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AFFINE GROUP ACTIONS ON EUCLIDEAN SPACE

A DISSERTATION APPROVED FOR THE

DEPARTMENT OF MATHEMATICS

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INTRODUCTION

Up to affine conjugacy, we describe rank two affine groups which act properly discontinuously on $\mathbb{R}^n$ (primarily $n = 2, 3$) in terms of "coordinates" of generators. In dimension two this coincides with all affine structures on the torus, $S^1 \times S^1$. Any such two-dimensional group G acts as "translation" along two transverse families of curves that each foliate the plane. A parameter space of normalized (two-dimensional) groups is defined and can be identified with $\mathbb{R}^2$ (where different points need not correspond to distinct groups), and a continuous homomorphism, $Res : \mathbb{R}^2 \to \mathbb{R}$ characterizes those groups that are isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and act properly discontinuously. In our main result we find necessary and sufficient conditions for two properly discontinuous rank two affine groups to be affinely conjugate. Consequences include (i) Each conjugacy class in the parameter space can be identified with the left cosets of H in $GL(2, \mathbb{Z})$, where H is the subgroup of upper triangular matrices; (ii) Any neighborhood of a point of $Ker(Res)$ contains a representative from each conjugacy class of properly discontinuous $G \cong \mathbb{Z} \oplus \mathbb{Z}$ (equivalently, for every affine structure of the torus there corresponds a point that lies in this open set); (iii) There are an uncountable number of conjugacy classes of properly discontinuous rank two affine groups (in all dimensions $n \geq 2$). If isometries are considered, there are a finite number of (affine) conjugacy classes (this is one of the Bieberbach theorems). Any two properly discontinuous affine groups $G \cong \mathbb{Z} \oplus \mathbb{Z}$ are conjugate in the larger group of quadratic polynomial maps: this is shown by first conjugating G by such a map to a translation group by "straightening out" the curves mentioned above, and then applying the fact that any two translation groups (generated by independent translations) are affinely conjugate. See also pp. 13-14 in [F-G].

In dimension three, if $G \cong \mathbb{Z} \oplus \mathbb{Z}$ acts properly discontinuously, then $Det(A) = Trace(A) - 1$ for each generator $\alpha = [A, a] \in G$. If in addition the matrix part of α has characteristic polynomial $(x-1)^3$, then G acts as a combination of "translation" along curves and "shearing" along surfaces. A parameter space of such α (normalized) is defined, and two elements of this space produce a properly discontinuous $\mathbb{Z} \oplus \mathbb{Z}$ action exactly when one of the elements does not lie on a curve kept invariant by the other (here as in dimension two, the parameter space is essentially the same as the space upon which the groups act). This generalizes the Residue.

Tartar and A. Yakir (1997) characterized affine conjugacy of single abstract affine maps using a formula that resembles the Rank-Nullity theorem for linear maps. See p. 156 in [T-Y].

In 1953 N. Kuiper showed that any 2-dimensional properly discontinuous affine group contains a finite-index subgroup that sits inside one of three particular affine groups. See p. 317 in [A].

In summary, we answer the following questions:

(a) When do $\alpha, \beta \in Affine(\mathbb{R}^2)$ generate a rank two properly discontinuous group?

(b) When are two such groups affinely conjugate?

(c) How many conjugacy classes are there?

(d) How are these conjugacy classes distributed in the parameter space of groups?

(e) Can the characterization in (a) be generalized to any dimension n?

1. **Notation**

(i) e_i: The ith unit vector (point) of $\mathbb{R}^n$

(ii) $Span(b)$: The set $\{tb : t \in \mathbb{R}\} \subseteq \mathbb{R}^n$

(iii) $span(b)^{\perp}$: $\{x \in \mathbb{R}^n : x \centerdot b = (0, 0, ..., 0)\}$

(iv) τ_i: The ith coordinate function, that is, $\tau_i(x_1, x_2, ..., x_n) = x_i$

(v) $\overline{X}$: The topological closure of the set X

(vi) $Homeo(X)$: The group of homeomorphisms of the topological space X

(vii) I: The $n \times n$ identity matrix

(viii) $[A, a]$: The affine map with matrix part $A \in GL(n, \mathbb{R})$ and translation part $a \in \mathbb{R}^n$, that is, $x \mapsto A(x) + a$

(iv) $Affine(\mathbb{R}^n)$: The group of all affine maps of $\mathbb{R}^n$

(x) α_i: The ith normalized affine map

(xi) $\beta = \beta_{(b,r)}$: The 2-dimensional normalized affine map which commutes with α_1

(xii) $Res(\beta)$: The real number $b - \frac{1}{2}r(r-1)$

(xiii) $\wp_{(b,r)}$: The base point parabola of $\beta = \beta_{(b,r)}$ (dimension 2)

(xiv) Ω_{α_1}: The parameter space $\{[b, r] : Res(\beta_{(b,r)}) \neq 0\}$

(xv) $\mathcal{C}_\beta$: The base point curve of β (dimension 3)

2. **Preliminaries**

When a group acts freely and properly discontinuously, the corresponding quotient space will be a manifold. A properly discontinuous action also implies that the group in question is discrete when given the compact-open topology (but not conversely). Definitions, some examples, and a lemma follow.

Definition 2.1. A group $G \leq Homeo(X)$ acts *properly discontinuously* on X if for each compact $E \subseteq X$ the set $\{g \in G : g(E) \cap E \neq \emptyset\}$ is finite.

When $X = \mathbb{R}^n$ this is equivalent to $g_i(E) \to \infty$ for all sequences $\{g_i\}$ in G of distinct elements and all compact sets E.

Example: Let $G_1 =< (x, y) \mapsto (x + 1, y) >$, $G_2 =< (x, y) \mapsto (x + 1, y), (x, y) \mapsto (x, y + 1) >$, $G_3 =< (x, y) \mapsto (2x, 2y) >$, $G_4 =< (x, y) \mapsto (2x, \frac{1}{2}y) >$ be subgroups of $Homeo(\mathbb{R}^2)$, and $G_5 =< (x, y, z) \mapsto (2x, 2y, z + 1) >$ be a subgroup of $Homeo(\mathbb{R}^3)$. These are all affine groups. G_1 and G_2 iv

act properly discontinuously on $\mathbb{R}^2$, while G_3 does not. G_3 does act properly discontinuously on $\mathbb{R}^2 - \{(0, 0)\}$, however. G_4 does not act properly discontinuously on $\mathbb{R}^2 - \{(0, 0)\}$ (use E=circle centered at $(0, 0)$). Since planes orthogonal to the z-axis are translated by the generator of G_5, this group also acts properly discontinuously (on $\mathbb{R}^3$).

Definition 2.2. A group $G \leq Homeo(X)$ acts *freely* if no nontrivial element of G fixes a point of X, that is, $g(x) = x$ for some $x \in X$ implies $g = identity$.

G_1 and G_2 in the previous examples act freely on $\mathbb{R}^2$, but G_3 does not. The group $< (x, y) \mapsto (2x, 2y), x \mapsto (x + 1, y) >$ does not act freely either, since $(0, 0)$ is fixed by the first generator (but not by each element of the group, which was the case with G_3).

If $G \leq Homeo(\mathbb{R}^n)$ acts freely, then G is torsion free (When $g \in G$ has order $m \in \mathbb{Z}^+$, g fixes $(x + g(x) + ... + g^{m-1}(x))$ for any $x \in X$ - see [C]), but not conversely (G_3 above is torsion free).

A group (of homeomorphisms) containing an infinite order element that has a fixed point cannot act properly discontinuously.

In particular, any torsion free group that acts properly discontinuously must act freely as well. The converse is false. For example, G_4 acts freely on $\mathbb{R}^2 - \{(0,0)\}$.

When an infinite cyclic affine group acts on all of $\mathbb{R}^n$, acting freely is equivalent to acting properly discontinuously.

A finite group of homeomorphisms acts properly discontinuously "by default".

Computations can be divided in half with the following lemma:

Lemma 2.3. *If $\{g_i\} \subseteq G \leq Homeo(\mathbb{R}^n)$ are distinct and E is a compact set, then $g_i(E) \to \infty$ implies $g_i^{-1}(E) \to \infty$.*

Proof.

Assume there is a compact set D such that $g_i^{-1}(D) \cap B(N,0) \neq \emptyset$ for all i, where $B(N,0)$ denotes the Euclidean (metric) ball centered at the origin having radius N. For each i let $x_i \in g_i^{-1}(D) \cap B(N,0)$. Since $\overline{B(N,0)}$ is compact, there is a subsequence (call the same) $\{x_i\} \to d \in \overline{B(N,0)}$. The set $T = \{x_i : i \in \mathbb{Z}^+\} \cup \{d\}$ is compact, so $g_i(T) \to \infty$ by hypothesis. But for each i, $g_i(x_i) \in g_i(T) \cap D$, which is a contradiction.

$\square$

Definition 2.4. Let X be compact. A group $G \leq Homeo(X)$ is *discrete* if there is no sequence $\{g_i\}$ in G such that g_i converges uniformly to some

$f \in Homeo(X)$. This is equivalent to the set G being discrete as a subspace of $Homeo(X)$ with the compact-open topology.

Remark 2.5. If a group acts properly discontinuously, it must be discrete. The converse is false. For example, the cyclic group G generated by $g = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is a discrete group. Although the sequence $g^{-1}, g^{-2}, g^{-3}, ...$ converges to $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, this limit map is not a homeomorphism. But $(0, 0)$ is fixed by every element of this infinite group, preventing a properly discontinuous action.

3. **General Facts About Affine Groups**

Let $\beta = [B, b]$ denote the affine map having matrix part $B \in GL(n, \mathbb{R})$ and translation part $b \in \mathbb{R}^n$, i.e., $\beta(x) = B(x) + b$ for $x \in \mathbb{R}^n$. β is bijective (a homeomorphism), and maps straight lines (planes, three-dimensional hyperplanes,...) onto straight lines (planes, three-dimensional hyperplanes,...), but does not map vector subspaces (of $\mathbb{R}^n$) onto vector subspaces unless its translation part is trivial. An affine map $\beta = [B, b]$ is an isometry of the Euclidean metric if and only if B is an orthogonal matrix.

The group $Affine(\mathbb{R}^n)$ is the semi-direct product $\mathbb{R}^n \rtimes GL(n, \mathbb{R})$ in a natural way. The multiplication (composition) is $(a, A)\cdot(b, B) = (A(b)+a, AB)$ and the map $\psi : GL(n, \mathbb{R}) \to Aut(\mathbb{R}^n)$ defined by $\psi(A)(x) = A(x)$ is a homomorphism, where $Aut(\mathbb{R}^n)$ is the group of automorphisms of $\mathbb{R}^n$. Here, the orders of the matrix part and the translation part have been reversed (notation).

Any two affine maps or affine groups will be considered the "same" when they are affinely conjugate, since the (geometric) properties of interest in this paper are preserved under conjugation. In particular:

(1) $g \in Affine(\mathbb{R}^n)$ has infinite order if and only if any conjugate of g has infinite order.

(2) $g \in Affine(\mathbb{R}^n)$ is a translation if and only if any (affine) conjugate of g is a translation. In particular, a subgroup G of $Affine(\mathbb{R}^n)$ is a pure translation group (consists of translations only) if and only if any conjugate of G is a pure translation group, and G contains no translations (besides the identity) if and only if any conjugate of G contains no translations.

(3) The fixed point set of fgf^{-1} is the set $\{f(x) : x \text{ is fixed by } g\}$. In particular, g has no fixed points if and only if any conjugate of g has no fixed points.

(4) G acts freely if and only if any conjugate of G acts freely.

(5) G is discrete if and only if any conjugate of G is discrete.

(6) G acts properly discontinuously if and only if any conjugate of G acts properly discontinuously.

(7) Conjugate affine groups are isomorphic (the conjugating map is an isomporphism), but isomorphic groups need not be conjugate. For instance,

$G =< [I, e_1], [I, e_2] >\cong \mathbb{Z} \oplus \mathbb{Z} \cong< [I, e_1], [I, \sqrt{2}e_1] >$, but they are not affinely conjugate (they're not conjugate by any homeomorphism either), since the first group acts properly discontinuously and the second does not.

The following facts are the result of a computation:

$(\alpha = [A, a], \beta = [B, b] \in Affine(\mathbb{R}^n))$

(i) For all positive integers j

$$\beta^j = [B^j, \sum_{i=0}^{j-1} B^i(b)]$$

(ii) $\alpha = \beta$ if and only if $A = B$ and $a = b$.

(iii) $\alpha\beta = \beta\alpha$ if and only if $AB = BA$ and $A(b) + a = B(a) + b$.

(iv) If $\alpha = [I, e_i]$, then (ii) above becomes $B(e_i) = e_i$.

(v) The affine group G contains a non-trivial translation if and only if there are $\alpha = [A, a], \beta = [B, b] \in G$ so that $A^k B^j = I$ and the translation part of $\alpha^k \beta^j$ is not the zero vector.

Below is a characterization of a properly discontinuous infinite cyclic group that is an immediate consequence of a normalized affine form given in a paper by H. Abels. See [A].

Lemma 3.1. *If $\beta \in Affine(\mathbb{R}^n)$ has infinite order, then the group $G =< \beta >$ acts properly discontinuously (on $\mathbb{R}^n$) if and only if β is fixed point-free.*

Proof.

Certainly infinite order β must act freely if the group it generates acts properly discontinuously. The converse is not so clear. β has no fixed points if and only if the equation $B(x)+b = x$ has no solution in $\mathbb{R}^n$, which is equivalent to $b \notin Image(I - B)$. This implies that B has a 1-eigenvector.

β is affine conjugate to some $[B_i, e_n]$, $B_i = \begin{bmatrix} C_i & 0 \\ 0 & J_{n-i} \end{bmatrix}$, $1 \leq i \leq n \quad (*)$

where $C_i \in GL(i-1, \mathbb{R})$ and J_{n-i} is the $(n-i+1) \times (n-i+1)$ matrix having 1's along the diagonal and along the super diagonal, with 0's everywhere else. $B_1 = J_{n-1}$. For any $x \in \mathbb{R}^n$, the nth coordinate of $\beta^k(x)$ grows arbitrarily large as $k \to +/-\infty$. Since the nth coordinates of any compact set E of $\mathbb{R}^n$ are bounded, $< \beta >$ acts properly discontinuously.

