PROPORTION DIFFERENCES USING

THE BETA-BINOMIAL DISTRIBUTION

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CHAPTER 1

INTRODUCTION

1.1   STATEMENT OF PROBLEM AND PURPOSE

The Binomial distribution is appropriate when analyzing a fixed number of

independent Bernoulli trials, where the random variable of interest is the number of

successes, or alternatively, p, the proportion of successes. For example, if a fair coin is

tossed n times and the random variable X denotes the number of heads, then X is a

binomial random variable, and the parameter of interest is often p = proportion of

successes. Many studies in the biological sciences use the binomial distribution. As an

example, consider experiments conducted to determine the effect of new drugs on small

animals. A common design involves T populations (treatments, groups, etc.) over which

the probabilities of response, p_i *(i = 1, 2, ...T)* are to be compared. Many statistical

methods for analyzing binary data such as this, including confidence intervals and

multiple comparisons, are based on the binomial model.

One of the key elements in a binomial experiment is the assumption of

independence between the n trials. However, in many experiments, additional factors can

induce extra-binomial variability.   Hence the binomial sampling assumption of

independence is invalid. In these cases, there is often correlation among the trials which

can lead to overdispersion.   To illustrate this, consider a group of highly correlated trials.

If one unit in the group responds, then the probability that another unit in the same group

responds increases.   Likewise, if one unit in the group does not respond, then the probability that another unit in the same group does not respond is higher. This results in overdispersion. Thus there is increased variability created by the more extreme probabilities.

A common sampling model for overdispersed zero – one data is the beta-binomial distribution. The beta-binomial distribution is derived by assuming a beta prior on the binomial probability p. (Derivation 1.1 in Appendix A) There are several parameterizations of the beta-binomial distribution. One such parameterization introduces the parameter Φ, which measures departure from binomial sampling. Hence, when $\Phi = 0$ there is no overdipersion and binomial sampling is retained. As Φ increases there is more correlation or overdispersion in the experiment, and departure from binomial sampling increases.

In reproductive toxicity studies extra-binomial variability is often present. Haseman and Piegorsch (1994) note that there are several important statistical issues associated with this type of data, including proper identification of the experimental unit, appropriate methodology for data analysis, and sample size considerations. In these experiments, the experimental unit is the parent that was exposed to some chemical, drug or treatment. This is typically the female rodent or her entire litter; it is not the individual conceptus. Thus individual conceptuses represent correlated observations (Piegorsch, 1993).   This correlation within a litter is often referred to as the "litter effect", or the intralitter correlation. The beta-binomial model can be reparameterized to yield this intralitter correlation: $\rho_i = \Phi_i / (1 + \Phi_i)$, where ρ_i is the intralitter correlation for the ith treatment (or dose).   This parameterization can be easily interpreted

in terms of correlation. As Φ_i increases, the correlation approaches one, illustrating that

Φ_i measures departure from binomial sampling. When $\Phi_i = 0$, the correlation is zero,

implying that we have independence among the trials, hence binomial sampling is

retained.

In most reproductive toxicity experiments, the parameters of interest are the

response rates p_i, associated with the ith treatment (or dose) level. The researcher is

often interested in determining if there are significant differences in the proportion of

successes at the various treatment (or dose) levels. This can be accomplished via

confidence intervals in the two population case, or via multiple comparisons in the k

population case.

There are several methods for estimating binomial proportion differences via

confidence intervals. In the two population case, the standard textbook method is based

on the maximum likelihood estimators (MLEs). If n_1 and n_2 are sufficiently large, then

an approximate $100(1-\alpha)\%$ confidence interval for $p_1 - p_2$ is

$$(\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2} \left(\frac{\hat{p}_1 (1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2 (1 - \hat{p}_2)}{n_2} \right)^{1/2} ,$$

where $\hat{p}_1$ and $\hat{p}_2$ are the MLEs and $z_{\alpha/2}$ is the upper $1 - \alpha/2$ percentage point of the

standard normal distribution. There are, however, problems with this interval when the

sample sizes are too small, and also when the proportions are close to zero or one. As a

result, many authors have proposed alternative methods for estimating binomial

proportions. In particular, Beal (1987) considered several intervals based on the

asymptotic properties of the MLEs and compared the results to the standard confidence

interval. He used a reparameterization, $a = p_1 + p_2$ and $b = p_1 - p_2$, then considered

estimators of *a* and *b*. In one particular interval, a Bayesian approach was used to develop the Jeffreys-Perks interval. In this case, the estimate of *a* is the posterior mean when using a symmetrical prior. This is the approach we will consider for the beta-binomial case.

If we extend analysis of binomial proportions to the *k* population case, a usual approach is to utilize a multiple comparison procedure. Most textbooks propose using methods such as Scheffè's, Tukey's *w*, Dunnett's, or Bonferroni's to determine if there are significant differences in the proportions at various levels of the treatment (or dose). Here, all pairwise comparisons, $p_i - p_j$, are considered based on linear contrasts (see Piegorsch, 1991).

These estimation procedures assume independent trials and do not allow for correlation among the trials. However, as discussed, in many situations there is correlation among the trials. The overdispersion brought about by the extra-binomial variability in the p_i's requires the construction of alternative modes and methods for analysis of the treatment effects (Piegorsch, 1991).

As a first step, the two population case will be considered. Many authors have proposed confidence intervals for the difference between two binomial proportions which correct the small sample problems associated with the interval based on the MLEs. This paper will examine possible adjustments or changes to these procedures when the underlying distribution is beta-binomial. In particular, the binomial intervals proposed by Beal (1987) will be considered. An additional problem for possible future work is to then extend these ideas to the *k* population case by way of multiple comparisons. The

basic idea is to reexamine corrections for the binomial model when the beta-binomial distribution is utilized.

The purpose of this paper is to determine how these adjusted confidence intervals perform for a variety of cases. No previous work appears to have been done for this specific case. However, related work has been done using distribution-free methods based on ranks and resampling. In addition, many authors have investigated other types of estimation using the beta-binomial distribution.

1.2 REVIEW OF LITERATURE

Many authors have investigated interval estimation with the binomial distribution. The standard textbook method is based on the asymptotic properties of the MLEs. This interval, however, performs poorly when the proportions are close to zero or one and also when the samples are small. Hence, other approaches are warranted. Beal (1987) constructed and evaluated several asymptotically-based confidence intervals for the difference between two binomial parameters and compared them to the standard textbook method. A reparameterization defined by $a = p_1 + p_2$ and $b = p_1 - p_2$ is used in each case. Beal finds that intervals presented by Mee (1984) and Miettinen & Nurminen (1985) (based on the above reparameterization) are good overall choices in that they provide a significant improvement over the standard textbook method. Beal also notes however, that these two intervals must be constructed numerically, and in general are hard to compute. Beal (1987) proposes a simpler approach by defining a Jeffreys-Perks interval. The Jeffreys-Perks interval uses a Bayesian approach to estimate $a = p_1 + p_2$. The Jeffreys-Perks interval proves much easier to compute and provides considerable improvement over the standard textbook method.

Other authors, such as Vollset (1993) and Peskun (1993), have compared various intervals with binomial proportions to the standard textbook method. Vollset (1993) constructs thirteen different binomial confidence intervals. The standard textbook method is strongly discouraged and exact intervals and score intervals are recommended. Peskun (1993) constructs a confidence interval based on the normal approximation for the difference between two binomial probabilities. Newcombe (1998) compares eleven binomial intervals. These include asymptotic methods with and without continuity

corrections, several numerically intense intervals given by Mee, Miettinen, and Nurminen, Beal's Jeffreys-Perks and Haldane intervals, a method combining Wilson score intervals, and a likelihood method. All are compared in terms of coverage probability. Agresti and Caffo (2000) construct an interval by adding two successes and two failures before calculating the MLES of p_1 and p_2.

Piegorsch (1991) extends the idea to simultaneous confidence intervals for binomial proportions.   Several dichotomous response models in which overdispersion was not present are considered. Construction of simultaneous confidence intervals were based on the asymptotic normality of the MLEs and these performed poorly for small samples. Thus, a reformulation of the pairwise comparisons using a construction noted by Beal (1987) was recommended for use with small-to-moderate sample sizes.

In all of the above cases, the approach taken is based on the binomial model where the response variates are independent. As stated earlier, the overdispersion brought on by the extra-binomial variability in many toxicity experiments requires the construction of alternative models and methods for analysis of the treatment effects. Piegorsch & Haseman (1991) suggest three possible approaches: 1) Distribution-free methods based on ranks. These have been investigated by authors such as Fry & Lee (1988) and Shirley (1987);   2) Resampling methods. Some common resampling techniques are "jackknifing" [Quenouille (1949), Schucany (1971)] and "bootstrapping" [Efron (1982)]; and 3) Parametric representations of proportional data that account for overdispersion. This approach, utilizing the beta-binomial distribution, will be the focus of this paper.

Several experimental situations where a beta-binomial distribution might be reasonable have been considered in the literature. For instance, Piegorsch (1993) used a regression approach to model the dose response, p, under the beta-binomial distribution. He assumed a logistic model on dose response and used the MLE of β_1. Significant positive departure from $\beta_1 = 0$ would suggest increasing dose response. Kuper, et al. (1986) also considered a logistic dose-response model that utilized the beta-binomial distribution. However, he did so by introducing varying degrees of intralitter correlation. The biases of the MLEs of β_0 and β_1 were examined as a function of intralitter correlation. Pack (1986) assumed a beta-binomial distribution and examined the properties of the likelihood ratio test for equality of dose responses under various assumptions on the $\Phi_i 's$. On a similar note, Paul & Islam (1994) derive two C(α) [Neyman, (1959)]statistics, based on the extended beta-binomial model for testing homogeneity of proportions in the presence of a common over-dispersion or under-dispersion and show, by simulation, that the size and power of these statistics are similar to those of the corresponding likelihood ratio test. Lui, et al. (1996) developed a closed-form interval estimate for the intraclass correlation using the beta-binomial distribution and showed it to have asymptotically correct coverage. Crowder (1978 & 1979) proposes a regression analysis of proportions based on the beta-binomial model and also proposes a technique for inferences about the intraclass correlation in the beta-binomial ANOVA for proportions. Paul (1982) compares the pseudo-t test, based on the jackknife method, and the likelihood ratio test, based on the beta-binomial model for testing the equality of two proportions. The comparison shows no definite advantage of one approach over the other. Other authors, such as Wypij and Santer (1990), Griffiths (1973), and Smith

(1983) have investigated estimation of the marginal probability of success using the beta-binomial distribution. Each of these papers lends insight into estimation using the beta-binomial distribution, but all address slightly different situations than the one of interest.

In addition to procedures for estimating the beta-binomial probabilities, several authors have presented methods for estimating the intraclass correlation. Ridout, Demetrio, and Firth (1999) compare several estimators via simulation. Some are specific estimators, while others are general methods such as pseudo-likelihood and extended quasi-likelihood. Estimators considered include an analysis of variance estimator, a moment estimator, and several estimators that are asymptotically equivalent. Lee (2004) uses an extended quasi-likelihood method to estimate the intraclass correlation for binary data. Zou (2004) obtains closed-form asymptotic variance formulas for three point estimators of intraclass correlation. Simulation results indicate that confidence intervals based on these estimators provide coverage levels close to nominal over a wide range of parameter combinations. Additionally, Bonett (2002) examines sample size requirements for estimating intraclass correlation.

As discussed earlier, the beta-binomial distribution is a common underlying distribution when analyzing correlated binary data. It is a useful distribution because it introduces a new parameter, Φ, which measures departure from binomial sampling. The focus of this paper is to estimate proportion differences when the underlying data is assumed to be beta-binomial. Many authors have investigated the use of the beta-binomial in the estimation of proportions and proportion differences. Moore (1997) investigated the proportion differences for two treatment levels where overdispersion was present. Confidence intervals were used to estimate the proportion difference $p_1 - p_2$

using the MLEs from the beta-binomial distribution. The sample sizes required for standard error rates were determined to be too large for reasonable reproductive toxicity experiments. Thus an alternate approach seems warranted.

Chen, et al. (2004) developed three closed-form confidence intervals for the difference between two proportions when overdispersion is present. With all three of these intervals, the concept of effective sample size is utilized to extend existing methods for independent data to account for overdispersion. In particular, assuming equal correlation in the two groups, Pearson's correlation coefficient was used to estimate ρ and then used to determine an estimate of the variance inflation factor. The first method is an extension of a binomial method presented by Newcombe (1998), the second method is based on the Jeffreys-Perks interval presented by Beal (1987), and the third method is an extension of the binomial model presented by Peskun(1993) and later by Feigin and Lumelskii(2000). Using empirical estimates of coverage bias, Chen, et al. (2004) find under simulation that each of the three methods utilizing effective sample size and the variance inflation factor provides significant improvements over the standard method based on the MLEs.

1.3   MOTIVATING EXAMPLES

There are many cases in which the analysis of binary data in the presence of overdipersion is warranted. Two specific examples are considered here. The first is noted by Haseman & Piegorsch (1994) with data and analysis presented by Chen, et al. (1991) and the second is presented by Chen, et al. (2004).

The first example is a study on the effects of the drug hydroxyurea. In humans, hydroxyurea is a drug used in the treatment of sickle cell anemia as well as certain types of melanoma, but is also known to have harmful effects to an unborn fetus. According to Chen, et al. (1991), in a study using female rodents, each female is exposed to the chemical/drug hydroxyurea at various dose levels. The data recorded is the developmental effect on the fetus. Specifically, the number of dead fetuses, the number of malformed fetuses, and the number of normal fetuses are recorded. One parameter of interest is the difference in the proportion of dead fetuses between any two dose levels of hydroxyurea. The beta-binomial model is appropriate here since the experimental unit is the female rodent and not the individual fetuses. Chen, et al. (1991) provide real data for the study where the experiment consisted of four treatement groups – control, low dose, medium dose, and high dose. Maximum likelihood estimates of the parameters are then obtained by fitting a Dirichlet-trinomial distribution. A likelihood ratio test is then used to determine if there are significant differences in the dose levels.   Using the data given by Chen, et al. (1991) at high & low dose levels for the study on hydroxyurea, confidence intervals can be constructed to estimate the difference in the proportions of dead fetuses at these two dose levels using the statistical procedures based on the beta-binomial developed in Chapter 2 of this paper.

The second example is based on a study of two treatment levels and their effect on lice infestation where the experimental unit is the household. In this example presented by Chen, et al. (2004), all patients in a household who were lice-infested were randomly assigned one of two treatments (high dose or low dose). Of interest were the treatment differences in lice-free rate between the two groups at 2 and 8 days after the treatment. Chen, Li, & Zhou construct three confidence intervals based on extensions of binomial cases that account for overdispersion and apply them to the data on lice-infestation. The confidence intervals constructed and applied to this data are three of the six intervals compared in Chapter 3 of this paper.

As mentioned earlier, the purpose of this paper is to construct and compare several intervals for estimating the difference between two proportions in situations where there is correlation or overdispersion, and thus six confidence intervals will be constructed. The first part of Chapter 2 addresses the theory in the binomial case, focusing on the derivation of several binomial intervals presented by Beal(1987). The second part of Chapter 2 extends the theory to the beta-binomial distribution to derive two new intervals and discusses four other existing intervals which account for overdispersion in various ways. Chapter 3 describes the methodology and simulations used to evaluate the intervals and presents and summarizes the simulation results. Chapter 4 discusses conclusions, recommendations, and possible extensions of this research.

CHAPTER 2

THEORY

2.1 BINOMIAL CASE

We first consider the two population binomial model. Assume that p_1 and p_2 are independent binomial parameters and n_1 and n_2 are the known samples sizes, respectively. Now consider a reparameterization of the parameters p_1, p_2, n_1 and n_2 introduced by Beal (1987):

$$a = p_1 + p_2,$$

$$b = p_1 - p_2,$$

$$u = \frac{1}{4}(1/n_1 + 1/n_2), \text{ and}$$

$$v = \frac{1}{4}(1/n_1 - 1/n_2).$$

Let $\hat{p}_1$, $\hat{p}_2$, $\hat{a}$, and $\hat{b}$ be the maximum likelihood estimates of p_1, p_2, a, and b, respectively. Since our goal is to estimate the proportion difference via a confidence interval, we need to determine the variance of $\hat{b}$. By the invariance property of maximum likelihood estimators, we know that the MLE of $\hat{b}$ is $\hat{p}_1 - \hat{p}_2$. So to derive the variance of $\hat{b}$,

$$\text{Var}(\hat{b}) = \text{Var}(\hat{p}_1 - \hat{p}_2)$$

$$= \text{Var}(\hat{p}_1) + \text{Var}(\hat{p}_2), \text{ since } \hat{p}_1 \text{ and } \hat{p}_2 \text{ are independent.}$$

Next we substitute $\dfrac{p_1(1-p_1)}{n_1}$ for the Var($\hat{p}_1$) and $\dfrac{p_2(1-p_2)}{n_2}$ for the Var($\hat{p}_2$). This

leads to the following expression for the variance of $\hat{b}$:

$$\text{Var}(\hat{b}) = \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}$$

$$= \frac{1}{2}\left(\frac{2p_1}{n_1} - \frac{2p_1^2}{n_1} + \frac{2p_2}{n_2} - \frac{2p_2^2}{n_2}\right)$$

$$= \frac{1}{2}\left(\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\left[p_1 + p_2 - p_1^2 - p_2^2\right] + \left(\frac{1}{n_1} - \frac{1}{n_2}\right)\left[p_1 - p_2 - p_1^2 + p_2^2\right]\right)$$

Next, we substitute u and v as defined earlier, and rewrite p_1 and p_2 in terms of a and b

and then simplify the result.

$$\text{Var}(\hat{b}) = 2u\left(p_1 + p_2 - p_1^2 - p_2^2\right) + 2v\left(p_1 - p_2 - p_1^2 + p_2^2\right)$$

$$= u\left(2p_1 + 2p_2 - p_1^2 - 2p_1p_2 - p_2^2 - p_1^2 + 2p_1p_2 - p_2^2\right)$$

$$+ 2v\left(p_1 - p_1^2 - p_1p_2 - p_2 + p_1p_2 + p_2^2\right)$$

$$= u\left(2(p_1 + p_2) - (p_1 + p_2)^2 - (p_1 - p_2)^2\right) + 2v(p_1 - p_2)(1 - p_1 - p_2)$$

$$= u\left(2a - a^2 - b^2\right) + 2vb(1 - a)$$

$$= u\left((2 - a)a - b^2\right) + 2v(1 - a)b$$

The above expression for the variance of $\hat{b}$, denoted by $V(a,b;u,v)$, is used throughout

the Beal (1987) article.   From here, we consider the following equation:

$$(b - \hat{b})^2 = cV(\tilde{a},\tilde{b};u,v) \tag{2.1}$$

where c is the α quantile of the $\chi^2(1)$ distribution, and $\tilde{a}$, $\tilde{b}$ are expressions for a and b

respectively. Equation (2.1) comes from the fact that $\dfrac{(b - \hat{b})}{\sqrt{V(\tilde{a},\tilde{b};u,v)}} = z$ since it

represents the true value minus a point estimate over the square root of the variance.

Thus, $\dfrac{(b - \hat{b})^2}{V(\tilde{a},\tilde{b};u,v)} = z^2 = \chi^2(1)$. All the intervals based on the binomial model

suggested by Beal (1987) are constructed by solving equation (2.1) for b.

2.1.1 The Wald Interval

The usual asymptotic interval(Wald Interval) with nominal confidence level α is

constructed by substituting $\tilde{a} = \hat{a}$ and $\tilde{b} = \hat{b}$ into equation (2.1) and then solving for b.

Equation (2.1) is quadratic in b and results in the familiar interval given by

$$\left| b - \hat{b} \right| \leq z \sqrt{(\hat{p}_1 \hat{q}_1 / n_1 + \hat{p}_2 \hat{q}_2 / n_2)}$$

where $z = \sqrt{c}$. This result is the standard binomial confidence interval presented in

many textbooks. (Derivation 2.1.1 in the Appendix A) As Beal (1987) notes, many other

authors have discovered problems with this interval when small samples are utilized, or

in cases were the population parameters p_1 and p_2 are close to zero or one.

2.1.2 The Jeffreys-Perks and Haldane Intervals

To address the problems with small samples or cases where the proportions are

near zero or one, Beal considers a class of simple intervals using a Baysian approach.

Good (1965) presents an estimate of a obtained by deriving the posterior mean of a using

the symmetrical prior density on (p_1, p_2) proportional to $(p_1 q_1 p_2 q_2)^\alpha$. To construct this interval, start with the symmetrical prior distribution given by

$$g(p_1, p_2) = \begin{cases} k\, p_1^{\,\alpha}\,(1-p_1)^{\alpha}\, p_2^{\,\alpha}\,(1-p_2)^{\alpha} & \text{if } 0 \le p_1 \le 1,\ \ 0 \le p_2 \le 1 \\[2ex] 0 & \text{otherwise} \end{cases}.$$

Next consider the joint density of the data given by

$$f(p_1,\, p_2 \mid x_1, x_2, n_1, n_2) = \binom{n_1}{x_1} p_1^{\,x_1}\,(1-p_1)^{n_1-x_1} \binom{n_2}{x_2} p_2^{\,x_2}\,(1-p_2)^{n_2-x_2},$$

where X_1 and X_2 are independent binomial random variables. Thus, the posterior density is derived as follows:

$$h(p_1, p_2 \mid x_1, x_2, n_1, n_2) =$$

$$\dfrac{k\, p_1^{\,\alpha}\,(1-p_1)^{\alpha}\, p_2^{\,\alpha}\,(1-p_2)^{\alpha} \binom{n_1}{x_1} p_1^{\,x_1}\,(1-p_1)^{n_1-x_1} \binom{n_2}{x_2} p_2^{\,x_2}\,(1-p_2)^{n_2-x_2}}{\displaystyle\int_0^1\!\!\int_0^1 k\, p_1^{\,\alpha}\,(1-p_1)^{\alpha}\, p_2^{\,\alpha}\,(1-p_2)^{\alpha} \binom{n_1}{x_1} p_1^{\,x_1}\,(1-p_1)^{n_1-x_1} \binom{n_2}{x_2} p_2^{\,x_2}\,(1-p_2)^{n_2-x_2}\, dp_1 dp_2}$$

$$= \dfrac{p_1^{\,\alpha+x_1}\,(1-p_1)^{\alpha+n_1-x_1}\, p_2^{\,\alpha+x_2}\,(1-p_2)^{\alpha+n_2-x_2}}{\displaystyle\int_0^1\!\!\int_0^1 p_1^{\,\alpha+x_1}\,(1-p_1)^{\alpha+n_1-x_1}\, p_2^{\,\alpha+x_2}\,(1-p_2)^{\alpha+n_2-x_2}\, dp_1\, dp_2}.$$

Now substitute $a_1 = \alpha + x_1 + 1$ and $b_1 = \alpha + n_1 - x_1 + 1$ to obtain

$$h(p_1, p_2 \mid x_1, x_2, n_1, n_2) =$$

$$\dfrac{p_1^{\,\alpha+x_1}\,(1-p_1)^{\alpha+n_1-x_1}\, p_2^{\,\alpha+x_2}\,(1-p_2)^{\alpha+n_2-x_2}}{\displaystyle\int_0^1 \left(p_2^{\,\alpha+x_2}\,(1-p_2)^{\alpha+n_2-x_2}\, \dfrac{\Gamma(a_1)\Gamma(b_1)}{\Gamma(a_1+b_1)} \int_0^1 \left(\dfrac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\, p_1^{\,a_1-1}\,(1-p_1)^{b_1-1} \right) dp_1 \right) dp_2}$$

$$\Uparrow$$
$$\text{(Beta pdf integrates to 1)}$$

$$= \frac{p_1^{\alpha+x_1}(1-p_1)^{\alpha+n_1-x_1} \, p_2^{\alpha+x_2}(1-p_2)^{\alpha+n_2-x_2}}{\dfrac{\Gamma(a_1)\Gamma(b_1)}{\Gamma(a_1+b_1)}\displaystyle\int_0^1 \left(p_2^{\alpha+x_2}(1-p_2)^{\alpha+n_2-x_2}\right)dp_2}$$

Next, substitute $a_2 = \alpha + x_2 + 1$ and $b_2 = \alpha + n_2 - x_2 + 1$ to obtain

$$h(p_1, p_2 \,|\, x_1, x_2, n_1, n_2) =$$

$$\frac{p_1^{\alpha+x_1}(1-p_1)^{\alpha+n_1-x_1} \, p_2^{\alpha+x_2}(1-p_2)^{\alpha+n_2-x_2}}{\dfrac{\Gamma(a_1)\Gamma(b_1)}{\Gamma(a_1+b_1)}\dfrac{\Gamma(a_2)\Gamma(b_2)}{\Gamma(a_2+b_2)}\displaystyle\int_0^1 \dfrac{\Gamma(a_2+b_2)}{\Gamma(a_2)\Gamma(b_2)}\left(p_2^{a_2-1}(1-p_2)^{b_2-1}\right)dp_2}$$

$$\Uparrow$$

(Beta pdf integrates to 1)

$$= \frac{p_1^{a_1-1}(1-p_1)^{b_1-1} \, p_2^{a_2-1}(1-p_2)^{b_2-1}}{\dfrac{\Gamma(a_1)\Gamma(b_1)}{\Gamma(a_1+b_1)}\dfrac{\Gamma(a_2)\Gamma(b_2)}{\Gamma(a_2+b_2)}}$$

$$= \left(\frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}p_1^{a_1-1}(1-p_1)^{b_1-1}\right)\left(\frac{\Gamma(a_2+b_2)}{\Gamma(a_2)\Gamma(b_2)}p_2^{a_2-1}(1-p_2)^{b_2-1}\right).$$

Thus, the posterior is a bi-variate BETA distribution with parameters a_1, b_1, a_2, and b_2.

Consequently, the posterior mean is given by

$$\left(\frac{a_1}{a_1+b_1}\right)\left(\frac{a_2}{a_2+b_2}\right) = \left(\frac{\alpha+x_1+1}{\alpha+x_1+1+\alpha+n_1-x_1+1}\right)\left(\frac{\alpha+x_2+1}{\alpha+x_2+1+\alpha+n_2-x_2+1}\right)$$

$$= \left(\frac{\alpha+x_1+1}{n_1+2(\alpha+1)}\right)\left(\frac{\alpha+x_2+1}{n_2+2(\alpha+1)}\right)$$

$$= \left(\frac{\alpha+1}{n_1+2(\alpha+1)}\right)+\left(\frac{x_1}{n_1+2(\alpha+1)}\right)+\left(\frac{\alpha+1}{n_2+2(\alpha+1)}\right)+\left(\frac{x_2}{n_2+2(\alpha+1)}\right)$$

$$= \left(\frac{\alpha + 1}{n_1 + 2(\alpha + 1)} \right) + \left(\frac{n_1}{n_1 + 2(\alpha + 1)} \right) \hat{p}_1$$

$$+ \left(\frac{\alpha + 1}{n_2 + 2(\alpha + 1)} \right) + \left(\frac{n_2}{n_2 + 2(\alpha + 1)} \right) \hat{p}_2 .$$

Now, for $\alpha \geq -1$, define the following estimate of a:

$$\hat{a}(\alpha) = \frac{n_1}{n_1 + 2(\alpha + 1)} \hat{p}_1 + \frac{\alpha + 1}{n_1 + 2(\alpha + 1)} + \frac{n_2}{n_2 + 2(\alpha + 1)} \hat{p}_2 + \frac{\alpha + 1}{n_2 + 2(\alpha + 1)} .$$

Notice that $\hat{a}(-1) = \hat{a}$ and as α increases $\hat{a}(\alpha)$ tends toward one. Using this estimate

of a, each value of α defines a separate confidence interval $I(\alpha)$: namely, use the

substitution $\tilde{a} = \hat{a}(\alpha)$ and $\tilde{b} = b$ in equation (2.1). Beal notes that when $v = 0$, (equal

sample sizes) as α increases, the length of $I(\alpha)$ also increases, but the center point

remains fixed. Thus for $\alpha > -1$, $I(\alpha)$ will always have confidence levels larger than

those of $I(-1)$. Several values of α were tried, and the choice $\alpha = -1/2$ seemed to

be an overall good choice (Beal, 1987). The interval $I(-1/2)$ will be referred to as the

Jeffreys-Perks interval since the prior with $\alpha = -1/2$ arises from the invariance

theories of Jeffreys and Perks (Good, 1965). The Jeffreys-Perks interval is evaluated by

Beal(1987) and shows significant improvement over the usual interval. In addition, the

interval $I(-1)$, referred to as the Haldane (1945) interval, is also evaluated by

Beal(1987).

One of the advantages of the intervals $I(\alpha)$ over the usual interval is that they do

not in general collapse to a single point when $\hat{p}_1 = \hat{p}_2 = 0$ or $\hat{p}_1 = \hat{p}_2 = 1$.

Specifically, assuming $v > 0$, the interval $I(-1)$ is given by $[0, 2cv/(1+cu)]$ when

$\hat{p}_1 = \hat{p}_2 = 0$ and given by $[-2cv/(1+cu), 0]$ when $\hat{p}_1 = \hat{p}_2 = 1$. In both of these cases

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however, if $v = 0$, the intervals are degenerate. However, if $\hat{p}_1 = 1$ and $\hat{p}_2 = 0$, the intervals are NOT degenerate.

