(I + K))$, that is, all $\mathbf{a}$ with $(I + K)\mathbf{a} \equiv \mathbf{0}$ and $a_2 \equiv -a_1$ modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, for $(I + K)\mathbf{v} \equiv \mathbf{0}$.
- $c_1 + c_2 = 0$, $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$
- $c_3 = 0$, must satisfy conditions in Table 3.9
- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}, A\tau), \beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B) \rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $b_4 = 0$, $\frac{1}{2}$
- Always has torsion

(6bc) $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B\tau) \rangle$

- $\mathcal{S} = nK - I$, where $K \in \mathrm{SL}(2, \mathbb{Z})$ with $\mathrm{tr}(K) = n > 2$ and $k_{12} = -k_{21}$.
- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\beta}) = -K$

• $H^1(\Psi; \text{Coker}(I - \mathcal{S})) \cong H^1(\langle A \rangle; \text{Coker}(I + K))$, that is, all $\mathbf{a}$ with $(I+K)\mathbf{a} \equiv \mathbf{0}$ and $a_2 \equiv -a_1$ modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, for $(I+K)\mathbf{v} \equiv \mathbf{0}$.

• $c_1 - c_2 = 0$, $(c_1, c_2) \in \text{Coker}(K^T - I)$

• $c_3 = 0$, must satisfy conditions in Table 3.7

• $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}, B\tau) \rangle$

• $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$

• Always has torsion

(6bd) $\Psi = \mathbb{Z}_2^3$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B), \bar{\gamma} = (\mathbf{t}_3^0, \tau) \rangle$

• $\mathcal{S} = nK - I$, where $K \in \text{SL}(2, \mathbb{Z})$ with $\text{tr}(K) = n > 2$ and $k_{12} = -k_{21}$.

• $\varphi(\bar{\alpha}) = A, \varphi(\bar{\beta}) = -K, \varphi(\bar{\gamma}) = I$

• $H^1(\Psi; \text{Coker}(I - \mathcal{S}))$ can be described as all $\mathbf{a}$, $\mathbf{r}$ with $(I+K)\mathbf{a} \equiv (I+K)\mathbf{r} \equiv \mathbf{0}$, $\mathbf{r} = (0,0)$ or $(\frac{1}{2}, \frac{1}{2})$ and $a_2 \equiv -a_1$,
$\mathbf{a}$ taken modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ where $(I+K)\mathbf{v} \equiv \mathbf{0}$.

• $(c_1, c_2) = (0,0)$ or $(c_1, c_2) = (\frac{1}{2}, \frac{1}{2})$, and $(c_1, c_2) \in \text{Coker}(I + K^T)$

• Unique choice for c_3 (up to isomorphism), found by solving $(\beta\gamma)^2 \in \widetilde{\Gamma}_{\mathcal{S}}$, this c_3 must satisfy the conditions in Tables 3.7 and 3.9

• $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B), \gamma = (\mathbf{t}_1^{t_1}\mathbf{t}_2^{t_2}\mathbf{t}_3^{0}, \tau) \rangle$

• $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$, $b_4 = 0$, $\frac{1}{2}$

• Always has torsion

Proof. First, observe that if a $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic group Π has

$$\begin{aligned}\mathbb{Z}_\Psi &= \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B\tau)\rangle \\ &= \langle \mathbf{t}_3, (\mathbf{t}_3^{\frac{1}{2}}, -A), (\mathbf{t}_3^{\frac{1}{2}}, B\tau)\rangle,\end{aligned}$$

then it is conjugate to a $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic group of type `(6bc)`, as conjugation by $\mathbf{t}_3^{-\frac{1}{4}}$ sends $(\mathbf{t}_3^{\frac{1}{2}}, -A)$ to $(\mathbf{t}_3^0, -A)$, and $-A$ and A are conjugate in $D_4 \times \mathbb{Z}_2$.

Also, note that

$$\begin{aligned}\mathbb{Z}_\Psi &= \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B), \bar{\gamma} = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle \\ &= \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^0, B\tau), \bar{\gamma} = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle\end{aligned}$$

is a lift of type `(6a)` from the Sol^3 classification, and it corresponds to type `(6af)` in Theorem 3.4.26.

The statements regarding $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ for cases `(6ba)`, `(6bb)`, and `(6bc)` follow from that of case `(6b)` in Theorem 3.2.21. The statement for case `(6bd)` follows from the cocycle conditions in Remark 3.2.17, together with application of Corollary 3.2.19.

The conditions on the c_i, as well as the choices for the exponents on $\mathbf{t}_4$, follow from our proceeding classification of $\mathrm{Sol}^3 \times \mathbb{R}$ groups with cyclic holonomy. In case `(6bd)`, from the cyclic cases, we obtain three conditions on (c_1, c_2)

$$\begin{aligned}&c_1 - c_2 = 0 \\ &c_1 = 0 \text{ or } \frac{1}{2}, c_2 = 0 \text{ or } \frac{1}{2} \\ &(c_1, c_2) \in \mathrm{Coker}(I + K^T),\end{aligned}$$

and the first two conditions force $(c_1, c_2) = (0, 0)$ or $(\frac{1}{2}, \frac{1}{2})$.

For case (6ba), notice that both $\langle\widetilde{\Gamma}_{\mathcal{S}}, \alpha\rangle$ and $\langle\widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta\rangle$ project to Sol^3 groups with torsion, whereas $\langle\widetilde{\Gamma}_{\mathcal{S}}, \beta\rangle$ is already torsion-free on $\mathrm{Sol}^3 \times \mathbb{R}$. These three groups are torsion-free only when $a_4 = \frac{1}{2}$ and $b_4 = 0$. Lemma 3.4.16 implies the remaining groups of this type all have torsion. □

Theorem 3.4.29 ((6bi) Family)**.** *The crystallographic groups of* $\mathrm{Sol}^3 \times \mathbb{R}$ *lifting an abstract kernel* $\mathbb{Z}_\Phi \to \mathrm{GL}(2, \mathbb{Z})$ *of case* (6bi) *of* Theorem 3.2.9,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ are}$$

(6bia) $\Psi = \mathbb{Z}_2^2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B)\rangle$

- $\mathcal{S} = nK - I$, where $K \in \mathrm{SL}(2, \mathbb{Z})$ with $\mathrm{tr}(K) = n > 2$ and $k_{11} = k_{22}$.
- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\beta}) = -K$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S})) \cong H^1(\langle A\rangle; \mathrm{Coker}(I + K))$, that is, all $\mathbf{a}$ with $(I + K)\mathbf{a} \equiv \mathbf{0}$ and $2a_1 \equiv 0$ taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$, where $(I + K)\mathbf{v} \equiv \mathbf{0}$.
- $c_2 = 0$ or $\frac{1}{2}$, $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$
- Two choices for c_3 (up to isomorphism), see Table 3.7
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = -a_2c_2 + \frac{i}{2}(i = 0, 1)$, $b_4 = 0, \frac{1}{2}$
- Torsion-free...
  - when $i = 0, b_4 = 0$, $a_1 = \frac{1}{2}$ and $a_2 \not\equiv \frac{(k_{11}-1)(2n+1)}{2k_{12}}$ for any $n \in \mathbb{Z}$.
  - when $i = 1, b_4 = 0$, always
  - when $i = 0, b_4 = \frac{1}{2}$, $a_1 = \frac{1}{2}$ and $a_2 \not\equiv \frac{(\sigma_{11}+1)(2n+1)}{2k_{12}}$ for any $n \in \mathbb{Z}$.

- when $i=1, b_4=\frac{1}{2}$, $\langle\widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta\rangle$ must be torsion-free, projected to Sol^3.

`(6bib)` $\Psi=\mathbb{Z}_2^2$, $\mathbb{Z}_\Psi=\langle \mathbf{t}_3, \bar{\alpha}=(\mathbf{t}_3^0, A\tau), \bar{\beta}=(\mathbf{t}_3^{\frac{1}{2}}, B)\rangle$

- $\mathcal{S}=nK-I$, where $K\in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(K)=n>2$ and $k_{11}=k_{22}$.
- $\varphi(\bar{\alpha})=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\beta})=-K$
- $H^1(\Psi;\mathrm{Coker}(I-\mathcal{S}))\cong H^1(\langle A\rangle;\mathrm{Coker}(I+K))$, that is, all $\mathbf{a}$ with $(I+K)\mathbf{a}\equiv\mathbf{0}$ and $2a_1\equiv 0$ taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$, where $(I+K)\mathbf{v}\equiv\mathbf{0}$.
- $c_1=0$ or $\frac{1}{2}$, $(c_1,c_2)\in \mathrm{Coker}(I+K^T)$
- $c_3=0$, must satisfy conditions in Table 3.9
- $\Pi=\langle\widetilde{\Gamma}_{\mathcal{S}}, \alpha=(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^0, A\tau), \beta=(\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B)\rangle$
- $H^2(\Psi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))=\mathbb{Z}_2$, $b_4=0, \frac{1}{2}$
- Torsion-free if and only if $a_1=\frac{1}{2}$ and $a_2\not\equiv\frac{(k_{11}-1)(2n+1)}{2k_{12}}$ for any $n\in\mathbb{Z}$.

`(6bic)` $\Psi=\mathbb{Z}_2^2$, $\mathbb{Z}_\Psi=\langle \mathbf{t}_3, \bar{\alpha}=(\mathbf{t}_3^0, A), \bar{\beta}=(\mathbf{t}_3^{\frac{1}{2}}, B\tau)\rangle$

- $\mathcal{S}=nK-I$, where $K\in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(K)=n>2$ and $k_{11}=k_{22}$.
- $\varphi(\bar{\alpha})=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\beta})=-K$
- $H^1(\Psi;\mathrm{Coker}(I-\mathcal{S}))\cong H^1(\langle A\rangle;\mathrm{Coker}(I+K))$, that is, all $\mathbf{a}$ with $(I+K)\mathbf{a}\equiv\mathbf{0}$ and $2a_1\equiv 0$ taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$, where $(I+K)\mathbf{v}\equiv\mathbf{0}$.
- $c_2=0$ or $\frac{1}{2}$, $(c_1,c_2)\in \mathrm{Coker}(K^T-I)$
- $c_3=0$, must satisfy conditions in Table 3.7
- $\Pi=\langle\widetilde{\Gamma}_{\mathcal{S}}, \alpha=(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^0\mathbf{t}_4^{a_4}, A), \beta=(\mathbf{t}_3^{\frac{1}{2}}, B\tau)\rangle$
- $H^2(\Psi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))=\mathbb{Z}_2$, $a_4=-a_2c_2+\frac{i}{2}(i=0,1)$
- Torsion-free...

- when $i=0$, $a_1=\frac{1}{2}$ and $a_2 \not\equiv \frac{(k_{11}-1)(2n+1)}{2k_{12}}$ for any $n \in \mathbb{Z}$.
- when $i=1$, $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta \rangle$ is torsion-free, projected to Sol^3.

(6bid) $\Psi = \mathbb{Z}_2^3$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B), \bar{\gamma} = (\mathbf{t}_3^0, \tau) \rangle$

- $\mathcal{S} = nK - I$, where $K \in \mathrm{SL}(2,\mathbb{Z})$ with $\mathrm{tr}(K) = n > 2$ and $k_{11} = k_{22}$.
- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\beta}) = -K, \varphi(\bar{\gamma}) = I$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is all $\mathbf{a}$, $\mathbf{r}$ with $(I+K)\mathbf{a} \equiv (I+K)\mathbf{r} \equiv \mathbf{0}$, $2a_1 \equiv 0$, and $2\mathbf{r} = 0$, $\mathbf{a}$ taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$, where $(I+K)\mathbf{v} \equiv \mathbf{0}$.
- $c_1 = 0$ or $\frac{1}{2}$, $c_2 = 0$ or $\frac{1}{2}$, $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$
- Unique choice for c_3 (up to isomorphism), found by solving $(\beta\gamma)^2 \in \widetilde{\Gamma}_{\mathcal{S}}$, this c_3 must satisfy the conditions in Tables 3.7 and 3.9
- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^0\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B), \gamma = (\mathbf{t}_1^{t_1}\mathbf{t}_2^{t_2}\mathbf{t}_3^0, \tau) \rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = -a_2c_2 + \frac{i}{2}(i = 0, 1)$, $b_4 = 0$, $\frac{1}{2}$
- Torsion-free when $\mathbf{r} \neq \mathbf{0}$ and when the torsion-free criteria for $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha, \beta \rangle$ (6bia), and $\langle \widetilde{\Gamma}_{\mathcal{S}}, \gamma\alpha, \beta \rangle$ (6bib) are satisfied.

Proof. First, observe that if a $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic group Π has

$$\begin{aligned} \mathbb{Z}_\Psi &= \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B\tau) \rangle \\ &= \langle \mathbf{t}_3, (\mathbf{t}_3^{\frac{1}{2}}, -A), (\mathbf{t}_3^{\frac{1}{2}}, B\tau) \rangle, \end{aligned}$$

then it is conjugate to a $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic group of type (6bic), similar to the situation in Theorem 3.4.28.

Also, note that

$$\begin{aligned}\mathbb{Z}_\Psi &= \langle \mathbf{t}_3, \bar\alpha = (\mathbf{t}_3^0, A), \bar\beta = (\mathbf{t}_3^{\frac{1}{2}}, B), \bar\gamma = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle \\ &= \langle \mathbf{t}_3, \bar\alpha = (\mathbf{t}_3^0, A), \bar\beta = (\mathbf{t}_3^0, B\tau), \bar\gamma = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle\end{aligned}$$

is a lift of type `(6a`i`)` from the Sol^3 classification, and it corresponds to type `(6a`i`f)` in Theorem 3.4.27.

See Theorem 3.2.21, case `(6b`i`)`, for $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ in cases `(6b`i`a)`, `(6b`i`b)`, and `(6b`i`c)`. For case `(6b`i`d)`, use the cocycle conditions in Remark 3.2.17, together with Corollary 3.2.19. The conditions on the c_i follow from previous cases. The computation of $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ is identical to that in Theorem 3.4.28, and so we obtain the stated exponents on $\mathbf{t}_4$.

In case `(6b`i`a)`, the subgroup $\langle \widetilde{\Gamma}_{\mathcal{S}}, \beta\rangle$ is torsion-free by Lemma 3.4.17.

$$\begin{aligned}
i = 0, b_4 = 0 &\implies \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\rangle, \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta\rangle \text{ must be torsion-free projected to } \mathrm{Sol}^3 \\
i = 1, b_4 = 0 &\implies \text{ already torsion-free} \\
i = 0, b_4 = \frac{1}{2} &\implies \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\rangle \text{ must be torsion-free projected to } \mathrm{Sol}^3 \\
i = 1, b_4 = \frac{1}{2} &\implies \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta\rangle \text{ must be torsion-free projected to } \mathrm{Sol}^3
\end{aligned}$$

Thus, when $i = 0, b_4 = 0$, we apply the criteria for case `(6b`i`)` in Theorem 3.2.21. When $i = 1, b_4 = 0$, both $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\rangle$ and $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta\rangle$ are already torsion-free. When $i = 0, b_4 = \frac{1}{2}$, $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta\rangle$ is torsion-free, and we apply the criteria for case `(3`i`)` in Theorem 3.2.21 to ensure that $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\rangle$ is torsion-free.