$\square$

The normalized form $(*)$ above is the cornerstone for the results in this paper. Therefore, the two and three dimensional versions are emphasized below:

Normalized Fixed Point Free Affine Maps

(i) Any two dimensional fixed point free affine map is affinely conjugate to one of $\alpha_1 = [A_1, e_2]$ or $\alpha_2 = [A_2, e_2]$,

where $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_0 & 0 \\ 0 & 1 \end{bmatrix}$, $a_0 \neq 0$.

(ii) Any three dimensional fixed point free affine map is affinely conjugate to one of $\alpha_1 = [A_1, e_3]$, $\alpha_2 = [A_2, e_3]$ or $\alpha_3 = [A_3, e_3]$,

Where $A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} a_0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $a_0 \neq 0$,

and $A_3 = \begin{bmatrix} A_0 & 0 \\ 0 & 1 \end{bmatrix}$, $A_0 \in GL(n, \mathbb{R})$.

In general, if $span(b)$ is transverse to a family of co-dimension one B-invariant subspaces, then the group generated by $\beta = [B, b]$ acts properly discontinuously. A more detailed proof of the above lemma in dimension two is given later as well.

Given the translation $[I, a]$, $a \neq 0$, the conjugate $C[I, a]C^{-1} = [I, e_i]$ for any matrix C such that $C(a) = e_i$. For this reason $[I, a] = [I, e_i]$ may be assumed.

Lemma 3.2. *Let $\alpha = [I, e_1]$, $\beta = [B, b] \in Affine(\mathbb{R}^n)$ with β fixed point-free and $\beta\alpha = \alpha\beta$. Then the group $< \alpha, \beta >$ is $GL(n, \mathbb{R})$ - conjugate to $< \alpha, \beta' >$, where $\beta' = [B', b']$ and there is a foliation by B' - invariant codimension one hyperplanes that are either orthogonal to $span(e_1)$ or $span(e_n)$, none containing span(b').*

Proof.

Conjugating $G =< \alpha, \beta >$ by a matrix that fixes each point in $span(e_1)$ will preserve the form of α. The form $(*)$ given in the proof of Lemma 3.1 implies there is a family of B-invariant parallel codimension one hyperplanes, none of which contain $span(b)$. If $span(e_1)$ is not contained in any of these hyperplanes, we can shear B (i.e., conjugate G) twice along $span(e_1)$ so that the resulting invariant hyperplanes are orthogonal to $span(e_1)$.

If some hyperpane from this family contains $span(e_1)$, then by rotating around $span(e_1)$, the resulting invariant hyperplanes will be orthogonal to $span(e_n)$.

□

Lemma 3.3 (B-invariant codimension one hyperplanes orthogonal to $span(e_n)$)**.** *Let $\beta = [B, b] \in Affine(\mathbb{R}^n)$ commute with $\alpha = [I, e_1]$, and suppose that each codimension one hyperplane that is orthogonal to the $span(e_n)$ is B-invariant,* *and that none contain span(b). Then* $G =< \alpha, \beta >\cong \mathbb{Z} \oplus \mathbb{Z}$*, acts freely and properly discontinuously on* $\mathbb{R}^n$*.*

Proof.

The matrix part of β will have the form $B = \begin{bmatrix} 1 & B_0 \\ 0 & 1 \end{bmatrix}$, with

$B_0 \in GL(n-1, \mathbb{R})$, where the two 1's are in the $(1,1)$ and (n,n) positions, and where the 0 in the corner represents $0's$ in the $(s,1)$ positions, $s = 2, 3, ..., n$ and the (n,t) positions, $t = 1, 2, ..., n-1$.

The translation part of β will have the form $b = (b_1, b_2, ..., b_n)$ where $b_n \neq 0$. Let $\{g_i\}$ be a sequence of distinct elements in $G =< \alpha, \beta >$. Then the powers of α or β are increasing (positive or negative). Assume the latter (the former is similar). $\alpha^k \beta^j (x_1, x_2, ..., x_n) = (*_1, *_2, ..., *_{(n-1)}, x_n + jb_n)$, where the first $(n-1)$ coordinates are complicated and don't need to be examined. Since $b_n \neq 0$, and the x_n coordinates of any compact set E are bounded, $g_i(E) \rightarrow \infty$ as $j \rightarrow \infty$. By Lemma 2.3, $g_i^{-1}(E) \rightarrow \infty$ as $j \rightarrow \infty$.

□

Lemma 3.4 (B-invariant codimension one hyperplanes orthogonal to the $span(e_1)$)**.** *Let* $\beta = [B, b] \in Affine(\mathbb{R}^n)$ *commute with* $\alpha = [I, e_1]$*, and suppose that each codimension one hyperplane that is orthogonal to span*(e_1) *is B-invariant, and that none contain span(b). If* $proj(b)|_{span(e_1)^\perp} \notin Image(I - B)$*, then* $G \cong \mathbb{Z} \oplus \mathbb{Z}$ *and acts freely and properly discontinuously on* $\mathbb{R}^n$*.*

$(span(e_1)^\perp = \{x \in \mathbb{R}^n : x \cdot e_1 = 0\})$

Proof.

Now $B = \begin{bmatrix} 1 & 0 \\ 0 & B_0 \end{bmatrix}$, $B_0 \in GL(n-1, \mathbb{R})$ and $b = (b_1, b_2, ..., b_n)$, $b_1 \neq 0$.

By hypothesis $proj(b)|_{span(e_1)^\perp} = (0, b_2, b_3, ..., b_n) \notin Image(I - B)$. This is equivalent to $(b_2, b_3, ..., b_n) \notin Image(I - B_0)$.

G can be conjugated further (leaving α intact) so that B_0 has the form given in Lemma 3.1 and with $b = (b_1, 0, 0, ..., 0, 1)$. This implies that $\beta^j(E) \to \infty$ as $j \to +/-\infty$ (independent of the α action). □

Proposition 3.5. *Let fixed point free $\beta = [B, b]$ commute with $\alpha = [I, e_1]$, where*

(1) B keeps invariant each codimension one hyperplane orthogonal to span(e_1), none containing span(b), or

(2) B keeps invariant each codimension one hyperplane orthogonal to span(e_n), none containing span(b). The following are equivalent:

(i) $G =< \alpha, \beta >\cong \mathbb{Z} \oplus \mathbb{Z}$ and acts properly discontinuously on $\mathbb{R}^n$

(ii) $proj(b)|_{span(e_1)^\perp} \notin Image(I - B)$

Proof.

To show that (i) implies (ii), assume that $G \cong \mathbb{Z} \oplus \mathbb{Z}$ and acts properly discontinuously on $\mathbb{R}^n$.

If (2) holds, then $proj(b)|_{span(e_1)^\perp} \notin Image(I - B)$. See the form of β in the proof of Lemma 3.3.

Suppose that (1) holds. If $proj(b)|_{span(e_1)^\perp} \in Image(I-B)$, then $(b_2, b_3, ..., b_n) \in Image(I - B_0)$ - see the form of β in the proof of Lemma 3.4. This implies that there is a point $(x_2, x_3, ..., x_n)$ fixed by $[B_0, (b_2, b_3, ..., b_n)]$, and so the line $\{(x, x_2, x_3, ..., x_n) : x \in \mathbb{R}\}$ is G invariant. Thus, either G is cyclic or G does not act properly discontinuously, a contradiction.

Conversely, assume that $proj(b)|_{span(e_1)^\perp} \notin Image(I - B)$. By Lemmas 3.3 or 3.4, $G \cong \mathbb{Z} \oplus \mathbb{Z}$ and acts properly discontinuously on $\mathbb{R}^n$.

$\square$

A proof will be given later in dimensions 2 in which the normalization condition is unnecessary.

4. A Look at Dimension One

A non-vertical/horizontal line in the plane viewed as the graph of a function $f : \mathbb{R} \to \mathbb{R}$ can be considered a 1-dimensional affine map, where the linear part is a 1-by-1 matrix (a real number). f will have no fixed points exactly when this line does not intersect the line y=x, i.e., when $f \neq$ identity and graph(f) has slope 1. The cyclic group $< f >$ will act properly discontinuously on $\mathbb{R}$ in this case (and the quotient space $\mathbb{R}/ < f >$ is homeomorphic to S^1).

Lemma 4.1. *If $f : \mathbb{R} \to \mathbb{R}$ is a non-constant linear map, then the cyclic group $G =< f >$ acts properly discontinuously on $\mathbb{R}$ if and only if $f(x) = x + b$ for some $b \in \mathbb{R}$*

Remark 4.2. When $b = 0$, the above group is $G =< identity >$, which acts properly discontinuously too.

5. **Actions on $\mathbb{R}^2$**

5.1. **Cyclic Groups ($\mathbb{Z}$-Actions).**

This was discussed in the general (n-dimensional) section. A more detailed proof of the cyclic case is given below.

$\beta = [B, b] \in Affine(\mathbb{R}^2)$ will have no fixed points when the equation $B(z) + b = z$ has no solution in $\mathbb{R}^2$, which is equivalent to $b \notin Image(I - B)$. This implies B has a 1-eigenvector. The Jordan form of B is one of

$$B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \text{ or } B_2 = \begin{bmatrix} b_0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $\beta_1 = [B_1, b]$ and $\beta_2 = [B_2, b]$, where $b = \binom{b_1}{b_2}$.

Then $\pi_i \beta_i \pi_i^{-1} = [B_i, e_2]$ where $\pi_i = [C, a_i]$, $C = \begin{bmatrix} \frac{1}{b_2} & 0 \\ 0 & \frac{1}{b_2} \end{bmatrix}$,

$a_1 = \binom{0}{\frac{1}{b_2}}$, and $a_2 = \binom{\frac{b_1}{b_2(b_0-1)}}{\frac{1}{b_2}}$ showing that any fixed point free $\beta \in Affine(\mathbb{R}^2)$ is affine conjugate to one of the $[B_i, e_2]$, $i = 1, 2$.

$$\text{Now } \beta_1^j = \left[\begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix}, \binom{\frac{j(j-1)}{2}}{j} \right] \text{ and } \beta_2^j = \left[\begin{bmatrix} b_0^j & 0 \\ 0 & 1 \end{bmatrix}, \binom{0}{j} \right].$$

If E is compact, the y-coordinate of $\beta_i^j(E)$ grows arbitrarily large as $j \to +\infty$ or $-\infty$. This proves:

Lemma 5.1. *If $\beta \in Affine(\mathbb{R}^2)$ has infinite order, then the group $G = <\beta>$ acts properly discontinuously on $\mathbb{R}^2$ if and only if β is fixed point free.*

With two generators, an affine group that acts freely need not act properly discontinuously (on $\mathbb{R}^2$). For example, the group

$G =< z \mapsto z + e_1, z \mapsto z + \sqrt{2}e_1 >$ acts freely, yet does not act properly discontinuously on $\mathbb{R}^2$ since it is not discrete.

5.2. $\mathbb{Z} \oplus \mathbb{Z}$ **Actions-Translation Extensions.**

An affine map $[B, b]$ is a translation when $B = I$ (in any dimension). A translation group G is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$ and acts properly discontinuously if and only if the translation parts of the two generators are linearly independent (otherwise, the group they generate is cyclic when the translation parts are rational multiples of each other, or isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and does not act properly discontinuously if one is an irrational multiple of the other). If only one map in a generating set is a translation, the other must be analyzed to determine whether the group they generate acts properly discontinuously. A subgroup G of $Affine(\mathbb{R}^n)$ will be called a *translation extension* if there is a generating set that contains a translation.

Lemma 5.2. *Let $\alpha = [I, e_1]$ and fixed point free $\beta = [B, b]$ be elements of $Affine(\mathbb{R}^2)$, and assume that $\alpha\beta = \beta\alpha$. The following are equivalent:*

(i) $G \cong \mathbb{Z} \oplus \mathbb{Z}$ and acts properly discontinuously (and therefore freely) on $\mathbb{R}^2$.

(ii) $B = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$ and $b = \binom{b_1}{b_2}$ with $b_2 \neq 0$.

Proof.

Since β commutes with α, $B(e_1) = e_1$ and so $B = \begin{bmatrix} 1 & r \\ 0 & s \end{bmatrix}$.

Thus the matrix part of $\alpha^k\beta^j$ is $B^j = \begin{bmatrix} 1 & r(1+s+...+s^{j-1}) \\ 0 & s^j \end{bmatrix}$,

and the translation part is $\binom{k+jb_1+b_2r(1+(1+s)+...+(1+s+s^2+...+s^{j-2}))}{b_2(1+s+...+s^{j-1})}$.

To show (i) implies (ii), assume that $s \neq 1$ or $b_2 = 0$. If $b_2 = 0$ then $span(e_1)$ is G invariant, prohibiting a properly discontinuous $\mathbb{Z}\oplus\mathbb{Z}$ action. Assume now that $s \neq 1$. The condition $b \notin Image(I - B)$ is equivalent to $\frac{b_1}{b_2} \neq \frac{r}{s-1}$. $(*)$

The matrix part of $\alpha^k\beta^j$ is $B^j = \begin{bmatrix} 1 & r(\frac{1-s^j}{1-s}) \\ 0 & s^j \end{bmatrix}$, which has bounded entries (assume that $|s| < 1$ by choosing β^{-1} if necessary). The translation part is $\binom{k+jb_1+b_2r(\frac{j-1}{1-s}-\frac{s(1-s^{j-1})}{(1-s)^2})}{b_2(\frac{1-s^j}{1-s})}$. Considering only parts that have growth potential, this may be written essentially as $\binom{k+jb_1+\frac{jb_2r}{1-s}}{constant} = \binom{k+j(b_1+\frac{b_2r}{1-s})}{constant}$. By $(*)$, $j_i, k_i \in \mathbb{Z}$ can be chosen so that the first coordinate converges.