2.1.3 Other Binomial Intervals

Beal then considers several other binomial intervals proposed by various authors in which equation (2.1) is used. Anbar (1983) and Mee (1984) constructed intervals in a similar fashion except that Anbar substituted $\tilde{a} = 2\hat{p}_1 - b$ and $\tilde{b} = b$ and Mee substituted $\tilde{a} = 2\hat{p}_2 + b$ and $\tilde{b} = b$. These are essentially the "same" substitutions, both of them derived from the reparameterization $a = p_1 + p_2$ and $b = p_1 - p_2$. Anbar took $a + b$ and solved for a in terms of p_1 and b; Mee took $a - b$ and solved for a in terms of p_2 and b. The result, however, is that these give two intervals that are generally different. Mee (1984) also suggested a more theoretical interval. Use $\tilde{b} = b$ and $\tilde{a} = a*(b)$, where $a*(b)$ is the maximum likelihood estimate of a when b is assumed to have known value. Using this substitution, equation (1) is no longer quadratic in b. However, it can be solved numerically to obtain two roots. The resulting interval is evaluated in the Beal (1987) article and performs significantly better than the standard textbook interval. Additionally, Miettinen and Nurminen (1985) describe an interval very similar to Mee's, the only difference being that for any given value of b, the variance of $\hat{b}$ is estimated by $V(a*(b), b; u, v)\left(\dfrac{N}{N-1}\right)$ where $N = n_1 + n_2$. This interval was also evaluated by Beal (1987) and performed very similarly to the Mee interval.

The disadvantage of the intervals presented by Mee (1984), and Miettinen and Nurminen (1985) is that they are in general hard to compute and much care is needed to find the numerical solution. A simpler approach can be found by utilizing the substitution $\tilde{a} = \hat{a}$ and $\tilde{b} = b$ into equation (2.1). Beal notes that the resulting interval represents a significant improvement over the usual interval. There are, however, a few values of p_1 and p_2 for which the confidence level is too small.

Other authors have constructed intervals for the difference of two binomial proportions using approaches different than Beal (1987). Newcombe (1998) constructs and compares eleven binomial intervals for the difference between two proportions. One such interval is an asymptotic method with continuity correction given by:

$$(\hat{p}_1 - \hat{p}_2) \pm \left[Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} + \frac{1}{2}\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \right]$$

where $X_1 \sim BIN(n_1, p_1)$, $X_2 \sim BIN(n_2, p_2)$, $\hat{p}_1 = \frac{x_1}{n_1}$, $\hat{p}_2 = \frac{x_2}{n_2}$. Agresti and Caffo (2000) also assume a binomial model and construct an interval by adding two successes and two failures. The resulting interval is given by:

$$(\tilde{p}_1 - \tilde{p}_2) \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{\tilde{p}_1(1-\tilde{p}_1)}{n_1 + 2} + \frac{\tilde{p}_2(1-\tilde{p}_2)}{n_2 + 2}}$$

where $\tilde{p}_1 = (x_1 + 1)/(n_1 + 1)$ and $\tilde{p}_2 = (x_2 + 1)/(n_2 + 1)$. A list of additional binomial intervals can be found in Appendix C.

2.2 EXTENSION TO THE BETA-BINOMIAL CASE

Beal's (1987) article and the other intervals discussed in section 2.1 assume a binomial model for all cases. However, the beta-binomial model is the focus of this paper, and thus we will attempt to extend some of the ideas and intervals presented by Beal and others to the case where we have correlation among the trials. We will again start with the two population case, where p_1, and p_2 are the population proportions and n_1 and n_2 are the known sample sizes. We also assume that there is correlation. This will imply the introduction of two more parameters, Φ_1 and Φ_2, likely unknown. Recall that

$$\rho = \frac{\Phi}{1 + \Phi}$$ and measures departure from binomial sampling. As a first step, we will derive the equivalent of equation (2.1) using the beta-binomial as the underlying distribution. Once this is accomplished, we can construct intervals similar to those presented by Beal (1987).

2.2.1 Wald Interval

Consider the two population beta-binomial model and assume that p_1 and p_2 are independent beta-binomial parameters and n_1 and n_2 are the known sample sizes, respectively. Additionally, let the overdispersion parameters for each population be denoted by Φ_1 and Φ_2. Since our goal is to estimate $b = p_1 - p_2$ using the MLEs of p_1 and p_2 we must first derive $\hat{p}_i$, $i = 1, 2$, assuming the underlying distribution is beta-binomial. Then we need to derive an expression for the variance of $\hat{b}$.

We begin with the log-likelihood function for a beta-binomial and simplify using the equality $\Gamma(a) = (a - 1)\Gamma(a - 1)$. The resulting log-likelihood is: (Derivation 2.2.1 in the Appendix A)

$$\ln(L(p_i, \Phi_i)) = \ln\left\{\prod_{j=1}^{J_i}\binom{n_{ij}}{y_{ij}}\left[\frac{\prod_{r=0}^{y_{ij}-1}(p_i + r\Phi_i)\prod_{r=0}^{n_{ij}-y_{ij}-1}(1 - p_i + r\Phi_i)}{\prod_{r=0}^{n_{ij}-1}(1 + r\Phi_i)}\right]\right\}.$$

To find the MLEs of p_i, we recognize that the log-likelihood function is proportional to the following summation:

$$\sum_{j=1}^{J_i}\left\{\sum_{r=0}^{y_{ij}-1}\ln(p_i + r\Phi_i) + \sum_{r=0}^{n_{ij}-y_{ij}-1}\ln(1 - p_i + r\Phi_i)\right\}.$$

Notice that the terms not involving the parameters p_i have been ignored since they will not play a role in the maximization. The MLEs are found by determining the values of p_i that maximize the above summation or by solving the equation

$\partial/\partial p_i\left(\ln L(p_i, \Phi_i)\right) = 0$ for p_i. Notice that one of the inner summations of the expression above is not defined when either $y_{ij} = n_{ij}$ or when $y_{ij} = 0$ as we have summations from zero to negative one. In these cases it follows that the summation that is not defined is actually zero. (See derivation 2.2.1a in the Appendix A for details.)

Next we derive the variance of $\hat{b}$. Since the two samples are assumed to be independent, the variance of $\hat{b} = \hat{p}_1 - \hat{p}_2$ is equal to Var($\hat{p}_1$) + Var($\hat{p}_2$). We can determine these quantities by utilizing the *Fisher information*, which will take on the form of a variance-covariance matrix I_n^{-1}, with the ith element on the diagonal giving

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the asymptotic variance of $\hat{p}_i$, henceforth referred to as $\text{var}(\hat{p}_i)$. (Derivation 2.2.2 in

the Appendix) Thus, from the *Fisher information* we have

$$\frac{1}{\hat{\text{var}}(\hat{p}_i)} = \sum_{j=1}^{J_i} E\left(\left[\frac{\partial \ln f(y; p_i, \Phi_i)}{\partial p_i}\right]^2\right).$$

Unfortunately, this expression depends on p_i and Φ_i which are assumed unknown.

However, since the MLEs are consistent estimators, $\hat{p}_i$ converges in probability to p_i

and $\hat{\Phi}_i$ converges in probability to Φ_i. Therefore, substituting the MLEs of p_i and Φ_i

into the asymptotic variance expression above results in a consistent estimator of the true

asymptotic variance, leaving the limiting distribution unchanged. This consistent

estimator will be denoted by $\hat{\text{var}}(\hat{p}_i)$.

Next we simplify $E\left(\left[\frac{\partial \ln f(y; p_i, \Phi_i)}{\partial p_i}\right]^2\right)$ from the above expression in order to

derive $\hat{\text{var}}(\hat{p}_i)$. We start with the log of the density function,

$$\ln(f(y_i; p_i, \Phi_i)) = \ln\left\{\binom{n_{ij}}{y_i}\left[\frac{\prod_{r=0}^{y_i}(p_i + r\Phi_i)\prod_{r=0}^{n_{ij}-y_i-1}(1 - p_i + r\Phi_i)}{\prod_{r=0}^{n_{ij}-1}(1 + r\Phi_i)}\right]\right\}$$

$$= \ln\binom{n_{ij}}{y_i} + \sum_{r=0}^{y_i-1}\ln(p_i + r\Phi_i) + \sum_{r=0}^{n_{ij}-y_i-1}\ln(1 - p_i + r\Phi_i) - \sum_{r=0}^{n_{ij}-1}\ln(1 + r\Phi_i)$$

Differentiating with respect to p_i gives the following:

$$\left(\frac{\partial}{\partial p_i}\right)(\ln f(y_i; p_i, \Phi_i)) = \sum_{r=0}^{y_i-1}\left(\frac{1}{p_i + r\Phi_i}\right) - \sum_{r=0}^{n_{ij}-y_i-1}\left(\frac{1}{1 - p_i + r\Phi_i}\right).$$

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Since $\displaystyle\sum_{j=1}^{J_i} E\left(\left[\frac{\partial \ln f(y;p_i,\Phi_i)}{\partial p_i}\right]^2\right)$ can be rewritten as $\displaystyle\sum_{j=1}^{J_i} E\left(-\left[\frac{\partial^2 \ln f(y;p_i,\Phi_i)}{\partial p_i^2}\right]\right)$,

we differentiate a second time with respect to p_i and obtain the following:

$$\left(\frac{\partial^2}{\partial p_i^2}\right)\left(\ln f(y_i;p_i,\Phi_i)\right) = \sum_{r=0}^{y_i-1}\frac{-1}{(p_i+r\Phi_i)^2} - \sum_{r=0}^{n_{ij}-y_i-1}\frac{1}{(1-p_i+r\Phi_i)^2}$$

Thus, from Hogg and Craig (1995) we have:

$$\frac{1}{\mathrm{var}(\hat{p}_i)} = \sum_{j=1}^{J_i} E\left(\left[\frac{\partial \ln f(y;p_i,\Phi_i)}{\partial p_i}\right]^2\right) = -\sum_{j=1}^{J_i} E\left(\left[\frac{\partial^2 \ln f(y;p_i,\Phi_i)}{\partial p_i^2}\right]\right)$$

$$= \sum_{j=1}^{J_i} E\left(\sum_{r=0}^{y_{ij}-1}\frac{1}{(p_i+r\Phi_i)^2} + \sum_{r=0}^{n_{ij}-y_{ij}-1}\frac{1}{(1-p_i+r\Phi_i)^2}\right).$$

Hence, the asymptotic variance of $\hat{b}$ is given by:

$$\mathrm{var}(\hat{b}) = \mathrm{var}(\hat{p}_1 - \hat{p}_2) = \mathrm{var}(\hat{p}_1) + \mathrm{var}(\hat{p}_2)$$

$$= \left\{\sum_{j=1}^{J_1}\left[E\left(\sum_{r=0}^{y_{1j}-1}\left(\frac{1}{(p_1+r\Phi_1)^2}\right) + \sum_{r=0}^{n_{1j}-y_{1j}-1}\left(\frac{1}{(1-p_1+r\Phi_1)^2}\right)\right)\right]\right\}^{-1}$$

$$+ \left\{\sum_{j=1}^{J_2}\left[E\left(\sum_{r=0}^{y_{2j}-1}\left(\frac{1}{(p_2+r\Phi_2)^2}\right) + \sum_{r=0}^{n_{2j}-y_{2j}-1}\left(\frac{1}{(1-p_2+r\Phi_2)^2}\right)\right)\right]\right\}^{-1}. \qquad (2.2)$$

Thus a $(1-\alpha)100\%$ Wald confidence interval for $p_1 - p_2$, based on the MLEs is

constructed as follows:

$$(\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}^2 \sqrt{Var(\hat{p}_1) + Var(\hat{p}_2)}$$

where $z_{\alpha/2}$ is the $(1 - \alpha/2)$ percentile of the standard normal distribution and the variance expression is given by equation (2.2) above with $\hat{p}_i$ and $\hat{\Phi}_i$ replacing p_i and Φ_i respectively.

2.2.2 Jeffreys-Perks and Haldane Intervals

To construct extensions of the Jeffreys-Perks and Haldane intervals presented by Beal from the binomial to the beta-binomial case, we must rewrite the var($\hat{b}$) in terms of a and b and utilize the beta-binomial as the underlying distribution. Recall Beal's reparameterization of a and b given by $a = p_1 + p_2$ and $b = p_1 - p_2$. Adding a and b and solving for p_1 yields $p_1 = \dfrac{a+b}{2}$, while subtracting a and b and solving for p_2 yields $p_2 = \dfrac{a-b}{2}$. Substituting these expressions into equation (2.2) derived earlier gives the following expression for the variance of $\hat{b}$ in terms of the parameters a and b:

$$
\operatorname{var}(\hat{b}) = \left\{ \sum_{j=1}^{J_1} E\left[\sum_{r=0}^{y_{1j}-1} \left(\frac{1}{\left(\frac{a+b}{2} + r\Phi_1 \right)^2} \right) + \sum_{r=0}^{n_{1j}-y_{1j}-1} \left(\frac{1}{\left(1 - \frac{a+b}{2} + r\Phi_1 \right)^2} \right) \right] \right\}^{-1}
$$

$$
+ \left\{ \sum_{j=1}^{J_2} E\left[\sum_{r=0}^{y_{2j}-1} \left(\frac{1}{\left(\frac{a-b}{2} + r\Phi_2 \right)^2} \right) + \sum_{r=0}^{n_{2j}-y_{2j}-1} \left(\frac{1}{\left(1 - \frac{a-b}{2} + r\Phi_2 \right)^2} \right) \right] \right\}^{-1} . \quad (2.3)
$$

Now we need to derive the equivalent of the estimator $\hat{a}(\alpha)$. Recall from Beal that $\hat{a}(\alpha)$ is the posterior mean of a using the symmetrical prior density on (p_1, p_2) proportional to $(p_1 q_1 p_2 q_2)^\alpha$. This prior density is given by

$$g(p_1, p_2) = \begin{cases} k\, p_1^\alpha\, (1-p_1)^\alpha\, p_2^\alpha\, (1-p_2)^\alpha & \text{if } 0 \le p_1 \le 1, \ 0 \le p_2 \le 1 \\ \\ 0 & \text{otherwise} \end{cases}.$$

Here the density of the data is bi-variate beta-binomial, rather than binomial as previously. Thus the density is given by:

$$f(y_1, y_2 \,|\, p_1, p_2, \Phi_1, \Phi_2, n_1, n_2) \;=$$

$$\binom{n_1}{y_1} \frac{\Gamma(y_1 + \frac{p_1}{\Phi_1}) \Gamma(\Phi_1^{-1}) \Gamma\!\left(n_1 - y_1 + \frac{1-p_1}{\Phi_1}\right)}{\Gamma\!\left(\frac{p_1}{\Phi_1}\right) \Gamma\!\left(\frac{1-p_1}{\Phi_1}\right) \Gamma(\Phi_1^{-1} + n_1)} \binom{n_2}{y_2} \frac{\Gamma(y_2 + \frac{p_2}{\Phi_2}) \Gamma(\Phi_2^{-1}) \Gamma\!\left(n_2 - y_2 + \frac{1-p_2}{\Phi_2}\right)}{\Gamma\!\left(\frac{p_2}{\Phi_2}\right) \Gamma\!\left(\frac{1-p_2}{\Phi_2}\right) \Gamma(\Phi_2^{-1} + n_2)}.$$

Therefore, the resulting posterior density is given by:

$$h(p_1, p_2 \,|\, y_1, y_2, \Phi_1, \Phi_2, n_1, n_2) \;=\; \frac{g(p_1, p_2)\, f(y_1, y_2)}{\int_0^1 \int_0^1 g(p_1, p_2) f(y_1, y_2)\, dp_1\, dp_2}.$$

Consequently, the posterior mean, $\hat{a}(\alpha)$ is given by:

$$\hat{a}(\alpha) \;=\; \int_0^1 \int_0^1 (p_1 + p_2)\, h(p_1, p_2 \,|\, y_1, y_2, \Phi, n_1, n_2)\, dp_1\, dp_2.$$

This double integral simplifies to the following: (derivation 2.2.3 in Appendix A)

$$\hat{a}(\alpha) \;=\; \frac{\int_0^1 p_1^{\alpha+1} (1-p_1)^\alpha\, , f(y_1)\, dp_1}{\int_0^1 p_1^\alpha (1-p_1)^\alpha\, , f(y_1)\, dp_1} \;+\; \frac{\int_0^1 p_2^{\alpha+1} (1-p_2)^\alpha\, , f(y_2)\, dp_2}{\int_0^1 p_2^\alpha (1-p_2)^\alpha\, , f(y_2)\, dp_2}. \qquad (2.4)$$

This estimator $\hat{a}(\alpha)$ is computed numerically using an IMSL subroutine in FORTRAN. We will utilize $\alpha = -1/2$ in equation (2.4) in forming the Jeffreys-Perks interval for the beta-binomial, and $\alpha = -1$ in forming the Haldane interval for the beta-binomial.

Now that $\hat{a}(\alpha)$ has been computed, we can construct our Jeffreys-Perks and Haldane intervals for $p_1 - p_2$. Consider again equation (2.1) from the Beal article,

$$(b - \hat{b})^2 = c\,Var(\tilde{a}, \tilde{b}, n_1, n_2),$$

where u and v have been suppressed and we now use n_1 and n_2 instead. To derive the Jeffreys-Perks and Haldane intervals for the beta-binomial case we utilize $\tilde{a} = \hat{a}(\alpha)$, $\tilde{b} = b$, and equation (2.3) for the variance of $\hat{b}$. After making these substitutions into equation (2.1) and rewriting, we get the following:

$$[b - (\hat{p}_1 - \hat{p}_2)]^2 - z_{\alpha/2}^2 \; Var(\hat{a}(\alpha), b) = 0. \tag{2.5}$$

The confidence interval is then constructed by solving equation (2.5) for b. To do this, we note that equation (2.5) is not quadratic in b but can be solved numerically to find two roots, b_l and b_u, on the interval [-1 , 1]. To find the upper root, b_u, we start by evaluating equation (2.5) at $b = (\hat{p}_1 - \hat{p}_2)$ and note that the left hand side of the equation should be negative. From hear we search iteratively by evaluating the left hand side of equation (2.5) for increasing values of b until the left hand side becomes positive. Once this happens this would indicate that somewhere between these last two values of b there is a root. From here the IMSL subroutine DZBREN can be utilized to find the upper root for equation (2.5). Similarly, we search iteratively going down from $b = (\hat{p}_1 - \hat{p}_2)$ to find a sign change and thus an interval for the lower root b_l. If we search iteratively going up and do not find a sign change, then b_u is set to 1, likewise if

no sign change is found searching iteratively downward, b_l is set to -1. These roots, b_l and b_u, are the upper and lower limits of the confidence interval for $p_1 - p_2$. Thus we have constructed the equivalent Jeffreys-Perks and Haldane intervals for $p_1 - p_2$ assuming the data are beta-binomial.

2.2.3 Beta-Binomial Intervals using a Variance Inflation factor

Three additional intervals, each a direct extension of existing methods for independent data, are constructed for comparison.   The three methods are presented by Chen, Li, and Zhou (2004). Each method is a direct mapping from independent cases through the concept of effective sample size and the introduction of a variance inflation factor. Each of these three confidence intervals constructed for $p_1 - p_2$ are in closed form and are based on asymptotic normality. Due to the correlation effect, the response rate inflates the usual variance and hence a variance inflation factor is used. The first method is an extension of Newcombe's (1998) method without continuity correction, the second method is an extension of Beal's (1987) Jeffreys-Perks method, and the third method is an extension of Peskun's (1993) method. Chen, et al. (2004) note that Peskun's(1993) original method is conservative and approximately guarantees exact coverage. The following notation will be used for the three intervals proposed by Chen, et al. (2004):

J_i = number of litters in treatment group i,

n_{ij} = size of litter j in treatment group i ,

$$y_{ijk} = \begin{cases} 1 & \textit{if kth member of litter } j, \textit{treatment i responded} \\ 0 & \textit{if kth member of litter } j, \textit{treatment i did not respond} \end{cases}$$

$$y_{ij} = \sum_{j=1}^{n_{ij}} y_{ijk} ,\ \text{the number of responses in litter j, treatment i,}$$

$$y_i = \sum_{i=1}^{J_i}\sum_{j=1}^{n_{ij}} y_{ijk} ,\ \text{the total number of responses in treatment } i,$$

$$n_i = \sum_{j=1}^{J_i} n_{ij} ,\ \text{the total sample size in treatment } i ,\ \text{and}$$

$$\hat{p}^*_i = \frac{y_i}{n_i} ,\ \text{an estimate of } p_i \text{ (note this is NOT the beta-binomial MLE of } p_i).$$

For the fixed litter size case with an equal number of litters in each population, the two groups are assumed to share the same ρ, which is jointly estimated using Pearson's correlation coefficient and denoted by $\hat{\rho}$.

$$\hat{\rho} = \frac{\sum_{i=1}^{2}\left[\dfrac{1}{Jn(n-1)}\sum_{j=1}^{J} y_{ij}(y_{ij}-1) - \hat{p}^{*2}_i \right]}{\left[\hat{p}^*_1(1-\hat{p}^*_1) + \hat{p}^*_2(1-\hat{p}^*_2) \right]} ,$$

where $J_i = J$ and $n_{ij} = n$. Once $\hat{\rho}$ is obtained, the variance inflation factor is estimated by $\hat{\lambda}_i = (1+(n-1))\hat{\rho}$. For the variable litter size case and/or unequal number of litters, the variance inflation factor is estimated by

$$\hat{\lambda}_i = \frac{\hat{v}(\hat{p}^*_i)}{\left[\dfrac{\hat{p}^*_i(1-\hat{p}^*_i)}{n_i} \right]} ,$$

where $\hat{v}(\hat{p}^*_i) = \dfrac{1}{J_i}\sum_{j=1}^{J_i}\left(\dfrac{n_{ij}}{\bar{n}_i} \right)^2 \dfrac{\left(\hat{p}^*_{ij} - \hat{p}^*_i \right)^2}{J_i - 1}$, $\bar{n}_i = n_i / J_i$, and $\hat{p}^*_{ij} = y_{ij} / n_{ij}$.

Based on $\hat{\lambda}_i$, the effective sample size in group i is defined to be $\hat{n}_i = \hat{\lambda}_i^{-1} n_i$ and the effective number of responders is $\hat{x}_i = \hat{\lambda}_i^{-1} x_i$.

Using the above estimators we can now construct the three intervals proposed by Chen, et al. (2004).

METHOD 1: (extension of Newcombe (1998)) The endpoints of the $(1 - \alpha)100\%$ confidence interval are given by $\hat{d} - \hat{\theta}_1$ and $\hat{d} + \hat{\theta}_2$ where $\hat{d} = \hat{p}_1 - \hat{p}_2$ and

$$\hat{\theta}_1 = z_{\alpha/2} \sqrt{\hat{p}_{11}(1-\hat{p}_{11})/\hat{n}_1 \; + \; \hat{p}_{22}(1-\hat{p}_{22})/\hat{n}_2}$$

$$\hat{\theta}_2 = z_{\alpha/2} \sqrt{\hat{p}_{12}(1-\hat{p}_{12})/\hat{n}_1 \; + \; \hat{p}_{21}(1-\hat{p}_{21})/\hat{n}_2}$$

Here, $\hat{p}_{i1}$ and $\hat{p}_{i2}$ are the two closed-form solutions of p from the equation

$(\hat{p}_i - p)^2 = z_{\frac{\alpha}{2}}^2 p(1 - p)/\hat{n}_i$, given by

$$p_{11} = \left.\left(\left(2\hat{p}_1 + \frac{z_{\alpha/2}^2}{\hat{n}_1}\right) - \sqrt{\left(-2\hat{p}_1 - \frac{z_{\alpha/2}^2}{\hat{n}_1}\right)^2 - 4\left(1 + \frac{z_{\alpha/2}^2}{\hat{n}_1}\right)\hat{p}_1^2}\right)\middle/ 2\left(1 + \frac{z_{\alpha/2}^2}{\hat{n}_1}\right)\right. ,$$

$$p_{12} = \left.\left(\left(2\hat{p}_1 + \frac{z_{\alpha/2}^2}{\hat{n}_1}\right) + \sqrt{\left(-2\hat{p}_1 - \frac{z_{\alpha/2}^2}{\hat{n}_1}\right)^2 - 4\left(1 + \frac{z_{\alpha/2}^2}{\hat{n}_1}\right)\hat{p}_1^2}\right)\middle/ 2\left(1 + \frac{z_{\alpha/2}^2}{\hat{n}_1}\right)\right. ,$$

$$p_{21} = \left.\left(\left(2\hat{p}_2 + \frac{z_{\alpha/2}^2}{\hat{n}_2}\right) - \sqrt{\left(-2\hat{p}_2 - \frac{z_{\alpha/2}^2}{\hat{n}_2}\right)^2 - 4\left(1 + \frac{z_{\alpha/2}^2}{\hat{n}_2}\right)\hat{p}_2^2}\right)\middle/ 2\left(1 + \frac{z_{\alpha/2}^2}{\hat{n}_2}\right)\right. , \text{ and}$$

$$p_{22} = \left.\left(\left(2\hat{p}_2 + \frac{z_{\alpha/2}^2}{\hat{n}_2}\right) + \sqrt{\left(-2\hat{p}_2 - \frac{z_{\alpha/2}^2}{\hat{n}_2}\right)^2 - 4\left(1 + \frac{z_{\alpha/2}^2}{\hat{n}_2}\right)\hat{p}_2^2}\right)\middle/ 2\left(1 + \frac{z_{\alpha/2}^2}{\hat{n}_2}\right)\right. .$$

(See Derivation 2.3.2 in the Appendix A for details.) Method 1 is a direct extension of Newcombe's method without continuity correction and degenerates to Newcombe's original method when the data are independent.

METHOD 2: (extension of Beal (1987)) The endpoints of the $(1-\alpha)100\%$ confidence

interval are given by $\dfrac{1}{(1+z_{\alpha/2}^2 u)}\left[\hat{d}+z_{\alpha/2}^2 v(1-2w)\ \pm\ z_{\alpha/2}\sqrt{f}\right]$, where u, v, w, and f

are defined as follows:

$$u = (1/\hat{n}_1 + 1/\hat{n}_2)/4,$$

$$v = (1/\hat{n}_1 - 1/\hat{n}_2)/4,$$

$$w = \left[\frac{(\hat{x}_1 + 1/2)}{(\hat{n}_1 + 1)} + \frac{(\hat{x}_2 + 1/2)}{(\hat{n}_2 + 1)}\right]\Big/2, \text{ and}$$

$$f = u\left[4w(1-w)-\hat{d}^2\right] + 2v(1-2w)\hat{d}\ +\ 4z_{\alpha/2}^2 u^2(1-w)w + z_{\alpha/2}^2 v^2(1-2w)^2.$$

Method 2 is a direct extension of Beal's method for binomial data and degenerates to

Beal's original method when the data are independent.

METHOD 3: (extension of Peskun (1993)) The endpoints of the $(1-\alpha)100\%$

confidence interval are given by

$$\frac{\hat{n}_1 + \hat{n}_2}{1 + z_{\alpha/2}^2}\left[\hat{d} + \pm\ z_{\alpha/2}\sqrt{g}\right] \text{ where } g = \frac{z_{\alpha/2}^2 + \hat{n}_1 + \hat{n}_2}{4\hat{n}_1\hat{n}_2} - \frac{\hat{d}^2}{\hat{n}_1 + \hat{n}_2}.$$

This method is an extension of Peskun's method and is conservative, asymptotically

guaranteeing exact coverage in the binomial case. Chen, et al. (2004) note that this

extension preserves this conservative property.

2.2.4 Other Beta-Binomial Intervals

Beal (1987) and other authors have proposed additional methods for the binomial

model that could be extended to the beta-binomial. In a procedure similar to the

construction of the Jeffreys-Perks and Haldane beta-binomial intervals discussed earlier

in section 2.3.2, Beal considers binomial methods proposed by Anbar (1984), Mee (1984), and Miettinen and Nurminen (1985). Each of these suggest a slightly different estimator of a for use in equation (2.1), yielding a new confidence interval. Beal notes however, that these methods are numerically intense and hard to compute in general. The interval by Mee(1984) was investigated for a possible extension to the beta-binomial case. In this case, Mee suggests substituting $\tilde{b} = b$ and $\tilde{a} = a*(b)$ into equation (2.1), where $a*(b)$ is the maximum likelihood estimate of a when b is assumed to have known value. Equation (2.1) is no longer quadratic but can be solved numerically to find two roots. For the beta-binomial case however, determining $a*(b)$ would be computationally intense as each evaluation of equation (2.5) would require the calculation of a new value for $a(\alpha)$. Thus this extension of the interval by Mee(1984) was not constructed.

CHAPTER 3

SIMULATIONS

3.1 INTRODUCTION

In order to compare methods, a simulation program was constructed using

FORTRAN. In the program, a random sample from the beta-binomial distribution is

generated and six confidence intervals for $p_1 - p_2$ are constructed. Coverage

probability and average interval width are computed for each of the six intervals. The six

intervals constructed and compared in the FORTRAN program are:

1) **Wald** interval – usual asymptotic method based on the MLEs,

2) **Jeffreys-Perks** interval – Bayesian approach with $\alpha = -1/2$, and described in section 2.2.2,

3) **Haldane** interval – Bayesian approach with $\alpha = -1$, and described in section 2.2.2,

4) **Extended Beal** interval – from Chen, et al. (2004), and described in 2.2.3,

5) **Extended Newcombe** interval – from Chen, et al. (2004), and described in 2.2.3, and

6) **Extended Peskun** interval – from Chen, et al. (2004), and described in 2.2.3.

