KANTOROVICH DUALITY AND OPTIMAL TRANSPORT PROBLEMS ON MAGNETIC GRAPHS

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Abstract. We consider Lipschitz- and Arens-Eells-type function spaces constructed for magnetic graphs, which are adapted to the magnetic setting from the classical area of optimal transport on discrete spaces. After establishing the duality between this spaces, we prove a characterization of the extreme points of the unit ball in the (magnetic) Lipschitz space as well as a semi-constructive result relating the (magnetic) Arens-Eells norm for functions defined on a magnetic graph to the (classical) Arens-Eells norm for functions defined on the so-called magnetic lift graph.

1. Introduction

1.1. **Background.** Let $G = (V(G), E(G))$ be an undirected, finite graph without loops or multiple edges (henceforth 'simple'), and suppose one has two mass (probability) distributions $\mu, \nu : V(G) \to \mathbb{R}$. A natural question concerns how one may transport the mass distribution μ to the distribution ν in a manner which is optimal with respect to certain quantities of interest like energy or cost. Such questions constitute the research area of optimal transport on discrete domains [3, 12], a topic which enjoys applications in a number of applied areas such as computer graphics and image processing [7, 6, 9], geometry [11], and physics [1]. One classical approach toward these problems is by way of Kantorovich duality, which in the formulation presented here relates Lipschitz-type and so-called Arens-Eells function spaces via duality.

One setting where discrete transport problems have, to this author's knowledge, not been posed is magnetic (or signed) graphs. These are essentially combinatorial graphs which have been equipped with an additional structure known as a signature, which can be viewed as a discrete analogue of a magnetic potential field [10]. These graphs have helped researchers model systems from discrete quantum mechanics [10] and chemistry [4], to even social psychology [2].

In some classical discrete transport problems, the 1-Wasserstein distance is (in some sense) reformulated via the Arens-Eells norm. In this paper, we will approach so-called magnetic transport not via 1-Wasserstein distance but rather directly through adapted Lipschitz- and Arens-Eells-type function spaces designed for magnetic graphs. After some preliminary remarks, we will establish the duality of these spaces using a form of Riesz representation in the manner of [13]. Then, we will prove a characterization of the extreme points of the unit ball in our (magnetic) Lipschitz space. Finally, we will approach the central problem of computing the σ-Arens-Eells norm via a so-called compression mapping, and then discuss a false conjecture of the author concerning an attempt to establish a further link between the σ-Arens-Eells space on a magnetic graph and the traditional Arens-Eells space on its magnetic lift.

Each of these arguments explores optimal transport on magnetic graphs from the functional analytical perspective, an approach which neglects to consider questions of physical significance. In the classical case, the functional analysis was rooted in the concrete problem of transporting mass distributions between adjacent vertices in a graph. Such concrete realizations of the problems in the magnetic setting remain elusive, and constitute an interesting open problem. Nevertheless, the graph theory and analysis still manage to surprise the researcher and supply interesting questions.

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1.2. **Graph theory preliminaries.** Throughout, $\mathbf{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle, and $\mathbf{S}^1_p = \{z \in \mathbb{C} : z^p = 1\}$ is the abelian group of p-th roots of unity, where $p \geqslant 1$. All graphs considered here are considered to be simple; that is, undirected, with a finite vertex set, no loops, and no multiple edges. If u, v are vertices, adjacency is indicated $u \sim v$. A graph is connected if there exists a path connecting any two of the vertices in the graph.

If $X = (V(X), E(X))$ is a simple graph, we define the *oriented edges* of X to be the set $E^{\text{or}}(X) := \big\{(u, v),\ (v, u) : \{u, v\} \in E(X)\big\}$.

A *signature* on X is a map $\sigma : E^{\text{or}}(X) \to \mathbf{S}^1 : (u, v) \mapsto \sigma_{uv}$, satisfying the algebraic property $\sigma_{vu} = \overline{\sigma_{uv}}$. A *magnetic graph* is a pair (X, σ). Throughout, (non)-magnetic graphs will be denoted with an ('G') 'X'. The trivial signature is defined to be 1 on every oriented edge. A magnetic graph (X, σ) is called *balanced* provided that the product of the values of the signature along any (directed) cycle is 1; otheriwse, X is *unbalanced*. If σ takes values in a finite subgroup of $\mathbf{S}^1$ and $\tau : V(X) \to \mathbf{S}^1_p$ is some function, then we may produce the *τ-switched signature* denoted σ^τ via

$$\sigma^\tau_{uv} := \tau(u)\sigma_{uv}\tau(v)^{-1} \tag{1}$$

Two distinct signatures related in this manner by some switching function are called *switching equivalent*. A signature σ is balanced if and only if it is switching equivalent to the trivial signature [8].