In case `(6b`i`b)`, Lemma 3.4.16 applies to show that the projection of Π to a Sol^3 group of type `(6b`i`)` must be torsion-free, so we apply the criteria for case `(6b`i`)` in Theorem 3.2.21.

In case `(6b`i`c)`, when $i = 0$, the projection of Π to Sol^3 must be torsion-free. When $i = 1$, $\langle\widetilde{\Gamma}_{\mathcal{S}}, \alpha\rangle$ is already, torsion-free, but we must ensure that $\langle\widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta\rangle$ is torsion-free when projected to Sol^3.

When Π is of type `(6bid)`, of course $\mathbf{r} \neq \mathbf{0}$ is necessary and sufficient for the group $\langle\widetilde{\Gamma}_{\mathcal{S}}, \gamma, \beta\rangle$ to be torsion-free. Then, Π will be torsion-free precisely when the groups $\langle\widetilde{\Gamma}_{\mathcal{S}}, \alpha, \beta\rangle$ and $\langle\widetilde{\Gamma}_{\mathcal{S}}, \gamma\alpha, \beta\rangle$ are torsion-free, as the union of these three $\mathbb{Z}_2^2$ holonomy $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic groups exhausts Π. See Example 3.4.34. □

Theorem 3.4.30 (`(7)` Family)**.** *The crystallographic groups of* $\mathrm{Sol}^3 \times \mathbb{R}$ *lifting an abstract kernel* $\mathbb{Z}_\Phi \to \mathrm{GL}(2, \mathbb{Z})$ *of case* `(7)` *of* Theorem 3.2.9*,*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ are}$$

`(7a)` $\Psi = D_4$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B)\rangle$

- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2, \mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) > 0$, and $k_{12} = -k_{21}$.
- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\beta}) = -K$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is all $\mathbf{a}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$ and $a_2 \equiv -a_1$, taken modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, where $(I + K)\mathbf{v} \equiv \mathbf{0}$.
- $c_1 - c_2 = 0$, $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$
- Two choices for c_3 (up to isomorphism), see Table 3.7
- $\Pi = \langle\widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^0\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$, $b_4 = 0$ or $\frac{1}{2}$
- Always has torsion

`(7b)` $\Psi = D_4$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B)\rangle$

- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2, \mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) > 0$, and $k_{12} = -k_{21}$.

- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\beta}) = -K$

- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is all $\mathbf{a}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$ and $a_2 \equiv -a_1$, taken modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, where $(I + K)\mathbf{v} \equiv \mathbf{0}$.

- $c_1 + c_2 = 0$, $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$

- $c_3 = 0$, must satisfy conditions in Table 3.9

- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}, A\tau), \beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B) \rangle$

- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $b_4 = 0$ or $\frac{1}{2}$

- Always has torsion

`(7c)` $\Psi = D_4$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B\tau) \rangle$

- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2, \mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) > 0$, and $k_{12} = -k_{21}$.

- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\beta}) = -K$

- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is all $\mathbf{a}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$ and $a_2 \equiv -a_1$, taken modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, where $(I + K)\mathbf{v} \equiv \mathbf{0}$.

- $c_1 - c_2 = 0$, $(c_1, c_2) \in \mathrm{Coker}(K^T - I)$

- $c_3 = 0$, must satisfy conditions in Table 3.7

- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}, B\tau) \rangle$

- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$

- Always has torsion

`(7d)` $\Psi = D_4$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B\tau) \rangle$

- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2, \mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) > 0$, and $k_{12} = -k_{21}$.

- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\beta}) = -K$

- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is all $\mathbf{a}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$ and $a_2 \equiv -a_1$, taken modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, where $(I + K)\mathbf{v} \equiv \mathbf{0}$.
- $c_1 + c_2 = 0$, $(c_1, c_2) \in \mathrm{Coker}(K^T - I)$
- $c_3 = 0$, must satisfy conditions in Table 3.9
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A\tau), \beta = (\mathbf{t}_3^{\frac{1}{2}}, B\tau)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_4$, four choices for a_4, the solutions of $(\alpha\beta)^4 = \mathbf{t}_4^j, (j = 0, 1, 2, 3)$.
- Always has torsion

(7e) $\Psi = D_4 \times \mathbb{Z}_2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B), \bar{\gamma} = (\mathbf{t}_3^0, \tau)\rangle$

- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2, \mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) > 0$, and $k_{12} = -k_{21}$.
- $\varphi(\bar{\alpha}) = A, \varphi(\bar{\beta}) = -K, \varphi(\bar{\gamma}) = I$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is all $\mathbf{a}$, $\mathbf{r}$ with $(I - \mathcal{S})\mathbf{a} \equiv (I - \mathcal{S})\mathbf{r} \equiv \mathbf{0}$, $a_2 \equiv -a_1$ and $(t_1, t_2) = (0, 0)$ or $(\frac{1}{2}, \frac{1}{2})$,
  $\mathbf{a}$ taken modulo $(v_1 - v_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, where $(I + K)\mathbf{v} \equiv \mathbf{0}$.
- $(c_1, c_2) = (0, 0)$ or $(\frac{1}{2}, \frac{1}{2})$, $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$
- Unique choice for c_3 (up to isomorphism), found by solving $(\beta\gamma)^2 \in \widetilde{\Gamma}_{\mathcal{S}}$, this c_3 must satisfy the conditions in Tables 3.7 and 3.9
- $\mathit{\Pi} = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B), \gamma = (\mathbf{t}_1^{t_1}\mathbf{t}_2^{t_2}\mathbf{t}_3^{0}, \tau)\rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = 0$ or $\frac{1}{2}$, $b_4 = 0$ or $\frac{1}{2}$
- Always has torsion

Proof. Observe that

$$\begin{aligned}\mathbb{Z}_\Psi &= \langle \mathbf{t}_3, \bar\alpha = (\mathbf{t}_3^0, A), \bar\beta = (\mathbf{t}_3^{\frac{1}{2}}, B), \bar\gamma = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle \\ &= \langle \mathbf{t}_3, (\mathbf{t}_3^0, A), (\mathbf{t}_3^0, B\tau), (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle,\end{aligned}$$

does not lift to a $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic group by Lemma 3.4.10.

The statements regarding $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ for cases `(7a)`, `(7b)`, `(7c)`, and `(7d)` follow from that of case `(7)` in Theorem 3.2.21. The statement for case `(7e)` follows from the cocycle conditions in Remark 3.2.17, together with Corollary 3.2.19. Note also that the conditions on the c_i follow from previous cases.

However, the holonomy Ψ is no longer a direct product of cyclic groups. In case `(7a)`, the holonomy $\Psi = D_4$ acts trivially on $\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})$, and with this trivial action,

$$H^2(D_4; \mathbb{Z}) = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Indeed, solving $\alpha^2 \in \widetilde{\Gamma}_{\mathcal{S}}$ and $\beta^2 \in \widetilde{\Gamma}_{\mathcal{S}}$ yield the choices $a_4 = 0$ or $a_4 = \frac{1}{2}$, and $b_4 = 0$ or $b_4 = \frac{1}{2}$.

In case `(7b)`, we set $a_4 = 0$ by Lemma 3.4.9, and obtain the two choices for b_4 by solving $\beta^2 \in \widetilde{\Gamma}_{\mathcal{S}}$. Case `(7c)` is similar, except that we take $b_4 = 0$. For `(7d)`, $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta\rangle$ is a $\mathrm{Sol}^3 \times \mathbb{R}$ group of type `(4a)`, so that we obtain 4 choices for a_4.

A $\mathrm{Sol}^3 \times \mathbb{R}$ group of type `(7a)` contains a subgroup of type `(4a)`, $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta\rangle$. In the notation of Theorem 3.4.24, $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta\rangle$ corresponds to the torsion case, that is, $i = 0$ or 2. Thus, all groups of type `(7a)` (and hence of type `(7e)`) have torsion.

The $\mathrm{Sol}^3 \times \mathbb{R}$ groups of type (7b) and (7c) have a subgroup of type (4b), $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta \rangle$, which has torsion.

For case (7d), Lemma 3.4.16 applied to the subgroup $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha \rangle$ shows that Π has torsion. □

Theorem 3.4.31 ((7i) Family)**.** *The crystallographic groups of* $\mathrm{Sol}^3 \times \mathbb{R}$ *lifting an abstract kernel* $\mathbb{Z}_\Phi \to \mathrm{GL}(2, \mathbb{Z})$ *of case* (7i) *of* Theorem 3.2.9,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \textit{ are}$$

(7ia) $\Psi = D_4$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B) \rangle$

- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2, \mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$, $k_{11} = k_{22}$.
- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\beta}) = -K$
- $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ is all $\mathbf{a}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$ and $2a_1 \equiv 0$, taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$, where $(I + K)\mathbf{v} \equiv \mathbf{0}$.
- $c_2 = 0$ or $\frac{1}{2}$, $(c_1, c_2) \in \mathrm{Coker}(I + K^T)$.
- Two choices for c_3 (up to isomorphism), see Table 3.7
- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B) \rangle$
- $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = -a_2c_2 + \frac{i}{2}(i = 0, 1)$, $b_4 = 0$ or $\frac{1}{2}$
- Always has torsion

(7ib) $\Psi = D_4$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B) \rangle$

- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2, \mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$, $k_{11} = k_{22}$.
- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\beta}) = -K$

- $H^1(\Psi;\mathrm{Coker}(I-\mathcal{S}))$ is all $\mathbf{a}$ with $(I-\mathcal{S})\mathbf{a} \equiv \mathbf{0}$ and $2a_1 \equiv 0$, taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$, where $(I+K)\mathbf{v} \equiv \mathbf{0}$.
- $c_1 = 0$ or $\frac{1}{2}$, $(c_1, c_2) \in \mathrm{Coker}(I+K^T)$.
- $c_3 = 0$, must satisfy conditions in Table 3.9
- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}, A\tau), \beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B)\rangle$
- $H^2(\Psi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $b_4 = 0$ or $\frac{1}{2}$
- Always has torsion

(7*i*c) $\Psi = D_4$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B\tau)\rangle$

- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2,\mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$, $k_{11} = k_{22}$.
- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\beta}) = -K$
- $H^1(\Psi;\mathrm{Coker}(I-\mathcal{S}))$ is all $\mathbf{a}$ with $(I-\mathcal{S})\mathbf{a} \equiv \mathbf{0}$ and $2a_1 \equiv 0$, taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$, where $(I+K)\mathbf{v} \equiv \mathbf{0}$.
- $c_2 = 0$ or $\frac{1}{2}$, $(c_1, c_2) \in \mathrm{Coker}(K^T - I)$.
- $c_3 = 0$, must satisfy conditions in Table 3.7
- $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}, B\tau)\rangle$
- $H^2(\Psi;\mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2$, $a_4 = -a_2c_2 + \frac{i}{2}(i = 0, 1)$
- Always has torsion

(7*i*d) $\Psi = D_4$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A\tau), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B\tau)\rangle$

- $\mathcal{S} = nK + I$, $K \in \mathrm{GL}(2,\mathbb{Z})$, $\det(K) = -1$, $\mathrm{tr}(K) = n > 0$, $k_{11} = k_{22}$.
- $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\beta}) = -K$

• $H^1(\Psi; \text{Coker}(I - \mathcal{S}))$ is all $\mathbf{a}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$ and $2a_1 \equiv 0$, taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$, where $(I + K)\mathbf{v} \equiv \mathbf{0}$.

• $c_1 = 0$ or $\frac{1}{2}$, $(c_1, c_2) \in \text{Coker}(K^T - I)$.

• $c_3 = 0$, must satisfy conditions in Table 3.9

• $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A\tau), \beta = (\mathbf{t}_3^{\frac{1}{2}}, B\tau)\rangle$

• $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_4$, four choices for a_4, the solutions of $(\alpha\beta)^4 = \mathbf{t}_4^j, (j = 0, 1, 2, 3)$.

• Torsion-free $j = 1, 3$ and $a_1 = \frac{1}{2}$ and $a_2 = -\frac{k_{21}+1}{2k_{11}} + \frac{i}{k_{11}}$ for $i = 0, \ldots, k_{11} - 1$

(7ie) $\Psi = D_4 \times \mathbb{Z}_2$, $\mathbb{Z}_\Psi = \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B), \bar{\gamma} = (\mathbf{t}_3^0, \tau)\rangle$

• $\mathcal{S} = nK + I$, $K \in \text{GL}(2, \mathbb{Z})$, $\det(K) = -1$, $\text{tr}(K) = n > 0$, $k_{11} = k_{22}$.

• $\varphi(\bar{\alpha}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varphi(\bar{\beta}) = -K, \varphi(\bar{\gamma}) = I$

• $H^1(\Psi; \text{Coker}(I - \mathcal{S}))$ is all $\mathbf{a}$, $\mathbf{r}$ with $(I - \mathcal{S})\mathbf{a} \equiv (I - \mathcal{S})\mathbf{r} \equiv \mathbf{0}$, $2a_1 \equiv 0$, $2\mathbf{r} \equiv 0$, $\mathbf{a}$ taken modulo $\begin{bmatrix} 0 \\ 2v_2 \end{bmatrix}$, where $(I + K)\mathbf{v} \equiv \mathbf{0}$.

• $c_1 = 0$ or $\frac{1}{2}$, $c_2 = 0$ or $\frac{1}{2}$, $(c_1, c_2) \in \text{Coker}(I + K^T)$.

• Unique choice for c_3 (up to isomorphism), found by solving $(\beta\gamma)^2 \in \widetilde{\Gamma}_{\mathcal{S}}$, this c_3 must satisfy the conditions in Tables 3.7 and 3.9

• $\Pi = \langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A), \beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B), \gamma = (\mathbf{t}_1^{t_1}\mathbf{t}_2^{t_2}\mathbf{t}_3^{0}, \tau)\rangle$

• $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}})) = \mathbb{Z}_2 \times \mathbb{Z}_2$, $a_4 = -a_2c_2 + \frac{i}{2}(i = 0, 1)$, $b_4 = 0$ or $\frac{1}{2}$

• Always has torsion

Proof. The group

$$\begin{aligned}\mathbb{Z}_\Psi &= \langle \mathbf{t}_3, \bar{\alpha} = (\mathbf{t}_3^0, A), \bar{\beta} = (\mathbf{t}_3^{\frac{1}{2}}, B), \bar{\gamma} = (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle \\ &= \langle \mathbf{t}_3, (\mathbf{t}_3^0, A), (\mathbf{t}_3^0, B\tau), (\mathbf{t}_3^{\frac{1}{2}}, \tau)\rangle,\end{aligned}$$

does not lift to a $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic group by Lemma 3.4.10.

See Theorem 3.2.21, case $(7i)$, for $H^1(\Psi; \mathrm{Coker}(I - \mathcal{S}))$ in cases $(7i\mathtt{a})$, $(7i\mathtt{b})$, $(7i\mathtt{c})$, and $(7i\mathtt{d})$. For case $(7i\mathtt{e})$ use the cocycle conditions in Remark 3.2.17, together with Corollary 3.2.19. The conditions on the c_i follow from previous cases. The computation of $H^2(\Psi; \mathcal{Z}(\widetilde{\Gamma}_{\mathcal{S}}))$ is identical to that in Theorem 3.4.30, and so we obtain the stated exponents on $\mathbf{t}_4$.