To show that (ii) implies (i), assume that $b_2 \neq 0$ and $s = 1$. Then the second coordinates of images grow arbitrarily large when a given sequence of elements of G have unbounded powers of β. Otherwise the action is essentially an α action since $\beta^{j_1}(E) \cup \beta^{j_2}(E) \cup ... \cup \beta^{j_t}(E)$ is compact when E is. $G \cong \mathbb{Z}\oplus\mathbb{Z}$ is clear. $\square$

5.3. **General $\mathbb{Z}\oplus\mathbb{Z}$ Actions.**

In 1953, N. Kuiper characterized properly discontinuous subgroups of $Affine(\mathbb{R}^2)$ as follows:

Theorem 5.3. *(Kuiper)*

Let $H = \{[A_t, \binom{0}{t}] : t \in \mathbb{R}\},\ A_t = \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix},$

$T = \{[I, \binom{s}{t}] : s, t \in \mathbb{R}\},$

and $P = \{[B_s, \binom{t}{s}] : s, t \in \mathbb{R}\},\ B_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}.$

Every discrete subgroup of $H, T,$ *or* P *acts properly discontinuously on* $\mathbb{R}^2$*, and any properly discontinuous subgroup* G *of* $Affine(\mathbb{R}^2)$ *contains a finite-index subgroup which is a subgroup of* $H, T,$ *or* P*.*

See [A].

The above theorem includes the possibility that G is non-abelian or contains finite-order elements. In this paper, the question of properly discontinuous actions will be restricted to groups $G \cong \mathbb{Z} \oplus \mathbb{Z}$. Since any element of $Affine(\mathbb{R}^2)$ can be considered a point in $\mathbb{R}^6$, such a map can be examined in terms of its "coordinates". It will turn out that the generators of interest will correspond to points of the smaller Euclidean space $\mathbb{R}^2$.

Suppose that $G \cong \mathbb{Z} \oplus \mathbb{Z}$ does not consist solely of translations. After normalizing a non-translation generator (by conjugation), the coordinates of the second generator will be simplified. Conditions for a properly discontinuous action are then determined. The effect of group elements on points of the plane can be described as "translation along two transverse families of curves that each foliate $\mathbb{R}^2$". (Affine) conjugacy of two such groups is closely related to the features that insure that each acts properly discontinuously.

Recall that $\alpha_1 = [A_1, e_2]$ and $\alpha_2 = [A_2, e_2]$,

where $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $A_2 = \begin{bmatrix} a_0 & 0 \\ 0 & 1 \end{bmatrix}$, $a_0 \neq 0$.

α_2 is a one-parameter family of maps that includes the conjugacy class of translations (which is the set of translations). Fixed point free $\beta = [B, b]$ is conjugate to α_1 if and only if $B(b) \neq b$. The group P in the above theorem by Kuiper is the set of all β that commute with α_1.

Lemma 5.4. *Suppose α_2 and β commute. If $G =< \alpha_2, \beta >\cong \mathbb{Z} \oplus \mathbb{Z}$ and acts properly discontinuously, then α_2 is a translation. In particular, any non-translation of a properly discontinuous rank two group is conjugate to α_1.*

Proof.

Let $\beta = [B, \binom{b_1}{b_2}]$ where $B = \begin{bmatrix} r_1 & r_2 \\ s_1 & s_2 \end{bmatrix}$.

$\alpha_2\beta = \beta\alpha_2$ implies $a_0 r_2 = r_2$, $a_0 s_1 = s_1$, $s_2 = 1$, and $a_0 b_1 = r_2 + b_1$. If $a_0 \neq 1$ then $s_1 = 0 = r_2$ and $b_1 = \frac{r_2}{a_0 - 1} = 0$. G cannot be isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and act properly discontinuously because each element keeps invariant the line $span(e_2)$.

□

5.3.1. *The Residue.*

Let $\beta = [B, b] \in Affine(\mathbb{R}^2)$ where $b = \binom{b_1}{b_2}$. The following fact will be used often:

If β commutes with α_1 then $A_1 B = B A_1$ and $A_1(b) + e_2 = B(e_2) + b$, implying that $B = \begin{bmatrix} 1 & b_2 \\ 0 & 1 \end{bmatrix}$. This also shows that β is fixed point free when $\beta \neq identity$.

Corollary 5.5. *If $G \cong \mathbb{Z} \oplus \mathbb{Z}$ is a properly discontinuous affine group, then $A \in SL(2, \mathbb{R})$ and $Trace(A) = 2$ for all $\alpha = [A, a] \in G$.*

Proof.

Follows from the previous computation and Lemma 5.4 since the determinant and trace are conjugation invariant (the result is clear in a translation group).

□

$\beta = \left[\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}, \binom{b}{r} \right] = \beta_{(b,r)}$ will be written when β commutes with α_1.

Since each point $(b, r) \in \mathbb{R}^2$ can be identified with the group $< \alpha_1, \beta_{(b,r)} >$, it is natural to ask which correspond to properly discontinuous $\mathbb{Z} \oplus \mathbb{Z}$ actions.

Example: Let $\beta = [B, \binom{2}{2}]$. Then $G =< \alpha_1, \beta >\cong \mathbb{Z} \oplus \mathbb{Z}$ acts freely and properly discontinuously on $\mathbb{R}^2$: $g_i = \alpha_1^{k_i} \beta^{j_i} = \left[\begin{bmatrix} 1 & (2j_i + k_i) \\ 0 & 1 \end{bmatrix}, \binom{\frac{1}{2}(2j_i+k_i)^2 - \frac{1}{2}k_i}{2j_i+k_i} \right]$. If the sequence $\{2j_i + k_i\}$ is unbounded, then the second coordinate of images under g_i grow arbitrarily large as $i \to \infty$ (g_i are distinct). Otherwise, the first coordinate will grow arbitrarily large as $i \to \infty$.

Example: Let $\beta = [B, \binom{1}{2}]$. Then group $G =< \alpha_1, \beta >=< \alpha_1 >\cong \mathbb{Z}$ since $\alpha^{-2} = \beta^{-1}$.

Example: Let $\beta = [B, \binom{1-\frac{1}{\sqrt{2}}}{\sqrt{2}}]$. Then $G =< \alpha_1, \beta > \cong \mathbb{Z} \oplus \mathbb{Z}$. Choose $j_i, k_i \in \mathbb{Z}$ so that the sequence $\{j_i\sqrt{2} + k_i\}$ converges (to 0, say). Then the sequence of distinct elements $g_i = \alpha_1^{k_i}\beta^{j_i}$ converges to the identity map, implying that G is not discrete and therefore does not act properly discontinuously.

Remark 5.6. If $\beta = [B, \binom{\frac{1}{2}r(r-1)}{r}]$ where $r = \frac{p}{q}$ is rational then the group $G =< \alpha_1, \beta >$ is cyclic with generator $\left[\begin{bmatrix} 1 & \frac{1}{q} \\ 0 & 1 \end{bmatrix}, \binom{\frac{1}{2q}(\frac{1}{q}-1)}{\frac{1}{q}}\right]$.

Definition 5.7. Call the number $b - \frac{1}{2}r(r-1)$ the *residue of* $\beta_{(b,r)}$, denoted by $Res(\beta)$.

Proposition 5.8. *Let* $\alpha_1\beta = \beta\alpha_1$.

$< \alpha_1, \beta > \cong \mathbb{Z} \oplus \mathbb{Z}$ *and acts properly discontinuously on* $\mathbb{R}^2$ *if and only if* $Res(\beta) \neq 0$.

In particular, the following are equivalent:

(i) The affine group $G \cong \mathbb{Z} \oplus \mathbb{Z}$ *is not a translation group and acts properly discontinuously on* $\mathbb{R}^2$

(ii) G *is affinely conjugate to* $< \alpha_1, \beta >$ *where* $Res(\beta) \neq 0$

Remark 5.9. For another viewpoint, see [C] page 8.

Proof.

If $r = 0$ and $\beta \neq I$ then $Res(\beta) \neq 0$. β is a translation and the group $G \cong \mathbb{Z} \oplus \mathbb{Z}$ acts properly discontinuously, since the second coordinate of $g_i = \alpha_1^{k_i}\beta^{j_i}$ grows arbitrarily large when $k_i \to \infty$. Otherwise, the action of g_i is essentially that of β since for any compact set E, the finite union $\cup\{\alpha_1^{k_i}(E) : i \in \mathbb{Z}\}$ is compact. So assume that $r \neq 0$.

Suppose that $G =< \alpha_1, \beta >\cong \mathbb{Z} \oplus \mathbb{Z}$ and acts properly discontinuously.

Since $< \alpha_1, \beta >=< \alpha_1, \beta^{-1} >$ and $\beta^{-1}_{(b,r)} = \beta_{(r^2-b,-r)}$, assume that $r < 0$.

Choose an infinite set $(j_i, k_i) \in \mathbb{Z}^+ \oplus \mathbb{Z}^+$ so that the sequence $\{j_i r + k_i\}$ converges to $t \in \mathbb{R}$ (including repetitions if $r \in \mathbb{Q}$).

$\alpha_1^{k_i}\beta^{j_i} = [A_1^{k_i}B^{j_i}, v]$, where $A_1^{k_i}B^{j_i} = \begin{bmatrix} 1 & (j_i r + k_i) \\ 0 & 1 \end{bmatrix}$,

and $v = \binom{j_i b + \frac{1}{2}r^2 j_i(j_i-1) + \frac{1}{2}k_i(k_i-1) + k_i j_i r}{j_i r + k_i}$.

The first coordinate of the translation part above can be
rewritten as $\frac{1}{2}(j_i r + k_i)^2 + j_i(b - \frac{1}{2}r^2) - \frac{1}{2}k_i$.

If $b = \frac{1}{2}r(r-1)$, then this becomes $\frac{1}{2}(j_i r + k_i)^2 - \frac{1}{2}(j_i r + k_i)$.

If $r \notin \mathbb{Q}$ then the sequence $\{g_i = \alpha_1^{k_i}\beta^{j_i}\}$ converges to the affine map $[C, c]$,

where $C = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ and $c = \binom{\frac{1}{2}t(t-1)}{t}$,

preventing a properly discontinuous action, while if $r \in \mathbb{Q}$ then G is cyclic - both contradictions.

Conversely, suppose $b \neq \frac{1}{2}r(r-1)$. Then $b - \frac{1}{2}r^2 = -\frac{1}{2}r + d$ for some $d \neq 0$.

To show proper discontinuity, let $\{g_i = \alpha_1^{k_i}\beta^{j_i}\}$ be a sequence of distinct elements of G. There are two cases:

(a) There exist $j_i, k_i \in \mathbb{Z}$ with $\{j_i r + k_i\}$ converging to some $t \in \mathbb{R}$

(b) There are no such j_i, k_i

Case (a) We may assume that $j_i, k_i \in \mathbb{Z}^+$, since $\{j_i r + k_i\}$ diverges when j_i and k_i are opposite in sign (because $r < 0$), while Lemma 2.3 implies that $j_i, k_i < 0$ need not be examined. Then $\alpha_1^{k_i}\beta^{j_i}(x, y)$ converges to $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \binom{x}{y} + \binom{\frac{1}{2}t^2 - \frac{1}{2}t + dj_i}{t}$. Since the x and y coordinates of any compact set E are bounded, the x coordinate of $g_i(E)$ will grow arbitrarily large.

Case (b) Again, it can be assumed that $j_i, k_i > 0$, getting $g_i(E) \to \infty$.

This finishes the first equivalence.

(i) implies (ii) follows from Lemma 5.4. (ii) implies (i) is immediate. $\square$

Lemma 5.10. *$Res : \{\beta : \beta\alpha_1 = \alpha_1\beta\} \to \mathbb{R}$ defined by $\beta \mapsto Res(\beta)$ is a surjective group homomorphism having the following properties*

(let $G =< \alpha_1, \beta >$, $Res(\beta) \neq 0$):

(i) $Res(G) \cong \mathbb{Z}$ with generator $Res(\beta)$.

(ii) If $g \in G$ and $Res(g)$ is a generator of $Res(G)$, then g is a generator of G (but not conversely) .

(iii) If $f, g \in G$, then $< \alpha_1, f >$ is a subgroup of $< \alpha_1, g >$ if and only if $< Res(f) >$ is a subgroup of $< Res(g) >$, and $\frac{<\alpha_1, g>}{<\alpha_1, f>} \cong \frac{<Res(g)>}{<Res(f)>}$, a finite cyclic group of order $[< Res(g) >:< Res(f) >]$.

Proof.

Suppose $\beta = (b, r)$ and $\overline{\beta} = (\overline{b}, \overline{r})$ commute with α_1.

Then $\beta\overline{\beta} = (b + \overline{b} + r\overline{r}, r + \overline{r})$ and so

$Res(\beta\overline{\beta}) = b + \overline{b} + r\overline{r} - \frac{1}{2}(r + \overline{r})(r + \overline{r} - 1) =$

$b + \overline{b} + r\overline{r} - \frac{1}{2}(r^2 + \overline{r}^2 + 2r\overline{r} - r - \overline{r}) = b + \overline{b} - \frac{1}{2}r^2 - \frac{1}{2}\overline{r}^2 + \frac{1}{2}r + \frac{1}{2}\overline{r}$

$= b - \frac{1}{2}r(r-1) + \overline{b} - \frac{1}{2}\overline{r}(\overline{r} - 1).$

If $t \in \mathbb{R}$, then $Res(\beta_{(t,0)}) = t$, showing that Res is onto.

Properties (i), (ii) and (iii) are immediate.

$\square$

When $G =< \alpha_1, \beta >$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, Proposition 5.8 implies that the following are equivalent:

(i) G acts properly discontinuously (and freely) on $\mathbb{R}^2$.

(ii) G is a discrete subgroup of Affine($\mathbb{R}^2$) with the compact-open topology.

The above equivalence does not hold in general (see remark after Definition 2.4).

The Parameter Space of Groups

Part (ii) in Proposition 5.8 states that the point $\binom{b}{r}$ does not lie on the parabola $x = \frac{1}{2}y(y-1)$, which will be denoted by $\wp_{(0,1)}$ ($\wp_{(0,1)} = Ker(Res)$). The points of $\mathbb{R}^2$ can be viewed as a parameter space of maps β that commute with α_1 (or as groups $G =< \alpha_1, \beta >$). Define an equivalence relation on $\mathbb{R}^2$

by $(b,r) \approx (b',r')$ if and only if $< \alpha_1, \beta_{(b,r)} >=< \alpha_1, \beta_{(b',r')} >$, and let $[(b,r)]$ denote the equivalence class of (b,r).

Definition 5.11. $\Omega_{\alpha_1} = \{[(b,r)] : (b,r) \notin \wp_{(0,1)}\}$. Ω_{α_1} corresponds to the set of distinct G which are isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and act properly discontinuously.