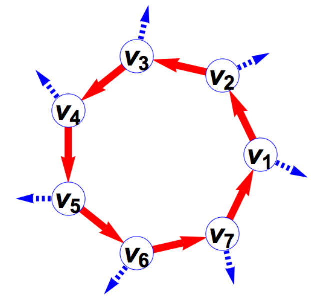
Given a magnetic graph (X, σ) whose signature takes values in some finite group $\mathbf{S}^1_p$ we may construct a related non-magnetic graph called the *magnetic lift*, denoted $\widehat{X}$, via vertex set $V(\widehat{X}) = V(X) \times \mathbf{S}^1_p$, and with the condition that two vertices $(u, \sigma_1), (v, \sigma_2)$ are adjacent if and only if $u \sim v$ in the original graph and $\sigma_2 = \sigma_1\sigma_{uv}$. The signature structure from the original graph is thus encoded in the edge structure of the new one, illustrated in Figure 1.

We will also have occasion to utilize the Hilbert space $\ell_2(V(X)) := \{f : V(X) \to \mathbb{C}\}$ with inner product structure given by

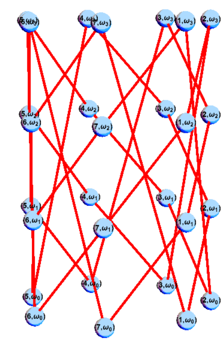
$$\langle f, g\rangle_{\ell_2} := \sum_{u \in V(X)} f(u)\overline{g(u)}.$$

Also, we will use the unit distributions $\delta_u \in \ell_2(X)$ given by

$$\delta_u(v) := \begin{cases} 0 & v \neq u \\ 1 & v = u \end{cases}.$$



(A) 7-cycle, with signature illustrated by the angular offset of the blue arrows from the red ones



(B) magnetic lift of the above cycle; this graph is isomorphic to a cycle on 28 vertices

FIGURE 1. A magnetic cycle graph and its lift.

1.3. **Classical Kantorovich duality, extreme points.** To complete this preliminary section, let us recall some results pertaining to non-magnetic graphs. If μ, ν are two mass distributions on the vertices of a connected graph G equipped with shortest-path metric d, we consider transport plans $\gamma : V(G) \times V(G) \to \mathbb{R}_{\geqslant 0}$ which are mass distributions on the Cartesian product of the vertex set whose marginals agree with μ and ν, and such that $\gamma(u, v)$ represents the amount of mass transported from vertex u to vertex v. $\Gamma(\mu, \nu)$ represents the set of all transport plans between μ and ν. The *1-Wasserstein distance* between μ, ν is then given by

$$W_1(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu, \nu)} \sum_{u,v} \gamma(u, v) d(u, v).$$

If one chooses an arbitrary but fixed base vertex, say $u_0 \in V(G)$, one can define the normed space

$$\text{Lip}_0(G) := \Big\{ f : V(G) \to \mathbb{R} \;\Big|\; f(u_0) = 0 \Big\}$$

where

$$||f||_{\mathrm{Lip}} = \max_{u\sim v} |f(u) - f(v)|. \tag{2}$$

Similarly, for each pair of adjacent vertices $u, v \in V(G)$ we may define the *combinatorial atom* $m_{uv} : V(G) \to \mathbb{R}$ defined by

$$m_{uv}(w) = \begin{cases} 1 & w = u \\ -1 & w = v \\ 0 & \text{otherwise} \end{cases} \quad \text{for each } x \in V(G).$$

Subsequently, we may construct the *Arens-Eells* space via

$$\text{Æ}(G) := \operatorname{span}_{\mathbb{R}}\{m_{uv} \,\Big|\, u, v \in V(G), \quad u \sim v\}$$

equipped with the norm

$$||m||_{\text{Æ}} := \inf \Big\{ \sum_{i=1}^{n} |a_i| \,\Big|\, m = \sum_{i=1}^{n} a_i m_{u_i v_i} \Big\}.$$

Viewing Æ(G) and $\mathrm{Lip}_0(G)$ as subspaces of the Hilbert space $\ell_2(G)$, one can prove via Riesz representation that Æ(G) is isometrically isomorphic to $\mathrm{Lip}_0(G)^*$, e.g. [13]. This is the so-called (classical) Kantorovich duality to which we dedicate a good part of the sequel.

2. Duality and extreme points

2.1. **Arens-Eells, signed Lipschitz spaces.** Let (X, σ) be a magnetic graph, endowed with usual shortest-path metric d. We define the signed Lipschitz function space

$$\mathrm{Lip}^{\sigma}(X) := \{f : V(X) \to \mathbb{C} \;\big|\; \exists\, C \geqslant 0 \text{ s.t. } |f(u) - \sigma_{uv} f(v)| \leqslant C \text{ for each } u \sim v\}$$

This definition leads to a natural choice of σ-Lipschitz norm, which we pair with an equivalent formulation. For each $f \in \mathrm{Lip}^{\sigma}(X)$, set

$$\begin{aligned} ||f||_{\mathrm{Lip}^{\sigma}} &:= \inf\{C \geqslant 0 \,\big|\; |f(u) - \sigma_{uv} f(v)| \leqslant C \text{ for each } u \sim v\} \\ &= \max_{u\sim v} |f(u) - \sigma_{uv} f(v)|. \end{aligned} \tag{3}$$