A $\mathrm{Sol}^3 \times \mathbb{R}$ group of type $(7i\mathtt{a})$ contains a subgroup of type $(4\mathtt{a})$, $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta \rangle$, which has torsion, as in Theorem 3.4.30. Thus, all groups of type $(7i\mathtt{a})$ (and hence of type $(7i\mathtt{e})$) have torsion.

Also, like in Theorem 3.4.30, groups of type $(7i\mathtt{b})$ and $(7i\mathtt{c})$, have a torsion subgroup of type $(4\mathtt{b})$, that is $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta \rangle$.

Only case $(7i\mathtt{d})$ remains. Our criteria for Π to be torsion-free are identical to the Sol_1^4 case (Theorem 3.3.21, case $(7i)$). First, the subgroup of type $(4\mathtt{a})$, $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha\beta \rangle$, must be torsion-free, which occurs when $j = 1, 3$ in $(\alpha\beta)^4 = \mathbf{t}_4^j$. By Lemma 3.4.17, the elements β and $\alpha\beta\alpha$ cannot contribute torsion. The only remaining elements are α and $\beta\alpha\beta$. That is, we need the groups $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha \rangle$ and $\langle \widetilde{\Gamma}_{\mathcal{S}}, \beta\alpha\beta \rangle$ to be torsion-free. They are torsion-free precisely when their projection to Sol^3 groups of type $(3i)$ are torsion-free by Lemma 3.4.16. The projections of α and $\beta\alpha\beta$ to Sol^3 are:

$$(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}, A),$$
$$(\mathbf{t}_1^{-k_{11}a_1-k_{12}a_2}\mathbf{t}_2^{-k_{21}a_1-k_{11}a_2}A, BAB)$$

The proof now proceeds exactly as in the Sol_1^4 case (Theorem 3.3.21, case (7i)). See Example 3.4.35. $\square$

Remark 3.4.32. In each case, one must explicitly check that a group Π exists with the lattice $\widetilde{\Gamma}_{\mathcal{S}}$ and given exponents on the $\mathbf{t}_i$. In particular, for generators A and B of Ψ, let α and β be generators of Π, projecting to A and B, respectively. For each relation $r(A, B) = e$ between A and B in Ψ, we need to check that $r(\alpha, \beta) \in \widetilde{\Gamma}_{\mathcal{S}}$.

In the special case when $c_1 = c_2 = c_3 = 0$, so that the lattice $\widetilde{\Gamma}_{\mathcal{S}}$ is a direct product $\Gamma_{\mathcal{S}} \times \mathbb{Z}$, once we have an abstract kernel $\Psi \to \mathrm{Out}(\widetilde{\Gamma}_{\mathcal{S}})$ these conditions are immediately satisfied, and an extension exists.

For example, suppose that

$$\Psi = \langle A, B\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Then for

$$\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A)$$
$$\beta = (\mathbf{t}_1^{b_1}\mathbf{t}_2^{b_2}\mathbf{t}_3^{b_3}\mathbf{t}_4^{b_4}, B)$$

With a_1, a_2, a_3, and a_4 given in the classification, we also need to check that

$$\alpha\beta\alpha^{-1}\beta^{-1} = \mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_3^{n_3}\mathbf{t}_4^{n_4}.$$

for integers n_i. In fact, n_1, n_2, n_3 are always integral (Theorem 3.2.16). But it must be checked that n_4 is integral for an extension to exist.

3.4.33 (Examples).

Example 3.4.34 ((`6b`i`d`)). Here we give an example of a torsion-free $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic group with $\mathbb{Z}_2^3$ holonomy. See Theorem 3.4.29. Take

$$K = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}. \text{ Then, } \mathcal{S} = \begin{bmatrix} 17 & 12 \\ 24 & 17 \end{bmatrix}.$$

Note that $k_{11} = k_{22}$ and $\mathcal{S}$ satisfy the conditions for case (`6b`i`d`) in Theorem 3.4.29. Form $\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$ and embed it in Sol^3 as a standard lattice, $\Gamma_{\mathcal{S}}$. We lift $\Gamma_{\mathcal{S}}$ to the lattice $\widetilde{\Gamma}_{\mathcal{S}} = \Gamma_{\mathcal{S}} \times \mathbb{Z}$ of $\mathrm{Sol}^3 \times \mathbb{R}$. Simply set $c_1 = c_2 = c_3 = 0$ in Notation 3.4.5. We have defined our lattice $\widetilde{\Gamma}_{\mathcal{S}}$ of Π. Now we must define

$$\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{0}\mathbf{t}_4^{a_4}, A),$$
$$\beta = (\mathbf{t}_3^{\frac{1}{2}}\mathbf{t}_4^{b_4}, B),$$
$$\gamma = (\mathbf{t}_1^{t_1}\mathbf{t}_2^{t_2}\mathbf{t}_3^{0}, \tau),$$

so that Π is torsion-free. Let us set

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\mathbf{r} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$
$$a_4 = \frac{1}{2}$$
$$b_4 = 0.$$

Notice that $\mathbf{a}$ and $\mathbf{r}$ satisfy all cocycle conditions. Checking the torsion criteria of Theorem 3.4.29, we see that the group $\langle \widetilde{\Gamma}_{\mathcal{S}}, \gamma, \beta \rangle$ is torsion-free since $\mathbf{r} \neq \mathbf{0}$.

The $\mathrm{Sol}^3 \times \mathbb{R}$ group of type `(6bia)`, $\langle \widetilde{\Gamma}_{\mathcal{S}}, \alpha, \beta \rangle$, is torsion-free by our choice of a_4 and b_4. Finally, we check that the $\mathrm{Sol}^3 \times \mathbb{R}$ group of type `(6bib)`, $\langle \widetilde{\Gamma}_{\mathcal{S}}, \gamma\alpha, \beta \rangle$, is torsion-free. Note that

$$\gamma\alpha = (\mathbf{t}_1^{\frac{1}{2}} \mathbf{t}_2^0 \mathbf{t}_3^0 \mathbf{t}_4^{\frac{1}{2}}, A\tau).$$

The exponent on $\mathbf{t}_1$ is $\frac{1}{2}$, and as for the exponent on $\mathbf{t}_2$,

$$0 \not\equiv \frac{(k_{11}-1)(2n+1)}{2k_{12}} = \frac{(2n+1)}{2} = n + \frac{1}{2},$$

for any $n \in \mathbb{Z}$. Therefore the criteria for $\langle \widetilde{\Gamma}_{\mathcal{S}}, \gamma\alpha, \beta \rangle$ to be torsion-free are satisfied. From this it follows that $\mathit{\Pi}$ is torsion-free.

So $\mathit{\Pi} \backslash (\mathrm{Sol}^3 \times \mathbb{R})$ defines an infra-solvmanifold of $\mathrm{Sol}^3 \times \mathbb{R}$ with $\mathbb{Z}_2^3$ holonomy.

This example is analogous to the following flat 4-manifold $\mathit{\Pi}' \backslash \mathbb{R}^4$ with $\mathbb{Z}_2^3$ holonomy. We define

$$\mathit{\Pi}' \subset \mathbb{R}^4 \rtimes \mathrm{O}(4, \mathbb{R})$$

by $\Pi' = \langle \mathbb{Z}^4, \alpha, \beta, \gamma \rangle$, where

$$\alpha = \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

$$\beta = \left(\begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

$$\gamma = \left(\begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right).$$

Example 3.4.35 (`(7id)`)**.** Here we give an example of a torsion-free $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic group with D_4 holonomy. See Theorem 3.4.31. We take

$$K = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad \text{therefore } n = k_{11} + k_{22} = 2, \quad \mathcal{S} = nK + I = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}.$$

As in Example 3.4.34, $\mathbb{Z}^2 \rtimes_{\mathcal{S}} \mathbb{Z}$ and embeds in Sol^3 as a standard lattice, $\Gamma_{\mathcal{S}}$. We lift $\Gamma_{\mathcal{S}}$ to the lattice $\widetilde{\Gamma}_{\mathcal{S}} = \Gamma_{\mathcal{S}} \times \mathbb{Z}$ of $\mathrm{Sol}^3 \times \mathbb{R}$ by setting $c_1 = c_2 = c_3 = 0$ in Notation 3.4.5. Now we must define

$$\alpha = (\mathbf{t}_1^{a_1} \mathbf{t}_2^{a_2} \mathbf{t}_3^{0} \mathbf{t}_4^{a_4}, A\tau),$$
$$\beta = (\mathbf{t}_3^{\frac{1}{2}}, B\tau),$$

so that Π is torsion-free. We compute that $(\alpha\beta)^4 = \mathbf{t}_4^{4a_4}$ with $c_i = 0$, and we define

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

$$a_4 = \frac{1}{4}.$$

Note that the cocycle conditions on $\mathbf{a}$ are satisfied; namely, $\mathbf{a} \in \operatorname{Coker}(I - \mathcal{S})$. Also the torsion-free criteria for case $(7id)$ of Theorem 3.4.31 are satisfied, as $(\alpha\beta)^4 = \mathbf{t}_4$, $a_1 = \frac{1}{2}$, and $a_2 = -\frac{k_{21}+1}{2k_{11}} \equiv 0$.

Thus $\Pi\backslash(\mathrm{Sol}^3 \times \mathbb{R})$ is an infra-solvmanifold of $\mathrm{Sol}^3 \times \mathbb{R}$ with D_4 holonomy.

3.5 Crystallographic Groups of $\mathrm{Sol}^4_{m,n}$

We have the family of solvable Lie groups of the form $\mathbb{R}^3 \rtimes_\phi \mathbb{R}$, where

$$\phi(u) = \begin{bmatrix} e^{\theta u} & 0 & 0 \\ 0 & e^u & 0 \\ 0 & 0 & e^{-(1+\theta)u} \end{bmatrix}, \theta > 1.$$

The parameter θ corresponds precisely to the isomorphism type of the Lie group. Since we are concerned with crystallographic groups, we need to assume that θ is chosen so that our Lie group admits a lattice. This is equivalent to $\phi(u_0)$ being conjugate to an element of $\mathrm{SL}(3, \mathbb{Z})$, for some u_0. Then the characteristic polynomial of $\phi(u_0)$ is $x^3 - mx^2 + nx - 1 = (x - e^{\theta u})(x - e^u)(x - e^{-(1+\theta)u})$ for $m, n \in \mathbb{Z}$, and we will denote the group $\mathbb{R}^3 \rtimes_\phi \mathbb{R}$ by $\mathrm{Sol}^4_{m,n}$. It is known that there are only countably many such θ [23]. For the rest of this section, fix such a θ.

Here we provide a classification of the crystallographic groups of $\mathrm{Sol}^4_{m,n}$.

Given a crystallographic group

$$\Pi \subset \mathrm{Sol}^4_{m,n} \rtimes \mathbb{Z}_2^3,$$

$\widetilde{\Gamma} = \Pi \cap \mathrm{Sol}^4_{m,n}$ is a lattice of $\mathrm{Sol}^4_{m,n}$ with finite quotient group $\Phi \subset \mathbb{Z}_2^3$. We have the short exact sequence

$$1 \to \widetilde{\Gamma} \to \Pi \to \Phi \to 1.$$

The arguments in this section are nearly identical to those used in our classification of Sol^3 crystallographic groups. For $u \in \mathbb{R}$, let E^u denote the action of $\mathbb{R}$ on $\mathbb{R}^3$ in $\mathrm{Sol}^4_{m,n}$.

3.5.1 (Lattices of $\mathrm{Sol}^4_{m,n}$). First we will describe the lattices of $\mathrm{Sol}^4_{m,n}$. Let $\mathcal{S} \in \mathrm{SL}(3,\mathbb{Z})$ have three distinct real eigenvalues $\lambda_1, \lambda_2, \frac{1}{\lambda_1\lambda_2}$, satisfying $\lambda_1 = \lambda_2^\theta$ and

$$\lambda_1 > \lambda_2 > 1 > \frac{1}{\lambda_1\lambda_2}.$$

Notation 3.5.2. We always take

$$\Delta = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \frac{1}{\lambda_1\lambda_2} \end{bmatrix},$$

with $\lambda_1 > \lambda_2 > 1 > \frac{1}{\lambda_1\lambda_2}$.

We form the semi-direct product $\mathbb{Z}^3 \rtimes_{\mathcal{S}} \mathbb{Z}$ and embed this group in $\mathrm{Sol}^4_{m,n}$ as a lattice. Let P be a matrix with $\det(P) = 1$ diagonalizing $\mathcal{S}$:

$$P\mathcal{S}P^{-1} = \Delta = E^{\ln(\lambda_2)},$$

where the last equality follows from $\lambda_2^\theta = \lambda_1$.

With such P and Δ for $\mathcal{S}$, we define a map

$$\begin{aligned} \phi : \mathbb{Z}^3 \rtimes_{\mathcal{S}} \mathbb{Z} \longrightarrow & \quad \mathrm{Sol}^4_{m,n} \\ (\mathbf{n}, u) \longmapsto & \ (P\mathbf{n}, u \ln(\lambda_2)) \end{aligned} \tag{3.20}$$

The equality $P\mathcal{S}P^{-1} = \Delta = E^{\ln(\lambda_2)}$ guarantees that ϕ is a homomorphism. The image is a lattice of $\mathrm{Sol}^4_{m,n}$, and it maps the generators as follows.

$$\begin{aligned} \mathbf{e}_1 &\mapsto \mathbf{t}_1 = P\mathbf{e}_1 \\ \mathbf{e}_2 &\mapsto \mathbf{t}_2 = P\mathbf{e}_2 \\ \mathbf{e}_3 &\mapsto \mathbf{t}_3 = P\mathbf{e}_3 \\ \mathbf{e}_4 &\mapsto \mathbf{t}_4 = (\mathbf{0}, \ln(\lambda_2)). \end{aligned} \tag{3.21}$$

We denote image of $\mathbb{Z}^3 \rtimes_{\mathcal{S}} \mathbb{Z}$ as $\Gamma_{\mathcal{S}}$:

$$\Gamma_{\mathcal{S}} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4 \rangle \subset \mathrm{Sol}^4_{m,n}.$$

Notation 3.5.3. A lattice of $\mathrm{Sol}^4_{m,n}$ generated by $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4$ of the form in assignment (3.21) is a *standard lattice* of $\mathrm{Sol}^4_{m,n}$.

Conversely, we show any lattice of $\mathrm{Sol}^4_{m,n}$ is isomorphic to such a $\Gamma_{\mathcal{S}}$ as the following proposition shows.

Proposition 3.5.4 (cf. Proposition 3.2.5)**.** *There is a one-one correspondence between the isomorphism classes of* $\mathrm{Sol}^4_{m,n}$*-lattices and the conjugacy classes of* $\mathcal{S} \in \mathrm{SL}(3, \mathbb{Z})$ *with three distinct real eigenvalues* $\lambda_1 > \lambda_2 > 1 > \frac{1}{\lambda_1 \lambda_2}$ *such that* $\lambda_2^\theta = \lambda_1$*. Therefore, any lattice of* $\mathrm{Sol}^4_{m,n}$ *is conjugate to* $\Gamma_{\mathcal{S}}$*, for some* $\mathcal{S}$*, by an inner automorphism of* $\mathrm{Sol}^4_{m,n}$*.*

Proof. We show that any lattice Γ is conjugate to $\Gamma_{\mathcal{S}}$ for some $\mathcal{S}$. The intersection of Γ with the nilradical $\mathbb{R}^3$, $\Gamma \cap \mathbb{R}^3$, is a lattice of $\mathbb{R}^3$, [32, Corollary 8.28].