Tables 1 – 23 give coverage probability when the nominal level is .95 and are based on

10,000 simulation runs. The tables are located in section 3.4 and in Appendix D. In

addition to coverage probability, the program calculates the average interval width for the

six confidence intervals to be compared. The FORTRAN compiler used here is *Absoft*

Pro Fortran: F90| F77| C/C++ Compilers & Tools, V6.2, © 2000 with IMSL version 4.0.

The parameter values needed for the FORTRAN program are: the number of

litters in each population, J_1 and J_2 (denoted by J1, J2 in the program); the litter size,

n_{ij} (denoted by N1Vec and N2Vec in the program); the population proportions, p_1 and

p_1 (denoted by P1 and P2 in the program); a measure of the correlation or

overdispersion in each population, Φ_1 and Φ_2 (denoted by PHI1 and PHI2 in the

program); and the nominal probability given by $1 - \alpha = .95$. The specific values of the

parameters used are described below and all combinations of these parameter values are

simulated.

For the sample size parameters J_1, J_2, and n_{ij}, we considered the following cases:

a) $J_1 = 5$ and $J_2 = 5$ d) $J_1 = 5$ and $J_2 = 10$

b) $J_1 = 10$ and $J_2 = 10$ e) $J_1 = 5$ and $J_2 = 25$

c) $J_1 = 25$ and $J_2 = 25$

For each of the above cases, we ran simulations with $n_{ij} = 5$ and $n_{ij} = 10$. In addition,

three large sample cases were considered. The first is when $J_1 = 50$, $J_2 = 50$, and

$n_{ij} = 5$, the second uses $J_1 = 150$, $J_2 = 150$, and $n_{ij} = 15$, and the third uses $J_1 = 250$,

$J_1 = 250$, and $n_{ij} = 15$. These cases were done in order to verify the asymptotic

properties of the intervals constructed.

For the population proportions p_1 and p_2 we considered the following cases:

a) $p_1 = .1$ and $p_2 = .1$ g) $p_1 = .3$ and $p_2 = .7$

b) $p_1 = .1$ and $p_2 = .3$ h) $p_1 = .5$ and $p_2 = .5$

c) $p_1 = .1$ and $p_2 = .7$ i) $p_1 = .9$ and $p_2 = .9$

d) $p_1 = .1$ and $p_2 = .9$ j) $p_1 = .9$ and $p_2 = .01$

e) $p_1 = .3$ and $p_2 = .3$ k) $p_1 = .05$ and $p_2 = .05$

f) $p_1 = .3$ and $p_2 = .5$ l) $p_1 = .05$ and $p_2 = .01$

For the overdispersion parameters Φ_1 and Φ_2 we considered the following cases:

a) $\Phi_1 = 1/9$ and $\Phi_2 = 1/9$ e) $\Phi_1 = 1/9$ and $\Phi_2 = 3/7$

b) $\Phi_1 = 3/7$ and $\Phi_2 = 3/7$ f) $\Phi_1 = 1/9$ and $\Phi_2 = 1$

c) $\Phi_1 = 1$ and $\Phi_2 = 1$ g) $\Phi_1 = 1/9$ and $\Phi_2 = 9$

d) $\Phi_1 = 9$ and $\Phi_2 = 9$ h) $\Phi_1 = 3/7$ and $\Phi_2 = 9$

Note that Φ values of 1/9, 3/7, 1, and 9 correspond to correlation values of .1, .3, .5, and

.9. This can be shown using the equation $\rho = \dfrac{\Phi}{1 + \Phi}$.

3.2 A RANDOM SAMPLE FROM THE BETA-BINOMIAL

To simulate a random sample from the Beta-Binomial distribution, the probability integral transform is utilized. According to Bain and Engelhardt (1992) the probability integral transform states:

"If X is continuous with CDF $F(x)$ then $U = F(x) \sim \text{UNIF}(0,1)$."

Furthermore, Bain and Engelhardt state:

"Let $F(x)$ be a CDF and let $g(u) = \min\{x \mid u \leq F(x)\}$. If $U \sim \text{UNIF}(0,1)$ then

$X = G(U) \sim F(x)$."

This is an important result and provides a useful technique for simulating a random variable from a specified distribution with known distribution $F(x)$. The procedure is as follows: first generate U from a UNIF(0,1) and find X by utilizing the formula,

$g(u) = \min\{x \mid u \leq F(x)\}$. The result will be a random variable X with distribution $F(x)$.

To illustrate this with a specific example, consider generating a random variable Y from a Beta-Binomial distribution with parameters $p = .7$, $\Phi = 1$, and $n = 5$. From Appendix A, the probability distribution function of the Beta-Binomial is given by:

$$f(y \mid p, \Phi, n) = \binom{n}{y} \frac{\Gamma(y + \frac{p}{\Phi})\Gamma(\Phi^{-1})\Gamma\left(n - y + \frac{1-p}{\Phi}\right)}{\Gamma\left(\frac{p}{\Phi}\right)\Gamma\left(\frac{1-p}{\Phi}\right)\Gamma(\Phi^{-1} + n)} \tag{3.1}$$

To use the probability integral transform, we must construct the CDF. This can be done for this specific case by determining the probabilities for each value of Y via the probability distribution function given in equation (3.1) above. From these calculations we get the following probabilities for $Y = 0, 1, 2, 3, 4, 5$: $f(0) = .106$, $f(1) = .086$,

$f(2) = .089$, $f(3) = .104$, $f(4) = .149$, $f(5) = .466$. From this we can

determine the values of the CDF: $F(0) = .106$, $F(1) = .192$, $F(2) = .281$,

$F(3) = .385$, $F(4) = .534$, $F(5) = 1.00$. Thus for this specific case, we would

generate U from a UNIF(0,1) and determine Y by the following rule: $Y = y_k$ if

$F(y_{k-1}) < U \leq F(y_k)$, or specifically,

$Y = 0$ if $0 < U \leq .106$,

$Y = 1$ if $.106 < U \leq .192$,

$Y = 2$ if $.192 < U \leq .281$,

$Y = 3$ if $.281 < U \leq .385$,

$Y = 4$ if $.385 < U \leq .534$, and

$Y = 5$ if $.534 < U \leq 1.0$.

For each simulation run, the Probability Integral Transform technique will be used to

generate a random sample from the Beta-Binomial distribution. The IMSL routines

RNSET, RNOPT, DRUNF will be used to generate a random number from UNIF(0,1).

DRNUNF generates a pseudorandom number from a UNIF(0,1) distribution where

RNOPT and RNSET select the number generator and initialize the seed. In the program,

the seed is set to zero so that a different value of U is computed each time using the

system clock.

3.3 MAXIMUM LIKLIHOOD ESTIMATORS FOR THE BETA-BINOMIAL

In each of the first three intervals constructed and compared in this paper (Wald, Jeffreys-Perks, and Haldane), maximum likelihood estimators are utilized. In the FORTRAN program, an algorithm by Smith (1983) was used to determine the maximum likelihood estimators of the parameters of the Beta-Binomial Distribution. The subroutine BBML calculates the MLEs for p and Φ using a damped Newton-Raphson technique as described by Dixon (1972) and uses moment estimates as initial estimators. Details for the FORTRAN program are in Appendix B.

3.4 SIMULATION RESULTS

A variety of simulations were performed using FORTRAN. The values of several parameters were varied in order to simulate many different cases, and the results are summarized in Tables 1-23. Several tables are presented here in section 3.4 to illustrate various trends in the data and as a way to compare intervals, while other tables that provide similar results are found in Appendix D. Each table gives coverage probability for each of the six intervals constructed assuming a nominal probability level of $1 - \alpha = .95$ for various values of the sample size parameters J_1 , J_2 , and n_{ij}. With each table, the sample size changes by either changing the number of litters or changing the litter sizes.   In general, increasing the litter size from $n_{ij} = 5$ to $n_{ij} = 10$ does not significantly change the results, however, changing the number of litters in each population does affect the coverage probabilities. This is not surprising as increasing the litter size essentially amounts to adding more correlated data and does not provide new information. Since increasing the litter size has little affect on the results, all tables with $n_{ij} = 10$ are presented in Appendix D as they provide essentially repeated information from the corresponding tables with $n_{ij} = 5$. By increasing the number of litters in each population, however, all six intervals generally improve as the coverage probabilities become closer to .95.

Within each table there are 48 different combinations of the vectors p and Φ discussed in section 3.1. Intervals with coverage probability within 2 standard errors of the nominal level of .9500 are bold-faced. The standard error here is given by

$$S.E. = \sqrt{\frac{(1 - \alpha)(\alpha)}{N}}$$, where $N = \#$ of simulations. Thus 2 standard errors are equal to

$2 * \sqrt{\dfrac{.95(.05)}{10000}} = .0044$, and hence, estimated coverage probabilities between .9456 and

.9544 are considered not significantly different than .9500. Using the information in the

tables we can compare and evaluate each of the six intervals constructed.

In general the Wald interval performs poorly in most cases, especially when the

sample sizes are small or the probabilities are close to zero or one. This is illustrated in

Table 1 with $J_1 = 5$ and $J_2 = 5$. Here we see coverage probabilities consistently

ranging from around .7300 to .8800.   The Wald is a particularly poor choice when the

proportions are close to zero as illustrated by the case when $p_1 = .05$ and $p_2 = .01$ in

Table 1. In this case the Wald has coverage probabilities of: .5529, .5840, .6417, and

.8789. Similar results can be seen in other tables when the sample sizes are small and/or

the proportions are near zero or one. Examples include Tables 10, 12, 13, and 19 found

in Appendix D.

The Wald interval generally improves as the sample size (number of litters) is

increased. However, the improvement is somewhat slow, and even for large samples

there are specific cases when the coverage probabilities are still much less than the

desired nominal level of .95. This is illustrated in Table 2 when $J_1 = 250$, $J_2 = 250$,

$p_1 = .05$, and $p_2 = .05$ with equal correlation between the populations. Here we find

coverage probabilities of .8730, .8891, .8991, and .8923. Additional cases can be found

in Tables 22 & 23 in Appendix D that show the Wald interval improving as the sample

size increases, but with specific cases where the coverage probability is significantly less

than the nominal level of .95.

Table 1: $J_1 = J_2 = 5$, $n_{ij} = 5 \ \ \forall_{i,j}$, $\alpha = .05$, equal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

		Confidence Levels when nominal level is .95									

p_1	p_2	.1	.3	.5	.9	p_1	p_2	.1	.3	.5	.9
.1	.1	.8684	.8933	.9183	.8864	.3	.7	.7676	.7471	.7783	.9028
		.9397	**.9523**	**.9529**	.9844			.8945	.9083	.9085	.9699
		.9068	.9131	.9203	.9786			.8202	.8065	.8267	.9244
		.9740	.9866	.9826	.9978			.9191	.9046	.9017	.9310
		.9890	.9808	.9834	.9881			.9318	.9116	.9082	.9431
		.9972	.9951	.9875	.9881			.9333	.9155	.9144	.9493
.1	.3	.7721	.7267	.7392	.8957	.5	.5	.8109	.7983	.7938	.8964
		.9097	.8997	.9036	.9799			.9008	.9046	.9106	.9742
		.8288	.8079	.8033	.9063			.8350	.8345	.8363	.9413
		.9397	.9180	.9256	.9811			.9209	.9040	.9068	.9667
		.9368	.9110	.9182	.9722			.9212	.9071	.9114	.9693
		.9559	.9705	.9766	.9935			.9222	.9104	.9157	.9753
.1	.7	.7826	.7486	.7765	.9053	.9	.9	.8682	.8879	.9196	.8824
		.8987	.8848	.9085	.9876			.9358	**.9499**	.9660	.9925
		.8309	.7983	.8090	.8929			.9078	.9061	.9222	.9725
		.9234	.9040	.8993	.8814			.9626	.9773	.9712	.9965
		.9262	.9008	.8955	.9115			.9685	.9888	.9720	.9968
		.9409	.9178	.9120	.9244			.9770	.9957	.9973	.9991
.1	.9	.7786	.7854	.7374	.9049	.9	.01	.7708	.7392	.7612	.7787
		.9055	.9187	.9421	.9844			.9008	.9013	.9017	.9191
		.8479	.8466	.8544	.8675			.8708	.8765	.8621	.8107
		.9142	.8840	.8435	.8980			.8817	.7926	.8239	.8755
		.9471	.9250	.9031	.9061			.8950	.8335	.9015	.9090
		.9539	.9339	.9142	.9132			.8933	.9124	.9046	.9243
.3	.3	.7764	.7395	.7862	.9029	.05	.05	.9037	.8590	.8661	.8958
		.8768	.8928	.9098	.9786			**.9540**	.9796	.9830	.9881
		.8207	.8026	.8295	**.9463**			.9614	**.9539**	.9631	.9692
		.9306	.9184	.9353	.9686			.9766	.9729	.9980	.9884
		.9365	.9229	.9388	.9806			.9785	.9973	.9882	.9891
		.9485	.9286	**.9521**	.9879			.9795	.9896	.9992	1.000
.3	.5	.8004	.7719	.7829	.8903	.05	.01	.5529	.5840	.6417	.8789
		.9076	.8962	.9078	.9724			.9799	.9709	.9895	.9682
		.8518	.8152	.8303	.9223			.9694	.9617	**.9517**	.9653
		.9241	.9048	.9138	**.9517**			.9895	.9894	.9899	.9898
		.9265	.9084	.9174	.9563			.9895	.9894	.9809	.9891
		.9314	.9186	.9291	.9638			.9995	.9899	.9989	.9981

Table 2: $J_1 = 250$, $J_2 = 250$, $n_{ij} = 15 \;\; \forall_{i,j}$, $\alpha = .05$, equal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

		Confidence Levels when nominal level is .95									
		ρ						ρ			
p_1	p_2	.1	.3	.5	.9	p_1	p_2	.1	.3	.5	.9
.1	.1	.9180	.9041	.9007	.9198	.3	.7	**.9487**	**.9497**	.9397	**.9541**
		.9263	.9089	**.9454**	.9398			**.9465**	**.9468**	**.9462**	**.9530**
		.9174	.8998	.9112	.9201			**.9478**	**.9488**	.9358	**.9530**
		.9465	**.9502**	**.9500**	**.9484**			**.9504**	**.9504**	**.9464**	**.9519**
		.9449	**.9515**	**.9539**	**.9459**			**.9496**	**.9516**	**.9461**	**.9518**
		.9981	.9994	.9991	.9973			**.9495**	**.9515**	**.9466**	**.9543**
.1	.3	**.9501**	**.9466**	.9331	**.9470**	.5	.5	.9325	**.9501**	**.9488**	.9275
		.9479	**.9469**	**.9464**	**.9480**			**.9459**	**.9497**	**.9482**	**.9463**
		.9458	**.9458**	.9318	**.9480**			.9350	**.9497**	**.9481**	.9337
		.9510	**.9532**	**.9540**	**.9516**			.9349	**.9515**	**.9530**	**.9460**
		.9511	**.9517**	**.9530**	**.9511**			.9348	**.9537**	**.9522**	**.9460**
		.9883	.9907	.9906	.9883			.9399	**.9533**	**.9519**	.9892
.1	.7	.9281	.9245	.9005	**.9471**	.9	.9	.9002	.8956	.9059	.9451
		.9481	**.9489**	**.9457**	**.9488**			.9132	.9015	**.9459**	**.9479**
		.9277	.9295	.9094	**.9486**			.9067	.9001	.9198	.9230
		.9329	**.9500**	.9330	**.9497**			**.9464**	**.9478**	.9555	.9561
		.9330	**.9525**	.9317	**.9500**			**.9464**	**.9465**	.9567	.9549
		.9484	**.9506**	**.9461**	**.9506**			.9997	.9973	.9993	.9977
.1	.9	.9333	.9256	.8997	.9331	.9	.01	.9200	.8940	.8891	.9239
		.9503	**.9466**	**.9494**	**.9468**			**.9470**	.9109	**.9505**	**.9469**
		.9345	.9280	.9005	.9321			.9183	.9006	.9156	.9331
		.9526	**.9540**	**.9484**	**.9467**			**.9540**	**.9523**	**.9464**	.9575
		.9526	**.9533**	**.9486**	**.9459**			**.9506**	**.9533**	**.9461**	.9547
		.9536	**.9536**	**.9487**	**.9466**			**.9539**	**.9540**	**.9481**	**.9542**
.3	.3	**.9523**	**.9511**	**.9478**	**.9463**	.05	.05	.8730	.8891	.8991	.8923
		.9498	**.9491**	**.9485**	**.9492**			.9170	.9225	**.9501**	**.9474**
		.9475	**.9466**	**.9481**	**.9482**			.9056	.9122	.9090	.9236
		.9531	**.9510**	9505	**.9540**			**.9500**	**.9469**	**.9467**	**.9487**
		.9533	**.9534**	**.9516**	**.9531**			**.9500**	**.9480**	**.9468**	**.9468**
		.9687	**.9507**	.9705	.9718			.9997	.9999	1.000	.9995
.3	.5	**.9456**	**.9523**	**.9497**	**.9487**	.05	.01	.8998	.8670	.8666	.9003
		.9491	**.9514**	**.9468**	**.9507**			.9003	**.9456**	.9194	**.9473**
		.9502	**.9507**	**.9465**	**.9502**			.8975	.9102	.8854	.9203
		.9498	**.9531**	**.9530**	**.9521**			.9358	**.9480**	**.9530**	**.9500**
		.9463	**.9518**	**.9519**	**.9530**			.9350	**.9465**	**.9516**	**.9466**
		.9492	**.9541**	**.9539**	**.9519**			.9991	.9999	.9998	1.000

The Haldane interval is, in general, an improvement over the Wald interval for most cases but still has several situations with less than desired coverage probability. As an example consider data from Table 3 for the case when $J_1 = 5$, $J_2 = 10$, $p_1 = .3$, and $p_2 = .7$. In this situation the coverage probabilities for the Haldane interval are .8472, .8360, .8526, and .9236. While each of these is an improvement over the corresponding Wald interval, these results are still considerably lower than the nominal level of .95.

The Jeffreys-Perks interval provides a significant improvement over the Wald and Haldane intervals as do the extended Newcombe and extended Beal intervals. This can be seen in a majority of the cases throughout the tables presented in this section as well as the tables found in Appendix D. Additionally, the Peskun interval generally has the highest coverage probability in most cases. As Chen, et al. (2004) note however, this was expected as this interval is an extension of a binomial method that is conservative and approximately guarantees exact coverage.

Tables 1 – 3 represent cases with equal correlations. As Table 4 illustrates, we get similar results in cases with unequal correlations. Table 4 presents the case when $J_1 = 5$, $J_2 = 25$ and the correlations are unequal (various values). In this case we see results similar to those when the correlations were equal, as the Wald again provides the lowest coverage, with the Jeffreys-Perks, extended Newcombe, and extended Beal providing the best coverage in the majority of cases. The remaining tables (Tables 5 – 23) are given in Appendix D.

Table 3: $J_1 = 5$, $J_2 = 10$, $n_{ij} = 5$ $\forall_{i,j}$, $\alpha = .05$, equal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

Confidence Levels when nominal level is .95											
		ρ						ρ			
p_1	p_2	.1	.3	.5	.9	p_1	p_2	.1	.3	.5	.9
.1	.1	.7721	.8244	.8885	.9507	.3	.7	.8074	.7871	.7997	.9048
		.9342	.9431	.9630	.9746			.9064	.9003	.9143	.9762
		.9111	.9146	.9239	.9778			.8472	.8360	.8526	.9236
		.8364	.7930	.7744	.9004			.9759	.9864	.9907	.9581
		.9098	.8918	.8973	.9580			.9799	.9989	.9958	.9730
		.9986	.9971	.9966	.9964			.9990	.9994	.9970	.9762
.1	.3	.8706	.8676	.8803	.9377	.5	.5	.8409	.8185	.8244	.8979
		.9338	.9385	**.9505**	.9680			.8889	.9000	.9129	.9648
		.9093	.8950	.8917	.9241			.8615	.8479	.8574	.9318
		.9591	.9147	.8829	.9304			.9983	.9881	.9728	.9669
		.9682	.9323	.9106	**.9538**			.9993	.9911	.9799	.9742
		.9969	.9840	.9668	.9798			.9994	.9928	.9836	.9764
.1	.7	.8385	.8443	.8428	.9134	.9	.9	.7761	.8173	.8845	**.9466**
		.9568	.9584	.9573	.9656			.9361	.9329	**.9500**	.9794
		.9023	.8923	.8885	.9108			.9166	.9068	.9281	.9774
		.9862	.9832	.9652	.9110			.9997	.9997	.9998	.9992
		.9982	.9908	.9792	.9373			1.000	1.000	.9998	.9993
		.9986	.9924	.9827	.9687			1.000	.9991	.9998	.9994
.1	.9	.7645	.7524	.7804	**.9456**	.9	.01	.7073	.7148	.7654	.8055
		.9520	**.9483**	**.9505**	.9686			.9326	**.9477**	.9250	**.9470**
		.8958	.8807	.8774	.8978			9321	.9170	.8943	.8638
		.9866	.9607	.9297	.8396			.9775	.9821	.9109	.8659
		.9989	.9914	.9739	.9159			.9891	.9827	.9728	.8790
		.9995	.9963	.9828	.9291			.9799	.9819	.9950	.8935
.3	.3	.8114	.7749	.7839	.9011	.05	.05	.8955	.9245	.9401	.9707
		.8686	.8669	.8930	.9689			.9662	.9561	.9740	.9886
		.8450	.8228	.8367	.9338			.9410	**.9461**	.9631	.9801
		.9578	.9296	.8972	.9329			.8272	.8792	.8534	.8908
		.9819	.9547	.9386	.9657			.9234	.8936	.8944	.9866
		.9936	.9798	.9688	.9768			.9991	.9891	.9994	.9989
.3	.5	.8143	.7867	.8024	.9110	.05	.01	.6153	.6255	.6312	**.9490**
		.9089	.8970	.9104	.9670			.9704	.9890	.9732	.9890
		.8513	8349	.8537	.9324			.9711	.9748	.9680	.9930
		.9871	.9836	.9656	**.9542**			.9611	.9099	.8930	9705
		.9972	.9963	.9850	.9673			.9882	.9637	**.9497**	.9901
		.9996	.9976	.9882	.9715			1.000	.9994	.9998	.9995

Table 4: $J_1 = 5$, $J_2 = 25$, $n_{ij} = 5$ $\forall_{i,j}$, $\alpha = .05$, unequal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

		$\rho_1 \mid \rho_2$						$\rho_1 \mid \rho_2$			
p_1	p_2	.1 | .3	.1 | .5	.1 | .9	.3 | .9	p_1	p_2	.1 | .3	.1 | .5	.1 | .9	.3 | .9
.1	.1	.6697	.7114	.8599	.8341	.3	.7	.8071	.8131	.8317	.7790
		.9321	.9330	**.9533**	**.9544**			.9394	.9126	.9233	.9047
		.9159	.9134	.9351	.9294			.8600	.8503	.8669	.8372
		.8626	.8254	.8906	.8801			.9667	.9932	.9776	**.9544**
		.8867	.8988	.9444	.9421			.9761	.9928	.9777	.9556
		.9948	.9963	.9978	.9970			.9962	.9929	.9779	.9565
.1	.3	.8361	.8661	.8884	.8828	.5	.5	.8267	.8369	.8565	.8266
		.9265	.9383	**.9543**	.9589			.8798	.8794	.9207	.9056
		.9081	.9150	.9236	.9184			.8488	.8592	.8795	.8545
		.9401	.9260	.9420	.9231			.9740	.9929	.9910	.9747
		.9418	**.9462**	**.9499**	.9412			.9939	.9930	.9912	.9742
		.9964	.9907	.9856	.9825			.9941	.9937	.9923	.9769
.1	.7	.8178	.8370	.8734	.8632	.9	.9	.8650	.8588	.8406	.6965
		.9523	**.9525**	.9603	.9725			.9348	**.9516**	**.9508**	**.9505**
		.9094	.9096	.9219	.9202			.9167	.9349	.9342	.9419
		.9907	.9811	**.9525**	.9448			.9998	.9999	.9982	.9995
		.9954	.9885	.9566	.9578			.9997	1.000	.9991	1.000
		.9974	.9941	.9704	.9722			1.000	.9999	.9999	1.000
.1	.9	.7298	.7585	.8007	.7740	.9	.01	.6465	.6683	.8578	.8497
		.9543	.9571	.9794	.9802			**.9505**	**.9477**	.9098	.9093
		.9059	.9054	.9232	.9257			.9419	.9423	.9072	.9064
		.9828	.9738	**.9500**	.9290			.9895	.9681	.9656	.9825
		.9964	.9934	.9753	.9675			.9991	.9991	.9997	.9982
		.9966	.9934	.9782	.9710			.9999	.9990	.9991	.9988
.3	.3	.7909	.8031	.8257	.7770	.05	.05	.7942	.8633	.9188	.9426
		.8788	.8969	.8910	.8755			**.9532**	**.9507**	.9689	.9689
		.8239	.8538	.8614	.8302			.9633	.9109	.9757	**.9518**
		.9633	.9642	.9428	.9315			.8711	.8997	.9124	.9017
		.9622	.9650	.9667	.9376			.9225	.9148	.9582	**.9499**
		.9772	.9802	.9794	.9601			.9799	.9899	.9998	.9887
.3	.5	.7983	.9161	.8346	.7877	.05	.01	.6195	.6133	.6298	.5997
		.8941	.9152	.9246	.9052			.9640	.9686	.9710	.9721
		.8355	.8505	.8715	.8395			**.9532**	**.9458**	.9676	.9662
		.9824	.9788	.9687	.9388			.9285	.9361	.9733	.9590
		.9847	.9819	.9723	**.9461**			.9898	.9875	.9952	.9955
		.9891	.9850	.9767	**.9533**			.9999	.9999	1.000	1.000

The Jeffreys-Perks, extended Newcombe, extended Beal, and extended Peskun intervals are all significant improvements over both the Wald interval and the Haldane interval. There are several cases where the Jeffreys-Perks, extended Newcombe, extended Beal, and extended Peskun have very similar coverage probabilities. However, there are also certain cases when one interval seems to perform slightly better than the others. For example, in most cases when the true difference between the two probabilities is large, the Jeffreys-Perks interval tends to perform better. To illustrate this, consider the following cases: CASE 1-- $p_1 = .1,\ p_2 = .9$ and $\Phi_1 = \Phi_2 = 1.0$; CASE 2-- $p_1 = .9,\ p_2 = .01$ and $\Phi_1 = \Phi_2 = 9.0$; and CASE 3-- $p_1 = .1,\ p_2 = .9$ and $\Phi_1 = 3/7,\ \Phi_2 = 9.0$. These three examples are illustrated by Figures $1 - 3$ and utilize data from Tables 1, 3, and $5 - 11$.