The operation $(x_1, y_1) * (x_2, y_2) = (x_1 + x_2 + y_1 y_2, y_1 + y_2)$ on points of $\mathbb{R}^2$ is simply composition of affine maps $\beta_{(x_1,y_1)}$ and $\beta_{(x_2,y_2)}$. $(\mathbb{R}^2, *)$ is a topological group and $Res : \mathbb{R}^2 \to \mathbb{R}$ is a continuous homomorphism ($\mathbb{R}^2$ and $\mathbb{R}$ having the Euclidean topology). A "set action" can be examined where the acting set(s) need not form a group. For example, the vertical line $\ell = \{(1,y) : y \in \mathbb{R}\}$ action on the line $m = \{(x, \frac{1}{2}x) : x \in \mathbb{R}\}$ can be described as "leveling off" the lines $(a,b) * m$ as the points (a,b) of ℓ move up ($b \to +\infty$), while slopes of the $(a,b) * m$ become much steeper (large positive) as $b \to -2^+$. More examples of this geometric viewpoint will be analyzed later (invariant sets and distortion of certain sets, for example).

Recall that the affine group $G =< \alpha = [A,a], \beta = [B,b] >$ contains a non-trivial translation if and only if $A^k B^j = I$ for some integers j,k where the translation part of $\alpha^k \beta^j$ is not the zero vector. In particular, $G =< \alpha_1, \beta_{(b,r)} > \cong \mathbb{Z} \oplus \mathbb{Z}$ and contains a translation if and only if $r \in \mathbb{Q}$ and $Res(\beta) \neq 0$. If this is the case, then G must be a translation extension:

Example: Let $\beta = \beta_{(0,\frac{1}{2})}$. Then $\alpha_1^{-1}\beta^2 = [I, \binom{\frac{1}{4}}{0}] = \pi$. Here, $< \alpha_1, \pi >$ is a proper subgroup of $< \alpha_1, \beta >$, but $< \beta, \pi >=< \alpha_1, \beta >$, so this group is a translation extension (it also acts properly discontinuously).

Example: Let $\beta = \beta_{(0,\frac{2}{3})}$. Then $< \alpha_1, \beta >=< \alpha_1^{-2}\beta^3, \alpha_1^3\beta^{-4} >$ and $\alpha_1^{-2}\beta^3 = [I, \binom{-2}{0}]$.

Lemma 5.12.

(i) $\alpha_1(\wp_{(0,1)}) = \wp_{(0,1)}$

(ii) Each parabola $\wp_{(0,1)} + te_1$ *is* α_1 *invariant.*

Proof.

(i) If $\binom{x_0}{y_0}$ lies on $\wp_{(0,1)}$, then $\alpha_1\binom{x_0}{y_0} = \binom{x_0+y_0}{y_0+1} = \binom{\frac{1}{2}y_0(y_0-1)+y_0}{y_0+1} = \binom{\frac{1}{2}y_0(y_0+1)}{y_0+1}$ also lies on $\wp_{(0,1)}$.

(ii) Immediate, since any point that lies on $\wp_{(0,1)}+te_1$ has the form $\binom{\frac{1}{2}y_0(y_0-1)+t}{y_0}$.

$\square$

There is a corresponding family of parabolas that are $\beta_{(b,r)}$-invariant, where β is not a translation (that is, $r \neq 0$). The "base point parabola" $\wp_{(b,r)}$ of $\beta_{(b,r)}$ (containing the point $\binom{b}{r}$) is $x = \frac{1}{2}y(y + \frac{2b}{r} - r)$.

Proposition 5.13.

(i) $\beta_{(b,r)}(\wp_{(b,r)}) = \wp_{(b,r)}$.

(ii) Each parabola $\wp_{(b,r)} + te_1$ *is* $\beta_{(b,r)}$ *invariant.*

Proof.

(i) Let $\binom{x_0}{y_0}$ lie on $\wp_{(b,r)}$. Then $x_0 = \frac{1}{2}y_0(y_0 + \frac{2b}{r} - r)$ and $\beta\binom{x_0}{y_0} = \binom{x_0+ry_0+b}{y_0+r} = \binom{\frac{1}{2}y_0(y_0+\frac{2b}{r}-r)+ry_0+b}{y_0+r}$.

The first coordinate can be rewritten as $\frac{1}{2}(y_0 + r)(y_0 + r + \frac{2b}{r} - r)$.

(ii) Immediate from the form of β.

$\square$

Thus, if $G \cong \mathbb{Z} \oplus \mathbb{Z}$ acts properly discontinuously, then the elements of G act as "translations" along two transverse families of curves that foliate $\mathbb{R}^2$: Let $G =< \gamma, \delta >$, $\pi\alpha_1\pi^{-1} = \gamma$ and $\pi\beta\pi^{-1} = \delta$ where $\pi \in Affine(\mathbb{R}^2)$. The two families are $\{\pi(\wp_{(0,1)} + te_1) : t \in \mathbb{R}\}$ and $\{\pi(\wp_{(b,r)} + te_1) : t \in \mathbb{R}\}$.

When $Res(\beta_{(b,r)}) \neq 0$ no two maps that lie on $\wp_{(b,r)}$ have the same residue. That is, $Res : \wp_{(b,r)} \rightarrow (\mathbb{R}, +)$ is an isomorphism.

Proposition 5.14. *(Geometric Viewpoint for Inverse Pairs)*

Let ℓ be the tangent line to $\wp_{(b,r)}$ at the origin. Any line parallel to ℓ that meets $\wp_{(b,r)}$ passes through inverse pairs.

Proof.

Let $\wp_{(b,r)}$ be the parabola containing the base points of the non-translations $\beta = \beta_{(b,r)}$ and $\beta^{-1} = \beta_{(b',r')}$. Write $\wp(y) = \frac{1}{2}y(y + \frac{2b}{r} - r)$.
$\binom{b'}{r'} = \binom{r^2-b}{-r}$ and $\wp'(y) = y + \frac{1}{2}(\frac{2b}{r} - r)$, showing that $\wp'(0) = \frac{b-b'}{r-r'}$.

$\square$

A Geometric Viewpoint of the Residue

The group $< \alpha_1, \beta > \cong \mathbb{Z} \oplus \mathbb{Z}$ (not a translation group and therefore not affine conjugate to a translation group) is conjugate to a translation group if the conjugating map is a "polynomial mapping", which generalizes a matrix of real numbers (linear transformation).

Definition 5.15. A *Polynomial Mapping* is a matrix having polynomial functions ($\mathbb{R} \rightarrow \mathbb{R}$) as entries. An $n \times n$ polynomial mapping acts as a function

$\mathbb{R}^n \to \mathbb{R}^n$. Composition is performed by the composition of the polynomial entries resulting from matrix multiplication.

Remark 5.16. In fact, a polynomial mapping generalizes an affine map.

In dimension two, $\pi = \left[\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \binom{c_1}{c_2}\right]$ in $Affine(\mathbb{R}^2)$ can be written as

$\begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix}$ where $g_1(x) = a_1x + c_1$, $g_2(x) = a_2x$, $g_3(x) = a_3x + c_2$, and $g_4(x) = a_4x$ are linear maps $\mathbb{R} \to \mathbb{R}$.

Using such a map, an alternate proof of Proposition 5.8 can be given:

Proof.

Let $\varphi = \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}$ where $f_1(x) = f_4(x) = x$, $f_2(x) = -\frac{1}{2}x(x-1)$, and $f_3(x) = 0$. Denote this by $\varphi = \begin{bmatrix} x & -\frac{1}{2}x(x-1) \\ 0 & x \end{bmatrix}$.

Then $\varphi : \binom{a}{b} \mapsto \binom{f_1(a)+f_2(b)}{f_3(a)+f_4(b)} = \binom{a-\frac{1}{2}b(b-1)}{b}$ is a homeomorphism. $\varphi(\wp_{(0,1)}) = span(e_2)$. φ maps the parabola $\wp_{(0,1)} + te_1$ onto the vertical line $x = t$. When $Res(\beta) \neq 0$, φ maps the parabola $\wp_{(b,r)}$ onto the non-vertical line $x = \frac{1}{2}(\frac{2b}{r} - r + 1)y$, and the parabola $\wp_{(b,r)} + te_1$ is sent to the line $x = \frac{1}{2}(\frac{2b}{r} - r + 1)y + t$. The group $H = \varphi < \alpha_1, \beta > \varphi^{-1} =< [I, e_2], \varphi\beta\varphi^{-1} >$ is therefore a translation group generated by two independent translations if and only if $Res(\beta) \neq 0$. In this case, $< \alpha_1, \beta > \cong \mathbb{Z} \oplus \mathbb{Z}$ and acts properly discontinuously on $\mathbb{R}^2$ (since H does).

$\square$

When $\beta = \beta_{(b,r)}$, $\varphi\binom{b}{r} = \binom{Res(\beta)}{r}$, that is, $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_1\binom{b}{r} = Res(\beta)$ and φ_2 is projection onto the second coordinate. Since φ_1 and φ_2 are homomorphisms, so is φ. See also [F-G], page:.

Corollary 5.17. *If $G \cong \mathbb{Z} \oplus \mathbb{Z}$ is a 2-dimensional properly discontinuous affine group, then G is conjugate to a translation group by a quadratic polynomial map.*

The group $Affine(\mathbb{R}^2)$ is a subgroup of the group of 2×2 polynomial homeomorphisms $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ of any degree (denote $Poly(\mathbb{R}^2)$). In this larger group there are a finite number of conjugacy classes of properly discontinuous affine groups $G \cong \mathbb{Z} \oplus \mathbb{Z}$, since translations are isometries, which divide into a finite number of affine conjugacy classes (Bieberbach). In fact, they are all conjugate, see [F,G].

5.3.2. ***Conjugacy Classes in terms of Residue.***

Conjugacy of individual affine maps (abstract and in any dimension) was characterized in the following theorem by L. Tartar and A. Yakir:

Theorem 5.18. *(Tartar, Yakir - 1997) The affine maps $[A, a]$ and $[B, b]$ are affine conjugate if and only if A is similar to B and $\tau(a, A) = \tau(b, B)$, where $\tau(a, A)$ is the least $k \in \mathbb{Z}^+$ such that $a \in Image(I - A) + Kernel(I - A)^k$.*

Such a number k always exists since for any linear map $A : V \to V$ there is an m so that $V = Image(I - A) + Kernel(I - A)^m$. Here, V is any vector space. See [T-Y].

For example, $\tau(e_2, A_1) = 2$ and $\tau(e_2, A_2) = 1$ (recall that $\alpha_1 = [A_1, e_2]$ and $\alpha_2 = [A_2, e_2]$).

If the groups $< \alpha_1, \beta > \neq < f, g >$ are conjugate, then $< f, g >$ must be conjugate to $< \alpha_1, \overline{\beta} >$ for some $\overline{\beta} \in Affine(\mathbb{R}^2)$ and so without loss of generality we may assume that $\pi < \alpha_1, \beta > \pi^{-1} =< \alpha_1, \overline{\beta} >$ for some π that does not commute with α_1.

Each matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{R})$ can be viewed as a map $M : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto \frac{ax+b}{cx+d}$ (a Mobius transformation).

Lemma 5.19. *Let $\beta = \beta_{(b,r)}$ and $\overline{\beta} = \beta_{(\bar{b},\overline{r})}$ both have non-zero residues.*

If $G =< \alpha_1, \beta >$ is affinely conjugate to $\overline{G} =< \alpha_1, \overline{\beta} >$, then $\varphi(\overline{r}) = r$ for some matrix $\varphi \in GL(2, \mathbb{Z})$.

Proof.

Suppose that $\pi G \pi^{-1} = \overline{G}$ for $\pi = [C, a]$ where $C = \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix}$ and $a = \binom{a_1}{a_2}$.

Then $\pi\alpha_1\pi^{-1} = \alpha_1^t\overline{\beta}^u$ (equivalently, $\pi\alpha_1 = \alpha_1^t\overline{\beta}^u\pi$) and $\pi\beta\pi^{-1} = \alpha_1^v\overline{\beta}^w$ (equivalently, $\pi\beta = \alpha_1^v\overline{\beta}^w\pi$) where $wt - uv = \pm 1$ (because $\pi\alpha_1\pi^{-1}$ and $\pi\beta\pi^{-1}$ generate $\overline{G}$).

The following equations are obtained:

$\pi\alpha_1 = [CA_1, \binom{a_1+c_2}{a_2+d_2}]$, where $CA_1 = \begin{bmatrix} c_1 & (c_1 + c_2) \\ d_1 & (d_1 + d_2) \end{bmatrix}$;

$$\alpha_1^t\overline{\beta}^u\pi = [A_1^t\overline{B}^uC, \binom{x_0}{a_2+u\overline{r}+t}], \text{ where } A_1^t\overline{B}^uC = \begin{bmatrix} (c_1+d_1(u\overline{r}+t)) & (c_2+d_2(u\overline{r}+t)) \\ d_1 & d_2 \end{bmatrix};$$

$$\pi\beta = [CB, \binom{c_1b+c_2r+a_1}{d_2r+a_2}], \text{ where } CB = \begin{bmatrix} c_1 & (c_1r+c_2) \\ d_1 & (d_1r+d_2) \end{bmatrix};$$

$$\alpha_1^v\overline{\beta}^w\pi = [A_1^v\overline{B}^wC, \binom{x_1}{a_2+w\overline{r}+v}], \text{ where } A_1^v\overline{B}^wC = \begin{bmatrix} (c_1+d_1(w\overline{r}+v)) & (c_2+d_2(w\overline{r}+v)) \\ d_1 & d_2 \end{bmatrix}.$$

The expressions x_0 and x_1 need not be specified at this juncture. The second coordinates of the translation equations imply that $d_2 = u\overline{r}+t$ and $d_2r = w\overline{r}+v$. The matrix equation $CA_1 = A_1^t\overline{B}^uC$ implies $d_1 = 0$ (and $c_1 = d_2^2$, which is used in Theorem 5.21) so that $d_2 \neq 0$, since C is invertible.

Therefore $r = \frac{w\overline{r}+v}{u\overline{r}+t} = \varphi(\overline{r})$ where $\varphi = \begin{bmatrix} w & v \\ u & t \end{bmatrix}$.