Let $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3 \in \mathbb{R}^3$ generate $\Gamma \cap \mathbb{R}^3$, and let $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}\}$ generate Γ. Then $\mathbf{t}$ is of the form $(\mathbf{a}, \ln(\lambda_2))$, for some λ_2. We can assume that $\lambda_2 > 1$ by taking the inverse of $\mathbf{t}$, if needed. Conjugation by $((E^{\ln(\lambda_2)} - I)^{-1}\mathbf{a}, 0)$ maps this lattice to

$$\Gamma = \langle (\mathbf{t}_1, 0),\ (\mathbf{t}_2, 0), (\mathbf{t}_3, 0)\ (\mathbf{0}, \ln(\lambda_2))\rangle.$$

Define P by $P\mathbf{e}_i = \mathbf{t}_i$, for $i = 1, 2, 3$, and set $\mathcal{S} = P^{-1}E^{\ln(\lambda_2)}P$. As $E^{\ln(\lambda_2)}$ acts on $\langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\rangle$ as an integral matrix, $\mathcal{S} \in \mathrm{SL}(3, \mathbb{Z})$ necessarily. Notice that, in case $\det(P) < 0$, we can make it positive by taking $-\mathbf{t}_1$ in place of $\mathbf{t}_1$. Furthermore, we can assume that $\det(P) = 1$. For conjugation by $(0, \frac{1}{\sqrt[3]{\det(P)}}I) \in \mathrm{Aff}(\mathrm{Sol}^4_{m,n})$ maps this lattice to another which has $\det(P) = 1$. Therefore Γ is conjugate to $\Gamma_{\mathcal{S}}$ in $\mathrm{Aff}(\mathrm{Sol}^4_{m,n})$.

We now show the isomorphism class statement. If $\mathcal{S}' = B\mathcal{S}B^{-1}$ with $B \in \mathrm{SL}(3, \mathbb{Z})$, then an isomorphism between $\mathbb{Z}^3 \rtimes_{\mathcal{S}} \mathbb{Z}$ and $\mathbb{Z}^3 \rtimes_{\mathcal{S}'} \mathbb{Z}$ is given by $\mathbf{e}_i \to B\mathbf{e}_i$ for $i = 1, 2, 3$ and $\mathbf{e}_4 \to \mathbf{e}_4$.

Conversely, suppose two lattices of $\mathrm{Sol}^4_{m,n}$ are isomorphic. We may assume these lattices are of the form $\Gamma_{\mathcal{S}}$ and $\Gamma_{\mathcal{S}'}$. We have an isomorphism

$$\phi : \mathbb{Z}^3 \rtimes_{\mathcal{S}} \mathbb{Z} \longrightarrow \mathbb{Z}^3 \rtimes_{\mathcal{S}'} \mathbb{Z}.$$

Since $\mathbb{Z}^3$ is the discrete nil-radical of both groups, ϕ restricts to an isomorphism on the $\mathbb{Z}^3$ factors, say $B \in \mathrm{GL}(3, \mathbb{Z})$. Also, ϕ induces an isomorphism ϵ on the quotients, so that $\epsilon = \pm 1$. Therefore, ϕ is of the form

$$\phi(\mathbf{x}, m) = (B\mathbf{x} + \eta(\mathbf{x}, m), \epsilon m).$$

Evidently, η is independent of $\mathbf{x}$. So $\eta(\mathbf{x}, m) = \eta(m)$. Then applying ϕ to

$(0,1)(\mathbf{x},0)(0,-1) \in \mathbb{Z}^3 \rtimes_{\mathcal{S}} \mathbb{Z}$ in two different ways, we get

$$\begin{aligned}
\phi((0,1)(\mathbf{x},0)(0,-1)) &= \phi(\mathcal{S}\mathbf{x},0) \\
&= (B(\mathcal{S}\mathbf{x}),0) \\
\phi(0,1)\phi(\mathbf{x},0)\phi(0,-1) &= (\eta(1),\epsilon)(B\mathbf{x},0)(\eta(1),\epsilon)^{-1} \\
&= (\mathcal{S}'^{\epsilon}B\mathbf{x},0)
\end{aligned}$$

Thus, $B\mathcal{S}\mathbf{x} = \mathcal{S}'^{\epsilon}B\mathbf{x}$ for every $\mathbf{x}$, and we have $\mathcal{S}'^{\epsilon} = B\mathcal{S}B^{-1}$. Now, $\epsilon = +1$ is forced since, by definition, both $\mathcal{S}$ and $\mathcal{S}'$ have two eigenvalues greater than 1, and one eigenvalue less than 1. Furthermore, by taking $-B$ in place of B if needed, we can assume that $B \in \mathrm{SL}(3,\mathbb{Z})$. Thus, $\mathcal{S}$ and $\mathcal{S}'$ are conjugate in $\mathrm{SL}(3,\mathbb{Z})$. □

3.5.5 (Classification of $\mathrm{Sol}^4_{m,n}$ Crystallographic Groups).

Proposition 3.5.6. *Any crystallographic group Π of $\mathrm{Sol}^4_{m,n}$ can be conjugated in $\mathrm{Aff}(\mathrm{Sol}^4_{m,n})$ to $\Pi \subset \mathrm{Sol}^4_{m,n} \rtimes \mathbb{Z}_2^3$ so that:*

(1) We have $\Pi \cap \mathrm{Sol}^4_{m,n} = \Gamma_{\mathcal{S}}$. That is, the translation subgroup of Π is a standard lattice of $\mathrm{Sol}^4_{m,n}$, generated by $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$, and $\mathbf{t}_4$ as in assignment (3.21) in Subsection 3.5.1 .

(2) The holonomy group Φ is generated by at most three elements of $\mathbb{Z}_2^3$, and therefore Π is generated by $\langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4 \rangle$, and at most three isometries of the form $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A)$, for $A \in \mathbb{Z}_2^3$ and real numbers a_i.

Proof. Any maximal compact subgroup of $\mathrm{Aut}(\mathrm{Sol}^4_{m,n})$ is conjugate to $\mathbb{Z}_2^3 \subset \mathrm{Aut}(\mathrm{Sol}^4_{m,n})$ (see Proposition 3.1.5). Observe that the affine conjugations in Proposition 3.5.4 leave this maximal compact subgroup $\mathbb{Z}_2^3$ fixed, while putting the lattice in standard form. □

We will assume any $\mathrm{Sol}^4_{m,n}$ crystallographic group is embedded in $\mathrm{Sol}^4_{m,n} \rtimes \mathbb{Z}_2^3$ as in Proposition 3.5.6. In fact, we will see that the group $\mathbb{Z}_2^3$ can never occur as a holonomy group of a $\mathrm{Sol}^4_{m,n}$ crystallographic group. In fact, the maximal holonomy of a $\mathrm{Sol}^4_{m,n}$ crystallographic group is $\mathbb{Z}_2^2$.

Notation 3.5.7. Let

$$\tau_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tau_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tau_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

denote the automorphisms generating the maximal compact subgroup $\mathbb{Z}_2^3$ of $\mathrm{Aut}(\mathrm{Sol}^4_{m,n})$ (see Proposition 3.1.5).

Note that $\Pi \cap \mathbb{R}^3 = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3 \rangle$ is a lattice of $\mathbb{R}^3$. Denote the quotient $\Pi/\mathbb{Z}^3$ by $\mathbb{Z}_\Phi$ so that we have the diagram:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{Z}^3 & = & \mathbb{Z}^3 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Gamma_{\mathcal{S}} & \longrightarrow & \Pi & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle /\mathbb{Z}^3} & & \downarrow{\scriptstyle /\mathbb{Z}^3} & & \| & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_\Phi & \longrightarrow & \Phi & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{3.22}$$

Note that the quotient $\mathbb{Z}$ in the above diagram is generated by $\mathbf{t}_4$, not $\mathbf{t}_3$, as was the case for Sol^3. First we will classify all possible $\mathbb{Z}_\Phi$, as well as abstract kernels $\varphi : \mathbb{Z}_\Phi \to \mathrm{GL}(3, \mathbb{Z})$, which can arise as quotients of $\mathrm{Sol}^4_{m,n}$ crystallographic groups. The holonomy Φ is a subgroup of $\mathbb{Z}_2^3 = \langle \tau_1, \tau_2, \tau_3 \rangle$.

Lemma 3.5.8 (cf. Lemma 3.2.8)**.** *For* $\bar{\alpha} = (\mathbf{t}_4^{a_4}, A) \in \mathbb{Z}_\Phi$*,* $a_4 = 0$ *or* $\frac{1}{2}$ (mod $\mathbb{Z}$).

Proof. Note that $\bar{A} = +1$ for all $A \in \mathbb{Z}_2^3$. Thus,

$$(\mathbf{t}_4^{a_4}, A)^2 = (\mathbf{t}_4^{2a_4}, I).$$

For this to be in $\mathbb{Z} = \langle \mathbf{t}_4 \rangle$, $a_4 = 0$ or $\frac{1}{2}$ is forced. □

Lemma 3.5.9 (cf. Lemma 3.2.10)**.** *If* $\mathbb{Z}_\Phi$ *is an extension of* $\mathbb{Z}$ *by* Φ *that is the quotient of a crystallographic group* Π *of* $\mathrm{Sol}^4_{m,n}$*, then* $\mathbb{Z}_\Phi$ *cannot contain an element of the form* $\bar{\alpha} = (\mathbf{t}_4^0, \pm\tau_i)$.

Proof. Recall that the lattice of Π is of the form $\Gamma_{\mathcal{S}}$. The trace of $\mathcal{S}$ must be integral:

$$\lambda_1 + \lambda_2 + \frac{1}{\lambda_1\lambda_2} \in \mathbb{Z}.$$

Note that the matrix

$$\varphi(\mathbf{t}_4)\varphi(\bar{\alpha}) = \mathcal{S}\varphi(\bar{\alpha}) = P^{-1}\Delta P P^{-1}(\pm\tau_i)P = P^{-1}\Delta(\pm\tau_i)P.$$

must be integral, and so must have integral trace. We will argue the claim for $\pm\tau_1$. Then,

$$\mp\lambda_1 \pm \lambda_2 \pm \frac{1}{\lambda_1\lambda_2} \in \mathbb{Z}.$$

Of course, the Rational Zero Theorem implies that all eigenvalues of $\mathcal{S}$ must be irrational. From the above equations, we infer that $2\lambda_1 \in \mathbb{Z}$, so that λ_1 is rational, a contradiction. Therefore, $\mathbb{Z}_\Phi$ cannot contain $(\mathbf{t}_4^0, \pm\tau_1)$. The proof of the claim for $\pm\tau_2$ and $\pm\tau_3$ is identical. □

Remark 3.5.10. From Theorem 3.2.16, an element of $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ fixes the exponents on the $\mathbf{t}_i$ in Proposition 3.5.6, and hence defines an abstract kernel $\Phi \to \mathrm{Out}(\Gamma_{\mathcal{S}})$. As $\Gamma_{\mathcal{S}}$ has trivial center, $H^2(\Phi; \mathcal{Z}(\Gamma_{\mathcal{S}}))$ is trivial, and we have a unique (up to isomorphism) extension of $\Gamma_{\mathcal{S}}$ by Φ, which we can also view as

$$1 \to \mathbb{Z}^3 \to \Pi \to \mathbb{Z}_\Phi \to 1.$$

Proposition 3.5.11. *Let* $\Pi = \langle \Gamma_{\mathcal{S}}, (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{\frac{1}{2}}, A)\rangle$ *be an extension in* Theorem 3.2.16 *defining a* $\mathrm{Sol}^4_{m,n}$ *crystallographic group with lattice* $\Gamma_{\mathcal{S}} = \langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4\rangle$. *Then* $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) = 0$. *Therefore, there exists* $\mathbf{t}_1^{v_1}\mathbf{t}_2^{v_2}\mathbf{t}_3^{v_3}$ *which conjugates* $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{\frac{1}{2}}, A)$ *to* $(\mathbf{t}_4^{\frac{1}{2}}, A)$, *and leaves* $\Gamma_{\mathcal{S}}$ *invariant.*

Proof. Note that A has order 2 and $\bar{A} = 1$. Further, $\varphi(\bar{\alpha}) = P^{-1}\sqrt{\Delta}AP$. As $I - \sqrt{\Delta}A$ is non-singular,

$$P^{-1}(I - \sqrt{\Delta}A)P = I - P^{-1}\sqrt{\Delta}AP = I - \varphi(\bar{\alpha})$$

is non-singular and we may take $\mathbf{v} = (I - \varphi(\bar{\alpha}))^{-1}\mathbf{a}$. Then our $\mathbf{v}$ conjugates $(\mathbf{t}_3^{\frac{1}{2}}, A)$ to $(\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{\frac{1}{2}}, A)$. It remains to show $\mathbf{v} \in (I - \mathcal{S})^{-1}\mathbb{Z}^3$. First observe that $\varphi(\bar{\alpha})$ is a square root of $\mathcal{S}$.

$$\begin{aligned}\varphi(\bar{\alpha})^2 &= P^{-1}\sqrt{\Delta}A\sqrt{\Delta}AP = P^{-1}\Delta A^2 P \text{ (since } A \text{ is diagonal)}\\ &= P^{-1}\Delta P = \mathcal{S}.\end{aligned}$$

Now

$$(I - \mathcal{S})\mathbf{v} = (I + \varphi(\bar{\alpha}))(I - \varphi(\bar{\alpha}))\mathbf{v} = (I + \varphi(\bar{\alpha}))\mathbf{a} \in \mathbb{Z}^3. \qquad \square$$

Lemma 3.5.12 (cf. Lemma 3.2.20)**.** *When* $\alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{a_4}, A) \in \Pi$ *satisfies* $a_4 = \frac{1}{2}$, $\gamma\alpha$ *is infinite order for all* $\gamma \in \Gamma_{\mathcal{S}}$.