Figure 1: Coverage probability for $p_1 = .1,\ p_2 = .9$, $\Phi_1 = \Phi_2 = 1.0$
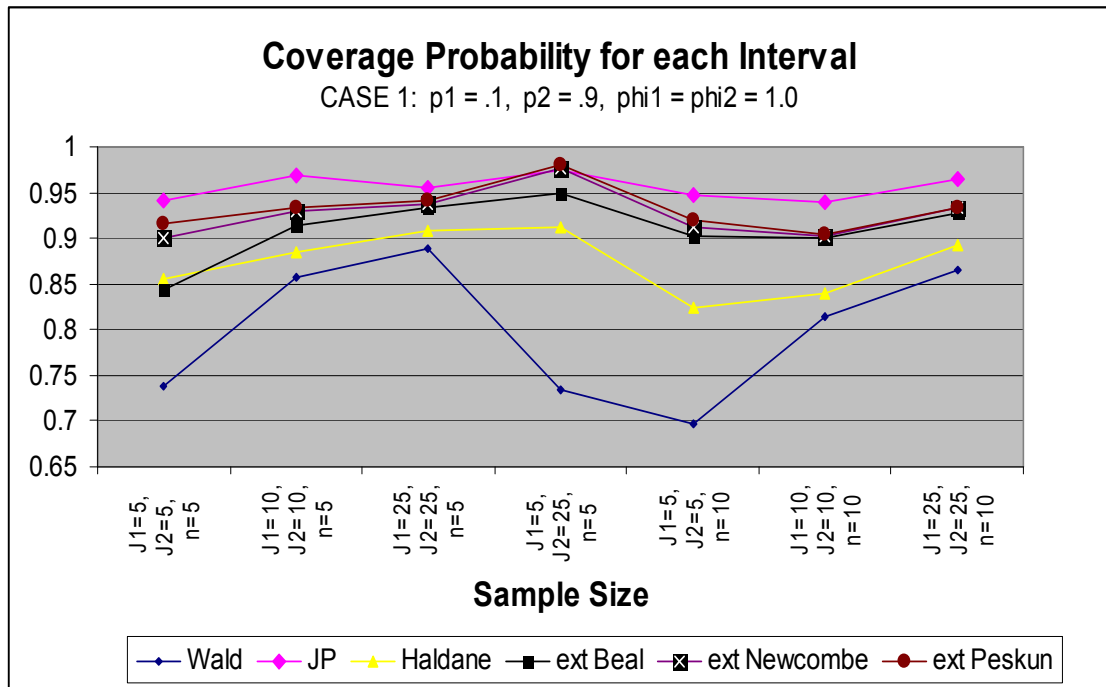
Figure 2: Coverage probability for $p_1 = .9$, $p_2 = .01$, $\Phi_1 = \Phi_2 = 9.0$

Coverage Probability for each Interval
CASE 2: p1 = .9, p2 = .01, phi1 = phi2 = 9.0
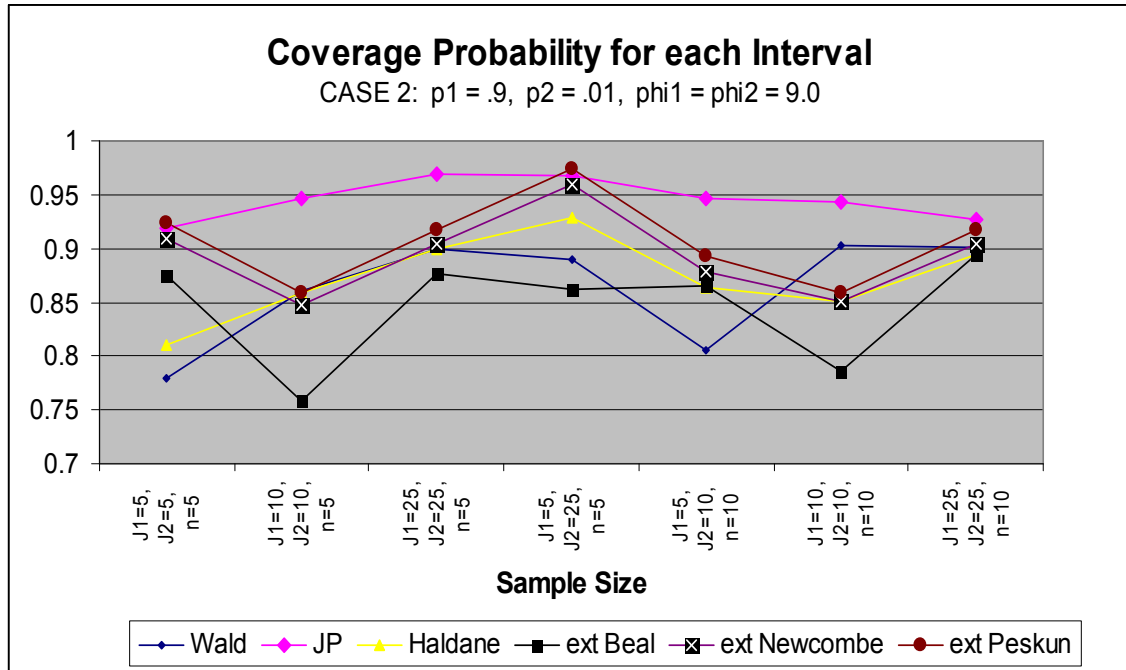
Figure 3: Coverage probability for $p_1 = .1$, $p_2 = .9$, $\Phi_1 = 3/7$, $\Phi_2 = 9.0$

Coverage Probability for each Interval
CASE 3: p1 = .1, p2 = .9, phi1 = 3/7, phi2 = 9.0
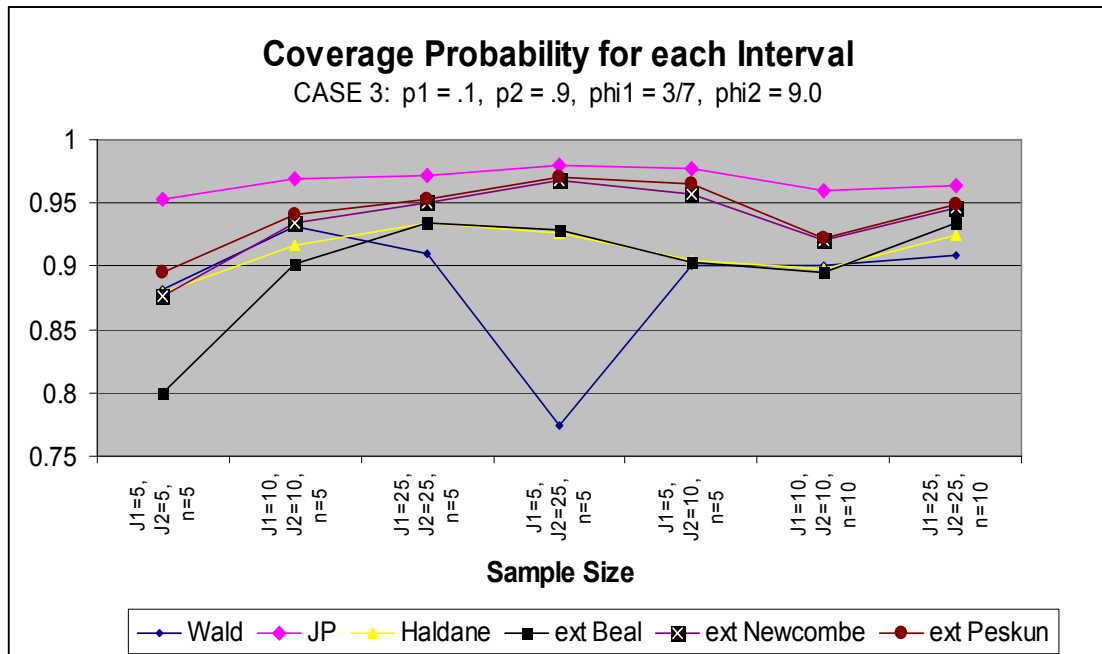
In each of the above cases we see that the Jeffreys-Perks interval has the highest coverage probability. From the tables we can also see this trend most of the time in the case when $p_1 = .1,\ p_2 = .7$. In general, it seems the Jeffreys-Perks interval is a good choice when the true difference between the proportions is large. We also see from the tables that as the sample size increases, the Jeffreys-Perks coverage probabilities improve, becoming closer to the nominal level of .95.

In cases where the true proportions are equal or close to each other, the extended intervals (Beal, Newcombe, & Peskun) tend to have the best coverage probability. Consider the following two cases: CASE 4-- $p_1 = .3,\ p_2 = .3$ and $\Phi_1 = 1/9,\ \Phi_2 = 1/9$; and CASE 5-- $p_1 = .1,\ p_2 = .3$ and $\Phi_1 = 1/9,\ \Phi_2 = 3/7$. These examples are illustrated by Figures 4 and 5 and utilize data from Tables $1, 3, 5 - 6, 8 - 9$, and 11.

While Figures 4 and 5 illustrate two specific cases where the three extended intervals appear to be the best choice, and Figures $1 - 3$ give examples of cases where the Jeffreys-Perks interval seems to be the best choice, there are also several cases where these intervals have very similar results. In general, the Wald and Haldane give the lowest coverage probability, while the Jeffreys-Perks, extended Beal, and extended Newcombe have comparable results with coverage closest to .95 in the majority of cases. The Peskun interval is the most conservative. One specific case is illustrated by Figure 6. There are several other cases in Tables $1 - 23$ where the Jeffreys-Perks, extended Newcombe, and extended Beal intervals have comparable results.

Figure 4: Coverage probability for $p_1 = .3$, $p_2 = .3$, $\Phi_1 = 1/9$, $\Phi_2 = 1/9$
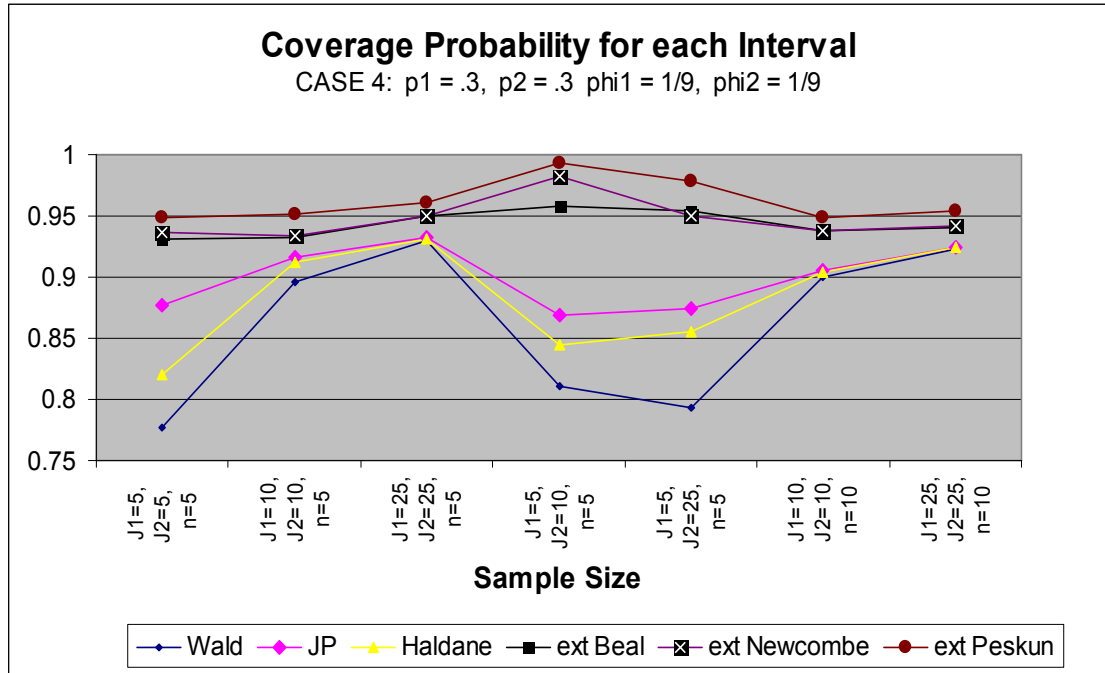
Figure 5: Coverage probability for $p_1 = .1$, $p_2 = .3$, $\Phi_1 = 1/9$, $\Phi_2 = 3/7$
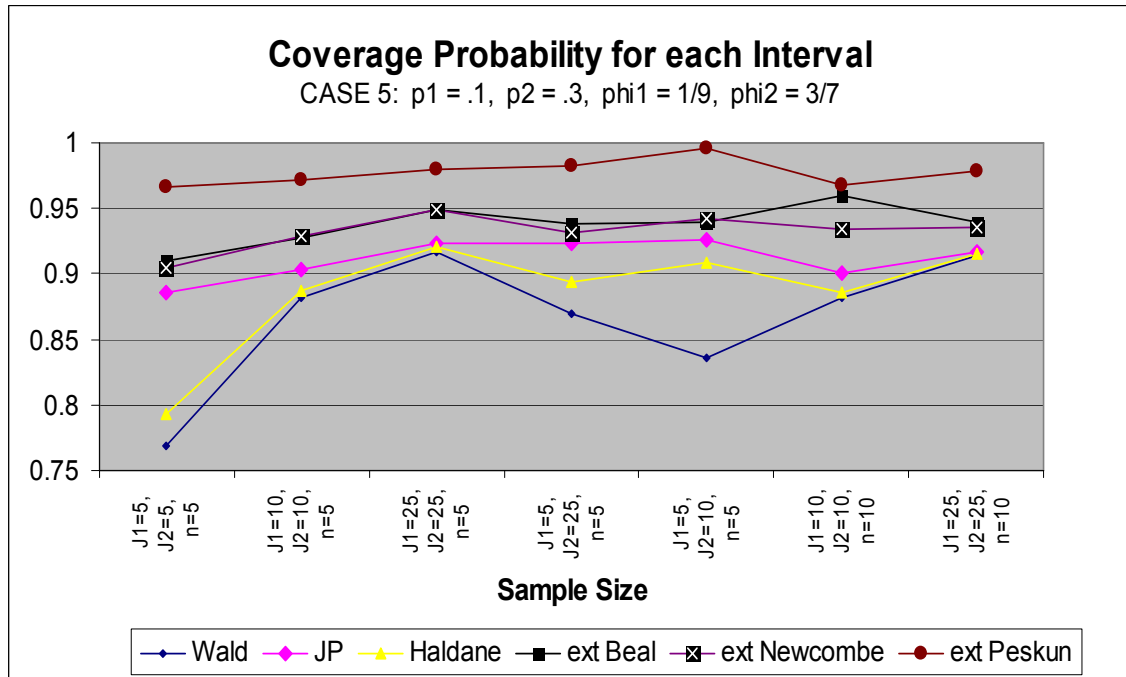
Figure 6: Coverage probability for $p_1 = .3$, $p_2 = .7$, $\Phi_1 = 1/9$, $\Phi_2 = 3/7$
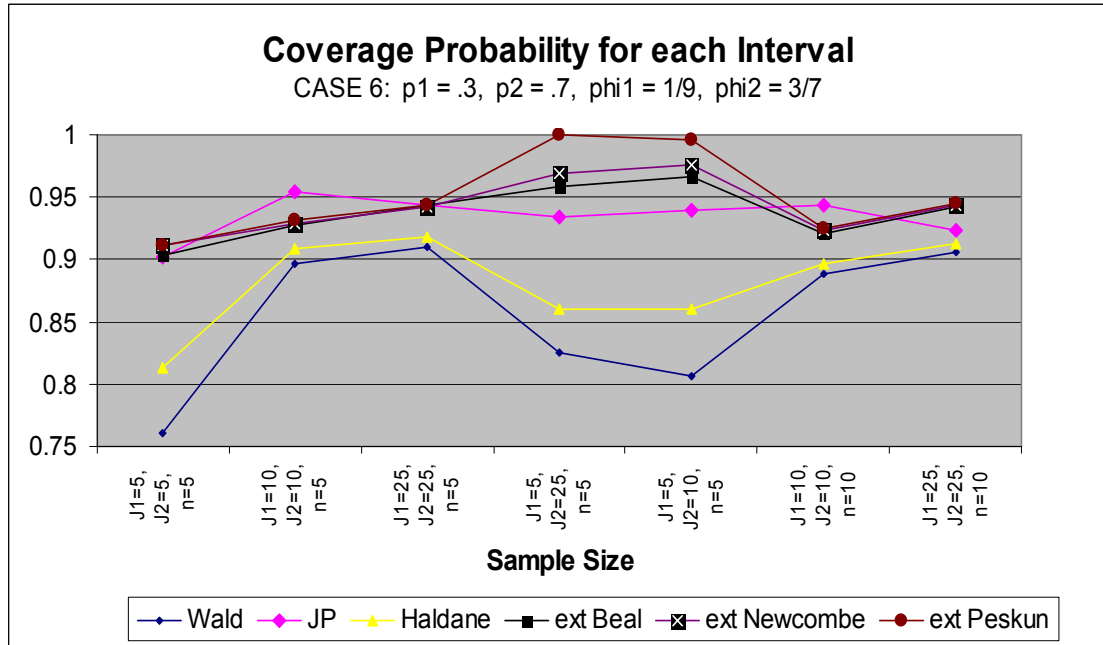
As a secondary means of evaluating and comparing the six intervals, average interval width was calculated. To some degree there will be a trade-off with average interval width and coverage probability. In general as the width of the interval increases, the coverage probability also increases. Ideally we would like an interval with coverage probability close to .95 and small interval width. Obviously sample size affects interval width, thus comparisons are made with fixed sample sizes. Three examples of average interval width are illustrated with Figures 7 – 9 and based on data in Tables 1, 4, and 9. From Figure 7, based on data from Table 1 where $J_1 = 5$, $J_2 = 5$, and $n_{ij} = 5$, we see that the Haldane and Wald intervals in general have the smallest interval width. However, as discussed earlier, these intervals also have the lowest coverage probability. In this case we also note that the Jeffreys-Perks and Peskun intervals have the largest interval width. Next we consider a case with larger sample sizes. This is done using Figure 8 which is based on the data in Table 9 where $J_1 = 25$, $J_2 = 25$, and $n_{ij} = 10$.

Figure 7: Average Interval Width for $J_1 = 5$, $J_2 = 5$, $n_{ij} = 5$, equal correlation
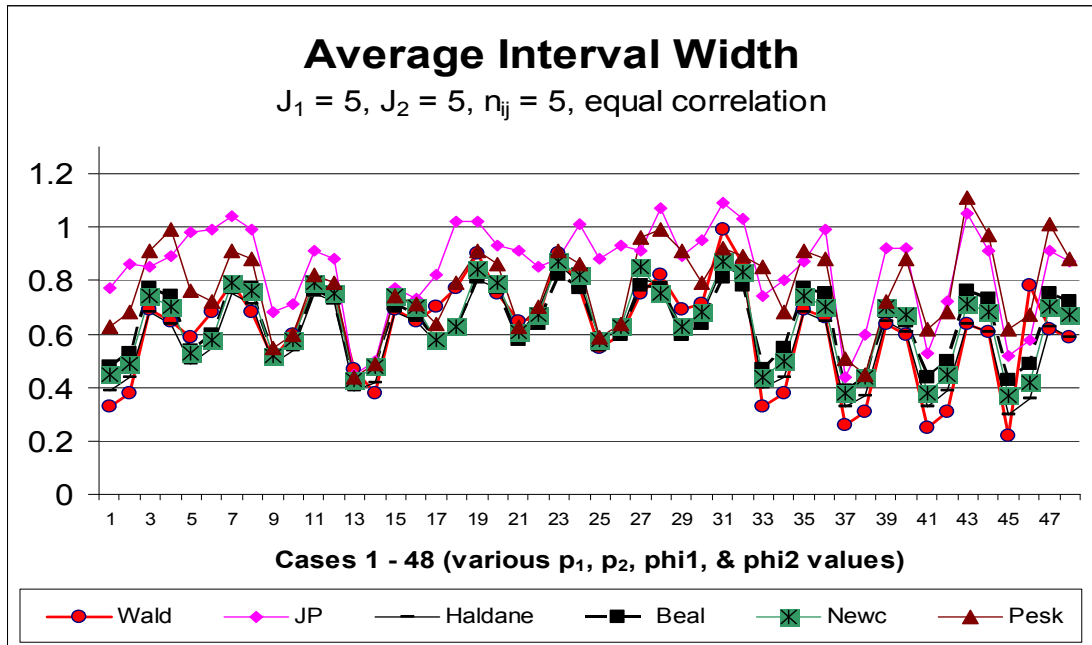
Figure 8: Average Interval Width for $J_1 = 25$, $J_2 = 25$, $n_{ij} = 10$, equal correlation
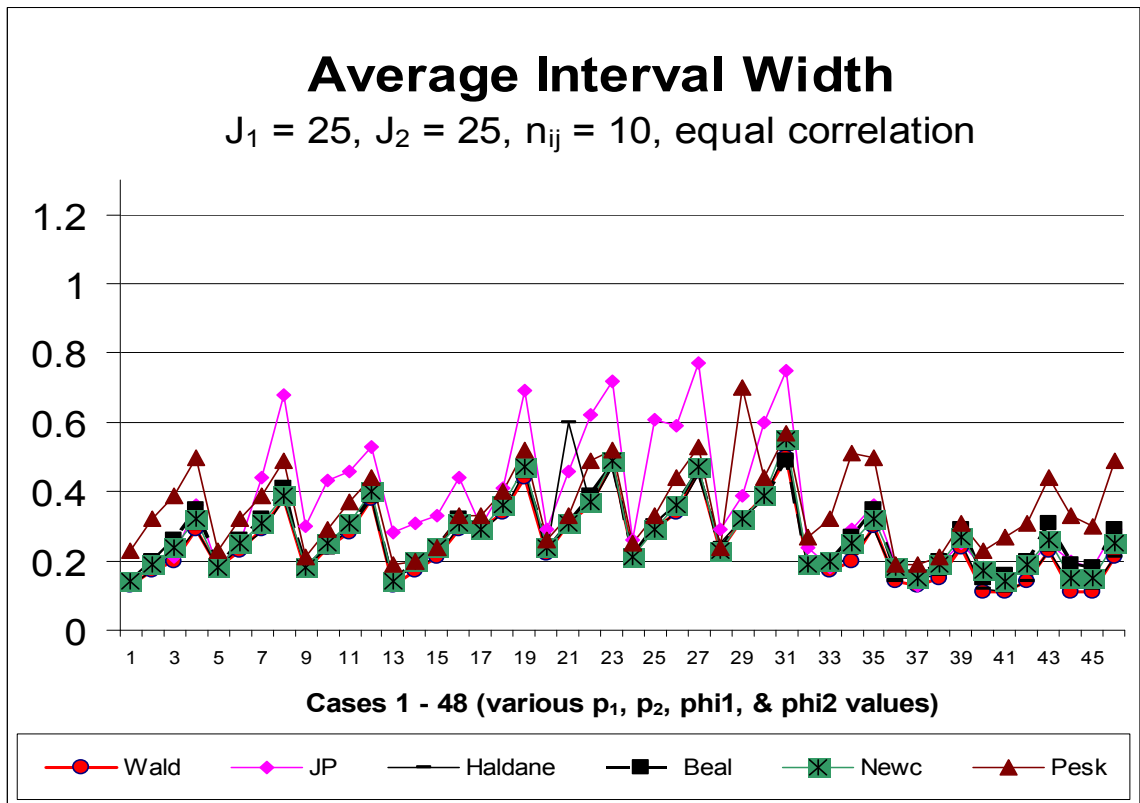
In Figure 8 we can see that by increasing the sample size we have decreased the average interval width for all six intervals. As in the previous case, the Jeffreys-Perks and Peskun intervals have slightly larger interval width than the other four intervals. As a final example illustrating average interval width we consider a case with unequal numbers of litters in the two populations and also unequal correlation. Figure 9 is based on data from Table 4.

Figure 9: Average Interval Width for $J_1 = 5$, $J_2 = 25$, $n_{ij} = 5$, unequal correlation
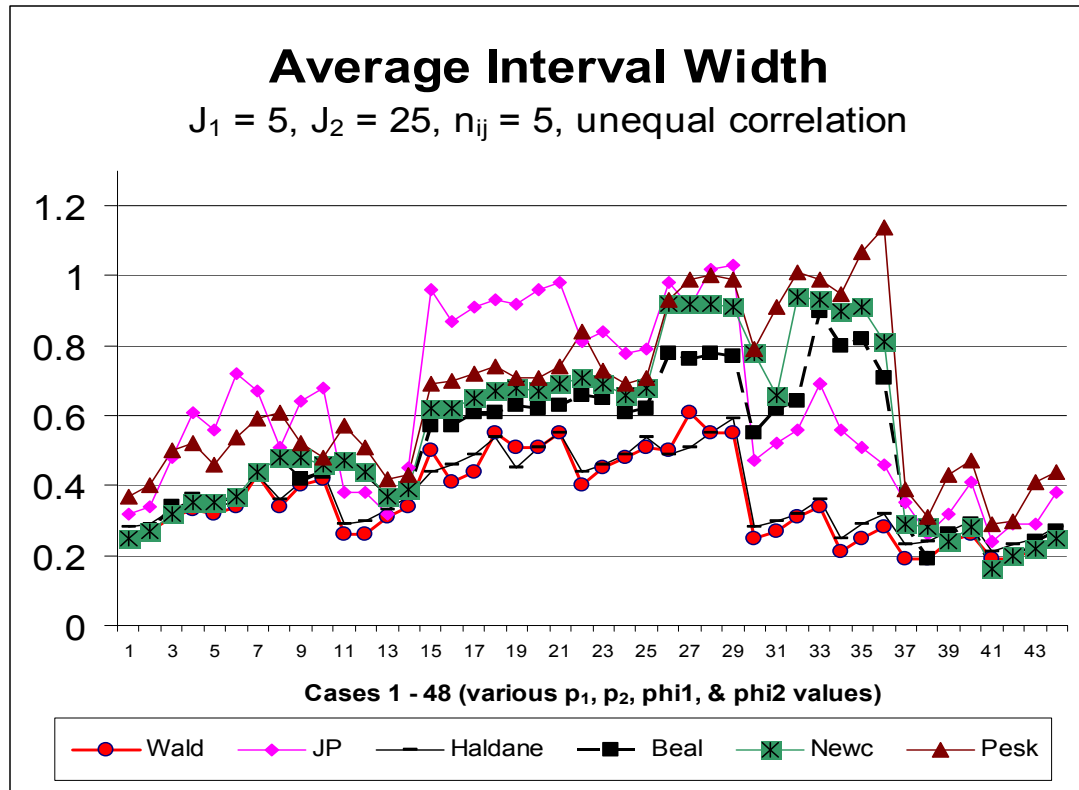
In this example there is much more variability in the average interval width. The Haldane and Wald interval again have the smallest width. In some cases the Jeffreys-Perks has the largest width where other times the extended intervals (Peskun, Beal, & Newcombe) have the largest width.

As a final analysis of the simulation results we investigate the asymptotic

properties of the six intervals. As discussed earlier in Chapter 2, all six intervals have

asymptotic properties. Thus, we would expect the coverage probabilities to become

closer to the nominal level of .95 as the sample sizes increase. Using the data in Tables

1, 2, 5, 6, 22 and 23, we can count the number of times each of the six intervals has

coverage probability that is within two standard errors of .95. Recall that the standard

error for this case is given by $S.E. = \sqrt{\dfrac{(1-\alpha)(\alpha)}{N}} = \sqrt{\dfrac{(.05)(.95)}{10000}} = .0022$ and each

result in the table that is within 2 standard errors is bold-faced.

Figure 10: Number of times out of 48 cases where the coverage probability for each

interval is within 2 standard errors of .95
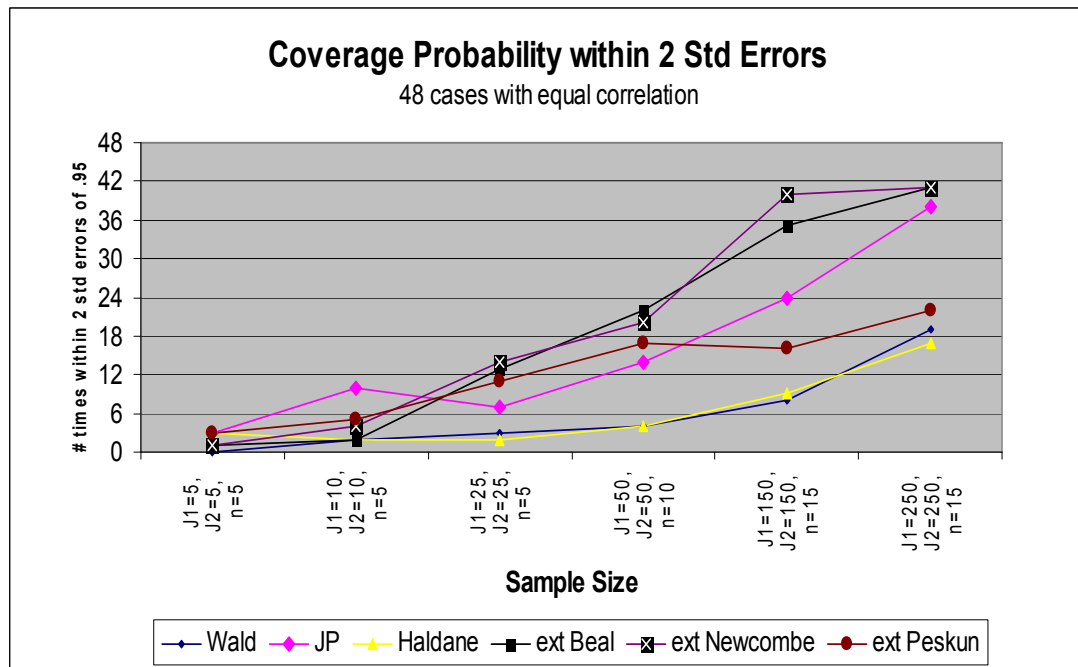
From Figure 10 we can see the asymptotic properties begin to take affect. As the

sample size increases, a larger proportion of the intervals have coverage probability

within 2 standard errors of the nominal level of .95. This convergence is rather slow,

particularly for the Wald, Haldane, and Peskun intervals. From Figure 10 we also note that the Jeffreys-Perks, extended Newcombe, and extended Beal intervals are, for the most part, better intervals as they have a larger number of cases with coverage probability near .95.

CHAPTER 4

CONCLUSIONS AND FUTURE WORK

The purpose of this paper was to construct intervals for the proportion difference between two populations where the data have a beta-binomial distribution.   Six intervals have been constructed and compared based on coverage probability and average interval width. The Wald interval is based on the MLEs using the beta-binomial as the underlying distribution. The Jeffreys-Perks and Haldane intervals are new intervals constructed as extensions of Beal's binomial intervals that account for overdispersion and correlation by assuming the beta-binomial distribution holds.   The final three intervals – the extended Beal, extended Newcombe, and the extended Peskun – are all extensions of binomial cases with the introduction of a variance inflation factor to account for overdispersion. These three intervals were originally presented by Chen, et al. (2004) and do not assume that the beta-binomial distribution holds. Each of these six intervals is evaluated via simulations using a variety of parameter values.