Lemma 2.1.1. *Let (X, σ) be an unbalanced magnetic graph. Then $||\cdot||_{Lip^{\sigma}}$ is a norm.*

Proof. Let $f, g \in \mathrm{Lip}^{\sigma}(X)$ and $\alpha \in \mathbb{C}$. Clearly $||\cdot||_{\mathrm{Lip}^{\sigma}} \geqslant 0$, and $||\alpha f||_{\mathrm{Lip}^{\sigma}} = |\alpha| ||f||_{\mathrm{Lip}^{\sigma}}$ from the definition. The triangle inequality is obtained as follows:

$$\begin{aligned} ||f+g||_{\mathrm{Lip}^{\sigma}} &= \max_{u\sim v} |(f+g)(u) - \sigma_{uv}(f+g)(v)| \leqslant \max_{u\sim v} |f(u) - \sigma_{uv} f(v)| + |g(u) - \sigma_{uv} g(v)| \\ &= \max_{u\sim v} |f(u) - \sigma_{uv} f(v)| + \max_{u\sim v} |g(u) - \sigma_{uv} g(v)| = ||f||_{\mathrm{Lip}^{\sigma}} + ||g||_{\mathrm{Lip}^{\sigma}}. \end{aligned}$$

For definiteness, let us assume that $||f||_{\mathrm{Lip}^{\sigma}} = 0$. The max formulation in equation (3) would imply that for each pair of adjacent vertices u, v one has $f(u) = \sigma_{uv} f(v)$, forcing either $f \equiv 0$ or $f = \lambda f_p$, where $|\lambda| > 0$ and $|f_p| \equiv 1$ is a switching function for each of the connected components of (X, σ), such that $\sigma^{f_p} \equiv 1$ as in equation (1). The latter case contradicts the assumption that (X, σ) is unbalanced so $f \equiv 0$. $\square$

Let us now consider two adjacent vertices $u, v \in V(X)$ and define the *magnetic atom* $m^{\sigma}_{uv} : V(X) \to \mathbb{C}$ as follows:

$$m^{\sigma}_{uv}(w) := \begin{cases} 1 & w = u \\ -\sigma_{uv} & w = v \\ 0 & \text{otherwise} \end{cases}.$$

We define the magnetic Arens-Eells space

$$\text{Æ}^{\sigma}(X) := \operatorname{span}_{\mathbb{C}}\{m^{\sigma}_{uv} \,\Big|\, u, v \in V(X); \quad u \sim v\}.$$

Elements of this space will be called magnetic molecules. We will use the next lemma to verify that Æ$^{\sigma}(X)$ indeed recovers all of $\ell_2(X)$ under the right condition.

Lemma 2.1.2. *Let* (X, σ) *be an unbalanced magnetic graph. Then* $\text{Æ}^\sigma(X) = \ell_2(X)$.

Proof. We will prove this lemma by showing that the orthogonal complement of $\text{Æ}^\sigma(X)$ in $\ell_2(X)$ is merely $\{0\}$. Let $f \in \ell_2(X)$ be such that $\langle f, \overline{m} \rangle_2 = 0$ for each $m \in \text{Æ}^\sigma(X)$. In particular, for each pair of adjacent vertices $u, v \in V(X)$ one has $\langle f, \overline{m_{uv}^\sigma} \rangle_2 = 0$. Explicitly, this means $f(u) = \sigma_{uv} f(v)$ forcing either $f \equiv 0$ or $f = \lambda f_p$, where $|\lambda| > 0$ and $|f_p| \equiv 1$ is a switching function for each of the connected components of (X, σ), such that $\sigma^{f_p} \equiv 1$ as in equation (1). The latter case contradicts the assumption that (X, σ) is unbalanced so $f \equiv 0$. □

We define for each molecule $m \in \text{Æ}^\sigma(X)$ the norm

$$||m||_{\text{Æ}^\sigma} := \inf \left\{ \sum_{i=1}^n |a_i| \;\middle|\; m = \sum_{i=1}^n a_i m_{u_i v_i}^\sigma;\ \{a_i\}_i \subset \mathbb{C};\ u_i \sim v_i \in V(X) \right\}.$$

Let us quickly check that this is actually a norm as claimed.