Proof. Let $\mathrm{pr} : \mathrm{Sol}^4_{m,n} \to \mathbb{R}$ denote the quotient homomorphism of $\mathrm{Sol}^4_{m,n}$ by its nil-radical $\mathbb{R}^3$. Write $\gamma \in \Gamma_{\mathcal{S}}$ as $\mathbf{t}_1^{n_1}\mathbf{t}_2^{n_2}\mathbf{t}_3^{n_3}\mathbf{t}_4^{n_4}$. Application of pr to $(\gamma\alpha)^2$

yields

$$\mathrm{pr}(\gamma\alpha)^2 = 2n_4 + 1,$$

from which we infer $\gamma\alpha$ is not torsion. $\square$

Theorem 3.5.13 (Classification of $\mathrm{Sol}^4_{m,n}$-geometry)**.** *The following is a complete list of crystallographic groups of* $\mathrm{Sol}^4_{m,n}$*. In each case, we give* Φ*,* $\mathbb{Z}_\Phi$*, the abstract kernel* $\varphi : \mathbb{Z}_\Phi \to \mathrm{GL}(3,\mathbb{Z})$ *(which gives the action of* $\mathbb{Z}_\Phi$ *on* $\langle \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\rangle$*), equations defining* $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$*, and* $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ *from* $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$*, and whether* Π *is torsion-free. Note that* $\varphi(\mathbf{t}_4) = \mathcal{S}$ *by definition, where* $\mathcal{S} \in \mathrm{SL}(3,\mathbb{Z})$ *has three distinct real eigenvalues*

$$\lambda_1 > \lambda_2 > 1 > \frac{1}{\lambda_1\lambda_2}$$

such that $\lambda_2^\theta = \lambda_1$*.*

(0) $\Phi = \text{trivial}$

- $\Pi = \Gamma_{\mathcal{S}}$
- Torsion-free

(1a) $\Phi = \mathbb{Z}_2$: $A = \tau_i$, $\mathbb{Z}_\Phi = \mathbb{Z} = \langle \mathbf{t}_4, \bar{\alpha} = (\mathbf{t}_4^{\frac{1}{2}}, A)\rangle$.

- $\varphi(\bar{\alpha}) = K$ is a square root of $\mathcal{S}$, with one negative eigenvalue.
- $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ is trivial so that $\mathbf{a} = \mathbf{0}$.
- $\Pi = \langle \Gamma_{\mathcal{S}}, \alpha = (\mathbf{t}_4^{\frac{1}{2}}, A)\rangle = \mathbb{Z}^3 \rtimes_K \mathbb{Z}$
- Torsion-free

(1b) $\Phi = \mathbb{Z}_2$: $A = -\tau_i$, $\mathbb{Z}_\Phi = \mathbb{Z} = \langle \mathbf{t}_4, \bar{\alpha} = (\mathbf{t}_4^{\frac{1}{2}}, A)\rangle$.

- $\varphi(\bar{\alpha}) = K$ is a square root of $\mathcal{S}$, with two negative eigenvalues.
- $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ is trivial so that $\mathbf{a} = \mathbf{0}$.
- $\Pi = \langle \Gamma_{\mathcal{S}}, \alpha = (\mathbf{t}_4^{\frac{1}{2}}, A)\rangle = \mathbb{Z}^3 \rtimes_K \mathbb{Z}$
- Torsion-free

(2a) $\Phi = \mathbb{Z}_2$: $A = -I$, $\mathbb{Z}_\Phi = \mathbb{Z} \times \mathbb{Z}_2 = \langle \mathbf{t}_4, \bar{\alpha} = (\mathbf{t}_4^0, A)\rangle$.

- $\varphi(\bar{\alpha}) = -I$
- $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) = \frac{\mathrm{Coker}(I-\mathcal{S})}{2\mathrm{Coker}(I-\mathcal{S})} \subseteq \mathbb{Z}_2^3$. That is, all $\mathbf{a}$ with $(I - \mathcal{S})\mathbf{a} \equiv \mathbf{0}$, taken modulo $2\mathbf{a}$.
- $\Pi = \langle \Gamma_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^0, A)\rangle$
- Always has torsion

(2b) $\Phi = \mathbb{Z}_2$: $A = -I$, $\mathbb{Z}_\Phi = \mathbb{Z} = \langle \mathbf{t}_4, \bar{\alpha} = (\mathbf{t}_4^{\frac{1}{2}}, A)\rangle$.

- $\varphi(\bar{\alpha}) = K$ is a square root of $\mathcal{S}$, with three negative eigenvalues.
- $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ is trivial so that $\mathbf{a} = \mathbf{0}$.
- $\Pi = \langle \Gamma_{\mathcal{S}}, \alpha = (\mathbf{t}_4^{\frac{1}{2}}, A)\rangle = \mathbb{Z}^3 \rtimes_K \mathbb{Z}$
- Torsion-free

(3a) $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = -I$, $B = \tau_i$,

$\mathbb{Z}_\Phi = \mathbb{Z} \times \mathbb{Z}_2 = \langle \mathbf{t}_4, \bar{\alpha} = (\mathbf{t}_4^0, A), \bar{\beta} = (\mathbf{t}_4^{\frac{1}{2}}, B)\rangle$.

- $\varphi(\bar{\alpha}) = -I$, $\varphi(\bar{\beta}) = K$ is a square root of $\mathcal{S}$, with one negative eigenvalue.
- $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) \cong \frac{\mathrm{Coker}(I-K)}{2\mathrm{Coker}(I-K)} \subseteq \mathbb{Z}_2^3$. That is, all $\mathbf{a}$ with $(I - K)\mathbf{a} \equiv \mathbf{0}$, taken modulo $2\mathbf{a}$.

- $\mathit{\Pi} = \langle \Gamma_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{0}, A), \beta = (\mathbf{t}_4^{\frac{1}{2}}, B)\rangle$
- Always has torsion

`(3b)` $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = -I$, $B = -\tau_i$,

$\mathbb{Z}_\Phi = \mathbb{Z} \times \mathbb{Z}_2 = \langle \mathbf{t}_4, \bar{\alpha} = (\mathbf{t}_4^0, A), \bar{\beta} = (\mathbf{t}_4^{\frac{1}{2}}, B)\rangle$.

- $\varphi(\bar{\alpha}) = -I$, $\varphi(\bar{\beta}) = K$ is a square root of $\mathcal{S}$, with two negative eigenvalues.
- $H^1(\Phi; \mathrm{Coker}(I-\mathcal{S})) \cong \frac{\mathrm{Coker}(I-K)}{2\mathrm{Coker}(I-K)} \subseteq \mathbb{Z}_2^3$. That is, all $\mathbf{a}$ with $(I-K)\mathbf{a} \equiv \mathbf{0}$, taken modulo $2\mathbf{a}$.
- $\mathit{\Pi} = \langle \Gamma_{\mathcal{S}}, \alpha = (\mathbf{t}_1^{a_1}\mathbf{t}_2^{a_2}\mathbf{t}_3^{a_3}\mathbf{t}_4^{0}, A), \beta = (\mathbf{t}_4^{\frac{1}{2}}, B)\rangle$
- Always has torsion

Proof. Straightforward application of Lemmas 3.5.8 and 3.5.9 show the various possibilities for $\mathbb{Z}_\Phi$. In particular, there is no $\mathbb{Z}_\Phi$ with $\Phi = \mathbb{Z}_2^3$, as any such $\mathbb{Z}_\Phi$ cannot occur as the quotient of a $\mathrm{Sol}^4_{m,n}$ crystallographic group by Lemma 3.5.9.

In cases `(1a)`, `(1b)`, and `(2b)`, recall from the proof of Proposition 3.5.11 that

$$\varphi(\bar{\alpha}) = P^{-1}\sqrt{\Delta}AP$$

is a square root of $\mathcal{S}$. When $A = \tau_i$, $-\tau_i$, and $-I$, then $\varphi(\bar{\alpha})$ has one, two, and three negative eigenvalues, respectively. We denote $\varphi(\bar{\alpha})$ by K. Proposition 3.5.11 shows that we may take $\mathbf{a} = \mathbf{0}$ in cases `(1a)`, `(1b)`, and `(2b)`. That is, $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ is trivial in these cases.

In cases `(2a)`, the cocycle condition on $\mathbf{a}$ vanishes since

$$(I + \varphi(\bar{\alpha})) = I - I = 0.$$

The coboundaries are

$$(I - \varphi(\bar{\alpha}))\mathbf{a} = 2\mathbf{a},$$

for $\mathbf{a} \in \mathrm{Coker}(I - \mathcal{S})$. Therefore,

$$H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) = \frac{\mathrm{Coker}(I - \mathcal{S})}{2\mathrm{Coker}(I - \mathcal{S})} \subseteq \mathbb{Z}_2^3,$$

where the last inclusion follows from the fact that $H^1(\Phi; \mathrm{Coker}(I - \mathcal{S}))$ is generated by at most 3 elements of order 2.

In cases `(3a)` and `(3b)`, we can set $\mathbf{b} = \mathbf{0}$ by Proposition 3.5.11. The commutator cocycle condition then forces $\mathbf{a} \in \mathrm{Coker}(I-K)$, so that $(I-K)\mathbf{a} \equiv \mathbf{0}$. Having fixed $\mathbf{b} = \mathbf{0}$, the coboundaries are

$$(I - \varphi(\bar{\alpha}))\mathbf{a} = 2\mathbf{a},$$

for $\mathbf{a} \in \mathrm{Coker}(I - K)$. Hence,

$$H^1(\Phi; \mathrm{Coker}(I - \mathcal{S})) = \frac{\mathrm{Coker}(I - K)}{2\mathrm{Coker}(I - K)} \subseteq \mathbb{Z}_2^3.$$

Lemma 3.5.12 shows that `(1a)`, `(1b)`, and `(2b)` all are torsion-free. Indeed, $\mathit{\Pi}$ is isomorphic to $\mathbb{Z}^3 \rtimes_K \mathbb{Z}$ in these cases. In case `(2a)`, $\alpha^2 = e$, and cases `(3a)` and `(3b)` also have torsion, as they contain a type `(2a)` subgroup. □

Remark 3.5.14. The quotient of $\mathrm{Sol}^4_{m,n}$ by a crystallographic group $\mathit{\Pi}$ of type `(0)`, `(1a)`, `(1b)`, or `(2b)` is the mapping torus of a linear self-diffeomorphism of T^3. These were known to be all the infra-solvmanifolds of $\mathrm{Sol}^4_{m,n}$ [23]. On the other hand, our classification here includes the crystallographic groups with torsion, and shows that $\mathbb{Z}_2^3$ cannot occur as the holonomy of any crystallographic group of $\mathrm{Sol}^4_{m,n}$.

4 Curvature of 4-Dimensional Infra-Solvmanifolds

Depending on choice of left invariant metric, a 4-dimensional solvable geometry can have different isometry groups and Ricci signatures. Here are the possible Ricci signatures on the 4-dimensional solvable geometries of the form $\mathbb{R}^3 \rtimes \mathbb{R}$ (all but ${\mathrm{Sol}_1}^4$.) [21]:

G	Possible Ricci signatures
$\mathbb{R}^4$	$(0, 0, 0, 0)$
$\mathrm{Nil}^3 \times \mathbb{R}$	$(0, +, -, -)$
Nil^4	$(0, +, -, -), (+, +, -, -), (+, -, -, -)$
$\mathrm{Sol}^3 \times \mathbb{R}$	$(0, 0, 0, -), (0, +, -, -), (+, +, -, -), (+, -, -, -)$
$\mathrm{Sol}^4_{m,n}$	$(0, 0, 0, -), (0, +, -, -), (+, +, -, -), (+, -, -, -)$
${\mathrm{Sol}_0}^4$	$(0, 0, 0, -), (0, +, -, -)$

Table 4.1: Ricci signatures

Suppose we only consider left invariant metrics on G that induce a particular Ricci signature. This may place restrictions on the size of the isometry group, and hence the compact isometric quotients of G may not account for all infra-solvmanifolds of G. Given an infra-solvmanifold M of G, a Ricci signature can be *realized* on M if there is a left invariant metric on G with prescribed Ricci signature such that $M = \Pi \backslash G$, for some $\Pi \subset \mathrm{Isom}(G)$. We will only consider 4-dimensional solvable geometries of the form $\mathbb{R}^3 \rtimes \mathbb{R}$. As there is only one possible Ricci signature for metrics on $\mathrm{Nil}^3 \times \mathbb{R}$, and Sol_0^4 has no infra-solvmanifolds (though it has compact forms), we examine which Ricci signatures can be realized on infra-solvmanifolds with Nil^4, $\mathrm{Sol}^4_{m,n}$, and $\mathrm{Sol}^3 \times \mathbb{R}$ geometry.

4.1 Ricci Curvature

Let M be a Riemannian n-manifold. First, we recall the definition of the Ricci quadratic form r and Ricci transformation $\hat{r}$, which are both measures of curvature at a point $p \in M$. We will use Milnor's definitions [27].

Definition 4.1.1. The *Ricci quadratic form* at p is a real valued quadratic function on $T_p(M)$. Fix an orthonormal basis $(e_1, \ldots, e_n)$ of $T_p(M)$. For $v \in T_p(M)$,

$$r(v) = \sum_{i=1}^{n} \langle R_{ve_i}(v), e_i \rangle.$$

If v is a unit vector, then $r(v)$ is the *Ricci curvature* in the direction of v. It is $n-1$ times the average of the sectional curvatures of the 2-planes containing v.

Definition 4.1.2. The *Ricci transformation* at $p \in M$ is the self-adjoint map $\hat{r} : T_p(M) \to T_p(M)$ defined by

$$\hat{r}(v) = \sum_{i=1}^{n} R_{e_i v}(e_i).$$

The Ricci quadratic form r and the Ricci transformation $\hat{r}$ are related by the identity

$$\langle \hat{r}(v), v \rangle = r(v).$$

Definition 4.1.3. The eigenvalues of the Ricci transformation $\hat{r}$ at p are the *principal Ricci curvatures* of the metric at p.

Definition 4.1.4. The collection of signs of the principal Ricci curvatures at p is the *Ricci signature* of the metric at p.

4.2 Computation of Ricci Signatures for Metric Lie Algebras

When G is a Lie group, any left invariant metric can be specified by declaring a basis $(v_1, v_2, \ldots, v_n)$ of the tangent space at the identity $T_e(G) \cong \mathfrak{g}$ to be orthonormal. We use left translation by g to propagate the metric across the tangent bundle: We declare $\ell_{g*}v_1, \ell_{g*}v_2, \ldots, \ell_{g*}v_n$ to be an orthonormal basis for $T_g(G)$.

Proposition 4.2.1 (cf. [34], Proposition 1.3)**.** *There is a one-one correspondence between the set of all left invariant metrics on a Lie group and the group of upper triangular matrices with positive diagonal entries.*

Proof. By choosing a basis for $\mathfrak{g}$, we can identify $\mathfrak{g}$ with $\mathbb{R}^n$, so that the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form an orthonormal basis for $\mathfrak{g}$. Any other metric on G can be specified by declaring $A\mathbf{e}_1, \ldots A\mathbf{e}_n$ to be an orthonormal basis, for $A \in \mathrm{GL}(n, \mathbb{R})$. We obtain the same metric if and only if A is orthogonal. By the Gram-Schmidt process, A can be written uniquely as $K\mathcal{B}$, where K is orthogonal, and $\mathcal{B}$ is an upper triangular matrix with positive entries. □

Left translation by $g \in G$ is an isometry carrying $e \in G$ to g. In particular, the Ricci signature at any point $g \in G$ is the same as the Ricci signature at the identity. So we may speak of the Ricci signature of the left invariant metric. The Ricci signature of a left invariant metric can be computed from the structure of the Lie algebra $\mathfrak{g}$ and the orthonormal basis $v_1, v_2, \ldots, v_n$ of $\mathfrak{g}$ [27].

Notation 4.2.2. Suppose a metric Lie algebra $\mathfrak{g}$ has a codimension 1 ideal $\mathfrak{u}$. Let b be a unit vector orthogonal to $\mathfrak{u}$, and let L denote the restriction of $\mathrm{ad}(b)$ to $\mathfrak{u}$:

$$L(u) = [b, u] .$$

Let S be the self-adjoint transformation $\frac{1}{2}(L + L^*)$. The ideal $\mathfrak{u}$ may also be thought of as a metric Lie algebra; let us denote the Riemannian connection for $\mathfrak{u}$ by $\overline{\nabla}$.