The Wald interval is the usual interval presented in many textbooks, but this interval has problems when the sample sizes are small or the proportions are near zero or one. The Haldane interval provides an improvement over the usual interval, but still has several cases with less than nominal coverage probability. The Jeffreys-Perks interval provides a significant improvement over the usual interval, even in cases with small

samples. Both the Haldane and Jeffreys-Perks intervals also have the property that the intervals do not in general collapse to a single point when $\hat{p}_1 = \hat{p}_2 = 0$ or $\hat{p}_1 = \hat{p}_2 = 1$. The extended intervals presented by Chen, et al. (2004) also provide a significant improvement over the Wald interval. The extended intervals do not collapse in the cases where $\hat{p}_1 = \hat{p}_2 = 0$ or $\hat{p}_1 = \hat{p}_2 = 1$ as the variance inflation factor is defined to be equal to one when $\hat{p}_i(1 - \hat{p}_i) = 0$. The Jeffreys-Perks interval tends to exhibit superior performance in cases where the difference between the proportions is large (greater than .6), whereas the extended intervals exhibit superior performance when the proportions are close together (less than .2). Additionally, the Jeffreys-Perks, extended Beal, and extended Newcombe intervals have very comparable results in many other cases. The extended Peskun interval has the highest overall coverage probability in most cases, but is very conservative.

This paper has examined the specific case of estimating the difference between two proportions when there is correlation or overdipersion among the trials. Future research in this area could involve extending some of these ideas to multiple comparisons for correlated binary data. The Jeffreys-Perks and Haldane intervals could be extended to include priors and estimates of the Φ values in the formulation of the problem. Additionally, the binomial intervals suggested by Mee (1984), Anbar (1984), and Miettinen and Nurminenm (1985) could be constructed for the beta-binomial case. This would involve a numerically intense procedure, however, as Beal (1987) notes in the binomial case, these intervals were a significant improvement over the usual interval. In addition to multiple comparisons and alternative confidence intervals, one may wish to investigate estimators for the overdispersion parameters Φ_1 and Φ_2. In the intervals

presented in this paper the MLEs were used to estimate these parameters, however, other

estimators may improve the intervals for $p_1 - p_2$.

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APPENDIXES

Derivation 1.1:   *The beta-binomial pdf:* (from "Bayesian Theory", by J. M. Bernardo & A. F. M. Smith, 1994)

The **beta-binomial** distribution is derived by assuming a beta prior on the binomial

parameter *p*.

A discrete random variable *X* has a beta-binomial distribution with parameters α , β ,

and *n* if its pdf is of the form

$$BB(x|\alpha, \beta, n) = \binom{n}{x}\frac{\Gamma(\alpha+\beta)\,\Gamma(\alpha+x)\Gamma(\beta+n-x)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\alpha+\beta+n)}$$

$\alpha > 0, \ \beta > 0, \ x = 0, \ 1, \ 2 ... n$

where $\Gamma(\cdot)$ is the gamma function.

$$MEAN = E(X) = n\left(\frac{\alpha}{\alpha + \beta}\right) \qquad VARIANCE = V(X) = \frac{n\alpha\beta}{(\alpha + \beta)^2}\left(\frac{\alpha + \beta + n}{\alpha + \beta + 1}\right)$$

The beta-binomial is a mixture of the binomial and beta pdfs:

$$BB(x|\alpha, \beta, n) = \int_0^1 BIN(x|\theta, n)\, BETA(\theta|\alpha, \beta)\, d\theta$$

$$= \int_0^1 \binom{n}{x} \theta^x (1-\theta)^{n-x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} d\theta$$

$$= \binom{n}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \theta^{\alpha+x-1} (1-\theta)^{n+\beta-x-1} d\theta$$

$$= \binom{n}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+x)\Gamma(n+\beta-x)}{\Gamma(\alpha+\beta+n)} \int_0^1 \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+x)\Gamma(n+\beta-x)} \theta^{\alpha+x-1}(1-\theta)^{n+\beta-x-1} d\theta$$

$$= \binom{n}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+x)\Gamma(n+\beta-x)}{\Gamma(\alpha+\beta+n)}$$

An alternative form of the beta-binomial can be expressed in terms of the parameter Φ.

Recall that Φ measures departure from binomial sampling and is often referred to as the

overdispersion parameter. When $\Phi = 0$, there is no overdispersion and binomial sampling

retained. As Φ increases, departure from binomial sampling increases. Also, Φ is

related to correlation by the equation $\Phi = \dfrac{\rho}{1+\rho}$. Letting $\alpha = \dfrac{p}{\Phi}$, and

$\beta = \dfrac{1-p}{\Phi}$, the pdf of the Beta-binomial takes on the form:

$$f(x \mid p, \Phi, n) = \binom{n}{x} \frac{\Gamma(x+\frac{p}{\Phi})\Gamma(\Phi^{-1})\Gamma\left(n-x+\frac{1-p}{\Phi}\right)}{\Gamma\left(\frac{p}{\Phi}\right)\Gamma\left(\frac{1-p}{\Phi}\right)\Gamma(\Phi^{-1}+n)}$$

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Derivation 2.1.1: *Derivation of the standard CI for* $p_1 - p_2$

Substituting $\tilde{a} = \hat{a}$ and $\tilde{b} = \hat{b}$ into equation (1), we get

$$(b - \hat{b})^2 = cV(\hat{a}, \hat{b}; u, v)$$

$$(b - \hat{b})^2 = c\left(u[(2 - \hat{a})\hat{a} - \hat{b}^2] + 2v(1 - \hat{a})\hat{b}\right)$$

$$(b - \hat{b})^2 = c\left(\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}\right)$$

$$(b - \hat{b}) = \pm c\left(\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}\right)^{\frac{1}{2}}$$

$$CI \text{ for } b \implies \hat{b} \pm c\left(\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}\right)^{\frac{1}{2}}$$

Derivation 2.2.1: *Derivation of Log-Likelihood*

Given that the data is drawn from a beta-binomial distribution with parameters p_i and Φ_i, the likelihood function is given by: (from derivation 1.1)

$$L(p_i, \Phi_i) = \prod_{j=1}^{J_i} \binom{n_{ij}}{y_{ij}} \frac{\Gamma(y_{ij} + \frac{p_i}{\Phi_i})\Gamma(\Phi_i^{-1})\Gamma\left(n_{ij} - y_{ij} + \frac{1-p_i}{\Phi_i}\right)}{\Gamma\left(\frac{p_i}{\Phi_i}\right)\Gamma\left(\frac{1-p_i}{\Phi_i}\right)\Gamma(\Phi^{-1} + n_{ij})}$$

$$= \prod_{j=1}^{J_i} \binom{n_{ij}}{y_{ij}} \left(\frac{\Gamma(y_{ij} + \frac{p_i}{\Phi_i})}{\Gamma\left(\frac{p_i}{\Phi_i}\right)}\right)\left(\frac{\Gamma(\Phi_i^{-1})}{\Gamma(\Phi_i^{-1} + n_{ij})}\right)\left(\frac{\Gamma\left(n_{ij} - y_{ij} + \frac{1-p_i}{\Phi_i}\right)}{\Gamma\left(\frac{1-p_i}{\Phi_i}\right)}\right)$$

$$= \prod_{j=1}^{J_i} \binom{n_{ij}}{y_{ij}} \left(\Phi_i^{y_{ij}}\frac{\Gamma(y_{ij} + \frac{p_i}{\Phi_i})}{\Gamma\left(\frac{p_i}{\Phi_i}\right)}\right)\left(\Phi_i^{-n_{ij}}\frac{\Gamma(\Phi_i^{-1})}{\Gamma(\Phi_i^{-1} + n_{ij})}\right)\left(\Phi_i^{n_{ij}-y_{ij}}\frac{\Gamma\left(n_{ij} - y_{ij} + \frac{1-p_i}{\Phi_i}\right)}{\Gamma\left(\frac{1-p_i}{\Phi_i}\right)}\right)$$

$$\textit{Note that: } \left(\Phi_i^{y_{ij}}\right)\left(\Phi_i^{-n_{ij}}\right)\left(\Phi_i^{n_{ij}-y_{ij}}\right) = 1$$

$$= \prod_{j=1}^{J_i} \binom{n_{ij}}{y_{ij}} (A_1)(A_2)(A_3)$$

Using the equality $\Gamma(a) = (a-1)\Gamma(a-1)$, we manipulate the expressions A_1, A_2, and A_3.

$$A_1 = \Phi_i^{y_{ij}}\left(\frac{\Gamma(y_{ij} + p_i/\Phi_i)}{\Gamma(p_i/\Phi_i)}\right) = \Phi_i^{y_{ij}}\left(\frac{(y_{ij} + p_i/\Phi_i - 1)\Gamma(y_{ij} + p_i/\Phi_i - 1)}{\Gamma(p_i/\Phi_i)}\right)$$

$$= \Phi_i^{y_{ij}}\left(\frac{(y_{ij} + p_i/\Phi_i - 1)(y_{ij} + p_i/\Phi_i - 2)\Gamma(y_{ij} + p_i/\Phi_i - 2)}{\Gamma(p_i/\Phi_i)}\right)$$

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$$= \Phi_i{}^{y_{ij}} \left(\frac{(y_{ij} + p_i/\Phi_i - 1)(y_{ij} + p_i/\Phi_i - 2)(y_{ij} + p_i/\Phi_i - 3)\cdots(p_i/\Phi_i)\Gamma(p_i/\Phi_i)}{\Gamma(p_i/\Phi_i)} \right)$$

$$= \Phi_i{}^{y_{ij}} \left((y_{ij} + p_i/\Phi_i - 1)(y_{ij} + p_i/\Phi_i - 2)(y_{ij} + p_i/\Phi_i - 3)\cdots(p_i/\Phi_i) \right)$$

$$= (p_i + (y_{ij} - 1)\Phi_i)(p_i + (y_{ij} - 2)\Phi_i)(p_i + (y_{ij} - 3)\Phi_i)\cdots(p_i + \Phi_i)p_i$$

$$= \prod_{r=0}^{y_{ij}-1} (p_i + r\Phi_i)$$

$$A_2 = \Phi_i{}^{-n_{ij}} \left(\frac{\Gamma(\Phi_i{}^{-1})}{\Gamma(\Phi_i{}^{-1} + n_{ij})} \right) = \Phi_i{}^{-n_{ij}} \left(\frac{\Gamma(\Phi_i{}^{-1})}{(\Phi_i{}^{-1} + n_{ij} - 1)\Gamma(\Phi_i{}^{-1} + n_{ij} - 1)} \right)$$

$$= \Phi_i{}^{-n_{ij}} \left(\frac{\Gamma(\Phi_i{}^{-1})}{(\Phi_i{}^{-1} + n_{ij} - 1)(\Phi_i{}^{-1} + n_{ij} - 2)\Gamma(\Phi_i{}^{-1} + n_{ij} - 2)} \right)$$

$$= \Phi_i{}^{-n_{ij}} \left(\frac{\Gamma(\Phi_i{}^{-1})}{(\Phi_i{}^{-1} + n_{ij} - 1)(\Phi_i{}^{-1} + n_{ij} - 2)\cdots(\Phi_i{}^{-1} + 1)\Phi_i{}^{-1}\,\Gamma(\Phi_i{}^{-1})} \right)$$

$$= \Phi_i{}^{-n_{ij}} \left(\frac{1}{(\Phi_i{}^{-1} + n_{ij} - 1)(\Phi_i{}^{-1} + n_{ij} - 2)\cdots(\Phi_i{}^{-1} + 1)\Phi_i{}^{-1}} \right)$$

$$= \left(\frac{1}{(1 + (n_{ij} - 1)\Phi_i)(1 + (n_{ij} - 2)\Phi_i)\cdots(1 + \Phi_i)} \right)$$

$$= \prod_{r=0}^{n_{ij}} \frac{1}{(1 + r\Phi_i)}$$

$$A_3 = \Phi_i{}^{n_{ij} - y_{ij}} \left(\frac{\Gamma\left(n_{ij} - y_{ij} + \frac{(1 - p_i)}{\Phi_i}\right)}{\Gamma\left(\frac{(1 - p_i)}{\Phi_i}\right)} \right) = \Phi_i{}^{n_{ij} - y_{ij}} \left(\frac{\left(n_{ij} - y_{ij} - 1 + \frac{(1 - p_i)}{\Phi_i}\right)\Gamma\left(n_{ij} - y_{ij} - 1 + \frac{(1 - p_i)}{\Phi_i}\right)}{\Gamma\left(\frac{(1 - p_i)}{\Phi_i}\right)} \right)$$

$$= \Phi_i^{n_{ij} - y_{ij}} \left[\frac{\left(n_{ij} - y_{ij} - 1 + \frac{(1 - p_i)}{\Phi_i} \right) \left(n_{ij} - y_{ij} - 2 + \frac{(1 - p_i)}{\Phi_i} \right) \cdots \left(1 + \frac{(1 - p_i)}{\Phi_i} \right) \left(\frac{(1 - p_i)}{\Phi_i} \right) \Gamma \left(\frac{(1 - p_i)}{\Phi_i} \right)}{\Gamma \left(\frac{(1 - p_i)}{\Phi_i} \right)} \right.$$

$$= \Phi_i^{n_{ij} - y_{ij}} \left(\left(n_{ij} - y_{ij} - 1 + \frac{(1 - p_i)}{\Phi_i} \right) \left(n_{ij} - y_{ij} - 2 + \frac{(1 - p_i)}{\Phi_i} \right) \cdots \left(1 + \frac{(1 - p_i)}{\Phi_i} \right) \left(\frac{(1 - p_i)}{\Phi_i} \right) \right)$$

$$= \left(\left(\left(n_{ij} - y_{ij} - 1 \right) \Phi_i + 1 - p_i \right) \left(\left(n_{ij} - y_{ij} - 2 \right) \Phi_i + (1 - p_i) \right) \cdots \left(\Phi_i + (1 - p_i) \right) \left((1 - p_i) \right) \right)$$

$$= \prod_{r=0}^{n_{ij} - y_{ij} - 1} (1 - p_i + r\Phi_i)$$

Putting A_1, A_2, & A_3 together, we get the following expression for the log-likelihood:

$$\ln(L(p_i, \Phi_i)) = \ln \left\{ \prod_{j=1}^{J_i} \binom{n_{ij}}{y_{ij}} \left[\frac{\prod_{r=0}^{y_{ij} - 1} (p_i + r\Phi_i) \prod_{r=0}^{n_{ij} - y_{ij} - 1} (1 - p_i + r\Phi_i)}{\prod_{r=0}^{n_{ij} - 1} (1 + r\Phi_i)} \right] \right\}$$

Derivation 2.2.1a: Log-Likelihood when $y_{ij} = 0$ or $y_{ij} = n_{ij}$

Note in derivation 2.2.1, that if $y_{ij} = n_{ij}$ then the product $\prod\limits_{r=0}^{n_{ij}-y_{ij}-1}(1 - p_i + r\Phi_i)$ is not

defined and similarly if $y_{ij} = 0$ then the product $\prod\limits_{r=0}^{y_{ij}-1}(p_i + r\Phi_i)$ is not defined.

However, if we go back to the original log-likelihood for these cases, we can derive the following results:

CASE 1: $y_{ij} = 0$

Start with the Log-Likelihood from derivation 2.2.1:

$$L(p_i, \Phi_i) = \prod_{j=1}^{J_i}\binom{n_{ij}}{y_{ij}}\left(\Phi_i^{y_{ij}}\left(\frac{\Gamma(y_{ij} + \dfrac{p_i}{\Phi_i})}{\Gamma\left(\dfrac{p_i}{\Phi_i}\right)}\right)\right)\left(\Phi_i^{-n_{ij}}\left(\frac{\Gamma(\Phi_i^{-1})}{\Gamma(\Phi_i^{-1} + n_{ij})}\right)\right)\left(\Phi_i^{n_{ij}-y_{ij}}\left(\frac{\Gamma\left(n_{ij} - y_{ij} + \dfrac{1-p_i}{\Phi_i}\right)}{\Gamma\left(\dfrac{1-p_i}{\Phi_i}\right)}\right)\right)$$

$$= \prod_{j=1}^{J_i}\left(\Phi_i^{-n_{ij}}\left(\frac{\Gamma(\Phi_i^{-1})}{\Gamma(\Phi_i^{-1} + n_{ij})}\right)\right)\left(\Phi_i^{n_{ij}-y_{ij}}\left(\frac{\Gamma\left(n_{ij} - y_{ij} + \dfrac{1-p_i}{\Phi_i}\right)}{\Gamma\left(\dfrac{1-p_i}{\Phi_i}\right)}\right)\right) \quad \text{since}$$

$y_{ij} = 0$

$$= \prod_{j=1}^{J_i}(A_2)(A_3) \quad \text{by derivation 2.2.1}$$

Thus by the results of derivation 2.2.1 we get the following when $y_{ij} = 0$:

$$\ln(L(p_i,\Phi_i)) = \ln\left\{\prod_{j=1}^{J_i}\left[\frac{\prod_{r=0}^{n_{ij}-y_{ij}-1}(1-p_i+r\Phi_i)}{\prod_{r=0}^{n_{ij}-1}(1+r\Phi_i)}\right]\right\}$$

CASE 2: $\quad y_{ij} = n_{ij}$

Again, start with the Log-Likelihood from derivation 2.2.1:

$$L(p_i,\Phi_i) = \prod_{j=1}^{J_i}\binom{n_{ij}}{y_{ij}}\left(\Phi_i^{y_{ij}}\left[\frac{\Gamma(y_{ij}+\frac{p_i}{\Phi_i})}{\Gamma\left(\frac{p_i}{\Phi_i}\right)}\right]\right)\left(\Phi_i^{-n_{ij}}\left(\frac{\Gamma(\Phi_i^{-1})}{\Gamma(\Phi_i^{-1}+n_{ij})}\right)\right)\left(\Phi_i^{n_{ij}-y_{ij}}\left[\frac{\Gamma\left(n_{ij}-y_{ij}+\frac{1-p_i}{\Phi_i}\right)}{\Gamma\left(\frac{1-p_i}{\Phi_i}\right)}\right]\right)$$

$$= \prod_{j=1}^{J_i}\left(\Phi_i^{y_{ij}}\left[\frac{\Gamma(y_{ij}+\frac{p_i}{\Phi_i})}{\Gamma\left(\frac{p_i}{\Phi_i}\right)}\right]\left(\Phi_i^{-n_{ij}}\left(\frac{\Gamma(\Phi_i^{-1})}{\Gamma(\Phi_i^{-1}+n_{ij})}\right)\right)\right) \quad \text{since } y_{ij} = n_{ij}$$

$$= \prod_{j=1}^{J_i}(A_1)(A_2) \quad \text{by derivation 2.2.1}$$

Thus by the results of derivation 2.2.1 we get the following when $y_{ij} = n_{ij}$:

$$\ln(L(p_i,\Phi_i)) = \ln\left\{\prod_{j=1}^{J_i}\binom{n_{ij}}{y_{ij}}\left[\frac{\prod_{r=0}^{y_{ij}-1}(p_i+r\Phi_i)}{\prod_{r=0}^{n_{ij}-1}(1+r\Phi_i)}\right]\right\}$$

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Derivation 2.2.2: *The Fisher Information and Variance-Covariance Matrix I_n^{-1}:*
(from Hogg & Craig (1995), pgs 372-385)

Start with the one parameter case where X is a random variable with pdf $f(x; \theta)$, and θ is a parameter. In the continuous case, (the discrete case can be handled in a similar manner) we have:

$$\int_{-\infty}^{\infty} f(x;\theta)\, dx \; = \; 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\partial f(x;\theta)}{\partial \theta}\, dx \; = \; 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\frac{\partial f(x;\theta)}{\partial \theta}}{f(x;\theta)}\, f(x;\theta)\, dx \; = \; 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\partial \ln(f(x;\theta))}{\partial \theta}\, f(x;\theta)\, dx \; = \; 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \left(\left[\frac{\partial^2 \ln(f(x;\theta))}{\partial \theta^2} \right] f(x;\theta) \; + \; \left[\frac{\partial \ln(f(x;\theta))}{\partial \theta}\, \frac{\partial f(x;\theta)}{\partial \theta} \right] \right) dx \; = \; 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \left(\left[\frac{\partial^2 \ln(f(x;\theta))}{\partial \theta^2} \right] f(x;\theta) \; + \; \left[\frac{\partial \ln(f(x;\theta))}{\partial \theta}\, \frac{\partial \ln(f(x;\theta))}{\partial \theta}\, f(x;\theta) \right] \right) dx \; = \; 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \left[\frac{\partial \ln(f(x;\theta))}{\partial \theta} \right]^2 f(x;\theta)\, dx \; = \; - \int_{-\infty}^{\infty} \left[\frac{\partial^2 \ln(f(x;\theta))}{\partial \theta^2} \right] f(x;\theta)\, dx$$

$$\Rightarrow I(\theta) \; = \; \int_{-\infty}^{\infty} \left[\frac{\partial \ln(f(x;\theta))}{\partial \theta} \right]^2 f(x;\theta)\, dx \; = \; - \int_{-\infty}^{\infty} \left[\frac{\partial^2 \ln(f(x;\theta))}{\partial \theta^2} \right] f(x;\theta)\, dx$$

$\Rightarrow I(\theta)$ is called the **Fisher Information.**

If $X_1, X_2, ...X_n$ is a random sample from a distribution with pdf $f(x; \theta)$, then we define

the **Fisher Information** as: $I_n(\theta) = E\left[\left[\frac{\partial \ln(L(\theta))}{\partial \theta}\right]^2\right] = -E\left[\left[\frac{\partial^2 \ln(L(\theta))}{\partial \theta^2}\right]\right]$,

where $L(\theta) = f(x_1;\theta)f(x_2;\theta)f(x_3;\theta)\cdots f(x_n;\theta)$ is the likelihood function. It

follows that $I_n(\theta) = \sum_{i=1}^{n} E\left[\left[\frac{\partial \ln(f(X_i,\theta))}{\partial \theta}\right]^2\right]$

Now consider the case with two or more parameters. Let $X_1, X_2, ...X_n$ is a random

sample from a distribution with pdf $f(x;\theta_1,\theta_2)$. The information of the matrix sample

is equal to: $I_n(\theta_1,\theta_2) = -n\begin{bmatrix} E\left[\left[\frac{\partial^2 \ln(f(X,\theta_1,\theta_2))}{\partial \theta_1^2}\right]\right] & E\left[\left[\frac{\partial^2 \ln(f(X,\theta_1,\theta_2))}{\partial \theta_2 \partial \theta_1}\right]\right] \\ E\left[\left[\frac{\partial^2 \ln(f(X,\theta_1,\theta_2))}{\partial \theta_1 \partial \theta_2}\right]\right] & E\left[\left[\frac{\partial^2 \ln(f(X,\theta_1,\theta_2))}{\partial \theta_2^2}\right]\right] \end{bmatrix}$

If $\hat{\theta}_1$ and $\hat{\theta}_2$ are the MLEs of θ_1 and θ_2 then $\hat{\theta}_1$ and $\hat{\theta}_2$ have approximate bivariate

normal distributions with means θ_1 and θ_2 and variance-covariance matrix I_n^{-1}.

Consequently, $I_n^{-1} \approx \begin{bmatrix} \text{var}(\hat{\theta}_1) & \text{cov}(\hat{\theta}_1,\hat{\theta}_2) \\ \text{cov}(\hat{\theta}_1,\hat{\theta}_2) & \text{var}(\hat{\theta}_2) \end{bmatrix}$, and thus the ith element of the

diagonal gives $\hat{\text{var}}(\hat{\theta}_1)$.

Derivation 2.2.3: *Simplifying the estimator* $\hat{a}(\alpha)$

Here we derive the estimator $\hat{a}(\alpha)$ assuming an underlying beta-binomial distribution.

Recall that $\hat{a}(\alpha)$ is the posterior mean of a using the symmetrical prior density on (p_1, p_2) proportional to $(p_1 \, q_1 \, p_2 \, q_2)^\alpha$. Therefore the prior density is given by

$$g(p_1, p_2) = k \, p_1^\alpha \, (1-p_1)^\alpha \, p_2^\alpha \, (1-p_2)^\alpha \qquad 0 \le p_1 \le 1 \; \& \; 0 \le p_2 \le 1,$$

and the density of the data is bi-variate beta-binomial Thus the density is given by:

$$f(y_1, y_2 \,|\, p_1, p_2, \Phi_1, \Phi_2, n_1, n_2) =$$

$$\binom{n_1}{y_1} \frac{\Gamma(y_1 + \frac{p_1}{\Phi_1})\Gamma(\Phi_1^{-1})\Gamma\left(n_1 - y_1 + \frac{1-p_1}{\Phi_1}\right)}{\Gamma\left(\frac{p_1}{\Phi_1}\right)\Gamma\left(\frac{1-p_1}{\Phi_1}\right)\Gamma(\Phi_1^{-1} + n_1)} \binom{n_2}{y_2} \frac{\Gamma(y_2 + \frac{p_2}{\Phi_2})\Gamma(\Phi_2^{-1})\Gamma\left(n_2 - y_2 + \frac{1-p_2}{\Phi_2}\right)}{\Gamma\left(\frac{p_2}{\Phi_2}\right)\Gamma\left(\frac{1-p_2}{\Phi_2}\right)\Gamma(\Phi_2^{-1} + n_2)}$$

.

Therefore, the resulting posterior density is given by:

$$h(p_1, p_2 \,|\, y_1, y_2, \Phi_1, \Phi_2, n_1, n_2) = \frac{g(p_1, p_2) \, f(y_1, y_2)}{\int_0^1 \int_0^1 g(p_1, p_2) f(y_1, y_2) \, dp_1 \, dp_2}$$

Consequently, the posterior mean, $\hat{a}(\alpha)$ is given by:

$$\hat{a}(\alpha) = \int_0^1 \int_0^1 (p_1 + p_2) h(p_1, p_2 \,|\, y_1, y_2, \Phi_1, \Phi_2, n_1, n_2) \, dp_1 \, dp_2$$

$$= \int_0^1 \int_0^1 (p_1 + p_2) \left[\frac{g(p_1, p_2) \, f(y_1, y_2)}{\int_0^1 \int_0^1 g(p_1, p_2) f(y_1, y_2) \, dp_1 \, dp_2} \right] dp_1 \, dp_2$$

$$= \int_0^1 \int_0^1 p_1 \left[\frac{g(p_1, p_2) \, f(y_1, y_2)}{\int_0^1 \int_0^1 g(p_1, p_2) f(y_1, y_2) \, dp_1 \, dp_2} \right] dp_1 \, dp_2$$

$$+ \int_0^1 \int_0^1 p_2 \left[\frac{g(p_1, p_2) \, f(y_1, y_2)}{\int_0^1 \int_0^1 g(p_1, p_2) f(y_1, y_2) \, dp_1 \, dp_2} \right] dp_1 \, dp_2$$

$$= \int_0^1 \int_0^1 p_1 \left[\frac{p_1^{\alpha} (1-p_1)^{\alpha} p_2^{\alpha} (1-p_2)^{\alpha} f(y_1, y_2)}{\int_0^1 \int_0^1 p_1^{\alpha} (1-p_1)^{\alpha} p_2^{\alpha} (1-p_2)^{\alpha} f(y_1, y_2) dp_1 dp_2} \right] dp_1 dp_2$$

$$+ \int_0^1 \int_0^1 p_2 \left[\frac{p_1^{\alpha} (1-p_1)^{\alpha} p_2^{\alpha} (1-p_2)^{\alpha} f(y_1, y_2)}{\int_0^1 \int_0^1 p_1^{\alpha} (1-p_1)^{\alpha} p_2^{\alpha} (1-p_2)^{\alpha} f(y_1, y_2) dp_1 dp_2} \right] dp_1 dp_2$$

Since Y_1 & Y_2 are independent, $f(y_1, y_2) = f_1(y) f_2(y_2)$, and thus we have:

$$\hat{a}(\alpha) = \int_0^1 \int_0^1 \left[\frac{p_1^{\alpha+1} (1-p_1)^{\alpha} f_1(y_1) p_2^{\alpha} (1-p_2)^{\alpha} f_2(y_2)}{\int_0^1 \int_0^1 p_1^{\alpha} (1-p_1)^{\alpha} f_1(y_1) p_2^{\alpha} (1-p_2)^{\alpha} f_2(y_2) dp_1 dp_2} \right] dp_1 dp_2$$

$$+ \int_0^1 \int_0^1 \left[\frac{p_1^{\alpha} (1-p_1)^{\alpha} f(y_1) p_2^{\alpha+1} (1-p_2)^{\alpha} f(y_2)}{\int_0^1 \int_0^1 p_1^{\alpha} (1-p_1)^{\alpha} f_1(y_1) p_2^{\alpha} (1-p_2)^{\alpha} f_2(y_2) dp_1 dp_2} \right] dp_1 dp_2$$

$$= \int_0^1 \int_0^1 \left[\frac{p_1^{\alpha+1} (1-p_1)^{\alpha} f_1(y_1) p_2^{\alpha} (1-p_2)^{\alpha} f_2(y_2)}{\left[\left(\int_0^1 p_1^{\alpha} (1-p_1)^{\alpha} f_1(y_1) dp_1 \right) \left(\int_0^1 p_2^{\alpha} (1-p_2)^{\alpha} f_2(y_2) dp_2 \right) \right]} \right] dp_1 dp_2$$

$$+ \int_0^1 \int_0^1 \left[\frac{p_1^{\alpha} (1-p_1)^{\alpha} f_1(y_1) p_2^{\alpha+1} (1-p_2)^{\alpha} f_2(y_2)}{\left[\left(\int_0^1 p_1^{\alpha} (1-p_1)^{\alpha} f_1(y_1) dp_1 \right) \left(\int_0^1 p_2^{\alpha} (1-p_2)^{\alpha} f_2(y_2) dp_2 \right) \right]} \right] dp_1 dp_2$$

$$= \frac{\left(\int_0^1 p_1^{\alpha+1} (1-p_1)^\alpha f_1(y_1) dp_1\right)\left(\int_0^1 p_2^\alpha (1-p_2)^\alpha f_2(y_2) dp_2\right)}{\left(\int_0^1 p_1^\alpha (1-p_1)^\alpha f_1(y_1) dp_1\right)\left(\int_0^1 p_2^\alpha (1-p_2)^\alpha f_2(y_2) dp_2\right)}$$

$$+ \frac{\left(\int_0^1 p_1^\alpha (1-p_1)^\alpha f_1(y_1) dp_1\right)\left(\int_0^1 p_2^{\alpha+1} (1-p_2)^\alpha f_2(y_2) dp_2\right)}{\left(\int_0^1 p_1^\alpha (1-p_1)^\alpha f_1(y_1) dp_1\right)\left(\int_0^1 p_2^\alpha (1-p_2)^\alpha f_2(y_2) dp_2\right)}$$