Lemma 2.1.3. *Let* (X, σ) *be an unbalanced magnetic graph. Then* $||\cdot||_{\text{Æ}^\sigma}$ *is a norm.*

Proof. The positivity, homogeneity, and triangle inequality for this norm are all easily checked. We only need argue for why $||\cdot||_{\text{Æ}^\sigma}$ is in fact definite. Suppose for some molecule one has $||m||_{\text{Æ}^\sigma} = 0$. For positive integer k, find some finite linear combination of atoms $\sum_i a_i^k m_{u_i^k v_i^k}^\sigma$ for which

$$m = \sum_i a_i^k m_{u_i^k v_i^k}^\sigma, \quad \sum_i |a_i^k| < \frac{1}{k}.$$

Then,

$$||m||_{\ell_2} = ||\sum_i a_i^k m_{u_i^k v_i^k}^\sigma||_{\ell_2} \leqslant \sum_i |a_i^k| ||m_{u_i^k v_i^k}^\sigma||_{\ell_2} \leqslant 2 \sum_i |a_i^k| < \frac{2}{k} \to 0 \text{ as } k \to \infty.$$

From the definiteness of the $||\cdot||_{\ell_2}$ norm, the claim is verified. □

We have constructed two Banach function spaces for unbalanced magnetic graphs, $\mathrm{Lip}^\sigma(X)$ and $\text{Æ}^\sigma(X)$ (we verified their structures as normed spaces, completeness follows from their finite dimension). In the sequel, we identify them as dual to one another.

Assuming σ takes values in a finite group $\mathbf{S}_p^1$, we consider its magnetic lift $\widehat{X}$ as in Figure 1. Recall from earlier the Lip_0 space associated to a fixed base point $(u_0, \xi_0) \in V(\widehat{X})$:

$$\mathrm{Lip}_0(\widehat{X}) = \left\{ f : V(\widehat{X}) \to \mathbb{C} \;\middle|\; f(u_0, \xi_0) = 0,\ \exists\, C \geqslant 0 \text{ s.t.} |f(u, \xi) - f(v, \xi')| \leqslant C\ \forall\ (u, \xi) \sim (v, \xi') \in V(\widehat{X}) \right\}.$$

For each $f \in \mathrm{Lip}_0(\widehat{X})$ we recall the norm $||f||_{\mathrm{Lip}}$ as in equation (2).

For each pair of adjacent vertices $(u, \xi), (v, \xi\sigma_{uv}) \in V(\widehat{X})$, let us define the (non-magnetic) atom $m_{(u,\xi),(v,\xi\sigma_{uv})} : V(\widehat{X}) \to \mathbb{C}$ by setting, for each $(w, \rho) \in V(\widehat{X})$,

$$m_{(u,\xi),(v,\xi\sigma_{uv})}(w, \rho) := \begin{cases} 1 & (w, \rho) = (u, \xi) \\ -1 & (w, \rho) = (v, \xi\sigma_{uv}) \\ 0 & \text{otherwise} \end{cases}.$$

We define the Arens-Eells space for $\widehat{X}$ by

$$\text{Æ}(\widehat{X}) := \mathrm{span}_{\mathbb{C}} \left\{ m_{(u,\xi),(v,\xi\sigma_{uv})} \;\middle|\; (u, \xi) \sim (v, \xi\sigma_{uv}) \in V(\widehat{X}) \right\},$$

the elements of which, as before, we term 'molecules' on $\widehat{X}$. For each molecule $m \in \text{Æ}(\widehat{X})$, let us put

$$||m||_{\text{Æ}} := \inf \left\{ \sum_{i=1}^n |a_i| \;\middle|\; m = \sum_{i=1}^n a_i m_{(u_i,\xi_i),(v_i,\xi_i\sigma_{u_i v_i})};\ \{a_i\}_i \subset \mathbb{C};\ (u_i, \xi_i), (v_i, \xi_i\sigma_{u_i v_i}) \in V(\widehat{X}) \right\}.$$

2.2. **Duality.** We will now adapt the classical duality result mentioned in the preliminary discussion to the function spaces designed for magnetic graphs. The argument is in the manner of Weaver [13].

Theorem 2.1. *Let (X, σ) be an unbalanced magnetic graph. Then $Æ^\sigma(X)^*$ is isometrically isomorphic to $Lip^\sigma(X)$.*

Proof. Let us define a linear mapping $T_1 : Æ^\sigma(X)^* \to \mathrm{Lip}^\sigma(X)$ in the following manner. Let $M \in Æ^\sigma(X)^*$, and notice that since $Æ^\sigma(X) = \ell_2(X)$ as finite dimensional vector spaces, M may be viewed as a continuous linear functional on $\ell_2(X)$. In turn, by using the finite-dimensional Riesz representation theorem on the space $\ell_2(X)$, we may obtain a representative function $f_M \in \ell_2(X)$ so that for each $m \in Æ^\sigma(X)$, one has $M(m) = \langle f_M, \overline{m}\rangle_{\ell_2}$. Put $T_1(M) = f_M$. Notice that for each pair of adjacent vertices $u, v \in V(X)$ we have

$$\begin{aligned} |f_M(u) - \sigma_{uv} f_M(v)| &= |\langle f_M, \overline{m^\sigma_{vu}}\rangle_{\ell_2}| \\ &= |M(m^\sigma_{vu})| \leqslant ||M||_{Æ^\sigma *} ||m^\sigma_{vu}||_{Æ^\sigma} \leqslant ||M||_{Æ^\sigma *}. \end{aligned}$$