Lemma 4.2.3 ([27, Lemma 5.5])**.** *For any $u, v \in \mathfrak{u}$, we have the following expressions for ∇:*

$$\nabla_b b = 0 \qquad \text{and} \qquad \nabla_b u = \tfrac{1}{2}(L - L^*)u$$
$$\nabla_u b = -Su \qquad \text{and} \qquad \nabla_u v = \overline{\nabla}_u v + \langle Su, v\rangle b.$$

Proposition 4.2.4 ([9])**.** *Suppose a Lie algebra $\mathfrak{g}$ has an ideal $\mathfrak{u}$ of codimension 1. Assume that $\mathfrak{u}$ is abelian, so that the Lie bracket and $\overline{\nabla}$ vanish on $\mathfrak{u}$, as well as $\mathrm{tr}(L) = 0$, so that $\mathrm{tr}(S) = 0$. Then, for $u \in \mathfrak{u}$, we have the following formula for the Ricci transformation:*

$$\hat{r}(u) = \frac{1}{2}\,[L, L^*]\,(u) = \frac{1}{2}(LL^* - L^*L)(u).$$

Proof. We compute, for an orthonormal basis $e_1, \ldots, e_n$ of $\mathfrak{g}$, with $e_n = b$,

$$\begin{aligned}
\hat{r}(u) &= \left(\sum_{i=1}^{n-1} \nabla_{[e_i,u]}e_i - \nabla_{e_i}\nabla_u e_i + \nabla_u \nabla_{e_i} e_i\right) + \nabla_{[b,u]}b - \nabla_b \nabla_u b + \nabla_u \nabla_b b \\
&= \left(\sum_{i=1}^{n-1} -\langle Su, e_i\rangle \nabla_{e_i} b\right) + \left(\sum_{i=1}^{n-1} \langle Se_i, e_i\rangle \nabla_u b\right) - SLu + \tfrac{1}{2}(L - L^*)Su \\
&= \left(\sum_{i=1}^{n-1} \langle Su, e_i\rangle Se_i\right) - \mathrm{tr}(S)Su - SLu + \tfrac{1}{2}(L - L^*)Su \\
&= SSu - SLu + \tfrac{1}{2}(L - L^*)Su \\
&= \tfrac{1}{4}(L + L^*)(L + L^*)u - \tfrac{1}{2}(L + L^*)Lu + \tfrac{1}{4}(L - L^*)(L + L^*)u \\
&= \tfrac{1}{2}\left(LL^* - L^*L\right)u = \tfrac{1}{2}\left[L, L^*\right]u.
\end{aligned}$$

□

Proposition 4.2.5 ([27, Lemma 2.3])**.** *Suppose a Lie algebra $\mathfrak{g}$ has an ideal $\mathfrak{u}$ of codimension 1. Let b be a unit vector orthogonal to $\mathfrak{u}$. Then*

$$\hat{r}(b) = -\mathrm{tr}(S^2)b,$$

where S^2 is the self-adjoint transformation $\frac{1}{2}(L + L^*)$.

Proof. Take an orthonormal basis $e_1, \ldots, e_n$ of $\mathfrak{g}$, with $e_n = b$.

$$\begin{aligned}
\hat{r}(b) &= \left(\sum_{i=1}^{n-1} \nabla_{[e_i,b]}e_i - \nabla_{e_i}\nabla_b e_i + \nabla_b \nabla_{e_i} e_i\right) + \nabla_{[b,b]}b - \nabla_b \nabla_b b + \nabla_b \nabla_b b \\
&= \sum_{i=1}^{n-1} \nabla_{-Le_i} e_i - \nabla_{e_i} \tfrac{1}{2}(L - L^*)e_i + \langle Se_i, e_i\rangle \nabla_b b \\
&= \sum_{i=1}^{n-1} \langle -SLe_i, e_i\rangle b - \langle Se_i, \tfrac{1}{2}(L - L^*)e_i\rangle b \\
&= \left(-\mathrm{tr}(SL) - \tfrac{1}{2}\mathrm{tr}(SL - SL^*)\right) b.
\end{aligned}$$

Now we have, using that $\mathrm{tr}(L) = \mathrm{tr}(L^*)$, $\mathrm{tr}(LL^*) = \mathrm{tr}(L^*L)$,

$$-\mathrm{tr}(SL) - \tfrac{1}{2}\mathrm{tr}(SL - SL^*) = \mathrm{tr}\left(-\tfrac{1}{2}LL - \tfrac{1}{2}L^*L - \tfrac{1}{4}LL - \tfrac{1}{4}L^*L + \tfrac{1}{4}LL^* + \tfrac{1}{4}L^*L^*\right)$$
$$= -\tfrac{1}{2}\mathrm{tr}(LL) - \tfrac{1}{2}\mathrm{tr}(L^*L).$$

Now note that

$$\begin{aligned}-\mathrm{tr}(S^2) &= -\tfrac{1}{4}\left(\mathrm{tr}(LL) + \mathrm{tr}(L^*L) + \mathrm{tr}(LL^*) + \mathrm{tr}(L^*L^*)\right)\\ &= -\tfrac{1}{4}\left(2\mathrm{tr}(LL) + 2\mathrm{tr}(L^*L)\right) = -\tfrac{1}{2}\mathrm{tr}(LL) - \tfrac{1}{2}\mathrm{tr}(L^*L). \qquad \square\end{aligned}$$

4.3 Ricci Signatures of the 4-Dimensional Solvable Geometries

From here on, we specialize to the case when $\mathfrak{g}$ is the Lie algebra of a 4-dimensional solvable geometry of the form $\mathbb{R}^3 \rtimes_\psi \mathbb{R}$. Note that $\mathbb{R}^3 \rtimes_\psi \mathbb{R}$ is completely determined by the matrix $\psi(1)$:

$$\begin{aligned}
&\mathbb{R}: \quad \psi(1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} &&\mathrm{Nil}^3 \times \mathbb{R}: \quad \psi(1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\\
&\mathrm{Nil}^4: \quad \psi(1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} &&\mathrm{Sol}^3 \times \mathbb{R}: \quad \psi(1) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\\
&\mathrm{Sol}_0^4: \quad \psi(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} &&\mathrm{Sol}_{m,n}^4: \quad \psi(1) = \begin{bmatrix} \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -(1+\theta) \end{bmatrix}
\end{aligned} \tag{4.1}$$

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ denote the standard basis vectors for $\mathfrak{g} \cong \mathfrak{u} \rtimes \mathbb{R}$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Any left-invariant metric on G can be uniquely specified by an upper triangular matrix:

$$\mathcal{B}' = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & 0 & b_{33} & b_{34} \\ 0 & 0 & 0 & b_{44} \end{bmatrix}, \tag{4.2}$$

with $b_{11}, b_{22}, b_{33}, b_{44} > 0$, and declaring the vectors $\mathbf{v}_i = \mathcal{B}'\mathbf{e}_i$ to be an orthonormal basis for $\mathfrak{g}$. Note that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form an orthonormal basis of the ideal $\mathfrak{u} = \mathbb{R}^3$ of $\mathfrak{g}$, while $\mathbf{v}_4$ is a unit vector orthogonal to this ideal.

Let L denote the transformation $\mathrm{ad}_{\mathbf{v}_4}$ restricted to the ideal $\mathfrak{u} = \mathbb{R}^3 = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$, and set

$$\mathcal{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}. \tag{4.3}$$

With respect to the basis $(\mathbf{v}_i)$, note that L has matrix form

$$\mathcal{B}^{-1}\psi(b_{44})\mathcal{B}.$$

By a theorem of Gordon and Wilson [13], with any left-invariant metric on a 4-dimensional solvable geometry,

$$\mathrm{Isom}(G) \subseteq G \rtimes K \subset \mathrm{Aff}(G),$$

where K is a maximal compact subgroup of $\mathrm{Aut}(G)$. That is, the group of isometries fixing the identity, $\mathrm{Isom}_e(G)$, must be a subgroup of $K \subset \mathrm{Aut}(G)$. Given $\alpha \in \mathrm{Aut}(G)$, let α_* denote the induced automorphism of $\mathfrak{g}$. Recall that we have identified $\mathfrak{g}$ with $\mathbb{R}^4$ by taking the standard basis for $\mathfrak{g} \cong \mathbb{R}^3 \rtimes \mathbb{R}$, and so we may view α_* as an element of $\mathrm{GL}(4, \mathbb{R})$. Of course, $\mathrm{Aut}(G) \cong \mathrm{Aut}(\mathfrak{g})$.

Lemma 4.3.1. *Suppose an orthonormal basis of $\mathfrak{g}$ is specified by such a matrix $\mathcal{B}'$ in* assignment (4.2)*. Let $\alpha_* \in \mathrm{Aut}(\mathfrak{g})$. Then α_* defines an isometry of $\mathfrak{g}$ if and only if α_* is orthogonal when expressed in terms of the basis*

$$\mathbf{v}_i = \mathcal{B}'\mathbf{e}_i.$$

That is,

$$\mathcal{B}'^{-1}\alpha_*\mathcal{B}' \in \mathrm{O}(4, \mathbb{R}).$$

We will restrict our attention to the ideal $\mathbb{R}^3$ of $\mathfrak{g} \cong \mathbb{R}^3 \rtimes \mathbb{R}$. This ideal is the nilradical of $\mathfrak{g}$, and so we have the restriction homomorphism

$$p : \mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(\mathbb{R}^3) = \mathrm{GL}(3, \mathbb{R}).$$

If $\alpha_* \in \mathrm{Aut}(\mathfrak{g})$ is an isometry of $\mathfrak{g}$, then α_* must restrict to an isometry of the ideal $\mathbb{R}^3 = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$. Therefore,

$$\mathcal{B}^{-1}p(\alpha_*)\mathcal{B} \in \mathrm{O}(3, \mathbb{R}).$$

The condition $\mathcal{B}^{-1}p(\alpha_*)\mathcal{B} \in \mathrm{O}(3, \mathbb{R})$ will put rather strict restrictions on $\mathcal{B}$, for certain choices of $p(\alpha_*)$ in the image of $p(\mathrm{Aut}(\mathfrak{g}))$. This in turn will place restrictions on the possible Ricci signatures. Recall in our discussion of the crystallographic groups of $\mathrm{Sol}^4_{m,n}$ that we have fixed the notation:

$$\tau_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tau_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tau_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We also will set

$$A = \begin{bmatrix} 0 & -a & 0 \\ \frac{1}{a} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Lemma 4.3.2.

(1) The condition $\mathcal{B}^{-1}(\pm\tau_1)\mathcal{B} \in \mathrm{O}(3,\mathbb{R})$ forces $b_{12} = b_{13} = 0$.

(2) The condition $\mathcal{B}^{-1}(\pm\tau_2)\mathcal{B} \in \mathrm{O}(3,\mathbb{R})$ forces $b_{12} = b_{23} = 0$.

(3) The condition $\mathcal{B}^{-1}(\pm\tau_3)\mathcal{B} \in \mathrm{O}(3,\mathbb{R})$ forces $b_{13} = b_{23} = 0$.

(4) The condition $\mathcal{B}^{-1}(\pm A)\mathcal{B} \in \mathrm{O}(3,\mathbb{R})$ forces $b_{12} = b_{13} = b_{23} = 0$.

Proof. For (1),

$$\mathcal{B}^{-1}(\pm\tau_1)\mathcal{B} = \pm \begin{bmatrix} -1 & -\frac{2b_{12}}{b_{11}} & -\frac{2b_{13}}{b_{11}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{O}(3,\mathbb{R})$$

forces $b_{12} = b_{13} = 0$.

For (2),

$$\mathcal{B}^{-1}(\pm\tau_2)\mathcal{B} = \pm \begin{bmatrix} 1 & \frac{2b_{12}}{b_{11}} & \frac{2b_{12}b_{23}}{b_{11}b_{22}} \\ 0 & -1 & -\frac{2b_{23}}{b_{22}} \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{O}(3,\mathbb{R})$$

forces $b_{12} = b_{23} = 0$.

For (3),

$$\mathcal{B}^{-1}(\pm\tau_3)\mathcal{B} = \pm \begin{bmatrix} 1 & 0 & \frac{2b_{13}b_{22}-2b_{12}b_{23}}{b_{11}b_{22}} \\ 0 & 1 & \frac{2b_{23}}{b_{22}} \\ 0 & 0 & -1 \end{bmatrix} \in \mathrm{O}(3,\mathbb{R})$$

forces $b_{13} = b_{23} = 0$.

For (4), we compute that

$$\mathcal{B}^{-1}(\pm A)\mathcal{B} = \pm \begin{bmatrix} -\frac{b_{12}}{ab_{22}} & -\frac{b_{12}^2}{ab_{11}b_{22}} - \frac{ab_{22}}{b_{11}} & -\frac{b_{12}b_{13}}{ab_{11}b_{22}} - \frac{ab_{23}}{b_{11}} + \frac{-b_{13}b_{22}+b_{12}b_{23}}{b_{11}b_{22}} \\ \frac{b_{11}}{ab_{22}} & \frac{b_{12}}{ab_{22}} & \frac{b_{13}}{ab_{22}} - \frac{b_{23}}{b_{22}} \\ 0 & 0 & 1 \end{bmatrix}.$$

As A has positive determinant, this matrix must be in $\mathrm{SO}(3,\mathbb{R})$. As the upper 2×2 diagonal block must be in $\mathrm{SO}(2,\mathbb{R})$, we must have $b_{12} = 0$. From the $(1,3)$ and $(2,3)$ entries of this matrix, we obtain the equations

$$-b_{13} - ab_{23} = 0,$$

$$b_{13} - ab_{23} = 0,$$

which force $b_{13} = b_{23} = 0$, as well. □

Definition 4.3.3. Given an infra-solvmanifold M of G, a Ricci signature can be *realized* on M if there is a left invariant metric on G with prescribed Ricci signature such that $M = \Pi\backslash G$, for some $\Pi \subset \mathrm{Isom}(G)$.

Suppose a basis of the Lie algebra $\mathfrak{g}$ is specified by $\mathcal{B}'$ as in assignment (4.2). Recall that L denotes the transformation $\mathrm{ad}_{\mathbf{v}_4}$ restricted to the ideal $\mathfrak{u} = \mathbb{R}^3 = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$. With respect to this basis $(\mathbf{v}_i)$, note that L has matrix form

$$N = \mathcal{B}^{-1}\psi(b_{44})\mathcal{B}.$$

By Lemma 4.2.5,

$$r(\mathbf{v}_4) = -\mathrm{tr}(S^2) < 0.$$

We claim that $\mathrm{tr}(S^2) > 0$ for all G of the form $\mathbb{R}^3 \rtimes \mathbb{R}$, except for the trivial case $G = \mathbb{R}^4$. To see this, simply note that the matrix form of S is $\frac{1}{2}(N + N^T)$, which is a symmetric matrix. Thus, the $\mathbf{v}_4$ direction always corresponds to negative Ricci curvature, and we always put it as the last sign in the Ricci signature notation:

$$(*, *, *, -).$$

By Proposition 4.2.4, with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of $\mathbb{R}^3$, the matrix form of $\hat{r}$, restricted to $\mathbb{R}^3$, is given by

$$\hat{r}\big|_{\mathbb{R}^3} = \frac{1}{2}\left[N, N^T\right] = \frac{1}{2}(NN^T - N^T N).$$

To obtain the Ricci signature of the metric with basis given by $\mathcal{B}'$, we need to compute the signs of the eigenvalues of $\hat{r}\big|_{\mathbb{R}^3}$.