Therefore the estimator $\hat{a}(\alpha)$ simplifies to the following:

$$\hat{a}(\alpha) = \frac{\int_0^1 p_1^{\alpha+1} (1-p_1)^\alpha, f(y_1) dp_1}{\int_0^1 p_1^\alpha (1-p_1)^\alpha, f(y_1) dp_1} + \frac{\int_0^1 p_2^{\alpha+1} (1-p_2)^\alpha, f(y_2) dp_2}{\int_0^1 p_2^\alpha (1-p_2)^\alpha, f(y_2) dp_2}$$

Derivation 2.3.2: Solving $\left(\hat{p}_i - p\right)^2 = z_{\frac{\alpha}{2}}^2 \, p(1-p)/\hat{n}_i$ for p

Start with: $\left(\hat{p}_i - p\right)^2 = z_{\frac{\alpha}{2}}^2 \, p(1-p)/\hat{n}_i$

$\rightarrow \quad \left(\hat{p}_i^2 - 2p\,\hat{p}_i + p^2\right) = \left(\dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_i}\right)p - \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_i}p^2$

$\rightarrow \quad p^2\left(1 + \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_i}\right) + \left(-2\hat{p}_i - \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_i}\right)p + \hat{p}_i^2 = 0$

Utilizing the quadratic formula we get:

$$\rightarrow \quad p = \dfrac{\left(2\hat{p}_i + \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_i}\right) \pm \sqrt{\left(-2\hat{p}_i - \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_i}\right)^2 - 4\left(1 + \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_i}\right)\hat{p}_i^2}}{2\left(1 + \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_i}\right)}$$

Therefore:

$$p_{11} = \left.\left(\left(2\hat{p}_1 + \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_1}\right) - \sqrt{\left(-2\hat{p}_1 - \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_1}\right)^2 - 4\left(1 + \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_1}\right)\hat{p}_1^2}\right)\middle/ 2\left(1 + \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_1}\right)\right.$$

$$p_{12} = \left.\left(\left(2\hat{p}_1 + \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_1}\right) + \sqrt{\left(-2\hat{p}_1 - \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_1}\right)^2 - 4\left(1 + \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_1}\right)\hat{p}_1^2}\right)\middle/ 2\left(1 + \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_1}\right)\right.$$

$$p_{21} = \left.\left(\left(2\hat{p}_2 + \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_2}\right) - \sqrt{\left(-2\hat{p}_2 - \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_2}\right)^2 - 4\left(1 + \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_2}\right)\hat{p}_2^2}\right)\middle/ 2\left(1 + \dfrac{z_{\frac{\alpha}{2}}^2}{\hat{n}_2}\right)\right.$$

$$p_{22} = \left[\left(2\hat{p}_2 + \frac{z_{\alpha/2}^2}{\hat{n}_2} \right) + \sqrt{ \left(-2\hat{p}_2 - \frac{z_{\alpha/2}^2}{\hat{n}_2} \right)^2 - 4\left(1 + \frac{z_{\alpha/2}^2}{\hat{n}_2} \right) \hat{p}_2^2 } \right] \Bigg/ 2\left(1 + \frac{z_{\alpha/2}^2}{\hat{n}_2} \right)$$

APPENDIX B – FORTRAN CODE

General Outline of Program CODE: Parameter values needed for the program are:

p_1, p_2, α, Φ_1, Φ_2, J_1, J_2, & n_{ij}. These are denoted in the program as P1, P2, Alpha,

PHI1, PHI2, J1, J2, and N1Vec/N2Vec. The vectors N1Vec(J), N2Vec(J), Y1Vec(J) and

Y2Vec(J) are defined as follows: N1Vec(J) denotes the jth litter size in population 1;

N2Vec(J) denotes the jth litter size in population 2;   Y1Vec(J) denotes the number of

responses in the jth litter of population 1; Y2Vec(J) denotes the number of responses in

the jth litter of population 2. Additionally, J_1 will represent the number of litters in

population one and J_2 will represent the number of litters in population two. Various

cases with different values for the parameters are simulated in the program. The program

will consist of the following basic parts.

1. **Generate data from two independent beta-binomial distributions.**   This will
   be done utilizing the probability integral transform. (described in section 3.2)
2. **Construct the extended Beal, extended Newcombe, and extended Peskun**
   intervals. These closed form intervals are given by Chen, Li, & Zhou(2004).
   (details are given in section 2.2.3)
3. **Calculate the MLEs $\hat{p}_1$ and $\hat{p}_2$.** This will be done by using an algorithm
   presented by Smith(1983).   The subroutine BBML calculates the MLEs for p

and Φ using a damped Newton-Raphson technique as described by Dixon(1972) and uses moment estimates as initial estimators. BBML utilizes several auxiliary routines. Namely, subroutine BBME calculates the moment estimates of p and Φ, subroutine SET determines the array RL, dimensioned by (MRL,3), which is an array of the coefficients of $\log(p + r\Phi)$, $\log(1 - p + r\Phi)$, $\log(1 + r\Phi)$. These quantities are needed in the calculation of the log likelihood and its derivative. MRL is the maximum litter size and the first dimension of the array RL. The subroutine BBL calculates the log likelihood and subroutine GDER calculates the derivative of the log likelihood.

4.   **Construct the Wald Interval.** This interval is based on the MLEs.

5. **Define the var($\hat{b}$)** as a function of a and b as derived in section 2.2.2.

6. **Calculate the posterior mean, $\hat{a}(\alpha)$.** This will be done by computing double integral $\hat{a}(\alpha) = \int_0^1 \int_0^1 (p_1 + p_2) h(p_1, p_2 \mid y_1 , y_2, \Phi, n_1, n_2) dp_1 dp_2$. The integral is first simplified, (details are found in section 2.2.2 and in derivation 2.3.1) and then evaluated using the IMSL subroutine DQDAG. The Haldane interval is based on $\hat{a}(\alpha)$ when $\alpha = -1$ and the Jeffreys-Perks interval is based on $\hat{a}(\alpha)$ when $\alpha = -1/2$.

7. **Solve the equation $(b - \hat{b})^2 = cVar(\tilde{a}, \tilde{b})$ for b.** First we substitute $\hat{b} = \hat{p}_1 - \hat{p}_2$, as calculated in step 3; next, we substitute the variance expression of $\hat{b}$ from step 5, and last we substitute $\tilde{b} = b$ and $\tilde{a} = \hat{a}(\alpha)$, as calculated in step 6. This equation is then solved by defining a function, Equation[b]. The function is first evaluated starting at $\hat{b} = \hat{p}_1 - \hat{p}_2$ and then using a loop is

evaluated going up and also going down until a sign change is determined. This produces a range for the location of the upper root and a range for the location of the lower root. Once an interval is determined that includes a sign change, the subroutine DZREN is used to locate the root. The upper and lower roots are determined. These roots are used as the upper and lower limits for the Haldane and Jeffreys-Perks intervals. . Thus we have an interval for $p_1 - p_2$.

```
FORTRAN Code

INTEGER     J1, J2, N, J1T, J2T

PARAMETER (J1=5)
PARAMETER (J2=5)

INTEGER          R, S

INTEGER          Y1VEC(J1), Y2VEC(J2), N1VEC(J1), N2VEC(J2)

INTEGER     MRL1, MRL2, ITER, IFAULT, RL1(J1,3), RL2(J2,3)

INTEGER          MAXFN, MAXFN2, ERRORCt

INTEGER          OneCount, ZeroCount

INTEGER          CountW, CountJP, CountHaldane

INTEGER          M1, M2, X1, X2, N1, N2, K1, K2

INTEGER          NewcombeCount, BealCount, PeskunCount

DOUBLE PRECISION     P1HtStar, P2HtStar, Radical1, Radical2, Theta1, Theta2

DOUBLE PRECISION     Temp1, Temp2, Sum1, Sum2

DOUBLE PRECISION     Rho, RhoNum, RhoDen, VarInflation1, VarInflation2

DOUBLE PRECISION     N1ht, N2ht, X1ht, X2ht

DOUBLE PRECISION     K1bar, K2bar

DOUBLE PRECISION     QuadA, QuadB, QuadC, p11, p22, p12, p21

DOUBLE PRECISION     UCLnewcombe, LCLnewcombe, UCLbeal, LCLbeal,
UCLpeskun, LCLpeskun

DOUBLE PRECISION     Zvalue, Bealint1, Bealint2, Bealint3

DOUBLE PRECISION     Peskun1, Peskun2, Peskun3

DOUBLE PRECISION     fstar, fstar1, fstar2, fstar3, fstar4

DOUBLE PRECISION     ustar, vstar, wstar, gstar, gstar1, gstar2
```

```
DOUBLE PRECISION        W1(J1), PP1(J1), W2(J2), PP2(J2), KRIT, SEM, SETH,
RNL

DOUBLE PRECISION        P1HT, PHI1HT, P2HT, PHI2HT, VARP1, VARP2

DOUBLE PRECISION        P1, P2, PHI1, PHI2, PROB

DOUBLE PRECISION        UCI, LCI, TrueDiff, Diff, StdError, Alpha

DOUBLE PRECISION        U, CUMPRB, TEST, TEST1, Bstar

DOUBLE PRECISION        A, B, P1star, P2star, ERRABS, ERRREL, ERREST

DOUBLE PRECISION        AALPHAJP, AALPHAHaldane

DOUBLE PRECISION        Integrand1, Integrand2, Integrand3, Integrand4

DOUBLE PRECISION        Integrand5, Integrand6, Integrand7, Integrand8

DOUBLE PRECISION        Result1, Result2, Result3, Result4

DOUBLE PRECISION        Result5, Result6, Result7, Result8

DOUBLE PRECISION        A1, B1, A2, B2

DOUBLE PRECISION        PHI1Vec(4), PHI2Vec(4), P1Vec(12), P2Vec(12)

DOUBLE PRECISION        Guess, START

DOUBLE PRECISION        BealLength, PeskunLength, NewcombeLength

DOUBLE PRECISION        JPlength, HaldaneLength, WaldLength

EXTERNAL  Integrand1, Integrand2, Integrand3, Integrand4

EXTERNAL  Integrand5, Integrand6, Integrand7, Integrand8

EXTERNAL PDFY, VARNC1, VARNC2, DBINOM, DQDAG, DZBREN

EXTERNAL EQUATION, EQUATION2

COMMON  Y1VEC, Y2VEC, N1VEC, N2VEC, P1HT, P2HT, PHI1HT, PHI2HT
COMMON  AALPHAJP, AALPHAHaldane, J1T, J2T, Alpha

LOGICAL  DONE
```

```
LOGICAL NRT

LOGICAL  FlagZeros, FlagOnes

OPEN (Unit=1,file='C:\Absoft62\SimPrgm5-5-5e.txt')


Call Rnset(0)
Call Rnopt(6)


!Assign values for p's and phi's

 P1Vec(1)=1.0D-1
 P2Vec(1)=1.0D-1
 P1Vec(2)=1.0D-1
 P2Vec(2)=3.0D-1
 P1Vec(3)=1.0D-1
 P2Vec(3)=7.0D-1
 P1Vec(4)=1.0D-1
 P2Vec(4)=9.0D-1
 P1Vec(5)=3.0D-1
 P2Vec(5)=3.0D-1
 P1Vec(6)=3.0D-1
 P2Vec(6)=5.0D-1
 P1Vec(7)=3.0D-1
 P2Vec(7)=7.0D-1
 P1Vec(8)=5.0D-1
 P2Vec(8)=5.0D-1
 P1Vec(9)=9.0D-1
 P2Vec(9)=9.0D-1
 P1Vec(10)=9.0D-1
 P2Vec(10)=1.0D-2
 P1Vec(11)=5.0D-2
 P2Vec(11)=5.0D-2
 P1Vec(12)=5.0D-2
 P2Vec(12)=1.0D-2

 PHI1Vec(1)=1.0D0/9.0D0
 PHI2Vec(1)=1.0D0/9.0D0
 PHI1Vec(2)=3.0D0/7.0D0
 PHI2Vec(2)=3.0D0/7.0D0
 PHI1Vec(3)=1.0D0
 PHI2Vec(3)=1.0D0
 PHI1Vec(4)=9.0D0
 PHI2Vec(4)=9.0D0
```

```fortran
      Alpha = 5.0D-2

      DO 500 I=1,J1
       N1VEC(I)=5
500 Continue

      DO 600 I=1,J2
       N2VEC(I)=5
600 Continue

      MRL1=5
      MRL2=5
      KRIT = 1.0D-6
      ITER = 1000

      J1T=J1
      J2T=J2


      DO 222 R = 1,12

      Write(*,*)'R = ',R

      P1=P1Vec(R)
      P2=P2Vec(R)


      DO 333 S = 1,4

      PHI1=PHI1Vec(S)
      PHI2=PHI2Vec(S)


      CountW = 0
      CountJP = 0
      CountHaldane = 0

      NewcombeCount=0
      BealCount = 0
      PeskunCount = 0

      OneCount = 0
      ZeroCount = 0
      ERRORCt=0
```

```
WaldLength=0.0D0
JPlength=0.0D0

HaldaneLength=0.0D0

NewcombeLength=0.0D0
BealLength=0.0D0
PeskunLength=0.0D0

DO 444 T = 1,10000


!  GENERATE DATA FROM POP1

1 Continue

FlagZeros=.TRUE.
FlagOnes=.TRUE.

DO 20 I=1,J1
    N=N1VEC(I)
    U = DRNUNF()
    DONE=.FALSE.
    CUMPRB=0.0D0

    DO 10 J=1,N1VEC(I)
     IF (DONE) THEN
       GOTO 15
     END IF

     CALL PDFY(PROB,J-1,P1,PHI1,N)
     TEST=CUMPRB+PROB

     IF ((CUMPRB.LT.U).AND.(U.LE.TEST)) THEN
       Y1VEC(I)=J-1
       DONE=.TRUE.
     ELSE IF (J.EQ.N1VEC(I)) THEN
       Y1VEC(I)=N1VEC(I)
     END IF

     CUMPRB=TEST

10      CONTINUE

15 Continue
```

```
IF (Y1VEC(I).NE.0) THEN
 FlagZeros=.FALSE.
END IF

IF (Y1VEC(I).NE.N1VEC(I)) THEN
 FlagOnes=.FALSE.
END IF

20    CONTINUE

IF (FlagZeros) THEN
 ZeroCount=ZeroCount+1
 GOTO 1
END IF

IF (FlagOnes) THEN
 OneCount=OneCount+1
 GOTO 1
END IF



!  GENERATE DATA FROM POP2

2 Continue

FlagZeros=.TRUE.
FlagOnes=.TRUE.

   DO 40 I=1,J2
     N=N2VEC(I)
     U = DRNUNF()
     DONE=.FALSE.
     CUMPRB=0.0D0

   DO 30 J=1,N2VEC(I)
     IF (DONE) THEN
       GOTO 35
     END IF

     CALL PDFY(PROB,J-1,P2,PHI2,N)
     TEST=CUMPRB+PROB

     IF ((CUMPRB.LT.U).AND.(U.LE.TEST)) THEN
       Y2VEC(I)=J-1
```

```
      DONE=.TRUE.
      ELSE IF (J.EQ.N2VEC(I)) THEN
       Y2VEC(I)=N2VEC(I)
      END IF

      CUMPRB=TEST

30      CONTINUE

35 Continue

IF (Y2VEC(I).NE.0) THEN
 FlagZeros=.FALSE.
END IF

IF (Y2VEC(I).NE.N2VEC(I)) THEN
 FlagOnes=.FALSE.
END IF

40    CONTINUE

IF (FlagZeros) THEN
 ZeroCount=ZeroCount+1
 GOTO 2
END IF

IF (FlagOnes) THEN
 OneCount=OneCount+1
 GOTO 2
END IF


!Construct 3 Intervals given by Chen, Li, & Zhou (2004)
!1st define parameters

M1=J1
M2=J2
X1=0
X2=0
N1=0
N2=0


DO 1175 I=1,J1
 X1=X1+Y1VEC(I)
```

```fortran
1175 Continue

DO 1185 I=1,J2
 X2=X2+Y2VEC(I)
1185 Continue

DO 1195 I=1,J1
 N1=N1+N1VEC(I)
1195 Continue

DO 1105 I=1,J2
 N2=N2+N2VEC(I)
1105 Continue

P1HtStar=DBLE(X1)/DBLE(N1)
P2HtStar=DBLE(X2)/DBLE(N2)

IF (J1.EQ.J2) THEN

 K1=N1VEC(1)
 K2=N2VEC(1)
 Sum1=0.0D0
 Sum2=0.0D0
 Temp1=0.0D0
 Temp2=0.0D0

 DO 1115 I=1,J1
   Temp1=Temp1+Y1VEC(I)*(Y1VEC(I)-1.0D0)
 1115 Continue
 Temp2=Temp1/(DBLE(M1)*DBLE(K1)*(DBLE(K1)-1.0D0))
 Sum1=Temp2-(P1HtStar**2.0D0)

 Temp1=0.0D0
 Temp2=0.0D0

 DO 1125 I=1,J2
   Temp1=Temp1+Y2VEC(I)*(Y2VEC(I)-1.0D0)
 1125 Continue
 Temp2=Temp1/(DBLE(M2)*DBLE(K2)*(DBLE(K2)-1.0D0))
 Sum2=Temp2-(P2HtStar**2.0D0)

 RhoNum=(Sum1+Sum2)
 RhoDen=(P1HtStar*(1.0D0-P1HtStar))+(P2HtStar*(1.0D0-P2HtStar))
 Rho=RhoNum/RhoDen

 IF (Rho.LT.0.0D0) THEN
```

```fortran
      Rho = 0.0D0
     END IF

     VarInflation1=(1.0D0+((DBLE(K1)-1.0D0)*Rho))
     VarInflation2=(1.0D0+((DBLE(K2)-1.0D0)*Rho))

   END IF

   IF (J1.NE.J2) THEN
    K1bar=N1/M1
    K2bar=N2/M2

    Sum1=0.0D0
    Sum2=0.0D0
    TEMP1=0.0D0
    TEMP2=0.0D0

    DO 1135 I = 1,M1
      TEMP1=(1.0D0/(M1-1.0D0))*((N1Vec(I)/K1bar)**(2.0D0))
      TEMP2=((Y1Vec(I)/N1Vec(I))-P1HtStar)**2.0D0
      SUM1 = SUM1+TEMP1*TEMP2
1135 Continue
    SUM1=SUM1*(1.0D0/M1)

    TEMP1=0.0D0
    TEMP2=0.0D0

    DO 1145 I = 1,M2
      TEMP1=(1.0D0/(M2-1.0D0))*((N2Vec(I)/K2bar)**(2.0D0))
      TEMP2=((Y2Vec(I)/N2Vec(I))-P2HtStar)**2.0D0
      SUM2 = SUM2+TEMP1*TEMP2
1145 Continue
    SUM2=SUM2*(1.0D0/M2)

    VarInflation1=(N1*Sum1)/((P1HtStar)*(1.0D0-P1HtStar))
    VarInflation2=(N2*Sum2)/((P2HtStar)*(1.0D0-P2HtStar))

    If (VarInflation1.LT.1.0D0) THEN
      VarInflation1=1.0D0
    END IF

    If (VarInflation2.LT.1.0D0) THEN
      VarInflation2=1.0D0
    END IF

   END IF
```

N1ht=(1.0D0/VarInflation1)*(DBLE(N1))
N2ht=(1.0D0/VarInflation2)*(DBLE(N2))

X1ht=(1.0D0/VarInflation1)*(DBLE(X1))
X2ht=(1.0D0/VarInflation2)*(DBLE(X2))

Zvalue=DNORIN(1.0D0-(ALPHA/2.0D0))

!Chen, Li, & Zhou (2004) Method 1: extension of Newcombe(1998) interval

QuadA=1.0D0+((Zvalue**2.0D0)/N1ht)
QuadB=(-2.0D0)*P1HtStar-((Zvalue**2.0D0)/N1ht)
QuadC=P1HtStar**2.0D0

p11=(-QuadB - Sqrt((QuadB**2.0D0)-(4.0D0)*QuadA*QuadC))/((2.0D0)*QuadA)
p12=(-QuadB + Sqrt((QuadB**2.0D0)-(4.0D0)*QuadA*QuadC))/((2.0D0)*QuadA)

QuadA=1.0D0+((Zvalue**2.0D0)/N2ht)
QuadB=(-2.0D0)*P2HtStar-((Zvalue**2.0D0)/N2ht)
QuadC=P2HtStar**2.0D0

p21=(-QuadB - Sqrt((QuadB**2.0D0)-(4.0D0)*QuadA*QuadC))/((2.0D0)*QuadA)
p22=(-QuadB + Sqrt((QuadB**2.0D0)-(4.0D0)*QuadA*QuadC))/((2.0D0)*QuadA)

Radical1=((p11*(1.0D0-p11))/N1ht)+((p22*(1.0D0-p22))/N2ht)
Radical2=((p12*(1.0D0-p12))/N1ht)+((p21*(1.0D0-p21))/N2ht)

Theta1=Zvalue*(Sqrt(Radical1))
Theta2=Zvalue*(Sqrt(Radical2))

LCLnewcombe=(P1HtStar-P2HtStar)-Theta1
UCLnewcombe=(P1HtStar-P2HtStar)+Theta2

NewcombeLength=NewcombeLength+(UCLnewcombe-LCLnewcombe)

TrueDiff=P1-P2
IF ((TrueDiff.GT.LCLnewcombe).AND.(TrueDiff.LT.UCLnewcombe)) THEN
 NewcombeCount=NewcombeCount + 1
END IF

!Chen, Li, & Zhou (2004) Method 2: extension of Beal(1987) interval

ustar=((1/N1ht)+(1/N2ht))/(4.0D0)

vstar=((1/N1ht)-(1/N2ht))/(4.0D0)
wstar=((X1ht+5.0D-1)/(N1ht+1.0D0)+(X2Ht+5.0D-1)/(N2Ht+1.0D0))/(2.0D0)

fstar1=ustar*((4.0D0)*wstar*(1.0D0-wstar)-(P1HtStar-P2HtStar)**(2.0D0))
fstar2=(2.0D0)*vstar*(1.0D0-(2.0D0)*wstar)*(P1HtStar-P2HtStar)
fstar3=(4.0D0)*(Zvalue**(2.0D0))*(ustar**(2.0D0))*(1.0D0-wstar)*wstar
fstar4=(Zvalue**(2.0D0))*(vstar**(2.0D0))*((1.0D0-(2.0D0)*wstar)**(2.0D0))

fstar=fstar1+fstar2+fstar3+fstar4

Bealint1 = (1.0D0)/(1.0D0+(Zvalue**(2.0D0))*ustar)
Bealint2 = (P1HtStar-P2HtStar)+(Zvalue**(2.0D0))*(vstar)*((1.0D0)-(2.0D0)*wstar)
Bealint3 = (Zvalue)*(Sqrt(fstar))

LCLbeal = Bealint1*(Bealint2-Bealint3)
UCLbeal = Bealint1*(Bealint2+Bealint3)

BealLength=BealLength+(UCLbeal-LCLbeal)

IF ((TrueDiff.GT.LCLbeal).AND.(TrueDiff.LT.UCLbeal)) THEN
  BealCount=BealCount + 1
END IF

!Chen, Li, & Zhou (2004) Method 3: extension of Peskun(1993) interval

gstar1=((Zvalue**(2.0D0))+N1Ht+N2Ht)/((4.0D0)*N1Ht*N2Ht)
gstar2=((P1HtStar-P2HtStar)**(2.0D0))/(N1Ht+N2Ht)

gstar=gstar1-gstar2

Peskun1=(1.0D0)/(1.0D0+((Zvalue**(2.0D0))/(N1Ht+N2Ht)))
Peskun2=(P1HtStar-P2HtStar)
Peskun3=(Zvalue)*(Sqrt(gstar))

LCLpeskun=Peskun1*(Peskun2-Peskun3)
UCLpeskun=Peskun1*(Peskun2+Peskun3)

PeskunLength=PeskunLength+(UCLpeskun-LCLpeskun)

IF ((TrueDiff.GT.LCLpeskun).AND.(TrueDiff.LT.UCLpeskun)) THEN
  PeskunCount=PeskunCount + 1
END IF

!Determine MLEs for P1 & Phi1 using algorithm from Smith (1983)

Call
BBML(J1T,Y1VEC,N1VEC,W1,PP1,RL1,MRL1,ITER,KRIT,P1HT,PHI1HT,SEM,SET
H,RNL,IFAULT)

!Determine variance of P1

Call VARNC1(VARP1,P1HT,PHI1HT,N1VEC,J1T,Y1VEC)

!Determine MLEs for P2 & Phi2 using algorithm from Smith (1983)

Call
BBML(J2T,Y2VEC,N2VEC,W2,PP2,RL2,MRL2,ITER,KRIT,P2HT,PHI2HT,SEM,SET
H,RNL,IFAULT)

!Determine variance of P2

Call VARNC2(VARP2,P2HT,PHI2HT,N2VEC,J2T,Y2VEC)

!Construct Wald interval

TrueDiff=P1-P2
Diff=P1HT-P2HT
StdError=Sqrt(VARP1 + VARP2)
UCI=Diff+DNORIN(1.0D0-(ALPHA/2.0D0))*StdError
LCI=Diff-DNORIN(1.0D0-(ALPHA/2.0D0))*StdError

WaldLength=WaldLength+(UCI-LCI)