In turn, by taking a max,

$$||f_M||_{\mathrm{Lip}^\sigma} = \max_{u\sim v} |f_M(u) - \sigma_{uv} f_M(v)| \leqslant ||M||_{Æ^\sigma *}$$

which implies that T_1 is a nonexpansive operator. As a note, the linearity of T_1 is inherited from the Riesz Representation. Let us now suggestively define a mapping

$$T_2 : \mathrm{Lip}^\sigma(X) \to Æ^\sigma(X)^* : f \mapsto M_f,$$

where for each $m \in Æ^\sigma(X)$ we set $M_f(m) = \langle f, \overline{m}\rangle_{\ell_2}$. We verify that given any $m \in Æ^\sigma(X)$, realized as a finite linear combination $\sum_i a_i m^\sigma_{u_i v_i}$, it holds that

$$\begin{aligned} |M_f(m)| = |\langle f, \overline{m}\rangle_{\ell_2}| &= \big|\langle f, \sum_i \overline{a_i m^\sigma_{u_i v_i}}\rangle_{\ell_2}\big| \\ &\leqslant \sum_i |a_i| \cdot |\langle f, \overline{m^\sigma_{u_i v_i}}\rangle_{\ell_2}| = \sum_i |a_i| \cdot |f(u) - \sigma_{uv} f(v)| \\ &\leqslant ||f||_{\mathrm{Lip}^\sigma} \cdot \sum_i |a_i|. \end{aligned}$$

By taking an inf over all possible expressions $\sum_i |a_i|$ which realize m, we obtain the inequality $|M_f(m)| \leqslant ||f||_{\mathrm{Lip}^\sigma} \cdot ||m||_{Æ^\sigma}$, which implies $||M_f||_{Æ^\sigma(X)^*} \leqslant ||f||_{\mathrm{Lip}^\sigma}$, showing that T_2 is a non-expansive linear operator as well. The composition $T_2 T_1 : Æ^\sigma(X)^* \to Æ^\sigma(X)^*$ is easily checked to be the identity mapping. Since the mapping T_1 and its inverse are nonexpansive and invertible, T_1 is a vector space isomorphism and an isometry of Banach spaces, finalizing the claim. ☐

2.3. **Extreme points.** Some contextual remarks are in order before presenting the result.

Definition 2.3.1. *Let G be simple graph, $f \in Lip_0(X)$, with $||f||_{Lip} \leqslant 1$. Then f is called an extreme point of the unit ball in $Lip_0(X)$, denoted B_{Lip}, provided that for any $g \in Lip_0(X)$, if*

$$\Big\{ f + tg \,\Big|\, t \in [-1, 1] \Big\} \subset B_{Lip},$$

then $g \equiv 0$.

Definition 2.3.2. *Let G be simple graph and $f \in B_{Lip^\sigma}$. We say an edge $\{u, v\} \in E(X)$ is satisfied by f if*

$$|f(u) - f(v)| = 1.$$

Farmer [5] proves a version of the following result.

Theorem 2.2 (Farmer 1994)**.** *Let G be a connected simple graph and $f \in B_{Lip^\sigma}$. Then f is an extreme point of B_{Lip} if and only if the graph H_f, constructed with vertex set $V(G)$ and edge set*

$$E(H_f) := \Big\{ \{u, v\} \in E(G) \,\Big|\, \{u, v\} \text{ is satisfied by } f \Big\}$$

is connected.

We present an adaptation of this result to the function spaces designed for magnetic graphs. First, two preliminary definitions in the spirit of the preceding remarks.

Definition 2.3.3. *Let (X, σ) be an unbalanced magnetic graph, $f \in Lip^{\sigma}(X)$, with $||f||_{Lip^{\sigma}} \leqslant 1$. Then f is called an extreme point of the unit ball in $Lip^{\sigma}(X)$, denoted $B_{Lip^{\sigma}}$, provided that for any $g \in Lip^{\sigma}(X)$, if*

$$\left\{ f + tg \,\middle|\, t \in [-1, 1] \right\} \subset B_{Lip^{\sigma}},$$

then $g \equiv 0$.

Definition 2.3.4. *Let (X, σ) be an unbalanced graph, and suppose $f \in B_{Lip^{\sigma}}$. We say an edge $\{u, v\} \in E(X)$ is σ-satisfied by f if*

$$|f(u) - \sigma_{uv} f(v)| = 1.$$

Note as before that the quantity $|f(u) - \sigma_{uv} f(v)|$ need not depend on the choice of orientation of the edge being evaluated, i.e. $|f(u) - \sigma_{uv} f(v)| = | - \sigma_{vu} f(u) + f(v)|$.