□

Definition 5.20. If $\pi \in Affine(\mathbb{R}^2)$ where $\pi\alpha_1\pi^{-1} = \alpha_1^t\overline{\beta}^u$ and $\pi\beta\pi^{-1} = \alpha_1^v\overline{\beta}^w$, then define $\varphi_\pi \in GL(2,\mathbb{Z})$ to be $\begin{bmatrix} w & v \\ u & t \end{bmatrix}$ and say "φ_π is induced by π".

Example: Let $\beta = \beta_{(1,\sqrt{2})}$, $\overline{\beta} = \beta_{(3,5)}$. Then the groups $< \alpha_1, \beta >$ and $< \alpha_1, \overline{\beta} >$ are not affinely conjugate because $\frac{5w+v}{5u+t}$ is rational for all integers w, v, u, t (there is no $\varphi \in GL(2,\mathbb{Z})$ so that $\varphi(5) = \sqrt{2}$).

Theorem 5.21. *(Necessary Condition for Conjugacy)*

Let $\beta = \beta_{(b,r)}$ *and* $\overline{\beta} = \beta_{(\overline{b},\overline{r})}$ *have non-zero residues. If* $G =< \alpha_1, \beta >$ *and* $\overline{G} =< \alpha_1, \overline{\beta} >$ *are conjugate via* $\pi \in Affine(\mathbb{R}^2)$ $(\pi G \pi^{-1} = \overline{G})$ *then the following two conditions are satisfied:*

(i) $\varphi_\pi(\overline{r}) = r$

(ii) $\varphi'_\pi(\overline{r}) = Det(\varphi_\pi)\left(\frac{Res(\beta)}{Res(\overline{\beta})}\right)^{\frac{2}{3}}$

If this is the case, then

(iii) φ_π *upper triangular implies* $G = \overline{G}$. *The converse holds when* G *contains no translations.*

(iv) There is a $C \in GL(2,\mathbb{R})$ *conjugating* $\overline{G}$ *to* G *with* $Det(\varphi_\pi)Det(C) = \frac{Res(\beta)}{Res(\overline{\beta})}$.

Proof.

(i) follows from Lemma 5.19, which can be written as

$$\frac{w\overline{r} + v}{u\overline{r} + t} = r \tag{5.1}$$

where $\varphi_\pi = \begin{bmatrix} w & v \\ u & t \end{bmatrix}$.

To show (ii), let $\pi = [C, a]$, where $C = \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix}$ and $a = \binom{a_1}{a_2}$.

It is being assumed that

$$\pi \alpha_1 \pi^{-1} = \alpha_1^t \overline{\beta}^u \tag{5.2}$$

(which is the same as $\pi\alpha_1 = \alpha_1^t\overline{\beta}^u\pi$)

and

$$\pi\beta\pi^{-1} = \alpha_1^v\overline{\beta}^w \tag{5.3}$$

(which is the same as $\pi\beta = \alpha_1^v\overline{\beta}^w\pi$).

In the proof of Lemma 5.19 it was shown that $C = \begin{bmatrix} (u\overline{r}+t)^2 & c_2 \\ 0 & (u\overline{r}+t) \end{bmatrix}$.

Thus $\alpha_1^t\overline{\beta}^u\pi$ has matrix part $\begin{bmatrix} (u\overline{r}+t)^2 & ((u\overline{r}+t)^2+c_2) \\ 0 & (u\overline{r}+t) \end{bmatrix}$ and translation part $\binom{u\overline{b}+\frac{1}{2}\overline{r}^2u(u-1)+tu\overline{r}+\frac{1}{2}t(t-1)+a_1+a_2(u\overline{r}+t)}{u\overline{r}+t+a_2}$,

$\pi\alpha_1$ has matrix part $\begin{bmatrix} (u\overline{r}+t)^2 & ((u\overline{r}+t)^2+c_2) \\ 0 & (u\overline{r}+t) \end{bmatrix}$ and translation part $\binom{c_2+a_1}{u\overline{r}+t+a_2}$,

and $\pi\beta$ has matrix part $\begin{bmatrix} (u\overline{r}+t)^2 & (r(u\overline{r}+t)^2+c_2) \\ 0 & (u\overline{r}+t) \end{bmatrix}$ and translation part $\binom{b(u\overline{r}+t)^2+c_2r+a_1}{r(u\overline{r}+t)+a_2}$.

The form of $\alpha_1^v\overline{\beta}^w\pi$ is the same as that of $\alpha_1^t\overline{\beta}^u\pi$ (replace t with v and u with w).

The first coordinates of the translation parts of 5.2 imply

$$c_2 = u\overline{b} + \frac{1}{2}\overline{r}^2u(u-1) + tu\overline{r} + \frac{1}{2}t(t-1) + a_2(u\overline{r}+t) \tag{5.4}$$

and the first coordinates of the translation parts of 5.3 imply

$$(5.5)\quad b(u\overline{r}+t)^2+c_2r=w\overline{b}+\frac{1}{2}\overline{r}^2w(w-1)+vw\overline{r}+\frac{1}{2}v(v-1)+a_2(w\overline{r}+v)$$

Consider the following equations:

(5.6)

$$b(u\overline{r}+t)^2+r(u\overline{b}+\frac{1}{2}\overline{r}^2u(u-1)+\frac{1}{2}t(t-1)+tu\overline{r})=w\overline{b}+\frac{1}{2}\overline{r}^2w(w-1)+\frac{1}{2}v(v-1)+vw\overline{r}$$

(5.7)

$$b=\frac{(\frac{1}{2}(w\overline{r}+v)^2+w(\overline{b}-\frac{1}{2}\overline{r}^2)-\frac{1}{2}(v-tr)-ur(\overline{b}-\frac{1}{2}\overline{r}^2)-\frac{1}{2}r(u\overline{r}+t)^2)}{(u\overline{r}+t)^2}$$

$$(5.8)\qquad Res(\beta)=\frac{(\overline{b}-\frac{1}{2}\overline{r}^2)(w-ur)-\frac{1}{2}(v-tr)}{(u\overline{r}+t)^2}$$

$$(5.9)\qquad \varphi'_\pi(\overline{r})=Det(\varphi)\left(\frac{Res(\beta)}{Res(\overline{\beta})}\right)^{\frac{2}{3}}$$

It will be shown that { 5.4 and 5.5} $\Rightarrow$ 5.6 $\Rightarrow$ 5.7 $\Rightarrow$ 5.8 $\Rightarrow$ 5.9.

Proof that { 5.4 and 5.5 } $\Rightarrow$ 5.6:

Replace c_2 in 5.5 with the right side of 5.4.

5.1 implies $a_2(u\overline{r}+t)r = a_2(w\overline{r}+v)$, which now appear on each side of the above equation and can be cancelled. This is 5.6. We may now assume that $\pi = [C, \binom{0}{0}]$.

Proof that 5.6 $\Rightarrow$ 5.7:

5.6 can be rewritten as:

$$b(u\overline{r}+t)^2 + ur(\overline{b}-\tfrac{1}{2}\overline{r}^2) + \tfrac{1}{2}r(u\overline{r}+t)^2 - \tfrac{1}{2}tr = w(\overline{b}-\tfrac{1}{2}\overline{r}^2) + \tfrac{1}{2}(w\overline{r}+v)^2 - \tfrac{1}{2}v.$$

Solve for b. This is 5.7.

Proof that 5.7 $\Rightarrow$ 5.8:

Subtract $\frac{1}{2}r(r-1) = \frac{1}{2}(\frac{w\overline{r}+v}{u\overline{r}+t})(\frac{w\overline{r}+v}{u\overline{r}+t} - \frac{u\overline{r}+t}{u\overline{r}+t}) = \frac{\frac{1}{2}(w\overline{r}+v)^2 - \frac{1}{2}(w\overline{r}+v)(u\overline{r}+t)}{(u\overline{r}+t)^2}$

$= \frac{\frac{1}{2}(w\overline{r}+v)^2 - \frac{1}{2}r(u\overline{r}+t)^2}{(u\overline{r}+t)^2}$ from both sides of 5.7:

$$b-\tfrac{1}{2}r(r-1) = \frac{(-\frac{1}{2}(w\overline{r}+v)^2+\frac{1}{2}r(u\overline{r}+t)^2+\frac{1}{2}(w\overline{r}+v)^2+w(\overline{b}-\frac{1}{2}\overline{r}^2)-\frac{1}{2}(v-tr)-ur(\overline{b}-\frac{1}{2}\overline{r}^2)-\frac{1}{2}r(u\overline{r}+t)^2)}{(u\overline{r}+t)^2}$$

$$= \frac{(w(\overline{b}-\frac{1}{2}\overline{r}^2)-\frac{1}{2}(v-tr)-ur(\overline{b}-\frac{1}{2}\overline{r}^2))}{(u\overline{r}+t)^2}$$

$= \frac{((\overline{b}-\frac{1}{2}\overline{r}^2)(w-ur)-\frac{1}{2}(v-tr))}{(u\overline{r}+t)^2}$. This is 5.8.

Proof that 5.8 $\Rightarrow$ 5.9:

5.1 implies $v - tr = \overline{r}(ur-w)$, which in turn implies that $Res(\beta) = b - \frac{1}{2}r(r-1) = \frac{(w-ur)(\overline{b}-\frac{1}{2}\overline{r}^2+\frac{1}{2}\overline{r})}{(u\overline{r}+t)^2} = \frac{(w-ur)Res(\overline{\beta})}{(u\overline{r}+t)^2}$.

From $w - ur = w(\frac{u\overline{r}+t}{u\overline{r}+t}) - u(\frac{w\overline{r}+v}{u\overline{r}+t}) = \frac{wt-uv}{u\overline{r}+t} = \frac{Det(\varphi_\pi)}{u\overline{r}+t}$ it follows that

$$\frac{Res(\beta)}{Res(\overline{\beta})} = \frac{Det(\varphi_\pi)}{(u\overline{r}+t)^3} \tag{5.10}$$

.

Since $\varphi'_\pi(\overline{r}) = \frac{Det(\varphi_\pi)}{(u\overline{r}+t)^2}$, then $\varphi'_\pi(\overline{r})^3 = \frac{Det(\varphi_\pi)}{(u\overline{r}+t)^6} = Det(\varphi_\pi)(\frac{Res(\beta)}{Res(\overline{\beta})})^2$.

This is equivalent to $\varphi'_\pi(\overline{r}) = Det(\varphi_\pi)\left(\frac{Res(\beta)}{Res(\overline{\beta})}\right)^{\frac{2}{3}}$, which is 5.9. It has been shown that $\pi G \pi^{-1} = \overline{G}$ implies $\{(i)$ and $(ii)\}$.

Now suppose that $\varphi = \varphi_\pi$, where $\varphi = \begin{bmatrix} w & v \\ u & t \end{bmatrix}$ (and $\pi G \pi^{-1} = \overline{G}$, where $G =< \alpha_1, \beta >$ and $\overline{G} =< \alpha_1, \overline{\beta} >$). From the proof that $\{$ 5.4 and 5.5 $\} \Rightarrow$ 5.6, we may also assume that $\pi = [C, \binom{0}{0}]$ where $C = \begin{bmatrix} (u\overline{r}+t)^2 & c_2 \\ 0 & (u\overline{r}+t) \end{bmatrix}$.

To show (iii), suppose that φ_π is upper triangular,

that is, $\varphi_\pi = \begin{bmatrix} w & v \\ u & t \end{bmatrix} = \begin{bmatrix} w_1 & m \\ 0 & w_2 \end{bmatrix}$, where $m \in \mathbb{Z}$ and $w_i \in \{1, -1\}$.

Since the matrix $\begin{bmatrix} -w_1 & -m \\ 0 & -w_2 \end{bmatrix}$ represents the same map $\mathbb{R} \to \mathbb{R}$, we may assume $w_2 = 1$.

Thus $u = 0$, $t = 1$ and $C = \begin{bmatrix} w_1 & \pm c_2 \\ 0 & 1 \end{bmatrix}$, where $w_1 \in \{1, -1\}$.

c_2 in 5.4 (with $a_2 = 0$) can be rewritten as $c_2 = \frac{1}{2}(u\overline{r}+t)^2 + u(\overline{b} - \frac{1}{2}\overline{r}^2) - \frac{1}{2}t = \frac{1}{2} - \frac{1}{2} = 0$. Therefore $C = I$ and so $G = \overline{G}$.

Conversely, assume that $G = \overline{G}$ contains no translations ($r, \overline{r} \notin \mathbb{Q}$ by (v) in General Facts About Affine Groups) and that φ_π is not upper triangular. Then $u \neq 0$ and $\overline{\beta} = \alpha_1^k \beta^j$ for some $j, k \in \mathbb{Z}$, and so $Res(\overline{\beta}) = Res(\alpha_1^k) + Res(\beta^j) = kRes(\alpha_1) + jRes(\beta) = jRes(\beta)$ since $Res(\alpha_1) = 0$. Therefore $\frac{1}{j} = \frac{Res(\beta)}{Res(\overline{\beta})} = \frac{Det(\varphi)}{(u\overline{r}+t)^3} = \frac{\pm 1}{(u\overline{r}+t)^3}$, a contradiction. This finishes the proof of (iii).

(iv) follows immediately, since $CGC^{-1} = \overline{G}$ is equivalent to $C^{-1}\overline{G}C = G$ and $Det(C^{-1}) = \frac{1}{Det(C)} = \frac{Res(\beta)}{Res(\overline{\beta})}$.

$\square$

Theorem 5.22. *(Sufficient Condition for Conjugacy)*

Let $\beta = \beta_{(b,r)}$ and $\overline{\beta} = \beta_{(\overline{b},\overline{r})}$ have non-zero residues.

Suppose there is a $\varphi = \begin{bmatrix} w & v \\ u & t \end{bmatrix}$ *in $GL(2,\mathbb{Z})$ for which*

(i) $\varphi(\overline{r}) = r$ and

(ii) $\varphi''(\overline{r}) = -2u\frac{Res(\beta)}{Res(\overline{\beta})}$

Then $G =< \alpha_1, \beta >$ and $\overline{G} =< \alpha_1, \overline{\beta} >$ are affinely conjugate.

In this case, there is a $\pi \in Affine(\mathbb{R}^2)$ so that $\pi G \pi^{-1} = \overline{G}$ and $\varphi = \varphi_\pi$.