IF ((TrueDiff.LT.UCI).AND.(TrueDiff.GT.LCI)) THEN
 CountW = CountW + 1
END IF

!Determine estimators a-ht of alpha

```fortran
A = 1.0D-4
B = 9.999D-1
ERRABS = 1.0D-6
ERRREL = 1.0D-6
IRULE = 2

Call DQDAG(Integrand1,A,B,ERRABS, ERRREL, IRULE, RESULT1, ERREST)
Call DQDAG(Integrand2,A,B,ERRABS, ERRREL, IRULE, RESULT2, ERREST)
Call DQDAG(Integrand3,A,B,ERRABS, ERRREL, IRULE, RESULT3, ERREST)
Call DQDAG(Integrand4,A,B,ERRABS, ERRREL, IRULE, RESULT4, ERREST)

AAlphaJP=(RESULT1/RESULT2)+(RESULT3/RESULT4)




A = 1.0D-4
B = 9.999D-1
ERRABS = 1.0D-6
ERRREL = 1.0D-6
IRULE = 2

Call DQDAG(Integrand5,A,B,ERRABS, ERRREL, IRULE, RESULT5, ERREST)
Call DQDAG(Integrand6,A,B,ERRABS, ERRREL, IRULE, RESULT6, ERREST)
Call DQDAG(Integrand7,A,B,ERRABS, ERRREL, IRULE, RESULT7, ERREST)
Call DQDAG(Integrand8,A,B,ERRABS, ERRREL, IRULE, RESULT8, ERREST)

AAlphaHaldane=(RESULT5/RESULT6)+(RESULT7/RESULT8)




!Determine roots to use for JP Interval

! find first root, b-upper
START=P1HT-P2HT
DONE=.FALSE.
NRT=.FALSE.

   Do 85 K = 1,200
     IF (DONE) GOTO 86
     IF ((START+(DBLE(K)/1.0D2)).GT.1.0D0) THEN
       NRT=.TRUE.
       B1=1.0D0
       GOTO 86
     END IF
```

```
        Guess=Equation((START+(DBLE(K)/1.0D2)))
         IF (Guess.GT.0.0D0) THEN
          A1=START+(DBLE(K-1)/1.0D2)
          B1=START+(DBLE(K)/1.0D2)
          DONE=.TRUE.
         END IF
     85 Continue

86 Continue

IF (DONE) THEN
  ERRABS = 0.0D0
  ERRREL = 1.0D-6
  MAXFN = 100
  IF ((Equation(A1))*(Equation(B1)).GT.0.0D0) THEN
   ERRORCt=ERRORCt+1
   GOTO 555
  END IF
  Call DZBREN(Equation,ERRABS, ERRREL, A1, B1, MAXFN)
END IF


!find second root b-lower
START=P1HT-P2HT
DONE=.FALSE.
NRT=.FALSE.

   Do 95 K = 1,200
     IF (DONE) GOTO 96
     IF ((START-(DBLE(K)/2.0D1)).LT.-1.0D0) THEN
      NRT=.TRUE.
      B2=-1.0D0
      GOTO 96
     END IF

     Guess=Equation((START-(DBLE(K)/1.0D2)))
      IF (Guess.GT.0.0D0) THEN
       A2=START-(DBLE(K)/1.0D2)
       B2=START-(DBLE(K-1)/1.0D2)
       DONE=.TRUE.
      END IF
    95 Continue

96 Continue

IF (DONE) THEN
```

```fortran
  ERRABS = 0.0D0
  ERRREL = 1.0D-6
  MAXFN = 100
  IF ((Equation(A2))*(Equation(B2)).GT.0.0D0) THEN
   ERRORCt=ERRORCt+1
   GOTO 555
  END IF
  Call DZBREN(Equation,ERRABS, ERRREL, A2, B2, MAXFN)
END IF

JPlength=JPlength+(B1-B2)

IF ((TrueDiff.LT.B1).AND.(TrueDiff.GT.B2)) THEN
 CountJP = CountJP + 1
END IF




!Determine roots to use for Haldane Interval

!find first root b-upper
START=P1HT-P2HT
DONE=.FALSE.
NRT=.FALSE.

  Do 185 K = 1,200
   IF (DONE) GOTO 186
   IF ((START+(DBLE(K)/1.0D2)).GT.1.0D0) THEN
    NRT=.TRUE.
    B1=1.0D0
    GOTO 186
   END IF

   Guess=Equation2((START+(DBLE(K)/1.0D2)))
    IF (Guess.GT.0.0D0) THEN
     A1=START+(DBLE(K-1)/1.0D2)
     B1=START+(DBLE(K)/1.0D2)
     DONE=.TRUE.
    END IF
  185 Continue

186 Continue

IF (DONE) THEN
 ERRABS = 0.0D0
```

```
      ERRREL = 1.0D-6
      MAXFN = 100
      IF ((Equation2(A1))*(Equation2(B1)).GT.0.0D0) THEN
        ERRORCt=ERRORCt+1
        GOTO 555
      END IF
      Call DZBREN(Equation2,ERRABS, ERRREL, A1, B1, MAXFN)
    END IF


!find second root b-lower
START=P1HT-P2HT
DONE=.FALSE.
NRT=.FALSE.

    Do 195 K = 1,200
      IF (DONE) GOTO 196
      IF ((START-(DBLE(K)/1.0D2)).LT.-1.0D0) THEN
        NRT=.TRUE.
        B2=-1.0D0
        GOTO 196
      END IF

      Guess=Equation2((START-(DBLE(K)/1.0D2)))
        IF (Guess.GT.0.0D0) THEN
          A2=START-(DBLE(K)/1.0D2)
          B2=START-(DBLE(K-1)/1.0D2)
          DONE=.TRUE.
        END IF
    195 Continue

196 Continue

IF (DONE) THEN
    ERRABS = 0.0D0
    ERRREL = 1.0D-6
    MAXFN = 100
    IF ((Equation2(A2))*(Equation2(B2)).GT.0.0D0) THEN
      ERRORCt=ERRORCt+1
      GOTO 555
    END IF
    Call DZBREN(Equation2,ERRABS, ERRREL, A2, B2, MAXFN)
END IF

HaldaneLength=HaldaneLength+(B1-B2)
```

```
IF ((TrueDiff.LT.B1).AND.(TrueDiff.GT.B2)) THEN
  CountHaldane = CountHaldane + 1
END IF




555 Continue


444 Continue

Write(1,*)'J1 = ',J1
Write(1,*)'J2 = ',J2
Write(1,*)'P1 = ',P1
Write(1,*)'P2 = ',P2
Write(1,*)'Phi1 = ',Phi1
Write(1,*)'Phi2 = ',Phi2
Write(1,*)'litter sizes equal and = 5'
Write(1,*)'        '

Write(1,*)'Count for Wald Interval ',CountW
Write(1,*)'Count for JP Interval ',CountJP
Write(1,*)'Count for Haldane Interval ',CountHaldane
Write(1,*)'              '

Write(1,*)'Count for extended Newcombe interval ',NewcombeCount
Write(1,*)'Count for extended Beal interval ',BealCount
Write(1,*)'Count for extended Peskun interval ',PeskunCount
Write(1,*)'              '

Write(1,*)'Count # times get all Zeros ',ZeroCount
Write(1,*)'Count # times get all Ones ',OneCount
Write(1,*)'Error Count ',ErrorCt
Write(1,*)'   '

Write(1,*)'Avg length Wald ',WaldLength/1.0D4
Write(1,*)'Avg length JP ',JPlength/1.0D4
Write(1,*)'Avg length Haldane ',HaldaneLength/1.0D4
Write(1,*)'Avg length Newcombe ',NewcombeLength/1.0D4
Write(1,*)'Avg length Beal ',BealLength/1.0D4
Write(1,*)'Avg length Peskun ',PeskunLength/1.0D4
Write(1,*)'-----------------------'
Write(1,*)'    '
Write(1,*)'    '
```

333 Continue

222 Continue

END

```fortran
DOUBLE PRECISION FUNCTION Integrand1(P1star)
DOUBLE PRECISION P1star, PROD, TEMP
INTEGER Y1VEC(5), N1VEC(5), Y2VEC(5),N2VEC(5),J1T, J2T
DOUBLE PRECISION P1HT, P2HT, PHI1HT, PHI2HT, AALPHAJP,
AALPHAHaldane, Alpha
COMMON  Y1VEC, Y2VEC, N1VEC, N2VEC, P1HT, P2HT, PHI1HT, PHI2HT
COMMON  AALPHAJP, AALPHAHaldane, J1T, J2T, Alpha
PROD=1.0D0
Do 60 I = 1,J1T
  CALL PDFY(TEMP,Y1VEC(I),P1star,PHI1HT,N1VEC(I))
   PROD=PROD*TEMP
60 Continue
Integrand1=(P1star**(5.0D-1))*((1.0D0-P1star)**(-5.0D-1))*PROD
RETURN
END
```

```fortran
DOUBLE PRECISION FUNCTION Integrand2(P1star)
DOUBLE PRECISION P1star, PROD, TEMP
INTEGER Y1VEC(5), N1VEC(5), Y2VEC(5),N2VEC(5),J1T, J2T
DOUBLE PRECISION P1HT, P2HT, PHI1HT, PHI2HT, AALPHAJP,
AALPHAHaldane, Alpha
COMMON  Y1VEC, Y2VEC, N1VEC, N2VEC, P1HT, P2HT, PHI1HT, PHI2HT
COMMON  AALPHAJP, AALPHAHaldane, J1T, J2T, Alpha
PROD=1.0D0
Do 65 I = 1,J1T
  CALL PDFY(TEMP,Y1VEC(I),P1star,PHI1HT,N1VEC(I))
   PROD=PROD*TEMP
65 Continue
Integrand2=(P1star**(-5.0D-1))*((1.0D0-P1star)**(-5.0D-1))*PROD
RETURN
```

```
END



DOUBLE PRECISION FUNCTION Integrand3(P2star)
DOUBLE PRECISION P2star, PROD, TEMP
INTEGER Y1VEC(5), N1VEC(5), Y2VEC(5),N2VEC(5),J1T, J2T

DOUBLE PRECISION P1HT, P2HT, PHI1HT, PHI2HT, AALPHAJP,
AALPHAHaldane, Alpha
COMMON  Y1VEC, Y2VEC, N1VEC, N2VEC, P1HT, P2HT, PHI1HT, PHI2HT
COMMON  AALPHAJP, AALPHAHaldane, J1T, J2T, Alpha
PROD=1.0D0
Do 70 I = 1,J2T
  CALL PDFY(TEMP,Y2VEC(I),P2star,PHI2HT,N2VEC(I))
  PROD=PROD*TEMP
70 Continue
Integrand3=(P2star**(5.0D-1))*((1.0D0-P2star)**(-5.0D-1))*PROD
RETURN
END



DOUBLE PRECISION FUNCTION Integrand4(P2star)
DOUBLE PRECISION P2star, PROD, TEMP
INTEGER Y1VEC(5), N1VEC(5), Y2VEC(5),N2VEC(5),J1T, J2T
DOUBLE PRECISION P1HT, P2HT, PHI1HT, PHI2HT, AALPHAJP,
AALPHAHaldane, Alpha
COMMON  Y1VEC, Y2VEC, N1VEC, N2VEC, P1HT, P2HT, PHI1HT, PHI2HT
COMMON  AALPHAJP, AALPHAHaldane, J1T, J2T, Alpha
PROD=1.0D0
Do 75 I = 1,J2T
  CALL PDFY(TEMP,Y2VEC(I),P2star,PHI2HT,N2VEC(I))
  PROD=PROD*TEMP
75 Continue
Integrand4=(P2star**(-5.0D-1))*((1.0D0-P2star)**(-5.0D-1))*PROD
RETURN
END



DOUBLE PRECISION FUNCTION Integrand5(P1star)
DOUBLE PRECISION P1star, PROD, TEMP
INTEGER Y1VEC(5), N1VEC(5), Y2VEC(5),N2VEC(5),J1T, J2T
DOUBLE PRECISION P1HT, P2HT, PHI1HT, PHI2HT, AALPHAJP,
AALPHAHaldane, Alpha
```

```fortran
COMMON  Y1VEC, Y2VEC, N1VEC, N2VEC, P1HT, P2HT, PHI1HT, PHI2HT
COMMON  AALPHAJP, AALPHAHaldane, J1T, J2T, Alpha
PROD=1.0D0
Do 60 I = 1,J1T
  CALL PDFY(TEMP,Y1VEC(I),P1star,PHI1HT,N1VEC(I))
   PROD=PROD*TEMP
60 Continue
Integrand5=((1.0D0-P1star)**(-1.0D0))*PROD
RETURN
END




DOUBLE PRECISION FUNCTION Integrand6(P1star)
DOUBLE PRECISION P1star, PROD, TEMP
INTEGER Y1VEC(5), N1VEC(5), Y2VEC(5),N2VEC(5),J1T, J2T
DOUBLE PRECISION P1HT, P2HT, PHI1HT, PHI2HT, AALPHAJP,
AALPHAHaldane, Alpha
COMMON  Y1VEC, Y2VEC, N1VEC, N2VEC, P1HT, P2HT, PHI1HT, PHI2HT
COMMON  AALPHAJP, AALPHAHaldane, J1T, J2T, Alpha
PROD=1.0D0
Do 65 I = 1,J1T
  CALL PDFY(TEMP,Y1VEC(I),P1star,PHI1HT,N1VEC(I))
   PROD=PROD*TEMP
65 Continue
Integrand6=(P1star**(-1.0D0))*((1.0D0-P1star)**(-1.0D0))*PROD
RETURN
END




DOUBLE PRECISION FUNCTION Integrand7(P2star)
DOUBLE PRECISION P2star, PROD, TEMP
INTEGER Y1VEC(5), N1VEC(5), Y2VEC(5),N2VEC(5),J1T, J2T
DOUBLE PRECISION P1HT, P2HT, PHI1HT, PHI2HT, AALPHAJP,
AALPHAHaldane, Alpha
COMMON  Y1VEC, Y2VEC, N1VEC, N2VEC, P1HT, P2HT, PHI1HT, PHI2HT
COMMON  AALPHAJP, AALPHAHaldane, J1T, J2T, Alpha
PROD=1.0D0
Do 70 I = 1,J2T
  CALL PDFY(TEMP,Y2VEC(I),P2star,PHI2HT,N2VEC(I))
   PROD=PROD*TEMP
70 Continue
Integrand7=((1.0D0-P2star)**(-1.0D0))*PROD
RETURN
END
```

```fortran
DOUBLE PRECISION FUNCTION Integrand8(P2star)
DOUBLE PRECISION P2star, PROD, TEMP
INTEGER Y1VEC(5), N1VEC(5), Y2VEC(5),N2VEC(5),J1T, J2T
DOUBLE PRECISION P1HT, P2HT, PHI1HT, PHI2HT, AALPHAJP,
AALPHAHaldane, Alpha
COMMON  Y1VEC, Y2VEC, N1VEC, N2VEC, P1HT, P2HT, PHI1HT, PHI2HT
COMMON  AALPHAJP, AALPHAHaldane, J1T, J2T, Alpha
PROD=1.0D0
Do 75 I = 1,J2T
  CALL PDFY(TEMP,Y2VEC(I),P2star,PHI2HT,N2VEC(I))
  PROD=PROD*TEMP
75 Continue
Integrand8=(P2star**(-1.0D0))*((1.0D0-P2star)**(-1.0D0))*PROD
RETURN
END




DOUBLE PRECISION FUNCTION EQUATION(Bstar)
DOUBLE PRECISION Bstar, P1HT, PHI1HT, P2HT, PHI2HT, AALPHAJP,
AALPHAHaldane, Alpha
DOUBLE PRECISION TPlus, TMinus, TEMP1, TEMP2, TEMPB, VARIANCE, Zvalue
INTEGER Y1VEC(5), N1VEC(5), Y2VEC(5),N2VEC(5),J1T, J2T, N, Y

COMMON  Y1VEC, Y2VEC, N1VEC, N2VEC, P1HT, P2HT, PHI1HT, PHI2HT
COMMON  AALPHAJP, AALPHAHaldane, J1T, J2T, Alpha

DOUBLE PRECISION  SUMVAR, SUM1, SUM2, SUMT
DOUBLE PRECISION  PROB, SUMEXP, VARP1, VARP2

SUMVAR=0.0D0
TPlus=(AALPHAJP+Bstar)/(2.0D0)
TMinus=(AALPHAJP-Bstar)/(2.0D0)

DO 195 J=1,J1T
 N=N1VEC(J)
 SUMEXP=0.0D0

 DO 194 K=0,N1VEC(J)
  Y=K
  SUM1=0.0D0
```

```fortran
      IF (Y.NE.0) THEN
        DO 191 L=0,Y-1
          TEMP1=TPlus+(DBLE(L)*PHI1HT)
          TEMP2=(1.0D0)/(TEMP1**2.0D0)
          SUM1=SUM1+TEMP2
        191 CONTINUE
      END IF

      SUM2=0.0D0


      IF (Y.NE.N) THEN
        DO 192 L=0,N-Y-1
          TEMP1=1.0D0-TPlus+(DBLE(L)*PHI1HT)
          TEMP2=(1.0D0)/(TEMP1**2.0D0)
          SUM2=SUM2+TEMP2
        192 CONTINUE
      END IF

      SUMT=SUM1+SUM2

      CALL PDFY(PROB,Y,TPlus,PHI1HT,N)

      SUMEXP=SUMEXP+(SUMT*PROB)

    194 CONTINUE

      SUMVAR=SUMVAR+SUMEXP

  195 CONTINUE

  VARP1=1.0D0/SUMVAR


  SUMVAR=0.0D0

  DO 295 J=1,J2T
    N=N2VEC(J)
    SUMEXP=0.0D0

    DO 294 K=0,N2VEC(J)
      Y=K
      SUM1=0.0D0

      IF (Y.NE.0) THEN
        DO 291 L=0,Y-1
```

```fortran
      TEMP1=TMinus+(DBLE(L)*PHI2HT)
      TEMP2=(1.0D0)/(TEMP1**2.0D0)
      SUM1=SUM1+TEMP2
  291 CONTINUE
    END IF

    SUM2=0.0D0

    IF (Y.NE.N) THEN
      DO 292 L=0,N-Y-1
        TEMP1=1.0D0-TMinus+(DBLE(L)*PHI2HT)
        TEMP2=(1.0D0)/(TEMP1**2.0D0)
        SUM2=SUM2+TEMP2
  292 CONTINUE
    END IF

    SUMT=SUM1+SUM2

    CALL PDFY(PROB,Y,TMinus,PHI2HT,N)

    SUMEXP=SUMEXP+(SUMT*PROB)

  294 CONTINUE

    SUMVAR=SUMVAR+SUMEXP

  295 CONTINUE

    VARP2=1.0D0/SUMVAR

    VARIANCE=VARP1+VARP2
    TEMPB=Bstar-(P1HT-P2HT)
    Zvalue=DNORIN(1.0D0-(Alpha/2.0D0))

    EQUATION=(TEMPB)**2.0D0 - ((Zvalue)**2.0D0)*(VARIANCE)
    RETURN

    END




    DOUBLE PRECISION FUNCTION EQUATION2(Bstar)
    DOUBLE PRECISION Bstar, P1HT, PHI1HT, P2HT, PHI2HT, AALPHAJP,
    AALPHAHaldane, Alpha
    DOUBLE PRECISION TPlus, TMinus, TEMP1, TEMP2, TEMPB, VARIANCE, Zvalue
    INTEGER Y1VEC(5), N1VEC(5), Y2VEC(5),N2VEC(5),J1T, J2T, N, Y
```

```fortran
COMMON  Y1VEC, Y2VEC, N1VEC, N2VEC, P1HT, P2HT, PHI1HT, PHI2HT
COMMON  AALPHAJP, AALPHAHaldane, J1T, J2T, Alpha

DOUBLE PRECISION  SUMVAR, SUM1, SUM2, SUMT
DOUBLE PRECISION  PROB, SUMEXP, VARP1, VARP2

SUMVAR=0.0D0
TPlus=(AALPHAHaldane+Bstar)/(2.0D0)
TMinus=(AALPHAHaldane-Bstar)/(2.0D0)

DO 195 J=1,J1T
 N=N1VEC(J)
 SUMEXP=0.0D0

 DO 194 K=0,N1VEC(J)
  Y=K
  SUM1=0.0D0

  IF (Y.NE.0) THEN
   DO 191 L=0,Y-1
    TEMP1=TPlus+(DBLE(L)*PHI1HT)
    TEMP2=(1.0D0)/(TEMP1**2.0D0)
    SUM1=SUM1+TEMP2
   191 CONTINUE
  END IF

  SUM2=0.0D0

  IF (Y.NE.N) THEN
   DO 192 L=0,N-Y-1
    TEMP1=1.0D0-TPlus+(DBLE(L)*PHI1HT)
    TEMP2=(1.0D0)/(TEMP1**2.0D0)
    SUM2=SUM2+TEMP2
   192 CONTINUE
  END IF

  SUMT=SUM1+SUM2

  CALL PDFY(PROB,Y,TPlus,PHI1HT,N)

  SUMEXP=SUMEXP+(SUMT*PROB)

 194 CONTINUE

 SUMVAR=SUMVAR+SUMEXP
```

```fortran
195 CONTINUE

    VARP1=1.0D0/SUMVAR


    SUMVAR=0.0D0

    DO 295 J=1,J2T
     N=N2VEC(J)
     SUMEXP=0.0D0

     DO 294 K=0,N2VEC(J)
      Y=K
      SUM1=0.0D0

      IF (Y.NE.0) THEN
       DO 291 L=0,Y-1
        TEMP1=TMinus+(DBLE(L)*PHI2HT)
        TEMP2=(1.0D0)/(TEMP1**2.0D0)
        SUM1=SUM1+TEMP2
       291 CONTINUE
      END IF

      SUM2=0.0D0

      IF (Y.NE.N) THEN
       DO 292 L=0,N-Y-1
        TEMP1=1.0D0-TMinus+(DBLE(L)*PHI2HT)
        TEMP2=(1.0D0)/(TEMP1**2.0D0)
        SUM2=SUM2+TEMP2
       292 CONTINUE
      END IF

      SUMT=SUM1+SUM2

      CALL PDFY(PROB,Y,TMinus,PHI2HT,N)

      SUMEXP=SUMEXP+(SUMT*PROB)

     294 CONTINUE

     SUMVAR=SUMVAR+SUMEXP

295 CONTINUE
```

VARP2=1.0D0/SUMVAR

VARIANCE=VARP1+VARP2
TEMPB=Bstar-(P1HT-P2HT)
Zvalue=DNORIN(1.0D0-(Alpha/2.0D0))

EQUATION2=(TEMPB)**2.0D0 - ((Zvalue)**2.0D0)*(VARIANCE)
RETURN

END

! SUBROUTINE TO CALCULATE THE DENSITY OF Y

    SUBROUTINE PDFY(PROB,Y,P,PHI,N)

    INTEGER Y, N

    DOUBLE PRECISION P, PHI, PROD1, PROD2, PROD3
    DOUBLE PRECISION PROB

    DOUBLE PRECISION DBINOM, NCHOOSEY

    EXTERNAL DBINOM

    NCHOOSEY=DBINOM(N,Y)

    PROD1=1.0D0
    IF (Y.NE.0) THEN
     DO 43 I=0,Y-1
      PROD1=PROD1*(P+(DBLE(I)*PHI))
43   CONTINUE
    END IF

    PROD3=1.0D0
    IF (Y.NE.N) THEN
     DO 44 I=0,N-Y-1
      PROD3=PROD3*(1.0D0-P+(DBLE(I)*PHI))
44   CONTINUE
    END IF

    PROD2=1.0D0

```
   DO 45 I=0,N-1
     PROD2=PROD2*(1.0D0+(DBLE(I)*PHI))
45  CONTINUE

   PROB=NCHOOSEY*(PROD1*PROD3)/PROD2
   RETURN
   END subroutine
```

!Subroutine to calculate the MLEs using algorithm from Smith (1983)

```
Subroutine
BBML(N,Y1VEC,N1VEC,W,P,RL,MRL,ITER,KRIT,P1HT,PHI1HT,SEM,SETH,RNL,
IFAULT)


DOUBLE PRECISION
        W(N),P(N),KRIT,P1HT,PHI1HT,SEM,SETH,RNL,INF,DUM
DOUBLE PRECISION        FD(2),SD(3),TD(4),UB(2),DEL,EPS,A,B,C,D,E,F

INTEGER Y1VEC(N),N1VEC(N),RL(MRL,3),LM(3),RD1(2,2),RD2(2,3),RD3(2,4)

LOGICAL MC

DATA INF /1.0D6/,RD1(1,1), RD1(2,1), RD1(1,2), RD1(2,2) /1,-1,1,1/
DATA RD2(1,1), RD2(2,1), RD2(1,2), RD2(2,2), RD2(1,3), RD2(2,3) /-1,-1,-1,1,-1,-1/
DATA RD3(1,1), RD3(2,1), RD3(1,2), RD3(2,2), RD3(1,3), RD3(2,3), RD3(1,4),
RD3(2,4) /2,-2,2,2,2,-2,2,2/

I=ITER
MC=.TRUE.
UB(1)=1.0D-2
UB(2)=1.0D-2


CALL SET(N,Y1VEC,N1VEC,RL,MRL,LM,IFAULT)
IF (IFAULT.NE.0) RETURN
  SEM = -1.0D0
  SETH = -1.0D0
  NND=0

CALL BBME(N,Y1VEC,N1VEC,W,P,INF,P1HT,PHI1HT)
IF (PHI1HT.EQ.INF) GOTO 150

105  IF (ITER.LE.I) GOTO 110
```

```
      IFAULT = 7
      GOTO 160

110 CALL GDER(P1HT,PHI1HT,RL,MRL,LM,2,RD1,FD)

   CALL GDER(P1HT,PHI1HT,RL,MRL,LM,3,RD2,SD)

   CALL GDER(P1HT,PHI1HT,RL,MRL,LM,4,RD3,TD)


   DUM = SD(1)*SD(3)-SD(2)*SD(2)
   IF (SD(1).LT.0.0.AND.DUM.GT.0.0) GOTO 115

   NND=NND+1

   A = P1HT-FD(1)/SD(1)
   B=PHI1HT
   IF (FD(2).NE.0.0) B=B+SIGN(UB(2),FD(2))
   IF (A.LE.0.0) A=1.0D-4
   IF (A.GE.1.0) A=9.999D-1
   IF (B.LT.0.0) B=0.0D0
   IF (B.GT.INF) B=INF
   CALL BBL(P1HT,PHI1HT,RL,MRL,LM,C)
   CALL BBL(A,B,RL,MRL,LM,D)
   IF (NND.GT.10.OR.C.GE.D) GOTO 140
   ITER=ITER+1
   P1HT=A
   PHI1HT=B
   GOTO 105

115 DEL=(FD(2)*SD(2)-FD(1)*SD(3))/DUM
   EPS=(FD(1)*SD(2)-FD(2)*SD(1))/DUM

   A=SD(2)*TD(2)-TD(1)*SD(3)
   B=SD(2)*TD(3)-TD(2)*SD(3)
   C=TD(1)*SD(2)-TD(2)*SD(1)
   D=SD(2)*TD(2)-SD(1)*TD(3)
   E=SD(2)*TD(4)-TD(3)*SD(3)
   F=TD(3)*SD(2)-TD(4)*SD(1)
   A=DEL*A+EPS*B
   C=DEL*C+EPS*D
   E=DEL*B+EPS*E
   F=DEL*D+EPS*F
   DUM=(A*A+C*C+E*E+F*F)/(DUM*DUM)
   IF (DUM.GE.1.0D0) GOTO 120
   IF (ABS(DEL).LE.KRIT.AND.ABS(EPS).LE.KRIT) MC=.FALSE.
```

```
   GOTO 145

120 A=FD(1)*FD(1)
   B=FD(2)*FD(2)
   C=A*SD(1)+2.0*SD(2)*FD(1)*FD(2)+B*SD(3)
   IF (C.NE.0.0) GOTO 125
   DEL=0.0
   IF (FD(1).NE.0.0D0) DEL=SIGN(UB(1),FD(1))
   EPS=0.0D0
   IF (FD(2).NE.0.0D0) EPS=SIGN(UB(2),FD(2))
   GOTO 130
125 C=-(A+B)/C
   DEL=C*FD(1)
   EPS=C*FD(2)
   IF (ABS(DEL).GT.UB(1)) DEL=SIGN(UB(1),DEL)
   UB(1)=2.0*ABS(DEL)
   IF (ABS(EPS).GT.UB(2)) EPS=SIGN(UB(2),EPS)
   UB(2)=2.0*ABS(EPS)

130 CALL BBL(P1HT,PHI1HT,RL,MRL,LM,C)
135 A=P1HT+DEL
   B=PHI1HT+EPS
   IF (A.LE.0.0D0) A=1.0D-4
   IF (A.GE.1.0D0) A=9.999D-1
   DEL=A-P1HT
   IF (B.LT.0.0D0) B=0.0D0
   IF (B.GT.INF) B=INF
   EPS=B-PHI1HT
   CALL BBL(A,B,RL,MRL,LM,D)

   IF (D.GT.C) GOTO 145
   DEL=DEL/2.0D0
   EPS=EPS/2.0D0
   IF (ABS(DEL).GT.KRIT.OR.ABS(EPS).GT.KRIT) GOTO 135
140 IFAULT = 8
   GOTO 160
145 ITER=ITER+1
   A=P1HT+DEL
   B=PHI1HT+EPS
   IF (A.GT.0.0D0.AND.A.LT.1.0D0.AND.B.GE.0.0D0.AND.B.LE.INF) GOTO 155
   IF (A.LE.0.0D0) P1HT=0.0D0
   IF (A.GE.1.0D0) P1HT=1.0D0
   IF (B.LT.0.0D0) PHI1HT=0.0D0
   IF (B.GT.INF) PHI1HT=INF
150 IFAULT = 6
```

```fortran
      GOTO 160

155 P1HT=A
    PHI1HT=B
    IF (MC) GOTO 105

    IF (SD(1).LT.0.0) SEM=SQRT(-1.0D0/SD(1))
    IF (SD(3).LT.0.0) SETH=SQRT(-1.0D0/SD(3))

160 CALL BBL(P1HT,PHI1HT,RL,MRL,LM,RNL)
    RETURN
    END

    SUBROUTINE BBME(N,Y1VEC,N1VEC,W,P,INF,P1HT,PHI1HT)

    DOUBLE PRECISION    W(N),P(N),INF,P1HT,PHI1HT,D1,D2,R,S,TP,WT
    INTEGER Y1VEC(N),N1VEC(N)
    LOGICAL J
    J=.FALSE.
    DO 205 I = 1,N
    W(I)=FLOAT(N1VEC(I))
    P(I)=FLOAT(Y1VEC(I))/W(I)
205 CONTINUE
210 WT=0.0D0
    TP=0.0D0
    DO 215 I=1,N
    WT=WT+W(I)
    TP=TP+W(I)*P(I)
215 CONTINUE
    TP=TP/WT
    S=0.0D0
    D1=0.0D0
    D2=0.0D0
    DO 220 I=1,N
    R=P(I)-TP
    S=S+W(I)*R*R
    R=W(I)*(1.0D0-W(I)/WT)
    D1=D1+R/FLOAT(N1VEC(I))
    D2=D2+R
220 CONTINUE
    S=FLOAT(N-1)*S/FLOAT(N)
    R=TP*(1.0D0-TP)
    IF (R.EQ.0.0D0) GOTO 230
    R=(S-R*D1)/(R*(D2-D1))
    IF (R.LT.0.0D0) R=0.0D0
    IF (J) GOTO 230
```

```
      DO 225 I=1,N
225 W(I)=W(I)/(1.0D0+R*(W(I)-1.0D0))
   J=.TRUE.
   GOTO 210
230 P1HT=TP
   IF (R.GE.1.0D0) GOTO 235
   PHI1HT=R/(1.0D0-R)
   IF (PHI1HT.LE.INF) RETURN
235 PHI1HT=INF
   RETURN
   END


   SUBROUTINE SET(N,Y1VEC,N1VEC,RL,MRL,LM,IFAULT)

   INTEGER Y1VEC(N),N1VEC(N),RL(MRL,3),LM(3)

   IF (N.GT.1) GOTO 305
   IFAULT=1
   RETURN
305 DO 310 I=1,N
   IF (Y1VEC(I).GT.0) GOTO 315
310 CONTINUE
   IFAULT=2
   RETURN
315 DO 320 I=1,N
   IF (Y1VEC(I).LT.N1VEC(I)) GOTO 325
320 CONTINUE
   IFAULT = 3
   RETURN

325 IFAULT=4
   DO 330 I=1,3
   LM(I)=0
   DO 330 J=1,MRL
   RL(J,I)=0
330 CONTINUE
   DO 365 I=I,N
   JJ=Y1VEC(I)
   MAR=1
   GOTO 345
335 JJ=N1VEC(I)-Y1VEC(I)
   MAR=2
   GOTO 345
340 JJ=N1VEC(I)
   MAR=3
```

```fortran
345 IF (JJ) 350,360,355
350 IFAULT=5
   RETURN
355 IF (JJ.GT.MRL) RETURN
   IF (JJ.GT.LM(MAR)) LM(MAR)=JJ
   RL(JJ,MAR)=RL(JJ,MAR)+1
360 GOTO (335,340,365),MAR
365 CONTINUE

   IFAULT=0

   DO 375 I=1,3
   JJ=LM(I)-1
   IF (JJ.LE.0) GOTO 375
   K=JJ
   DO 370 J=1,JJ
   RL(K,I)=RL(K,I)+RL(K+1,I)
   K=K-1
370 CONTINUE
375 CONTINUE
   RETURN
   END


   SUBROUTINE BBL(P1HT,PHI1HT,RL,MRL,LM,RNL)

   DOUBLE PRECISION      P1HT,PHI1HT,RNL,A
   INTEGER RL(MRL,3),LM(3)
   RNL=0.0
   MLM=LM(3)
   DO 405 I = 1,MLM
   A=FLOAT(I-1)*PHI1HT
   IF (I.LE.LM(1)) RNL=RNL+FLOAT(RL(I,1))*ALOG(P1HT+A)
   IF (I.LE.LM(2)) RNL=RNL+FLOAT(RL(I,2))*ALOG(1.0D0-P1HT+A)
   RNL=RNL-FLOAT(RL(I,3))*ALOG(1.0D0+A)
 405 CONTINUE
   RETURN
   END

   SUBROUTINE GDER(P1HT,PHI1HT,RL,MRL,LM,IDER,RD,PD)

   DOUBLE PRECISION      P1HT,PHI1HT,PD(IDER),A,B,C,D
   INTEGER RL(MRL,3),LM(3),RD(2,IDER)
   MLM=LM(3)
   KK=IDER-1
   DO 505 I=1,IDER
```

```
 505 PD(I)=0.0D0
  DO 545 I=1,MLM
  C=FLOAT(I-1)
  A=C*PHI1HT
  DO 540 J=1,3
  IF (I.GT.LM(J)) GOTO 540
  GOTO (510,515,520), J
510 D=P1HT+A
  GOTO 525
515 D=1.0D0-P1HT+A
  GOTO 525
520 D=1.0D0+A
525 B=FLOAT(RL(I,J))/D**KK
  IF (J.EQ.3) GOTO 535
  DO 530 K=1,IDER
  PD(K)=PD(K)+FLOAT(RD(J,K))*B
  B=B*C
530 CONTINUE
  GOTO 540
535 D=-FLOAT(RD(1,1))*B*C**KK
  PD(IDER)=PD(IDER)+D
540 CONTINUE
545 CONTINUE
  RETURN
  END




SUBROUTINE VARNC1(VARP1,P1HT,PHI1HT,N1VEC,J1,Y1VEC)

    INTEGER J1, N, Y, K, L

    INTEGER Y1VEC(J1), N1VEC(J1)

    DOUBLE PRECISION   P1HT, PHI1HT, SUMVAR, SUM1, SUM2, SUMT

    DOUBLE PRECISION   PROB, SUMEXP, VARP1

    SUMVAR=0.0D0

    DO 90 J=1,J1
     N=N1VEC(J)
```

```fortran
      SUMEXP=0.0D0

    DO 89 K=0,N1VEC(J)
     Y=K
     SUM1=0.0D0

     IF (Y.NE.0) THEN
      DO 87 L=0,Y-1
        SUM1=SUM1+(1.0D0/((P1HT+(DBLE(L)*PHI1HT))**2.0D0))
      87 CONTINUE
     END IF

     SUM2=0.0D0

     IF (Y.NE.N) THEN
      DO 88 L=0,N-Y-1
        SUM2=SUM2+(1.0D0/((1.0D0-P1HT+(DBLE(L)*PHI1HT))**2.0D0))
      88 CONTINUE
     END IF

     SUMT=SUM1+SUM2

     CALL PDFY(PROB,Y,P1HT,PHI1HT,N)

     SUMEXP=SUMEXP+(SUMT*PROB)

    89 CONTINUE

     SUMVAR=SUMVAR+SUMEXP

   90 CONTINUE

   VARP1=1.0D0/SUMVAR

   RETURN

   END



   SUBROUTINE VARNC2(VARP2,P2HT,PHI2HT,N2VEC,J2,Y2VEC)

     INTEGER J2, N, Y, K, L

     INTEGER Y2VEC(J2), N2VEC(J2)
```

```fortran
      DOUBLE PRECISION   P2HT, PHI2HT, SUMVAR, SUM1, SUM2, SUMT
      DOUBLE PRECISION   PROB, SUMEXP, VARP2

      SUMVAR=0.0D0


      DO 95 J=1,J2
       N=N2VEC(J)
       SUMEXP=0.0D0

       DO 94 K=0,N2VEC(J)
        Y=K
        SUM1=0.0D0

        IF (Y.NE.0) THEN
         DO 91 L=0,Y-1
          SUM1=SUM1+(1.0D0/((P2HT+(DBLE(L)*PHI2HT))**2.0D0))
   91    CONTINUE
        END IF

        SUM2=0.0D0

        IF (Y.NE.N) THEN
         DO 92 L=0,N-Y-1
          SUM2=SUM2+(1.0D0/((1.0D0-P2HT+(DBLE(L)*PHI2HT))**2.0D0))
   92    CONTINUE
        END IF

        SUMT=SUM1+SUM2

        CALL PDFY(PROB,Y,P2HT,PHI2HT,N)

        SUMEXP=SUMEXP+(SUMT*PROB)

   94  CONTINUE

       SUMVAR=SUMVAR+SUMEXP

   95 CONTINUE

      VARP2=1.0D0/SUMVAR

      RETURN

      END
```