Theorem 2.3. *Let (X, σ) be an unbalanced graph, and $f \in B_{Lip^{\sigma}}$. Then f is an extreme point of $B_{Lip^{\sigma}}$ if and only if the magnetic graph H_f defined by the vertex set $V(X)$, the edge set*

$$E(H_f) := \left\{ \{u, v\} \in E(X) \,\middle|\, \{u, v\} \text{ is } \sigma\text{-satisfied by } f \right\},$$

and which we equip with the same signature structure σ as on X, is unbalanced on each of its connected components.

Proof. Let us begin with the converse by supposing that H_f is unbalanced on each of its connected components, and that some $g \in \mathrm{Lip}^{\sigma}(X)$ satisfies

$$\left\{ f + tg \,\middle|\, t \in [-1, 1] \right\} \subset B_{\mathrm{Lip}^{\sigma}}.$$

This implies that for every edge $\{u, v\} \in E(X)$ σ-satisfied by f, it holds for every $t \in [-1, 1]$

$$|f(u) - \sigma_{uv} f(v) + t \left(g(u) - \sigma_{uv} g(v) \right)| \leqslant 1.$$

Knowing that $|f(u) - \sigma_{uv} f(v)| = 1$, the only way that the inequality above can hold for every $t \in [-1, 1]$ is if $|g(u) - \sigma_{uv} g(v)| = 0$ at every σ-satisfied edge; that is, $g(u) = \sigma_{uv} g(v)$. As we have seen before, this would imply that on each connected component of H_f, g must be either identically 0 or a scalar multiple of a switching function for σ associated to the trivial signature. Since the latter implication contradicts our assumption that (H_f, σ) is unbalanced on each of these connected components, it must hold that $g \equiv 0$, implying that f is an extreme point for $B_{\mathrm{Lip}^{\sigma}}$.

Next, let us assume that $f \in B_{\mathrm{Lip}^{\sigma}}$ is an extreme point for $B_{\mathrm{Lip}^{\sigma}}$, and that (H_f, σ) has some disjoint balanced connected components, say $A_1, A_2, ..., A_k \subset V(X)$, $k \geqslant 1$. For each of these components, viewed themselves as magnetic graphs with edge sets induced by H_f and the same signature structure σ, we may find a switching function $g_i : A_i \to \mathbf{S}^1$ associated to the trivial signature; that is, at each $\{u, v\} \in E(A_i)$, it holds $g_i(u) = \sigma_{uv} g_i(v)$. We also acknowledge that $E(H_f)$ need not contain every edge in the original graph, so let us identify

$$\epsilon := \max \left\{ |f(u) - \sigma_{uv} f(v)| \,\middle|\, \{u, v\} \in E(X), |f(u) - \sigma_{uv} f(v)| < 1 \right\} < 1.$$

If this set happens to be empty, choose $\epsilon := 0$. Let us stitch these functions g_i together and scale appropriately; i.e. put $g \in \mathrm{Lip}^{\sigma}(X)$ defined by

$$g(x) := \begin{cases} \frac{1-\epsilon}{2} g_i(x) & \text{if } x \in A_i \text{ for some } 1 \leqslant i \leqslant k \\ 0 & \text{if } x \notin A_i \text{ for each } 1 \leqslant i \leqslant k \end{cases}$$

for each $x \in V(X)$. Having chosen the switching functions and ϵ appropriately, and guaranteeing the existence of at least one switching function on a connected component of H_f, we note that $g \not\equiv 0$. Let us check that $\{f + tg \,\big|\, t \in [-1,1]\} \subset B_{\mathrm{Lip}^\sigma}$. If $\{u,v\} \in E(X)$, one of four possible cases holds. For the first two cases, if both $u, v \in A_i$ for some $1 \leqslant i \leqslant k$, or if $u, v \notin A_i$ for every $1 \leqslant i \leqslant k$, it holds that

$$\begin{aligned} |f(u) - \sigma_{uv} f(v) + t\,(g(u) - \sigma_{uv} g(v))\,| &\leqslant |f(u) - \sigma_{uv} f(v)| + |t| \cdot |\,(g(u) - \sigma_{uv} g(v))\,| \\ &\leqslant |f(u) - \sigma_{uv} f(v)| + |\,(g(u) - \sigma_{uv} g(v))\,| \leqslant 1 + 0 = 1. \end{aligned}$$

Third, if $u \in A_i$ for some $1 \leqslant i \leqslant k$, and $v \notin A_i$ for every $1 \leqslant i \leqslant k$, it holds $|g(u) - \sigma_{uv} g(v)| = \frac{1-\epsilon}{2} < 1$. In turn,

$$\begin{aligned} |f(u) - \sigma_{uv} f(v) + t\,(g(u) - \sigma_{uv} g(v))\,| &= |f(u) - \sigma_{uv} f(v)| + |t|\,|\,(g(u) - \sigma_{uv} g(v))\,| \\ &\leqslant |f(u) - \sigma_{uv} f(v)| + |\,(g(u) - \sigma_{uv} g(v))\,| \leqslant \epsilon + \frac{1-\epsilon}{2} \\ &< \epsilon + 1 - \epsilon = 1. \end{aligned}$$