If in addition G contains no translations, then $G = \overline{G}$ if and only if $u = 0$.

Proof.

We claim that $\varphi = \varphi_\pi$ where $\pi = [C, \binom{0}{0}]$ for $C = \begin{bmatrix} (u\overline{r}+t)^2 & c_2 \\ 0 & (u\overline{r}+t) \end{bmatrix}$

and $c_2 = u\overline{b} + \frac{1}{2}\overline{r}^2 u(u-1) + \frac{1}{2}t(t-1) + tu\overline{r} = \frac{1}{2}(u\overline{r}+t)^2 + u(\overline{b} - \frac{1}{2}\overline{r}^2) - \frac{1}{2}t$.

It will be shown that $\pi\alpha_1\pi^{-1} = \alpha_1^t \overline{\beta}^u$ and $\pi\beta\pi^{-1} = \alpha_1^v \overline{\beta}^w$.

$$\pi\alpha_1\pi^{-1} = \left[\begin{bmatrix} (u\overline{r}+t)^2 & c_2 \\ 0 & (u\overline{r}+t) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{(u\overline{r}+t)^2} & -\frac{c_2}{(u\overline{r}+t)^3} \\ 0 & \frac{1}{(u\overline{r}+t)} \end{bmatrix}, C\binom{0}{1}\right]$$

$$= \left[\begin{bmatrix} 1 & (-\frac{c_2(u\overline{r}+t)^2}{(u\overline{r}+t)^3} + \frac{(u\overline{r}+t)^2+c_2}{(u\overline{r}+t)}) \\ 0 & 1 \end{bmatrix}, \binom{c_2}{u\overline{r}+t}\right]$$

$$= \left[\begin{bmatrix} 1 & (u\overline{r}+t) \\ 0 & 1 \end{bmatrix}, \binom{u\overline{b}+\frac{1}{2}\overline{r}^2u(u-1)+\frac{1}{2}t(t-1)+tu\overline{r}}{u\overline{r}+t}\right] = \alpha_1^t\overline{\beta}^u.$$

$$\pi\beta\pi^{-1} = \left[\begin{bmatrix} (u\overline{r}+t)^2 & c_2 \\ 0 & (u\overline{r}+t) \end{bmatrix} \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{(u\overline{r}+t)^2} & -\frac{c_2}{(u\overline{r}+t)^3} \\ 0 & \frac{1}{(u\overline{r}+t)} \end{bmatrix}, C\binom{b}{r}\right]$$

$$= \left[\begin{bmatrix} (u\overline{r}+t)^2 & (r(u\overline{r}+t)^2+c_2) \\ 0 & (u\overline{r}+t) \end{bmatrix} \begin{bmatrix} \frac{1}{(u\overline{r}+t)^2} & -\frac{c_2}{(u\overline{r}+t)^3} \\ 0 & \frac{1}{(u\overline{r}+t)} \end{bmatrix}, \binom{b(u\overline{r}+t)^2+rc_2}{r(u\overline{r}+t)}\right]$$

$$= \left[\begin{bmatrix} 1 & (-\frac{c_2(u\overline{r}+t)^2}{(u\overline{r}+t)^3} + \frac{r(u\overline{r}+t)^2+c_2}{(u\overline{r}+t)}) \\ 0 & 1 \end{bmatrix}, \binom{b(u\overline{r}+t)^2+rc_2}{r(u\overline{r}+t)}\right]$$

$$= \left[\begin{bmatrix} 1 & r(u\overline{r}+t) \\ 0 & 1 \end{bmatrix}, \binom{b(u\overline{r}+t)^2+r(\frac{1}{2}(u\overline{r}+t)^2+u(\overline{b}-\frac{1}{2}\overline{r}^2)-\frac{1}{2}t)}{r(u\overline{r}+t)}\right]. \qquad (*)$$

And $\alpha_1^v\overline{\beta}^w = \left[\begin{bmatrix} 1 & (w\overline{r}+v) \\ 0 & 1 \end{bmatrix}, \binom{w\overline{b}+\frac{1}{2}\overline{r}^2w(w-1)+\frac{1}{2}v(v-1)+vw\overline{r}}{(w\overline{r}+v)}\right].$

The matrix parts (same as second coordinates of translation parts) of $\pi\beta\pi^{-1}$ and $\alpha_1^v\overline{\beta}^w$ are equal because $\frac{w\overline{r}+v}{u\overline{r}+t} = \varphi(\overline{r}) = r$.

Since $\varphi''(\overline{r}) = \frac{-2uDet(\varphi)}{(u\overline{r}+t)^3}$, our assumption (ii) implies $\frac{Res(\beta)}{Res(\overline{\beta})} = \frac{Det(\varphi)}{(u\overline{r}+t)^3}$, which can be rewritten as

$$b = \frac{1}{2}r(r-1) + \frac{Det(\varphi)(\overline{b}-\frac{1}{2}\overline{r}(\overline{r}-1))}{(u\overline{r}+t)^3} = \frac{1}{2}(\frac{w\overline{r}+v}{u\overline{r}+t})(\frac{w\overline{r}+v}{u\overline{r}+t} - \frac{u\overline{r}+t}{u\overline{r}+t}) + \frac{Det(\varphi)(\overline{b}-\frac{1}{2}\overline{r}(\overline{r}-1))}{(u\overline{r}+t)^3}.$$

By replacing b in $(*)$ with the rightmost side of the above string of equations, the first coordinate of the translation part of $\pi\beta\pi^{-1}$ becomes

$(\frac{1}{2}(\frac{w\overline{r}+v}{u\overline{r}+t})(\frac{w\overline{r}+v}{u\overline{r}+t} - \frac{u\overline{r}+t}{u\overline{r}+t}) + \frac{Det(\varphi)(\overline{b}-\frac{1}{2}\overline{r}(\overline{r}-1))}{(u\overline{r}+t)^3})(u\overline{r}+t)^2+$

$\frac{w\overline{r}+v}{u\overline{r}+t}(u\overline{b}+\frac{1}{2}\overline{r}^2u(u-1)+tu\overline{r}+\frac{1}{2}t(t-1))$

$= \frac{1}{2}((w\overline{r}+v)^2 - (w\overline{r}+v)(u\overline{r}+t)) + \frac{Det(\varphi)(\overline{b}-\frac{1}{2}\overline{r}(\overline{r}-1))}{(u\overline{r}+t)}$

$+(\frac{w\overline{r}+v}{u\overline{r}+t})(\frac{1}{2}(u\overline{r}+t)^2 + u(\overline{b}-\frac{1}{2}\overline{r}^2) - \frac{1}{2}t)$

$= \frac{1}{2}(w\overline{r}+v)^2 + \frac{Det(\varphi)(\overline{b}-\frac{1}{2}\overline{r}(\overline{r}-1))}{(u\overline{r}+t)} + r(u(\overline{b}-\frac{1}{2}\overline{r}^2) - \frac{1}{2}t) \qquad (**)$

(r has replaced $\frac{w\overline{r}+v}{u\overline{r}+t}$ in $(**)$ above).

The first coordinate of the translation part of $\alpha_1^v\overline{\beta}^u$ is

$w\overline{b}+\frac{1}{2}\overline{r}^2w(w-1)+\frac{1}{2}v(v-1)+vw\overline{r}$, which can be rewritten as

$\frac{1}{2}(w\overline{r}+v)^2 + w(\overline{b}-\frac{1}{2}\overline{r}^2) - \frac{1}{2}v$.

It must be shown that

$\frac{1}{2}(w\overline{r}+v)^2+\frac{Det(\varphi)(\overline{b}-\frac{1}{2}\overline{r}(\overline{r}-1))}{(u\overline{r}+t)}+r(u(\overline{b}-\frac{1}{2}\overline{r}^2)-\frac{1}{2}t) = \frac{1}{2}(w\overline{r}+v)^2+w(\overline{b}-\frac{1}{2}\overline{r}^2)-\frac{1}{2}v,$

or equivalently, $\frac{Det(\varphi)(\overline{b}-\frac{1}{2}\overline{r}(\overline{r}-1))}{(u\overline{r}+t)} = (w-ur)(\overline{b}-\frac{1}{2}\overline{r}^2) - \frac{1}{2}(v-tr)$.

This is the same as showing

$\frac{Det(\varphi)(\overline{b}-\frac{1}{2}\overline{r}(\overline{r}-1))}{(u\overline{r}+t)^3} = \frac{1}{(u\overline{r}+t)^2}((w-ur)(\overline{b}-\frac{1}{2}\overline{r}^2) - \frac{1}{2}\overline{r}(ur-w))$

which equals $\frac{1}{(u\overline{r}+t)^2}((w-ur)(\overline{b}-\frac{1}{2}\overline{r}^2+\frac{1}{2}\overline{r}))$.

Since $w-ur = \frac{Det(\varphi)}{u\overline{r}+1}$ (see the calculation just above line 5.10 in the proof of Theorem 5.19), it follows that $\frac{1}{(u\overline{r}+t)^2}((w-ur)(\overline{b}-\frac{1}{2}\overline{r}^2+\frac{1}{2}\overline{r})) = \frac{Det(\varphi)Res(\overline{\beta})}{(u\overline{r}+t)^3}$.

Therefore $\varphi = \varphi_\pi$.

When G contains no translations ($r \notin \mathbb{Q}$) $G = \overline{G}$ if and only if φ is upper triangular (same as $u = 0$) follows from Theorem 5.19.

$\square$

Corollary 5.23. *Let $\beta = \beta_{(b,r)}$ and $\overline{\beta} = \beta_{(\overline{b},\overline{r})}$ have non-zero residues, and set $G =< \alpha_1, \beta >$, $\overline{G} =< \alpha_1, \overline{\beta} >$. The following are equivalent:*

1. G and $\overline{G}$ are affinely conjugate

2. There is a $\varphi = \begin{bmatrix} w & v \\ u & t \end{bmatrix}$ in $GL(2, \mathbb{Z})$ so that
(i) $\varphi(\overline{r}) = r$ and

(ii) $\varphi''(\overline{r}) = -2u\frac{Res(\beta)}{Res(\overline{\beta})}$

Proof.

By Theorem 5.22 we only need to show that *1* implies *2*.

If $\pi G \pi^{-1} = \overline{G}$ for some $\pi \in Affine(\mathbb{R}^2)$, then by

Theorem 5.19 there is a $\varphi = \varphi_\pi = \begin{bmatrix} w & v \\ u & t \end{bmatrix}$ so that $\varphi(\overline{r}) = r$. In the proof of that theorem, it was shown that $\frac{Res(\beta)}{Res(\overline{\beta})} = \frac{Det(\varphi)}{(u\overline{r}+t)^3}$ (see line 5.10).

It follows that $\varphi''(\overline{r}) = \frac{-2uDet(\varphi)}{(u\overline{r}+t)^3} = -2u\frac{Res(\beta)}{Res(\overline{\beta})}$.

$\square$

Lemma 5.24. *If $G =< \alpha_1, \beta_{(b,r)} >=< \alpha_1, \overline{\beta} = \beta_{(\overline{b},\overline{r})} >= \overline{G}$, then $r - \overline{r} \in \mathbb{Z}$ or $r + \overline{r} \in \mathbb{Z}$.*

Proof.

Assume the above hypothesis. Then $\beta = \alpha_1^r \overline{\beta}^s$ and $\overline{\beta} = \alpha_1^k \beta^j$, implying that $sj = 1$ (since $Res(\beta) = sRes(\overline{\beta})$ and $Res(\overline{\beta}) = jRes(\beta)$).

$$\overline{\beta} = \left[\begin{bmatrix} 1 & \overline{r} \\ 0 & 1 \end{bmatrix}, \binom{\bar{b}}{\overline{r}} \right] \text{ and } \alpha_1^k \beta^{\pm 1} = \left[\begin{bmatrix} 1 & k \pm r \\ 0 & 1 \end{bmatrix}, \binom{\frac{1}{2}k(k-1)+b \pm kr}{k \pm r} \right].$$

This implies that $\overline{r} = k \pm r$, and so $r - \overline{r} \in \mathbb{Z}$ or $r + \overline{r} \in \mathbb{Z}$.

□

Lemma 5.25. *If $G =< \alpha_1, \beta >$ contains no translations, $\overline{G} =< \alpha_1, \overline{\beta} >$ and $Res(\beta) = Res(\overline{\beta})$, then G conjugate to $\overline{G}$ implies $G = \overline{G}$.*

Proof.

If $Res(\beta) = Res(\overline{\beta})$ and G is conjugate to $\overline{G}$ via φ, then by 5.10 in the proof of Theorem 5.21, $1 = \frac{Res(\beta)}{Res(\overline{\beta})} = \frac{Det(\varphi)}{(u\overline{r}+t)^3} = \frac{\pm 1}{(u\overline{r}+t)^3}$ forces $u = 0$ ($r \notin \mathbb{Q}$ implies $\overline{r} \notin \mathbb{Q}$). So φ is upper triangular and $G = \overline{G}$.

□

Proposition 5.26. *There are uncountably many affine conjugacy classes of affine groups isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ that act properly discontinuously on $\mathbb{R}^2$.*

Proof.

Let T be a maximal set of the form $T = \{\{r, \overline{r}\} \subseteq (\mathbb{R} - \mathbb{Q}) \cap (0, \frac{1}{2})$: no two sets have a common element$\}$ (such a T is not unique). Then T is uncountable. Let $T_0 = \{\{(b, r), (\bar{b}, \overline{r})\} : \{r, \overline{r}\} \in T, b - \frac{1}{2}r(r-1) = \bar{b} - \overline{r}(\overline{r}-1) \text{ and } r \pm \overline{r} \notin \mathbb{Z}\}$. Then T_0 is uncountable as well. Let $G =< \alpha_1, \beta_{(b,r)} >$ and $\overline{G} =< \alpha_1, \beta_{(\bar{b},\overline{r})} >$, where $\{(b, r), (\bar{b}, \overline{r})\} \in T_0$, and suppose (for a contradiction) G is conjugate to

$\overline{G}$. Lemma 5.25 implies $G = \overline{G}$, which in turn implies $r \pm \overline{r} \in \mathbb{Z}$ (Lemma 5.24), a contradiction. Therefore there must be uncountably many conjugacy classes of properly discontinuous groups isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

$\square$

In the case where the maps are (two-dimensional) isometries, there are two affine conjugacy classes (Bieberbach). See [C], page 41.