4.3.4 (Nil^4)**.** The crystallographic groups Π of Nil^4, and their associated infra-nilmanifolds $\Pi\backslash\mathrm{Nil}^4$ are classified in [7]. The maximal holonomy of an infra-nilmanifold of Nil^4 is $\mathbb{Z}_2^2$. With standard basis for the Lie algebra of Nil^4,

$$\mathrm{Isom}(\mathrm{Nil}^4) = \mathrm{Nil}^4 \rtimes \mathbb{Z}_2^2,$$

where $\mathbb{Z}_2^2$ is as defined in equation (2.1) in section 2.3. Note that $\mathbb{Z}_2^2$ consists of diagonal matrices.

Theorem 4.3.5. *If M is an infra-nilmanifold of* Nil^4*, then any of the three Ricci signatures* $(0, +, -, -)$*,* $(+, +, -, -)$*,* $(+, -, -, -)$ *can be realized on M.*

Proof. Declare the basis of the Lie algebra of Nil^4 given by

$$\mathcal{B}' = \begin{bmatrix} b_{11} & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 \\ 0 & 0 & b_{33} & 0 \\ 0 & 0 & 0 & b_{44} \end{bmatrix},$$

to be orthonormal. That is, declare $\mathbf{v}_i = \mathcal{B}'\mathbf{e}_i$ to be orthonormal As $\mathcal{B}'$ is diagonal,

$$\mathrm{Isom}(\mathrm{Nil}^4) = \mathrm{Nil}^4 \rtimes \mathbb{Z}_2^2,$$

where $\mathbb{Z}_2^2$ is the maximal compact subgroup in equation (2.1) in section 2.3, regardless of the diagonal entries of $\mathcal{B}'$.

Recall that with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, the matrix form of $\hat{r}$ restricted to $\mathbb{R}^3$ is given by

$$\hat{r}\big|_{\mathbb{R}^3} = \frac{1}{2}\left[N, N^T\right] = \frac{1}{2}(NN^T - N^TN), \text{ where } N = \mathcal{B}^{-1}\psi(b_{44})\mathcal{B}.$$

We compute that the eigenvalues of this matrix are

$$\left\{\frac{b_{22}^2 b_{44}^2}{2b_{11}^2}, -\frac{b_{33}^2 b_{44}^2}{2b_{22}^2}, \frac{1}{2}\left(-\frac{b_{22}^2 b_{44}^2}{b_{11}^2} + \frac{b_{33}^2 b_{44}^2}{b_{22}^2}\right)\right\}.$$

By choosing different values for b_{11}, b_{22}, b_{33}, and b_{44}, the third eigenvalue can be positive, negative, or vanishing. Therefore, any of the three Ricci signatures $(0,+,-,-)$, $(+,+,-,-)$, $(+,-,-,-)$ can be realized on M. □

4.3.6 ($\mathrm{Sol}^4_{m,n}$)**.** Recall that we have the family of Lie algebras $\mathbb{R}^3 \rtimes_{\psi(t)} \mathbb{R}$, where

$$\psi(1) = \begin{bmatrix} \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -(1+\theta) \end{bmatrix}.$$

The parameter $\theta > 1$ corresponds precisely to the isomorphism type of the Lie algebra. Let an orthonormal basis of $\mathfrak{g}$ be specified by $\mathcal{B}'$ in assignment (4.2). To realize metrics on certain infra-solvmanifolds of $\mathrm{Sol}^4_{m,n}$, we will see that $\mathcal{B}$ must have a certain form. For these various forms of $\mathcal{B}$, we record the eigenvalues of $\hat{r}\big|_{\mathbb{R}^3}$ in Table 4.2.

$\mathcal{B}$	Eigenvalues of $\hat{r}\big\vert_{\mathbb{R}^3} =$ $\frac{1}{2}\left[N, N^T\right] = \frac{1}{2}(NN^T - N^TN)$, where $N = \mathcal{B}^{-1}\psi(b_{44})\mathcal{B}$
$\begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}$	$\left\{0, -\frac{b_{23}\sqrt{b_{22}^2+b_{23}^2}b_{44}^2(2+\theta)^2}{2b_{22}^2}, \frac{b_{23}\sqrt{b_{22}^2+b_{23}^2}b_{44}^2(2+\theta)^2}{2b_{22}^2}\right\}$
$\begin{bmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}$	$\left\{0, -\frac{b_{13}\sqrt{b_{11}^2+b_{13}^2}b_{44}^2(1+2\theta)^2}{2b_{11}^2}, \frac{b_{13}\sqrt{b_{11}^2+b_{13}^2}b_{44}^2(1+2\theta)^2}{2b_{11}^2}\right\}$
$\begin{bmatrix} b_{11} & b_{12} & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}$	$\left\{0, -\frac{b_{12}\sqrt{b_{11}^2+b_{12}^2}b_{44}^2(-1+\theta)^2}{2b_{11}^2}, \frac{b_{12}\sqrt{b_{11}^2+b_{12}^2}b_{44}^2(-1+\theta)^2}{2b_{11}^2}\right\}$

Table 4.2: Eigenvalues of $\hat{r}\big\vert_{\mathbb{R}^3}$ for $\mathrm{Sol}^4_{m,n}$

Lemma 4.3.7. *Let*

$$M = \Pi\backslash\mathrm{Sol}^4_{m,n}$$

for $\Pi \subset \mathrm{Sol}^4_{m,n} \rtimes \mathrm{Aut}(\mathrm{Sol}^4_{m,n})$ *be an infra-solvmanifold of* $\mathrm{Sol}^4_{m,n}$*, diffeomorphic to an infra-solvmanifold* $M' = \Pi'\backslash\mathrm{Sol}^4_{m,n}$ *of type* `(1a)` *(respectively,* `(1b)`*) in* Theorem 3.5.13.

If the image of the holonomy of Π' *in* $\mathrm{Aut}(\mathbb{R}^3)$ *is generated by* τ_i *(respectively,* $-\tau_i$*), then the image of the holonomy of* Π *in* $\mathrm{Aut}(\mathbb{R}^3)$ *must also be generated by* τ_i *(respectively,* $-\tau_i$*).*

Proof. Since Π and Π' are isomorphic, Theorem 1.1.5 implies that Π and Π' are conjugate in $\mathrm{Aff}(\mathrm{Sol}^4_{m,n})$. Hence the holonomy groups must be conjugate in $\mathrm{Aut}(\mathrm{Sol}^4_{m,n})$. But the image of $\mathrm{Aut}(\mathrm{Sol}^4_{m,n})$ (and hence also that of $\mathrm{Aut}(\mathfrak{g})$) in $\mathrm{Aut}(\mathbb{R}^3)$ under restriction is the group of diagonal matrices $(\mathbb{R}^*)^3$, and $\pm\tau_i$ is central in $(\mathbb{R}^*)^3$. □

Theorem 4.3.8. *For each type of torsion-free* $\mathrm{Sol}^4_{m,n}$ *crystallographic group* Π *in* Theorem 3.5.13, *we list the Ricci signatures which can be realized on the infra-solvmanifold* $\Pi\backslash\mathrm{Sol}^4_{m,n}$.

Infra-$\mathrm{Sol}^4_{m,n}$	Holonomy	Realizable Ricci Signatures
`(0)`	$\{e\}$	(0,0,0,-), (0,+,-,-), (+,+,-,-), (+,-,-,-)
`(1a)`	$\mathbb{Z}_2$	(0,0,0,-), (0,+,-,-)
`(1b)`	$\mathbb{Z}_2$	(0,0,0,-), (0,+,-,-)
`(2b)`	$\mathbb{Z}_2$	(0,0,0,-), (0,+,-,-), (+,+,-,-), (+,-,-,-)

Table 4.3: Realizable Ricci signatures on infra-$\mathrm{Sol}^4_{m,n}$ manifolds

Proof. In cases `(0)` and `(2b)`, Lemma 4.3.2 does not provide any restriction on $\mathcal{B}$. Of course, Table 4.2 shows that $(0,0,0,-)$ and $(0,+,-,-)$ are realizable on M. In fact, we can explicitly realize the Ricci signature $(+,+,-,-)$ with

$$\mathcal{B}' = \begin{bmatrix} b_{11} & 0 & b_{13} & 0 \\ 0 & b_{22} & b_{23} & 0 \\ 0 & 0 & b_{33} & 0 \\ 0 & 0 & 0 & b_{44} \end{bmatrix}.$$

With this choice of $\mathcal{B}'$, the determinant of $\hat{r}\big|_{\mathbb{R}^3} = \frac{1}{2}(NN^T - N^TN)$ is:

$$-\frac{b_{13}^2 b_{23}^2 b_{44}^6 (-1+\theta)^2 (2+\theta)^2 (1+2\theta)^2}{8 b_{11}^2 {b_{22}}^2} < 0.$$

Since $\theta > 1$, when we take b_{13} and b_{23} to be nonzero, the three eigenvalues of $\hat{r}\big|_{\mathbb{R}^3}$ must have signs $(+,+,-)$, as the trace of $\frac{1}{2}(NN^T - N^TN)$ vanishes. On the other hand, we realize the Ricci signature $(+,-,-,-)$ with

$$\mathcal{B}' = \begin{bmatrix} b_{11} & b_{12} & b_{13} & 0 \\ 0 & b_{22} & 0 & 0 \\ 0 & 0 & b_{33} & 0 \\ 0 & 0 & 0 & b_{44} \end{bmatrix}.$$

With this choice of $\mathcal{B}'$, the determinant of $\hat{r}\big|_{\mathbb{R}^3} = \frac{1}{2}(NN^T - N^TN)$ is:

$$\frac{b_{12}^2 b_{13}^2 b_{44}^6 (-1+\theta)^2(2+\theta)^2(1+2\theta)^2}{8b_{11}^4} > 0.$$

When b_{12} and b_{13} are nonzero,the eigenvalues of $\hat{r}\big|_{\mathbb{R}^3}$ must have signs $(+,-,-)$, as the trace of $\frac{1}{2}(NN^T - N^TN)$ vanishes.

By Lemma 4.3.7, the image of the holonomy in $\mathrm{Aut}(\mathbb{R}^3)$ of any infra-solvmanifold diffeomorphic to an infra-sovlmanifold of type `(1a)` (or `(1b)`) is generated by τ_i (or $-\tau_i$).

In case `(1a)`, when the holonomy of Π is generated by τ_1, τ_2, or τ_3, Lemma 4.3.2 forces $b_{12} = b_{13} = 0$, $b_{12} = b_{23} = 0$, or $b_{13} = b_{23} = 0$, respectively. Table 4.2 shows that the eigenvalues of $\hat{r}\big|_{\mathbb{R}^3}$ can only have signs $(0,0,0)$ or $(0,+,-)$.

Likewise, in case `(1b)`, when the holonomy of Π is generated by $-\tau_1$, $-\tau_2$, or $-\tau_3$, Lemma 4.3.2 forces $b_{12} = b_{13} = 0$, $b_{12} = b_{23} = 0$, or $b_{13} = b_{23} = 0$, respectively. Again, Table 4.2 shows that the eigenvalues of $\hat{r}\big|_{\mathbb{R}^3}$ can only have signs $(0,0,0)$ or $(0,+,-)$. Consequently, in both cases `(1a)` and `(1b)`, only the Ricci signatures $(0,0,0,-)$ and $(0,+,-,-)$ can be realized on $\Pi\backslash\mathrm{Sol}^4_{m,n}$. $\square$

4.3.9 ($\mathrm{Sol}^3\times\mathbb{R}$)**.** When $G = \mathrm{Sol}^3\times\mathbb{R}$, the image of $\mathrm{Aut}(\mathfrak{g})$ under the restriction homomorphism

$$\mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(\mathbb{R}^3),$$

is the subgroup of $\mathrm{GL}(3,\mathbb{R})$ consisting of matrices of the form

$$\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & k \end{bmatrix},$$

where A is of the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ or $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$. See Proposition 3.1.7.

Lemma 4.3.10. *Let*

$$M = \Pi\backslash(\mathrm{Sol}^3 \times \mathbb{R})$$

and

$$M' = \Pi'\backslash(\mathrm{Sol}^3 \times \mathbb{R})$$

for $\Pi, \Pi' \subset \mathrm{Sol}^3 \times \mathbb{R} \rtimes \mathrm{Aut}(\mathrm{Sol}^3 \times \mathbb{R})$ *be two diffeomorphic infra-solvmanifolds of* $\mathrm{Sol}^3 \times \mathbb{R}$*.*

(1) If τ_i (respectively, $-\tau_i$), for $i = 1, 2$, is in the image of the holonomy of Π' in $\mathrm{Aut}(\mathbb{R}^3)$, then τ_j (respectively, $-\tau_j$), for $j = 1, 2$, is in the image of the holonomy of Π in $\mathrm{Aut}(\mathbb{R}^3)$.

(2) If τ_3 (respectively, $-\tau_3$) is in the image of the holonomy of Π' in $\mathrm{Aut}(\mathbb{R}^3)$, then τ_3 (respectively, $-\tau_3$) is in the image of the holonomy of Π in $\mathrm{Aut}(\mathbb{R}^3)$ (In our classification of $\mathrm{Sol}^3 \times \mathbb{R}$ crystallographic groups, we denoted τ_3 by τ).

(3) If $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is in the image of the holonomy of Π' in $\mathrm{Aut}(\mathbb{R}^3)$, then $\begin{bmatrix} 0 & -a & 0 \\ \frac{1}{a} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, for some real number a, is in the image of the holonomy of Π in $\mathrm{Aut}(\mathbb{R}^3)$.

Proof. Since Π and Π' are isomorphic, Theorem 1.1.5 implies that Π and

$\mathit{\Pi}'$ are conjugate in $\mathrm{Aff}(\mathrm{Sol}^3 \times \mathbb{R})$. Hence their holonomy groups must be conjugate in $\mathrm{Aut}(\mathrm{Sol}^3 \times \mathbb{R})$. Moreover, the image of the holonomy groups in $\mathrm{Aut}(\mathbb{R}^3)$ must be conjugate in the image of $\mathrm{Aut}(\mathrm{Sol}^3 \times \mathbb{R})$ in $\mathrm{Aut}(\mathbb{R}^3)$.

For (1), note that the only conjugates of τ_1 and $-\tau_1$ in the image of $\mathrm{Aut}(\mathrm{Sol}^3 \times \mathbb{R})$ in $\mathrm{Aut}(\mathbb{R}^3)$, are τ_2 and $-\tau_2$, respectively.

For (2), note that τ_3 is central in the image of $\mathrm{Aut}(\mathrm{Sol}^3 \times \mathbb{R})$ in $\mathrm{Aut}(\mathbb{R}^3)$, and so has no conjugates.