Lastly, if $u \in A_i$ for some i, and $v \in A_\ell$ for some $\ell \neq i$, then it holds

$$|g(u) - \sigma_{uv} g(v)| = |g_i(u) - \sigma_{uv} g_\ell(v)| \leqslant |g_i(u)| + |g_\ell(v)| = 2\frac{1-\epsilon}{2} = 1 - \epsilon,$$

which, with f, implies

$$\begin{aligned} |f(u) - \sigma_{uv} f(v) + t\,(g(u) - \sigma_{uv} g(v))\,| &= |f(u) - \sigma_{uv} f(v)| + |t|\,|\,(g(u) - \sigma_{uv} g(v))\,| \\ &\leqslant |f(u) - \sigma_{uv} f(v)| + |\,(g(u) - \sigma_{uv} g(v))\,| \leqslant \epsilon + 1 - \epsilon = 1. \end{aligned}$$

This confirms that at every possible edge $\{u,v\} \in E(X)$ and $t \in [-1,1]$, we have $|(f+tg)(u) - \sigma_{uv}(f + tg)(v)| \leqslant 1$, which forces

$$\{f + tg \,\big|\, t \in [-1,1]\} \subset B_{\mathrm{Lip}^\sigma},$$

and contradicts our assumption that f was an extreme point. This means H_f must be unbalanced on each of its connected components, and completes the proof. ☐

2.4. **Compression.** The final result we will present concerns an approach into the computation of the norm $||\cdot||_{\mathrm{Lip}^\sigma}$. We wish to say something about how the magnetic transport norm for molecules on the original magnetic graph may be related to the classical transport norm for molecules on the magnetic lift graph. First we need a way of translating between spaces.

Definition 2.4.1. *Let (X, σ) be a magnetic graph. We define the linear compression mapping*

$$\mathcal{C} : \text{Æ}(\widehat{X}) \to \text{Æ}^\sigma(X)$$

by the following. For each $m \in \text{Æ}(\widehat{X}), u \in V(X)$, put

$$(\mathcal{C}m)(u) = \sum_{\xi \in \mathbf{S}^1_p} \xi m(u, \xi).$$

Theorem 2.4. *Let (X, σ) be a magnetic graph. Then $\mathcal{C}$ is a surjective contraction from $\text{Æ}(\widehat{X})$ onto $\text{Æ}^\sigma(X)$.*

Proof. First, let us verify the surjectivity of the mapping $\mathcal{C}$. Suppose we take some $m^\sigma \in \text{Æ}^\sigma(X)$, which we represent with a finite linear combination of magnetic atoms in the form

$$m^\sigma = \sum_{i=1}^{n} a_i m^\sigma_{u_i v_i}.$$

By choosing $m^* \in \text{Æ}(\widehat{X})$ to be

$$m^* = \sum_{i=1}^{n} a_i m_{(u_i,1),(v_i \sigma_{u_i v_i})},$$

we may compute

$$(\mathcal{C}m^*)(u) = \sum_{\xi \in \mathbf{S}^1_p} m^*(u,\xi) = \sum_{\xi \in \mathbf{S}^1_p} \sum_{i=1}^n a_i m_{(u_i,1),(v_i\sigma_{u_iv_i})}(u,\xi)$$
$$= \sum_{i=1}^n a_i \delta_{u_i}(u) - \sigma_{u_iv_i}\delta_{v_i}(u) = \sum_{i=1}^n a_i m^\sigma_{u_iv_i}(u) = m^\sigma(u)$$

which verifies the onto claim; notice that from this computation, we obtain the general relation that for each adjacent pair of vertices $(u,\omega),(v,\omega\sigma_{uv})$ in the lift, we have

$$\mathcal{C}(m_{(u,\omega),(v,\omega\sigma_{uv})}) = \omega m^\sigma_{uv}. \tag{4}$$

Now let $m \in \text{Æ}(\widehat{X})$ be given, and suppose we represent it by an optimal linear combination of atoms in the form

$$m = \sum_{\ell=1}^m b_\ell m_{(u_\ell,\omega_\ell),(v_\ell,\omega_\ell\sigma_{u_\ell v_\ell})}, \quad \text{where } ||m||_{\text{Æ}} = \sum_{\ell=1}^m |b_\ell|.$$

In turn, from (4), one has

$$\mathcal{C}m = \sum_{\ell=1}^m b_\ell \omega_\ell m^\sigma_{u_\ell v_\ell}.$$

Applying inequalities, we see

$$||\mathcal{C}m||_{\text{Æ}^\sigma} \leqslant \sum_{\ell=1}^m |b_\ell\omega_\ell| = \sum_{\ell=1}^m |b_\ell| = ||m||_{\text{Æ}}$$

which implies that $\mathcal{C}$ is a contraction. □

As a simple corollary to the preceding theorem, we have the following equation.