Proposition 5.27. *Let $G =< \alpha_1, \beta >$, $\overline{G} =< \alpha_1, \overline{\beta} >$ and $\hat{G} =< \alpha_1, \hat{\beta} >$.*

(i) $\varphi_{(\pi^{-1})} = \varphi_\pi^{-1}$.

(ii) $\varphi_\pi \varphi_\tau = \varphi_{\tau\pi}$

(iii) If G contains no translations, $\pi G \pi^{-1} = \overline{G}$ and $\tau \overline{G} \tau^{-1} = G$, then $\varphi_{\tau\pi}$ is upper triangular.

Proof.

(i) and (ii) follow from Corollary 5.23: $\pi G \pi^{-1} = \overline{G}$ implies $\pi^{-1} \overline{G} \pi = \pi^{-1} \overline{G} (\pi^{-1})^{-1} = G$, while $\{\pi G \pi^{-1} = \overline{G}$ and $\tau \overline{G} \tau^{-1} = \hat{G}\}$ imply $\tau\pi G (\tau\pi)^{-1} = \hat{G}$. To show (iii), if $\pi G \pi^{-1} = \overline{G}$ and $\tau \overline{G} \tau^{-1} = G$, then $\tau\pi G (\tau\pi)^{-1} = G$ and $\varphi_{\tau\pi}$ is upper triangular by Theorem 5.22.

$\square$

If $G =< \alpha_1, \beta >$ contains no translations (and $Res(\beta) \neq 0$), then $GL(2, \mathbb{Z})$ acts on its conjugacy class by $\varphi \overline{G} = \hat{G}$ where $\pi \hat{G} \pi^{-1} = \overline{G}$ and $\varphi = \varphi_\pi$. We can extend this $GL(2,\mathbb{Z})$ action to $\{(b,r) \in \mathbb{R}^2 - \wp_{(0,1)} : r \notin \mathbb{Q}\}$ as well (when $r = \frac{p}{q}$ is rational, $\varphi(r) = \infty$ where $\varphi = \begin{bmatrix} w & v \\ q & -p \end{bmatrix}$).

The kernel of this action is the subgroup H of upper triangular matrices.

Corollary 5.28. *Suppose $G =< \alpha_1, \beta >$ contains no translations (and $Res(\beta) \neq 0$).*

There is a natural one-to-one correspondence between each normalized member of the conjugacy class of G and the left cosets of H in $GL(2,\mathbb{Z})$.

Proof.

We are assuming that all groups contain α_1 as a generator.

Let $\pi G \pi^{-1} = \overline{G}$ and $\tau G \tau^{-1} = \hat{G}$.

Assume first that $\varphi_\pi H = \varphi_\tau H$.

The conjugacy equations above imply $\tau^{-1}\hat{G}\tau = G$ and $\overline{G} = \pi\tau^{-1}\hat{G}\tau\pi^{-1} = \pi\tau^{-1}\hat{G}(\pi\tau^{-1})^{-1}$.

Since $\varphi_\tau^{-1}\varphi_\pi = \varphi_{\tau^{-1}}\varphi_\pi = \varphi_{\pi\tau^{-1}}$ is upper triangular and $r \notin \mathbb{Q}$, Theorem 5.22 implies $\overline{G} = \hat{G}$.

Now assume that $\overline{G} = \hat{G}$. Then $\overline{G} = \pi\tau^{-1}\hat{G}(\pi\tau^{-1})^{-1} = \hat{G}$. Again, Theorem 5.22 implies $\varphi_{\pi\tau^{-1}} = \varphi_{\tau^{-1}}\varphi_\pi$ is upper triangular ($\varphi_\pi H = \varphi_\tau H$).

$\square$

Another Look at the Kernel of the Residue

Recall that an affine group that contains a non-translation is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and acts properly discontinuously if and only if it is conjugate to some $G =< \alpha_1, \beta_{(b,r)} >$ where $(b,r) \notin Ker(Res) = \wp_{(0,1)}$, and the parameter space

$\Omega_{\alpha_1} = \{[(b,r)] : (b,r) \notin \wp_{(0,1)}\}$ corresponds to the set of such distinct G.

Theorem 5.29. *(Conjugacy Classes Limit to $\wp_{(0,1)}$)*

Any open ϵ-disc centered at $(x_0, y_0) \in \wp_{(0,1)}$ contains a representative from each conjugacy class of properly discontinuous affine $G \cong \mathbb{Z} \oplus \mathbb{Z}$ that contain a non-translation.

Proof.

Let $\epsilon > 0$ and assume that $D = D_\epsilon((x_0, y_0))$ is an open disc centered at $(x_0, y_0) \in \wp_{(0,1)}$. Let $n = inf\{y : (x,y) \in D \cap \wp_{(0,1)}\}$ and $N = sup\{y : (x,y) \in D \cap \wp_{(0,1)}\}$. Suppose $G \cong \mathbb{Z} \oplus \mathbb{Z}$ is not a pure translation group, acts properly discontinuously and is conjugate to $< \alpha_1, \overline{\beta} >$ where $\overline{\beta} = \beta_{(\bar{b},\bar{r})}$. Then $\bar{r} \notin \mathbb{Q}$. Choose $\frac{w}{u} \in \mathbb{Q}$, $n < \frac{w}{u} < N$. We may assume that $\frac{w}{u}$ is reduced, that is, $gcd(w,u) = 1$. Then there exist $t, v \in \mathbb{Z}$ so that $wt - uv = 1$.

Let $\varphi = \begin{bmatrix} w & v \\ u & t \end{bmatrix}$. φH corresponds to a group $< \alpha_1, \beta >$ that is conjugate to $< \alpha_1, \overline{\beta} >$, and is therefore conjugate to G (recall that H is the subgroup of upper triangular matrices of $GL(2, \mathbb{Z})$).

Specifically, for each $m \in \mathbb{Z}$, $< \alpha_1, \beta > = < \alpha_1, \beta_{(b_m, r_m)} >$,

where $r_m = \frac{w\bar{r}+v+mw}{u\bar{r}+t+mu}$ and $b_m = \frac{Res(\overline{\beta})}{(u\bar{r}+t+mu)^3} + \frac{1}{2}r_m(r_m - 1)$.

Thus $r_m \to \frac{w}{u}$ as $m \to \pm\infty$.

$Res(\overline{\beta})$ is fixed, which implies that $\{\frac{Res(\overline{\beta})}{(u\bar{r}+t+mu)^3}\} \to 0$ as $m \to \pm\infty$. Therefore $(b_m, r_m) \in D$ for large enough m.

□

As a consequence of Theorem 5.29, if we let V be the "$\epsilon-$neighborhood of $\wp_{(0,1)}$", then $\{[(b,r)] : (b,r) \in V - \wp_{(0,1)}\} = \Omega_{\alpha_1}$.

6. **Actions in** $\mathbb{R}^3$

When two generators are involved, acting freely (on $\mathbb{R}^3$) is not sufficient for a properly discontinuous action as there are non-discrete examples (see the 2-dimensional case).

However, being discrete will not insure a properly discontinuous action either.

Example: Let $G =< \alpha, \beta >$, where α is the Euclidean translation by e_1 and $\beta = [B, b]$ with $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $b = (\sqrt{2}, 1, 0)$.

$G \leq Affine(\mathbb{R}^3)$ and is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, acts freely, and is discrete. Yet this group does not act properly discontinuously on $\mathbb{R}^3$:

The element $g = \alpha^k \beta^j$ maps (x, y, z) to $(x + j\sqrt{2} + k, y + j(z+1), z)$. Choose a sequence $\{j_i, k_i\}$ of pairs of integers with $j_i \to \infty$ and $(j_i\sqrt{2} + k_i) \to r \in \mathbb{R}$ as $i \to \infty$. Each $g_i = \alpha^{k_i}\beta^{j_i}$ keeps the plane $z = -1$ invariant. The sequence $\{g_i\}$ converges uniformly to $f : (x, y, z) \mapsto (x + r, y, z)$ when restricted to this plane. Set $h_i = g_i g_{i+1}^{-1}$ and let U be any open set of $\mathbb{R}^3$ meeting the plane $z = -1$. $h_i(U) \cap (U) \neq \emptyset$ for large i, showing that G does not act discontinuously (and hence not properly discontinuously) on $\mathbb{R}^3$. Since G does act discontinuously at any point not in this plane, it is a discrete group.

6.1. $\mathbb{Z} \oplus \mathbb{Z}$ Actions (The General Case).

Properly discontinuous $G \cong \mathbb{Z} \oplus \mathbb{Z}$ are characterized in a similar manner to those in dimension 2: One generator is normalized, the form of the other generator simplifies from the commuting condition, and a properly discontinuous action is determined by the coordinates of the latter generator. In the case in which the matrix part of a generator has characteristic polynomial $(x-1)^3$, the group acts as "translation" along curves or "shearing" along surfaces.

Recall that $\alpha_i = [A_i, e_3]$ where $A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$,

$A_2 = \begin{bmatrix} a_0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $a_0 \neq 0$ and $A_3 = \begin{bmatrix} A_0 & 0 \\ 0 & 1 \end{bmatrix}$, $A_0 \in GL(2, \mathbb{R})$.

The cases are divided as follows:

(i) There is a generator that is conjugate to α_1

(ii) There is a generator that is conjugate to α_2

(iii) Both generators are conjugate to α_3

Also, unless stated otherwise, it will always be assumed that $\beta = [B, b]$.

By the theorem of Tartar and Yakir stated earlier:

(i) β is affine conjugate to α_1 if and only if B is similar to A_1 and $\tau(b, B) = \tau(e_3, A_1) = 3$

(ii) β is affine conjugate to α_2 if and only if B is similar to A_2 and $\tau(b, B) = \tau(e_3, A_2) = 2$

(iii) β is affine conjugate to α_3 if and only if B is similar to A_3 and $\tau(b, B) = \tau(e_3, A_3) = 1$.

Since α_1 and the families represented by α_2 and α_3 are specific, another description can be made:

Lemma 6.1. *Let $\beta = [B, b] \in Affine(\mathbb{R}^3)$ be fixed point free.*

(i) β is (affine) conjugate to α_1 if and only if $B(b) \neq b$ and there is no 2-dimensional subspace that is β-invariant.

(ii) β is conjugate to α_2 if and only if $B(b) \neq b$ and there is a 2-dimensional β-invariant subspace.

(iii) β is conjugate to α_3 if and only if $B(b) = b$

(The conjugacy class of translations (the translations themselves) are a subset of all maps of the form α_3.)

Proof.

The equation $B(b) = b$ holds if and only if β is conjugate to α_3 from $\tau(b, B) = 1$ being equivalent to $B(b) = b$.

(i) Suppose α_1 keeps a plane through the origin invariant. This plane would then be A_1-invariant implying that it contains $span(e_3) = z - axis$. Since the xy-plane is A_1-invariant, the intersection of these planes is A_1-invariant (a one dimensional subspace, say $\ell = span(re_1 + se_2)$, $r, s \in \mathbb{R}$). Let $v = (tr, ts, 0)$ be a point on ℓ. Now $A_1(v) = v + se_1$, but also $A_1(v) = (dr, ds, 0)$ implying that $s = 0$ and so ℓ is the x-axis. But the xz-plane is not α_1-invariant.

(ii) and (iii) are immediate.

$\square$

Lemma 6.2. *If $<\alpha_i, \beta> \cong \mathbb{Z} \oplus \mathbb{Z}$ then the following simplifications occur:*

(i) $i = 1$: $B = \begin{bmatrix} 1 & r_2 & r_3 \\ 0 & 1 & r_2 \\ 0 & 0 & 1 \end{bmatrix}$ and $b = (b_1, r_3, r_2)$.

This also implies β is fixed point free.

(ii) $i = 2$: $B = \begin{bmatrix} r_1 & 0 & 0 \\ s_1 & 1 & s_3 \\ 0 & 0 & 1 \end{bmatrix}$ and $b = (b_1, b_2, s_3)$.

If $a_0 \neq 1$, then $s_1 = 0$ and $b_1 = 0$.

(iii) If in addition β is conjugate to α_3 and $b \notin span(e_3)$, then α_3 further conjugates to $[A, e_3]$ where

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \text{ Then } B = \begin{bmatrix} r_1 & r_2 & 0 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & 1 \end{bmatrix}.$$

In this setting the following possibilities arise:

(a) If $a_1 \neq 1$ then $r_2 = s_3 = b_1 = t_1 = t_2 = 0$ (so $b_2 \neq 0$), $s_2 = 1$, and $b(r_1 - 1) = s_1(a_1 - 1)$. In this case, either $r_1 \neq 1$ or β is a translation.

(b) If $a_1 = 1$ and α_3 is not a translation, then $r_2 = t_2 = 0$, $r_1 = s_2 = 1$, and $b_1 = \frac{s_3}{a_2}$. Also either $s_3 = 0$ or both $b_3 = -\frac{s_1}{a_2}$ and $b_1 t_1 = 0$.

Proof.

(i) and (ii) follow from the computations $AB = BA$ and $A(b)+e_3 = B(e_3)+b$.

(iii) Let $B = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{bmatrix}$ and $b = (b_1, b_2, b_3)$.

$BA(e_3) = AB(e_3)$ implies $A_0(r_3, s_3) = (r_3, s_3)$. If $r_3 \neq 0$ or $s_3 \neq 0$, then A is the identity on a plane containing $span(e_3)$.

If $r_3 = s_3 = 0$, then $A(b) + e_3 = B(e_3) + b$ imply $A_0(b_1, b_2) = (b_1, b_2)$, which is not $(0, 0)$ by assumption (this last computation also implies $t_3 = 1$). Thus A has a 2-dimensional 1-eigenspace containing the $span(e_3)$. Now rotate (conjugate) this plane to $span(e_1)^{\perp}$.

The form of β follows from $AB = BA$, $A(b) + e_3 = B(e_3) + b$, $Det(B) = Trace(B) - 2$, and $B(b) = b$.

□

In (iii), if $b \in span(e_3)$, then the group G would keep the $span(e_3)$ invariant and therefore not act properly discontinuously.