For (3), note that in the image of $\mathrm{Aut}(\mathrm{Sol}^3 \times \mathbb{R})$ in $\mathrm{Aut}(\mathbb{R}^3)$, $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is conjugate to any matrix of the form $\begin{bmatrix} 0 & -a & 0 \\ \frac{1}{a} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $a > 0$. □

Theorem 4.3.11. *For each type of torsion-free* $\mathrm{Sol}^3 \times \mathbb{R}$ *crystallographic group* $\mathit{\Pi}$ *in* Section 3.4, *we list the Ricci signatures which can be realized on the infra-solvmanifold* $\mathit{\Pi} \backslash (\mathrm{Sol}^3 \times \mathbb{R})$.

Infra-$\text{Sol}^3 \times \mathbb{R}$	Holonomy	Realizable Ricci Signatures
`(0a)`	$\{e\}$	(0,0,0,-), (0,+,-,-), (+,+,-,-), (+,-,-,-)
`(0b)`, `(0c)`	$\mathbb{Z}_2$	(0,0,0,-), (0,+,-,-)
`(1a)`, `(1b)`	$\mathbb{Z}_2$	(0,0,0,-), (0,+,-,-)
`(1c)`	$\mathbb{Z}_2^2$	(0,0,0,-)
`(2aa)`	$\mathbb{Z}_2$	(0,0,0,-), (0,+,-,-)
`(2ad)`	$\mathbb{Z}_2^2$	(0,0,0,-), (0,+,-,-)
`(2ba)`	$\mathbb{Z}_2$	(0,0,0,-), (0,+,-,-)
`(2bb)`	$\mathbb{Z}_2$	(0,0,0,-), (0,+,-,-), (+,+,-,-), (+,-,-,-)
`(2bc)`	$\mathbb{Z}_2^2$	(0,0,0,-), (0,+,-,-)
`(3a)`	$\mathbb{Z}_2$	(0,0,0,-), (0,+,-,-), (+,+,-,-), (+,-,-,-)
(3ia), (3ib)	$\mathbb{Z}_2$	(0,0,0,-), (0,+,-,-), (+,+,-,-), (+,-,-,-)
(3ic), (3id)	$\mathbb{Z}_2^2$	(0,0,0,-), (0,+,-,-)
`(4a)`	$\mathbb{Z}_4$	(0,0,0,-)
`(5a)`, `(5b)`	$\mathbb{Z}_2^2$	(0,0,0,-)
(6aia), (6aib)	$\mathbb{Z}_2^2$	(0,0,0,-), (0,+,-,-)
(6aie)	$\mathbb{Z}_2^3$	(0,0,0,-), (0,+,-,-)
`(6ba)`	$\mathbb{Z}_2^2$	(0,0,0,-), (0,+,-,-)
(6bia), (6bib)	$\mathbb{Z}_2^2$	(0,0,0,-), (0,+,-,-)
(6bic)	$\mathbb{Z}_2^2$	(0,0,0,-), (0,+,-,-), (+,+,-,-), (+,-,-,-)
(6bid)	$\mathbb{Z}_2^3$	(0,0,0,-), (0,+,-,-)
(7id)	D_4	(0,0,0,-)

Table 4.4: Realizable Ricci signatures on infra-($\text{Sol}^3 \times \mathbb{R}$) manifolds

Proof. We can view $\text{Sol}^3 \times \mathbb{R}$ as $\text{Sol}^4_{m,n}$ with $\theta = -1$. Therefore, Table 4.2 applies.

The image of the holonomy in $\text{Aut}(\mathbb{R}^3)$ of any infra-solvmanifold of $\text{Sol}^3 \times \mathbb{R}$ diffeomorphic to an infra-solvmanifold of type `(1c)` must contain τ_3 and either τ_1 or τ_2, by Lemma 4.3.10. Then Lemma 4.3.2 forces the condition $b_{12} = b_{13} =$

$b_{23} = 0$ on $\mathcal{B}$. Now apply Table 4.2 with $\theta = -1$. Therefore, only the Ricci signature $(0, 0, 0, -)$ can be realized.

By Lemma 4.3.10, the image of the holonomy in $\mathrm{Aut}(\mathbb{R}^3)$ of any infra-solvmanifold diffeomorphic to an infra-solvmanifold of type `(4a)` or `(7`i`d)` contains an element of the form

$$\begin{bmatrix} 0 & -a & 0 \\ \frac{1}{a} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now Lemma 4.3.2 places the condition $b_{12} = b_{13} = b_{23} = 0$ on the matrix $\mathcal{B}$. Thus Table 4.2 shows that the eigenvalues of $\hat{r}\big|_{\mathbb{R}^3}$ all vanish in these cases. Therefore, only the Ricci signature $(0, 0, 0, -)$ can be realized in cases `(4a)` and `(7`i`d)`.

The image of the holonomy in $\mathrm{Aut}(\mathbb{R}^3)$ of any infra-solvmanifold of $\mathrm{Sol}^3 \times \mathbb{R}$ diffeomorphic to an infra-solvmanifold of type `(5a)` must contain $-\tau_3$, τ_1, and τ_2. For type `(5b)`, the image of the holonomy in $\mathrm{Aut}(\mathbb{R}^3)$ must contain $-\tau_3$, $-\tau_1$, and $-\tau_2$. In both cases, $b_{12} = b_{13} = b_{23} = 0$ by Lemma 4.3.2. Now apply Table 4.2 with $\theta = -1$ to see that only the Ricci signature $(0, 0, 0, -)$ can be realized.

The image of the holonomy in $\mathrm{Aut}(\mathbb{R}^3)$ of any infra-solvmanifold of $\mathrm{Sol}^3 \times \mathbb{R}$ diffeomorphic to an infra-solvmanifold of type `(2ad)` or `(2bc)`, must contain both τ_3, and $-\tau_3$. Then Lemma 4.3.2 forces the condition $b_{13} = b_{23} = 0$ on $\mathcal{B}$. We can make the signs of the eigenvalues of $\hat{r}\big|_{\mathbb{R}^3}$ either $(0, +, -)$ or $(0, 0, 0)$, by choice of b_{12} (Table 4.2 with $\theta = -1$).

The argument for cases `(0b)`, `(0c)`, `(1a)`, `(1b)`, `(2aa)`, and `(2ba)` is similar. In these cases, any infra-solvmanifold of $\mathrm{Sol}^3 \times \mathbb{R}$ diffeomorphic to an infra-solvmanifold of the above types must have a holonomy element projecting to τ_i or $-\tau_i$ in $\operatorname{Aut}(\mathbb{R}^3)$. Lemma 4.3.2 places conditions on $\mathcal{B}$, and Table 4.2 with $\theta = -1$ yields the desired result. In case `(2bb)`, no condition on $\mathcal{B}$ is forced, as the image of the holonomy in $\operatorname{Aut}(\mathbb{R}^3)$ is $-I$.

It remains to argue for cases `(3a)`, `(3`i`a)`, and `(3`i`b)`. We claim that any of the four Ricci signatures of $\mathrm{Sol}^3 \times \mathbb{R}$ are realizable. In fact, with basis

$$\mathcal{B}' = \begin{bmatrix} 4 & -3 & b_{13} & 0 \\ 0 & 5 & b_{13} & 0 \\ 0 & 0 & b_{33} & 0 \\ 0 & 0 & 0 & b_{44} \end{bmatrix}, \tag{4.4}$$

we compute that

$$\mathcal{B}'^{-1} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \mathcal{B}' = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 & 0 \\ \frac{4}{5} & -\frac{3}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \in \mathrm{O}(4, \mathbb{R}).$$

Thus, with any orthonormal basis of this form, the isometry group contains $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, in the notation of Theorems 3.4.22 and 3.4.23. With $\mathcal{B}'$ of this form, the determinant of $\hat{r}\big|_{\mathbb{R}^3}$ is

$$\frac{1}{800} b_{13}^2 \left(75 - b_{13}^2\right) b_{44}^6.$$

As $\hat{r}\big|_{\mathbb{R}^3}$ has vanishing trace, the eigenvalues of $\hat{r}\big|_{\mathbb{R}^3}$ have signs $(+, -, -)$ and $(+, +, -)$, when $75 - b_{13}^2 > 0$ ($b_{13} \neq 0$) and $75 - b_{13}^2 < 0$, respectively. Of course, when $b_{13}^2 = 75$, the eigenvalues of $\hat{r}\big|_{\mathbb{R}^3}$ have signs $(0, +, -)$. This proves that the four Ricci signatures can be realized in cases `(3a)` and `(3`i`a)`. For case

(3*i*b), with basis

$$\mathcal{B}' = \begin{bmatrix} 4 & 3 & -b_{13} & 0 \\ 0 & 5 & b_{13} & 0 \\ 0 & 0 & b_{33} & 0 \\ 0 & 0 & 0 & b_{44} \end{bmatrix}, \tag{4.5}$$

we compute that

$$\mathcal{B}'^{-1} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \mathcal{B}' = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \in \mathrm{O}(4, \mathbb{R}).$$

Thus, with orthonormal basis of this form, the isometry group contains $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tau$, in the notation of case (3*i*b) of Theorem 3.4.23. With $\mathcal{B}'$ of this form, the determinant of $\hat{r}\big|_{\mathbb{R}^3}$ is again

$$\frac{1}{800} b_{13}^2 \left(75 - b_{13}^2\right) b_{44}^6.$$

So, all four Ricci signatures realized in cases (3a) and (3*i*a) are also realized in case (3*i*b).

In cases (3*i*c), (3*i*d), (6a*i*a), (6a*i*b), (6a*i*e), (6ba), (6b*i*a), (6b*i*b), and (6b*i*d), $b_{13} = b_{23} = 0$, as the holonomy must contain either τ_3 or $-\tau_3$. Only two Ricci signatures are realizable depending on choice of b_{12}. In fact, in these cases, we can use the basis (4.4) with $b_{13} = 0$, to realize the Ricci signature $(0, +, -, -)$.

For case (6b*i*c), the holonomy generator $B\tau$ in Theorem 3.4.29 restricts to $-I$ in $\mathrm{GL}(3, \mathbb{R})$, and does impose any restriction on $\mathcal{B}$, and we argue just like in cases (3a) and (3*i*a). □

5 Topics for Further Research

There are remaining questions about the topology and geometry of 4-dimensional infra-solvmanifolds (Problems 5.1.1 and 5.2.1 below). Another direction for further research is to extend known partial results for Conjecture 1.2.1 to certain infra-solvmanifolds (Problems 5.3.3 and 5.3.2 below). Another interesting project is to characterize which solvable Lie groups satisfy the First Bieberbach Theorem (Problem 5.4.1 below).

5.1 Geometric Description of Infra-Solvmanifolds

Hillman provides a geometric classification of infra-solvmanifolds of $\mathrm{Sol}^3 \times \mathbb{R}$ in [18] as Seifert fiberings over 2-dimensional flat orbifolds. We have provided a complete classification of all crystallographic groups of $\mathrm{Sol}^3 \times \mathbb{R}$. The torsion-free crystallographic groups Π correspond to infra-solvmanifolds $\Pi \backslash G$. However, we do not give a correspondence between our classification and Hillman's classification.

Problem 5.1.1. *Find the correspondence between the torsion-free crystallographic groups of* $\mathrm{Sol}^3 \times \mathbb{R}$ *described here, and the classification Hillman provides in* [18]. *Also describe the infra-solvmanifolds of* Sol_1^4*, as Hillman does for* $\mathrm{Sol}^3 \times \mathbb{R}$ *in* [18].

5.2 $Pin^{\pm}$ Structures on Infra-Solvmanifolds

The group $Spin(n)$ is the double cover of $SO(n)$. The groups $Pin^{+}(n)$ and $Pin^{-}(n)$ are two double covers of $O(n)$. They are the same topologically, but have different group structures. The groups $Pin^{\pm}(n)$ allow a generalization of *Spin* structures to non-orientable manifolds; an orientation together with a Pin^{+} (or Pin^{-}) structure is equivalent to a *Spin* structure. An orientable Riemannian manifold M has a *Spin* structure precisely when the Stiefel Whitney class $\omega_2(M)$ vanishes. A Riemannian manifold M has a Pin^{+} structure if and only if $\omega_2(M)$ vanishes and has a Pin^{-} structure if and only if $\omega_2(M)+\omega_1^2(M)$ vanishes.

Hillman begins to explore which 4-dimensional infra-solvmanifolds admit Pin^{+} and Pin^{-} structures in [19]. This is related to a question of Ron Stern: is there an orientable aspherical surface bundle over the torus T^2 which does not admit a *Spin* structure? A 4-dimensional infra-solvmanifold may provide such an example. Putrycz and Szczepanski determined that only three orientable flat 4-manifolds are not *Spin* [31]. However, these are not surface bundles over the torus.

Problem 5.2.1. *Determine which 4-dimensional infra-solvmanifolds admit* Pin^{+} *and* Pin^{-} *structures.*

5.3 Bounding Problem for Infra-Solvmanifolds

Since all 4-dimensional infra-solvmanifolds bound, some n-dimensional infranilmanifolds bound, and all closed flat n-manifolds bound, it is reasonable to ask if n-dimensional infra-solvmanifolds bound.

Conjecture 5.3.1. *Let $M = \Pi\backslash G$ be an n-dimensional infra-solvmanifold. Then there exists a compact $(n+1)$-dimensional manifold W with $\partial W = M$.*

Hamrick and Royster used translational involutions to show that closed flat n-manifolds bound [15]. Marc Gordon introduced translational involutions in his proof that certain closed flat manifolds bound [11]. When G is solvable and has non-trivial center, translational involutions can also be used, as was done in here in the proof of Theorem 2.4.5.

Problem 5.3.2. *Let $M = \Pi\backslash G$ be an n-dimensional infra-solvmanifold. Assume that G has non-trivial center. Is there a compact $(n+1)$-dimensional manifold W with $\partial W = M$?*

On the other hand, Conjecture 5.3.1 may be false. To give a counterexample to Conjecture 5.3.1 or (Conjecture 1.2.1), we would need an infra-solvmanifold (or infra-nilmanifold) with a non vanishing Stiefel-Whitney number. In [42, 20] there are examples of flat manifolds with non-trivial Stiefel Whitney classes and this should be useful to approach:

Problem 5.3.3. *Give an example of an infra-solvmanifold or infra-nilmanifold which does not bound.*

5.4 Bieberbach's First Theorem on Solvable Lie Groups

It is not well understood when a solvable Lie group satisfies Bieberbach's First Theorem (Theorem 1.1.4). Even a 4-dimensional solvable geometry, Sol_0^4, does not satisfy it. See [8] for more examples of solvable Lie groups which do not satisfy Bieberbach's First Theorem.

Problem 5.4.1. *Characterize the solvable Lie groups for which Bieberbach's first theorem holds.*

Good progress has been made by Dekimpe, Lee, and Raymond, who give a sufficient, but not necessary, condition for a solvable Lie group to satisfy Bieberbach's first theorem [8]. In [3], Buser gives a geometric proof of Bieberbach's First Theorem for $\mathbb{R}^n$, inspired by Gromov's work on almost flat manifolds. Essentially, the idea is to show that the isometries in a crystallographic group of $\mathbb{R}^n$ with very small rotation part are in fact pure translations. Examining Buser's argument when $\mathbb{R}^n$ is replaced with a solvable Lie group should provide a simple characterization of solvable Lie groups satisfying Bieberbach's First Theorem.

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