Corollary 2.4.1. *Let $m^\sigma \in \text{Æ}^\sigma(X)$. We have the equation*

$$||m^\sigma||_{\text{Æ}^\sigma} = \min\left\{||m||_{\text{Æ}} \,\middle|\, m \in \text{Æ}(\widehat{X}); \mathcal{C}m = m^\sigma\right\}.$$

Proof. Knowing that $\mathcal{C}$ is surjective, the set on the right is nonempty; and, knowing also that $\mathcal{C}$ is a contraction, we may write

$$||m^\sigma||_{\text{Æ}^\sigma} = \inf\left\{||m||_{\text{Æ}} : m \in \text{Æ}(\widehat{X}); \mathcal{C}m = m^\sigma\right\}.$$

To justify the use of a min in the equation, we wish to check that the norm $||m^\sigma||_{\text{Æ}^\sigma}$ is attained in the set somewhere. To this end, fix some $m^\sigma \in \text{Æ}^\sigma(X)$, and realize it as an optimal linear combination of magnetic atoms, i.e.

$$m^\sigma = \sum_{i=1}^n a_i m^\sigma_{u_iv_i}, \quad \text{where } ||m^\sigma|| = \sum_{i=1}^n |a_i|.$$

Notice that, as in the preceding proof, the molecule $m^* \in \text{Æ}(\widehat{X})$ given by

$$m^* = \sum_{i=1}^n a_i m_{(u_i,1),(v_i\sigma_{u_iv_i})}$$

satisfies $\mathcal{C}m^* = m^\sigma$. Since the expression above is one realization of m^* as a linear combination of (non-magnetic) atoms, it holds that $||m^*||_{\text{Æ}} \leqslant \sum_i |a_i| = ||m^\sigma||_{\text{Æ}^\sigma}$. Moreover, since $\mathcal{C}m^* = m^\sigma$ and $\mathcal{C}$ is a contraction, it holds that $||m^\sigma||_{\text{Æ}^\sigma} \leqslant ||m^*||_{\text{Æ}}$. Putting the two inequalities together, i.e.

$$||m^*||_{\text{Æ}} \leqslant \sum_i |a_i| = ||m^\sigma||_{\text{Æ}^\sigma} \leqslant ||m^*||_{\text{Æ}}$$

we find $||m^*||_{\text{Æ}} = ||m^\sigma||_{\text{Æ}^\sigma}$ as desired. This completes the proof. □

The author had hoped to improve this result via a conjecture concerning the link of simple so-called *path molecules* defined on the original graph as well as the lift.

Definition 2.4.2. *Let $u, v \in V(X)$ be any two distinct vertices, and suppose*

$$P(u;v) := \{(u =: u_0, u_1), (u_1, u_2), ..., (u_{n-2}, u_{n-1}), (u_{n-1}, u_n := v)\} \subset E^{or}(X)$$

is a sequence of incident edges forming a path between u, v. We define the magnetic path molecule $m_{P(u;v)} \in Æ^{\sigma}(X)$ by the formula

$$m^{\sigma}_{P(u;v)}(w) := \begin{cases} 1 & w = u \\ -\prod_{i=0}^{n-1} \sigma_{u_i u_{i+1}} & w = v \\ 0 & \textit{otherwise} \end{cases}.$$

Definition 2.4.3. *Let G be a connected simple graph and let $u, v \in V(G)$ be any two distinct vertices, and suppose*

$$Q(u;v) := \{(u =: u_0, u_1), (u_1, u_2), ..., (u_{n-2}, u_{n-1}), (u_{n-1}, u_n := v)\} \subset E^{or}(G)$$

is a sequence of incident edges forming a path between u, v. We define the path molecule $m_{Q(u;v)} \in Æ(X)$ by the formula

$$m_{Q(u;v)}(w) := \begin{cases} 1 & w = u \\ -1 & w = v \\ 0 & \textit{otherwise} \end{cases}.$$

The author (falsely) conjectured that

$$||m^{\sigma}_{P(u;v)}||_{Æ^{\sigma}} = ||m_{Q((u,1);(v,\prod_{i=0}^{n-1} \sigma_{u_i u_{i+1}}))}||_{Æ},$$

where $m_{Q((u,1);(v,\prod_{i=0}^{n-1} \sigma_{u_i u_{i+1}}))} \in Æ(\widehat{X})$ is such that $\mathcal{C} m_{Q((u,1);(v,\prod_{i=0}^{n-1} \sigma_{u_i u_{i+1}}))} = m^{\sigma}_{P(u;v)}$, so the preceding corollary remains the most constructive effort towards the calculation of $||\cdot||_{\mathrm{Lip}^{\sigma}}$ thus far. The absence of such a strong relationship between these types of molecules, and by extension their ambient spaces, hints at a broader nontriviality present in understanding their relationship analytically.

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