A necessary condition for a properly discontinuous $\mathbb{Z} \oplus \mathbb{Z}$ action which does not rely on normalizing the group is given by:

Corollary 6.3. *Let $G \cong \mathbb{Z} \oplus \mathbb{Z}$ be a three dimensional affine group that contains a non-translation. If G acts properly discontinuously, then Det(A)=Trace(A)-2 for any generator $\alpha = [A, a] \in G$.*

Proof. Follows immediately from Lemma 6.2 since the trace and determinant are conjugation invariant.

□

The following two facts follow from a computation.

(1) For all integers k,

$$A_1^k = \begin{bmatrix} 1 & k & \frac{k(k-1)}{2} \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}, \ A_2^k = \begin{bmatrix} a_0^k & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } A_3^k = \begin{bmatrix} A_0^k & 0 \\ 0 & 1 \end{bmatrix}.$$

(2) For all integers k,

(i) $\alpha_1^k = [A_1^k, a_k]$ where $a_k = (\frac{k(k-1)(k-2)}{6}, \frac{k(k-1)}{2}, k)$.

(ii) $\alpha_2^k = [A_2^k, a_k]$ where $a_k = (0, \frac{k(k-1)}{2}, k)$.

(iii) $\alpha_3^k = [A_3^k, ke_3]$.

6.2. **A General Notion of Residue.**

In dimension two, $Res(\beta_{(b,r)}) \neq 0$ is equivalent to $(b,r) \notin \wp_{(0,1)}$, that is, the translation part of β does not lie on a certain parabola. In dimension three the analogous condition will be that the translation part of β not lie on a non-planar curve in $\mathbb{R}^3$. This condition generalizes in any dimension (See General Facts About Affine Groups)

Proposition 6.4. *Let* $G = < \alpha_1, \beta > \cong \mathbb{Z} \oplus \mathbb{Z}$*, where* $\beta = [B, b]$*. The following are equivalent:*

(i) G acts properly discontinuously (and freely) on $\mathbb{R}^3$

(ii) b does not lie on the curve $\{(\frac{1}{6}z(z-1)(z-2), \frac{1}{2}z(z-1), z) : z \in \mathbb{R}\}$

Proof.

Let $B = \begin{bmatrix} 1 & r_2 & r_3 \\ 0 & 1 & r_2 \\ 0 & 0 & 1 \end{bmatrix}$ and $b = (b_1, r_3, r_2)$.

Two proofs will be given. The first will involve coordinates, and the second will use a polynomial map.

Proof 1:

Suppose $b \neq (\frac{1}{6}r_2(r_2-1)(r_2-2), \frac{1}{2}r_2(r_2-1), r_2)$. $(*)$

Assume that all $j, j_i \in \mathbb{Z}^+$ and $k, k_i \in \mathbb{Z}$.

β^j has matrix part $B^j = \begin{bmatrix} 1 & jr_2 & (jr_3 + \frac{r_2^2 j(j-1)}{2}) \\ 0 & 1 & jr_2 \\ 0 & 0 & 1 \end{bmatrix}$ and translation part $(jb_1 + r_2r_3j(j-1) + \frac{1}{6}r_2^3j(j-1)(j-2), jr_3 + \frac{1}{2}r_2^2j(j-1), jr_2)$.

Then $\alpha_1^k\beta^j = [A_1^kB^j, (c_1, c_2, c_3)]$ where $A_1^kB^j = \begin{bmatrix} 1 & c_3 & c_2 \\ 0 & 1 & c_3 \\ 0 & 0 & 1 \end{bmatrix}$,

$c_3 = jr_2 + k$,

$c_2 = jr_3 + \frac{1}{2}r_2^2j(j-1) + r_2jk + \frac{1}{2}k(k-1)$, and

$c_1 = jb_1 + r_2r_3j(j-1) + \frac{1}{6}r_2^3j(j-1)(j-2) + k(jr_3 + \frac{1}{2}r_2^2j(j-1)) + \frac{1}{2}r_2jk(k-1) + \frac{1}{6}k(k-1)(k-2)$.

Suppose there is a sequence $g_i = \alpha_1^{k_i}\beta^{j_i}$ in G in which $\{t_i = j_ir_2 + k_i\}$ converges to t (otherwise $g_i(E) \to \infty$ via the third coordinate for all compact $E \subseteq \mathbb{R}^3$).

There are two cases (1) $r_3 \neq \frac{1}{2}r_2(r_2-1)$ and (2) $r_3 = \frac{1}{2}r_2(r_2-1)$.

(1) Then $c_2 = \frac{1}{2}(jr_2+k)^2 - \frac{1}{2}(jr_2+k) + d$, $d \neq 0$ and $g_i(E) \to \infty$ in the second coordinate.

(2) By $(*)$, $b_1 \neq \frac{1}{6}r_2(r_2-1)(r_2-2)$ and so $b_1 = \frac{1}{6}r_2^3 - \frac{1}{2}r_2^2 + \frac{1}{3}r_2 + d$ where $d \neq 0$.

c_1 above can be rewritten as

$\frac{1}{6}(jr_2+k)^3 + jb_1 - \frac{1}{2}r_2^3j - \frac{1}{2}r_2^2j^2 + \frac{1}{2}r_2^2j + \frac{1}{3}r_2^3j - r_2jk - \frac{1}{2}k^2 + \frac{1}{3}k$

$= \frac{1}{6}(jr_2+k)^3 - \frac{1}{2}(jr_2+k)^2 + j(b_1 - \frac{1}{6}r_2^3 + \frac{1}{2}r_2^2) + \frac{1}{3}k$

$= \frac{1}{6}(jr_2+k)^3 - \frac{1}{2}(jr_2+k)^2 + \frac{1}{3}(jr_2+k) + jd$, $(**)$

implying that $g_i(E) \to \infty$ as $i \to \infty$.

By Lemma 2.3 the inverse sequence need not be examined.

A free action follows from G being torsion free.

Conversely, suppose that $b = (\frac{1}{6}r_2(r_2-1)(r_2-2), \frac{1}{2}r_2(r_2-1), r_2)$.

Choose j_i, k_i so that $(j_ir_2 + k_i) \to t \in \mathbb{R}$ and then $\alpha_1^{k_i}\beta^{j_i}$ converges to the affine map having matrix part $\begin{bmatrix} 1 & t & \frac{1}{2}t(t-1) \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$ and translation part

$(\frac{1}{6}t(t-1)(t-2), \frac{1}{2}t(t-1), t)$. G does not act properly discontinuously.

Proof 2:

Let $\varphi = \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix}$ where $f_1(x) = f_5(x) = f_9(x) = x$, $f_2(x) = f_4(x)$ $= f_7(x) = f_8(x) = 0$, $f_6(x) = -\frac{1}{2}x(x-1)$, and $f_3(x) = -\frac{1}{6}x(x-1)(x-2)$.

That is, $\varphi = \begin{bmatrix} x & 0 & -\frac{1}{6}x(x-1)(x-2) \\ 0 & x & -\frac{1}{2}x(x-1) \\ 0 & 0 & x \end{bmatrix}$. This is a homeomorphism.

Then $\varphi G \varphi^{-1} = \varphi < \alpha_1, \beta > \varphi^{-1} =< [I, e_3], \varphi\beta\varphi^{-1} >$ is generated by two independent translations when $r_3 = \frac{1}{2}r_2(r_2-1)$ and $b_1 \neq \frac{1}{6}r_2(r_2-1)(r_2-2)$. When $r_3 \neq \frac{1}{2}r_2(r_2-1)$, the action of $\varphi\beta\varphi^{-1}$ is "translation" along parallel parabolas not orthogonal to the y-axis. In this case the y-coordinate or the z-coordinate of $g_j(E)$ grows arbitrarily large for any sequence $\{g_j\}$ of distinct elements of $\varphi G \varphi^{-1}$ and compact set E (and $G \cong \mathbb{Z} \oplus \mathbb{Z}$). When b lies on the curve $\mathcal{C}_{\alpha_1}$, either $\varphi G \varphi^{-1}$ does not act properly discontinuously or is cyclic. $\square$

Corollary 6.5. *Suppose $\alpha = [A, a], \beta \in Affine(\mathbb{R}^3)$ are fixed point free and commute, and that the characteristic polynomial of A is $(x-1)^3$. The following are equivalent:*

(i) $G =< \alpha, \beta >$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and acts properly discontinuously.

(ii) The translation part of β does not lie on a curve $\wp_\alpha$ kept invariant by α.

When G is conjugated so that the matrix of α is in Jordan form, the curve $\wp_\alpha$ becomes $\{(\frac{1}{6}z(z-1)(z-2), \frac{1}{2}z(z-1), z) : z \in \mathbb{R}\}$.

Remark 6.6. This result holds in any dimension. The normal form of $\wp_\alpha$ is then $\{(\frac{1}{n!}x_1, \frac{1}{(n-1)!}x_2, ..., \frac{1}{2}x_{n-1}, x) : x \in \mathbb{R}\}$, where $x_i = x(x-1)(x-2)...(x-(n-i))$.

The elements of $< \alpha_1, \beta >$ keep invariant families of curves or surfaces (proofs are omitted):

(i) α_1 keeps the curve $\mathcal{C}_{\alpha_1} = \{(\frac{1}{6}z(z-1)(z-2), \frac{1}{2}z(z-1), z) : z \in \mathbb{R}\}$ invariant.

(ii) Each curve $\mathcal{C}_t = \{(\frac{1}{6}(z(z-1)(z-2)+t, \frac{1}{2}z(z-1), z) : z \in \mathbb{R}\} = \mathcal{C}_{\alpha_1} + te_1$ is α_1 invariant.

(iii) Each curve $\mathcal{C}_{t,s} = \{(\frac{1}{6}z(z-1)(z-2)+t, \frac{1}{2}z(z-1)+s, z) : z \in \mathbb{R}\} = \mathcal{C}_{\alpha_1} + te_1 + se_2$ is sheared s units in the $span(e_1)$ direction by α_1 about the surface

$\cup\{\mathcal{C}_t : t \in \mathbb{R}\}$.

In particular, each surface $\cup\{\mathcal{C}_{t,s} : t \in \mathbb{R}\}$ is α_1 invariant.

(iv) The curve $\mathcal{C}_\beta = \{(\frac{1}{6}z^3 + (\frac{r_3}{r_2} - \frac{r_2}{2})z^2 + (\frac{b_1}{r_2} + \frac{1}{3}r_2^2 - r_3)z, \frac{1}{2}z^2 + (\frac{r_3}{r_2} - \frac{1}{2}r_2)z, z) : z \in \mathbb{R}\}$ is β invariant.

(v) Each curve $\mathcal{C}_\beta + te_1$ is β invariant and each curve $\mathcal{C}_\beta + te_1 + se_2$ is sheared sr_2 units in the $span(e_1)$ direction by β.

In particular, each surface $\cup\{\mathcal{C}_\beta + te_1 + se_2 : t \in \mathbb{R}\}$ is β invariant.

Lemma 6.7. *Let $G =< \alpha_2, \beta >\cong \mathbb{Z} \oplus \mathbb{Z}$ with $a_0 \neq 1$ and fixed point free β having the form given in Lemma 6.2. The following are equivalent:*

(i) G acts freely and properly discontinuously on $\mathbb{R}^3$

(ii) $b_2 \neq \frac{1}{2}s_3(s_3 - 1)$

When this occurs, the behavior of G is essentially the same as the subgroup $< \alpha_1, \beta >$ of $Affine(\mathbb{R}^2)$ with $Res(\beta) \neq 0$.

Specifically, $\alpha_2^k\beta^j(x, y, z) = (*, \alpha_1^k\beta^j(y, z))$ where the α_1 and β on the right side of the equation are two-dimensional.

Proof. Assume that all $j, j_i \in \mathbb{Z}^+$ and $k, k_i \in \mathbb{Z}$.

$$\beta^j = [B^j, (0, jb_2 + \tfrac{1}{2}s_3^2 j(j-1), js_3)] \text{ where } B^j = \begin{bmatrix} r_1^j & 0 & 0 \\ 0 & 1 & js_3 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\alpha_2^k\beta^j \text{ has matrix part } A_2^k B^j = \begin{bmatrix} a_0^k r_1^j & 0 & 0 \\ 0 & 1 & (js_3 + k) \\ 0 & 0 & 1 \end{bmatrix} \text{ and}$$

translation part $(0, jb_2 + \frac{1}{2}s_3^2 j(j-1) + jks_3 + \frac{1}{2}k(k-1), js_3 + k)$.

Suppose first that $b_2 \neq \frac{1}{2}s_3(s_3 - 1)$. Then $b_2 - \frac{1}{2}s_3^2 = -\frac{1}{2}s_3 + d$, $d \neq 0$.

Given a sequence $\{g_i = \alpha_2^{k_i}\beta^{j_i}\}$, assume that $\{j_i s_3 + k_i\}$ converges to $t \in \mathbb{R}$ (else done).

The second coordinate of the translation part of $\alpha_2^k \beta^j$ can be

rewritten as $jb_2 + \frac{1}{2}(js_3 + k)^2 - \frac{1}{2}s_3^2 j - \frac{1}{2}k = \frac{1}{2}(js_3 + k)^2 + j(b_2 - \frac{1}{2}s_3^2) - \frac{1}{2}k$

$= \frac{1}{2}(js_3 + k)^2 - \frac{1}{2}(js_3 + k) + jd.$

This implies that $g_i(E) \to \infty$ as $i \to \infty$ for any compact E.

Then G torsion free implies G acts freely.

Conversely, assume that $b_2 = \frac{1}{2}s_3(s_3 - 1)$ and choose j_i, k_i so that $\{j_i s_3 + k_i\} \to t \in \mathbb{R}$.

The translation part of $\alpha_2^{k_i}\beta^{j_i}$ converges to $(0, \frac{1}{2}t(t-1), t)$. If E is the closure of any neighborhood of $(0,0,0)$ then $g_i(E) \cap E \neq \emptyset$ for infinitely many i.

$\square$

If $s_3 = 0$, the conclusion of the lemma still holds ($b_2 \neq 0$ is then equivalent to β having no fixed points).

Corollary 6.8. *There are uncountably many affine conjugacy classes of 3-dimensional $G \cong \mathbb{Z} \oplus \mathbb{Z}$ that act properly discontinuously.*

Proof.

A 2-dimensional copy of $< \alpha_1, \beta >$ sits inside the 3-dimensional $< \alpha_2, \beta >$ above. Apply Theorem 5.21. $\blacksquare$

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