APPENDIX C – Binomial Intervals

*Agresti, A., and Caffo, B. (2000), "Simple and Effective Confidence Intervals for Proportions and Differences of Proportions Result from Adding Two Successes and Two Failures," *The American Statistician*, 54, 280-288. (cited: 34)

$$X_1 \sim BIN(n_1, p_1), \ X_2 \sim BIN(n_2, p_2), \ \tilde{p}_1 = (x_1 + 1)/(n_1 + 1), \ \tilde{p}_2 = (x_2 + 1)/(n_2 + 1)$$

$$(\tilde{p}_1 - \tilde{p}_2) \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{\tilde{p}_1(1 - \tilde{p}_1)}{n_1 + 2} + \frac{\tilde{p}_2(1 - \tilde{p}_2)}{n_2 + 2}} \ )$$

*Chan, I., and Zhang, Z. (1999), "Test-Based Exact Confidence Intervals for the Difference of Two Binominal Proportions," *Biometrics*, 55, 1202-1209. (cited: 34)

$(S = \hat{P}_1 - \hat{P}_2)$. The exact confidence interval based on the δ-*Projected Z statistic*

$(Z_2 = \dfrac{\hat{P}_1 - \hat{P}_2 - \delta_0}{\{\tilde{P}_1(1 - \tilde{P}_1)/N_1 + \tilde{P}_2(1 - \tilde{P}_2)/N_2\}^{1/2}}$, δ_0 is a specified value for δ) without restricted

search for the nuisance parameter performs best in almost every case. The *Binonial Z*

statistic $(Z_1 = \dfrac{\hat{P}_1 - \hat{P}_2}{\{\tilde{P}_1(1 - \tilde{P}_1)/N_1 + \tilde{P}_2(1 - \tilde{P}_2)/N_2\}^{1/2}}$) and *Likelihood ratio statistic*

$(LR = \log\{\Pr(X=I, \ Y=j | \delta, \ P)\})$ are generally wider than those based on the δ-*Projected Z statistic* (Z_2).

*Chen, Xun. (2002). "A quasi-exact method for the con_dence intervals of the di_erence of two independent binomial proportions in small sample cases," *Statistics in Medicine*, 21:943–956. (cited: 8)

*Newcombe, Robert G. (1998), "Interval estimation for the difference between independent proportions: comparison of eleven methods," *Statistics in Medicine*, 17, 873 – 890. (cited: 121)

$$X_1 \sim BIN(n_1, p_1), \ X_2 \sim BIN(n_2, p_2), \ \hat{p}_1 = \frac{x_1}{n_1}, \ \hat{p}_2 = \frac{x_2}{n_2}, \ \hat{p}_1 - \hat{p}_2 = \frac{x_1}{n_1} - \frac{x_2}{n_2}$$

C. I. for $p_1 - p_2$:

(1) Simple asymptotic method, no continuing correction

$$(\hat{p}_1 - \hat{p}_2) \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} \qquad \text{Wald}$$

(2) Simple asymptotic method with continuing correction

$$(\hat{p}_1 - \hat{p}_2) \pm \left[Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} + \frac{1}{2}\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \right]$$

(3) Beal's Haldane method, limits are $\theta^* \pm w$

$$\theta^* = \frac{\hat{p}_1 - \hat{p}_2 + Z_{\frac{\alpha}{2}}^2 v(1 - 2\tilde{\psi})}{1 + Z_{\frac{\alpha}{2}}^2 u},$$

$$w = \frac{Z_{\frac{\alpha}{2}}}{1 + Z_{\frac{\alpha}{2}}^2} \sqrt{u[4\tilde{\psi}(1-\tilde{\psi}) - \theta^2] + 2v(1-2\hat{\psi})\hat{\theta} + 4Z_{\frac{\alpha}{2}}^2 u^2 (1-\tilde{\psi})\tilde{\psi} + Z_{\frac{\alpha}{2}}^2 v^2 (1-2\tilde{\psi})^2}$$

$$\hat{\theta} = \hat{p}_1 - \hat{p}_2, \hat{\psi} = \frac{1}{2}(\hat{p}_1 + \hat{p}_2), u = \frac{1}{4}\left(\frac{1}{n_1} + \frac{1}{n_2}\right), v = \frac{1}{4}\left(\frac{1}{n_1} - \frac{1}{n_2}\right)$$

(4) Beal's Jeffreys-Perks method, same with (3) and limits are $\theta^* \pm w$, but

$$\hat{\psi} = \frac{1}{2}\left(\frac{x_1 + 0.5}{n_1 + 1} + \frac{x_2 + 0.5}{n_2 + 1}\right)$$

(5)~(9)

(10) $\hat{\theta} = \hat{p}_1 - \hat{p}_2$, $L = \hat{\theta} - \delta$, $U = \hat{\theta} + \delta$

$$\delta = \sqrt{(\hat{p}_1 - l_1)^2 + (u_2 - \hat{p}_2)^2} = Z_{\frac{\alpha}{2}} \sqrt{\frac{l_1(1-l_1)}{n_1} + \frac{u_2(1-u_2)}{n_2}},$$

$$\varepsilon = \sqrt{(u_1 - \hat{p}_1)^2 + (\hat{p}_2 - l_2)^2} = Z_{\frac{\alpha}{2}} \sqrt{\frac{u_1(1-u_1)}{n_1} + \frac{l_2(1-l_2)}{n_2}}$$

l_1 and u_1 are roots of $|p_1 - \hat{p}_1| = Z_{\frac{\alpha}{2}} \sqrt{p_1(1-p_1)/n_2}$, l_2 and u_2 are roots of

$|p_2 - \hat{p}_2| = Z_{\frac{\alpha}{2}} \sqrt{p_2(1-p_2)/n_2}$

(11) $\hat{\theta} = \hat{p}_1 - \hat{p}_2$, $L = \hat{\theta} - \delta$, $U = \hat{\theta} + \delta$

$$\delta = \sqrt{(\hat{p}_1 - l_1)^2 + (u_2 - \hat{p}_2)^2} = Z_{\frac{\alpha}{2}} \sqrt{\frac{l_1(1-l_1)}{n_1} + \frac{u_2(1-u_2)}{n_2}},$$

$$\varepsilon = \sqrt{(u_1 - \hat{p}_1)^2 + (\hat{p}_2 - l_2)^2} = Z_{\frac{\alpha}{2}} \sqrt{\frac{u_1(1-u_1)}{n_1} + \frac{l_2(1-l_2)}{n_2}}$$

l_1 and u_1 are roots of $|p_1 - \hat{p}_1| - \frac{1}{2n_1} = Z_{\frac{\alpha}{2}} \sqrt{p_1(1-p_1)/n_2}$, l_2 and u_2 are roots of

$$|p_2 - \hat{p}_2| - \frac{1}{2n_2} = Z_{\frac{\alpha}{2}} \sqrt{p_2(1-p_2)/n_2}$$

*Pan, Wei. (2002), "Approximate confidence intervals for one proportion and difference of two proportions," *Computational Statistics & Data Analysis*, 40, 143–157. (cited: 2)

$$X_1 \sim BIN(n_1, p_1), \ X_2 \sim BIN(n_2, p_2), \ \hat{p}_1 = \frac{x_1 + 1}{n_1 + 2}, \ \hat{p}_2 = \frac{x_2 + 1}{n_2 + 2},$$

C. I. for $p_1 - p_2$:

$$(\tilde{p}_1 - \tilde{p}_2) \pm t_{d, \frac{\alpha}{2}} \sqrt{\frac{\tilde{p}_1(1 - \tilde{p}_1)}{n_1 + 2} + \frac{\tilde{p}_2(1 - \tilde{p}_2)}{n_2 + 2}}), \ d \approx \frac{2\left[\dfrac{\tilde{p}_1(1 - \tilde{p}_1)}{n_1 + 2} + \dfrac{\tilde{p}_2(1 - \tilde{p}_2)}{n_2 + 2}\right]}{\Omega(\tilde{p}_1, n_1 + 2) + \Omega(\tilde{p}_2, n_2 + 2)}$$

$$\Omega(p, n) = \frac{p - p^2}{n^3} + \frac{1}{n^5}\left[p + (6n - 7)p^2 + 4(n-1)(n-3)p^2 - 2(n-1)(2n-3)p^3\right] - \frac{2}{n^4}\left[p + (2n-3)p^2 - 2(n-1)p^3\right]$$

*Santner, T. J. and Snell, M. K. (1980), "Small-Sample Confidence Intervals for p_1–p_2 and p_1/ p_2 in 2×2 Contingency Tables," *Journal of the American Statistical Association*, 75, 386-394. (cited: 70)

APPENDIX D – Tables

Table 5: $J_1 = J_2 = 10$, $n_{ij} = 5$ $\forall_{i,j}$, $\alpha = .05$, equal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

			ρ						ρ		
p_1	p_2	.1	.3	.5	.9	p_1	p_2	.1	.3	.5	.9
.1	.1	.8992	.9144	.9046	.9574	.3	.7	.8972	.8910	.8846	.9132
		.9230	.9258	.9322	.9872			**.9504**	**.9544**	.9580	.9764
		.9102	.9116	.9106	.9780			.9084	.9074	.8980	.9338
		.9658	.9738	.9834	.9974			.9318	.9278	.9208	.9408
		.9666	.9742	.9836	.9978			.9298	.9288	.9262	**.9524**
		.9982	.9974	.9990	.9899			.9318	.9310	.9288	.9592
.1	.3	.9066	.8836	.8832	.9290	.5	.5	.8992	.8966	.8890	.9152
		.9290	.9088	.9194	.9776			.9134	.9326	.9386	.9612
		.9094	.8888	.8842	.9362			.9088	.9108	.9106	.9418
		.9446	.9378	.9398	.9708			.9314	.9254	.9268	**.9480**
		.9386	.9290	.9312	.9642			.9326	.9276	.9290	**.9508**
		.9818	.9772	.9754	.9798			.9340	.9306	.9392	**.9536**
.1	.7	.8942	.8768	.8790	.9342	.9	.9	.8950	.9050	.9194	.9794
		.9612	**.9540**	**.9538**	.9902			.9260	.9244	.9264	.9846
		.9076	.8976	.9002	.9262			.9114	.9056	.9066	.9736
		.9360	.9168	.9308	.9304			.9678	.9730	.9840	.9974
		.9338	.9142	.9274	.9406			.9682	.9736	.9842	.9978
		.9398	.9314	**.9481**	.9568			.9974	.9872	.9908	.9996
.1	.9	.8708	.8482	.8578	.9434	.9	.01	.9052	.8895	.8842	.8609
		.9532	.9678	**.9462**	.9888			.9154	.9197	.9042	**.9461**
		.9130	.8898	.8840	.9032			.9152	.8984	.8819	.8587
		.9394	.9214	.9132	.8816			.9080	.8867	.8428	.7579
		.9458	.9324	.9298	.9154			.9240	.9167	.8894	.8476
		.9428	.9356	.9328	.9196			.9328	.9297	.9026	.8595
.3	.3	.8962	.8966	.8944	.9172	.05	.05	.9221	.9376	**.9505**	**.9489**
		.9160	.9180	.9322	.9682			.9646	.9661	.9696	.9849
		.9126	.9032	.9028	.9404			**.9495**	**.9531**	.9977	.9806
		.9330	.9312	.9368	.9610			.9921	.9965	.9978	.9993
		.9332	.9328	.9394	.9630			.9921	.9965	.9977	.9994
		.9520	**.9538**	.9608	.9720			.9999	1.000	1.000	.9999
.3	.5	.9010	.8892	.8922	.9176	.05	.01	.8775	.8454	.8421	.9681
		.9420	**.9478**	.9588	.9764			.9637	**.9521**	**.9537**	.9732
		.9198	.9040	.9114	.9410			.9290	.8980	.8990	.9812
		.9400	.9262	.9316	**.9520**			.9991	.9995	.9998	.9999
		.9411	.9292	.9362	**.9542**			.9980	.9998	.9995	1.000
		.9466	.9390	.9434	.9636			1.000	.9999	1.000	1.000

Table 6: $J_1 = J_2 = 25$, $n_{ij} = 5$ $\forall_{i,j}$, $\alpha = .05$, equal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

Confidence Levels when nominal level is .95											
		ρ						ρ			
p_1	p_2	.1	.3	.5	.9	p_1	p_2	.1	.3	.5	.9
.1	.1	.9277	.9170	.9161	**.9539**	.3	.7	.9247	.9172	.9178	.9313
		.9330	.9159	.9133	**.9529**			.9318	**.9541**	.9645	.9753
		.9284	.9068	.9011	**.9460**			.9301	.9236	.9277	.9426
		.9539	**.9543**	.9618	.9819			.9384	.9408	.9413	**.9460**
		.9539	**.9539**	.9616	.9832			.9393	.9442	.9434	**.9494**
		.9891	.9988	.9897	.9992			.9403	.9447	.9439	**.9509**
.1	.3	.9243	.9210	.9165	.9318	.5	.5	.9323	.9244	.9221	.9334
		.9268	.9223	.9201	**.9456**			.9354	.9297	.9296	.9616
		.9250	.9194	.9176	.9318			.9354	.9296	.9296	.9434
		.9464	.9445	**.9502**	.9553			.9449	.9439	.9409	**.9467**
		.9412	.9415	**.9474**	**.9501**			.9450	.9444	.9420	**.9477**
		.9824	.9824	.9845	.9862			.9452	.9451	.9435	**.9501**
.1	.7	.9206	.9121	.9081	.9286	.9	.9	.9243	.9153	.9143	**.9530**
		.9611	.9642	.9634	.9798			.9287	.9148	.9084	.9583
		.9276	.9266	.9210	.9377			.9236	.9074	.8967	**.9472**
		.9393	.9415	.9418	**.9466**			**.9514**	**.9538**	.9616	.9813
		.9392	.9412	.9390	.9473			**.9514**	**.9537**	.9616	.9820
		.9476	**.9492**	**.9470**	.9569			.9984	.9893	.9892	.9985
.1	.9	.9134	.8909	.8895	.9144	.9	.01	.9205	.9002	.9006	.8999
		.9668	**.9531**	**.9544**	.9827			.9295	.9350	.9183	.9693
		.9286	.9081	.9089	.9409			.9294	.9103	.8969	.8992
		.9422	.9252	.9333	.9402			.9406	.9298	.9088	.8772
		.9458	.9319	.9379	.9583			.9376	.9253	.9127	.9050
		.9461	.9332	.9406	.9609			**.9474**	.9349	.9233	.9177
.3	.3	.9293	.9226	.9230	.9328	.05	.05	.9083	.9298	**.9461**	.9698
		.9319	.9268	.9287	**.9473**			.9401	.9373	.9339	.9809
		.9315	.9260	.9267	.9406			.9269	.9132	.9126	.9716
		.9498	.9425	.9444	**.9490**			.9634	.9786	.9905	.9975
		.9495	.9431	**.9462**	**.9505**			.9634	.9783	.9902	.9976
		.9613	.9614	.9639	.9696			.9999	.9999	.9989	.9999
.3	.5	.9314	.9217	.9195	.9269	.05	.01	.8833	.8506	.8342	.9144
		.9329	.9270	.9384	.9705			.9090	.9003	.9014	**.9457**
		.9327	.9260	.9260	.9369			.8857	.8799	.8958	.9124
		.9431	**.9455**	.9444	**.9456**			.9720	.9638	.9434	.9850
		.9444	**.9457**	.9453	**.9461**			.9306	.9191	.9254	.9812
		.9502	**.9508**	**.9501**	**.9532**			.9899	.9811	.9980	.9990

Table 7: $J_1 = J_2 = 5$, $n_{ij} = 10 \ \forall_{i,j}$, $\alpha = .05$, equal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

		Confidence Levels when nominal level is .95									
		ρ						ρ			
p_1	p_2	.1	.3	.5	.9	p_1	p_2	.1	.3	.5	.9
.1	.1	.8537	.8192	.8994	.8709	.3	.7	.8077	.7608	.8125	.9007
		.8967	.9195	.9354	**.9482**			.9154	.9036	.9003	**.9499**
		.8923	.9270	.9027	.9103			.8276	.8013	.8567	.9357
		.9517	.9454	.9578	.9782			.9206	.8975	.9122	.9221
		.9519	**.9459**	.9586	.9809			.9146	.9164	.9083	**.9510**
		.9955	.9866	.9961	.9980			.9152	.9089	.9159	**.9531**
.1	.3	.7997	.7474	.7147	.8476	.5	.5	.8501	.7783	.9001	.8932
		.8939	.8905	.9101	.9506			.9007	.9051	.9560	.9765
		.8199	.8754	.8001	.8994			.8159	.8622	.9315	.9688
		.9160	.8703	.9160	.9235			.9091	.9015	.9618	.9821
		.9075	.8960	.8622	.9128			.9315	.9077	.9658	.9889
		.9691	.9356	.9445	.9672			.9201	.9208	.9822	.9901
.1	.7	.7815	.7417	.7163	.8991	.9	.9	.8682	.8879	.9197	.8924
		.9038	.9001	.8879	.9321			.9159	**.9499**	.9669	.9889
		.8236	.8019	.7925	.8713			.9078	.9061	.9222	.9725
		.9185	.8832	.8586	.8934			.9625	.9681	.9712	.9982
		.9068	.8666	.8432	.8892			.9585	.9888	.9737	.9968
		.9103	.8821	.8627	.9083			.9770	.9827	.9973	.9991
.1	.9	.7518	.7061	.6970	.8698	.9	.01	.9185	.8902	.9011	.8939
		.9005	.9099	**.9480**	**.9515**			.9295	.9350	.9183	.9591
		.8351	.8325	.8233	.9059			.9294	.9103	.8969	.9092
		.9344	.9155	.9031	.8865			.9409	.9298	.9088	.8972
		.9328	.9186	.9118	.8929			.9376	.9191	.9201	.9103
		.9344	.9225	.9206	.9161			.9387	.9221	.9327	.9161
.3	.3	.8104	.7680	.7371	.8484	.05	.05	.9019	.9301	.9009	.9267
		.8734	.9066	.8818	.9004			.9381	.9373	.9213	.9714
		.8266	.7992	.7841	.8994			.9229	.9132	.9126	.9716
		.9233	.8768	.9167	.9281			**.9521**	.9691	.9708	.9975
		.9238	.9078	.9165	.9892			.9598	.9692	.9891	.9939
		.9432	.9067	.9398	.9738			.9979	.9989	.9989	.9999
.3	.5	.7946	.7900	.7861	.8732	.05	.01	.8793	.8606	.8592	.9003
		.8900	.8861	.8997	**.9492**			.9009	.9191	.9059	.9297
		.8331	.8388	.8239	.9143			.8929	.8799	.8958	.9124
		.9185	.9032	.9143	.9224			.9692	**.9529**	.9434	.9850
		.9190	.9030	.8946	.9356			.9306	.9208	.9301	.9815
		.9255	.9005	.9257	.9578			.9819	.9792	.9967	.9981

Table 8: $J_1 = J_2 = 10$, $n_{ij} = 10$ $\forall_{i,j}$, $\alpha = .05$, equal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

		Confidence Levels when nominal level is .95						

p_1	p_2	.1	.3	.5	.9	p_1	p_2	.1	.3	.5	.9
.1	.1	.8942	.8671	.8751	.9182	.3	.7	.8907	.8902	.8838	.9020
		.8989	.8765	.8895	.9669			.9336	**.9515**	.9523	.9685
		.8869	.8458	.8432	**.9494**			.8987	.8988	.9031	.9223
		.9418	**.9513**	.9781	.9851			.9261	.9219	.9256	.9412
		.9418	.9615	.9786	.9857			.9271	.9244	.9281	**.9499**
		.9872	.9797	.9983	.9989			.9280	.9263	.9315	.9574
.1	.3	.8904	.8874	.8840	.9158	.5	.5	.8999	.8905	.8890	.8992
		.9039	.8974	.9066	.9642			.9060	.9157	.9406	.9640
		.8894	.8842	.8820	.9221			.9060	.9050	.9074	.9259
		.9285	.9372	.9426	.9658			.9289	.9262	.9280	.9377
		.9259	.9320	.9363	.9615			.9300	.9275	.9298	.9411
		.9725	.9779	.9801	.9896			.9314	.9298	.9327	.9452
.1	.7	.8865	.8683	.9203	.8666	.9	.9	.8973	.8818	.8979	.9245
		.9479	.9459	.9784	.9531			.8985	.8931	.9017	.9668
		.8962	.8862	.9178	.9028			.8883	.8504	.8842	**.9491**
		.9170	.9204	.9370	.9273			.9417	.9567	.9726	.9951
		.9153	.9222	.9449	.9238			.9419	**.9529**	.9829	.9953
		.9302	.9323	.9599	.9347			.9958	.9876	.9882	.9986
.1	.9	.8666	.8270	.8146	.9068	.9	.01	.8794	.8382	.8208	.9035
		.9431	.9157	.9399	.9615			.9088	.8909	.9005	.9429
		.8928	.8557	.8387	.9029			.8889	.8486	.8351	.8500
		.9173	.8978	.9001	.8923			.9206	.8863	.8572	.7862
		.9238	.9054	.9023	.9197			.9196	.8873	.8738	.8505
		.9247	.9074	.9039	.9157			.9289	.9013	.8873	.8594
.3	.3	.9004	.8928	.8908	.9045	.05	.05	.8963	.9263	.9542	.9689
		.9056	.9030	.9201	.9560			.9153	.9124	.9231	.9829
		.9047	.8970	.8952	.9191			.8924	.8801	.8864	.9746
		.9379	.9254	.9333	.9565			.9705	.9898	.9971	.9996
		.9385	.9276	.9348	.9584			.9706	.9898	.9972	.9997
		.9480	.9450	.9575	.9688			.9999	.9998	.9989	.9999
.3	.5	.8925	.8919	.8871	.9015	.05	.01	.8156	.7552	.7510	.9529
		.9033	.9383	**.9495**	.9674			.8636	.8253	.8678	.9681
		.8999	.9031	.9031	.9277			.8303	.7803	.7770	.9150
		9253	.9259	.9315	.9444			.9788	.9870	.9934	.9996
		.9261	.9268	.9329	**.9489**			.9369	.9321	**.9461**	.9941
		.9327	.9338	.9401	.9566			1.000	.9999	.9999	1.000

Table 9: $J_1 = J_2 = 25$, $n_{ij} = 10$ $\forall_{i,j}$, $\alpha = .05$, equal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

			ρ						ρ		
p_1	p_2	.1	.3	.5	.9	p_1	p_2	.1	.3	.5	.9
.1	.1	.9068	.9001	.8933	.9463	.3	.7	.9231	.9124	.9108	.9211
		.9158	.9035	.9408	.9391			.9259	.9352	**.9537**	**.9509**
		.9127	.8850	.8799	.9283			.9259	.9179	.9229	.9345
		.6461	**.9509**	.9577	.9790			.9398	.9424	.9419	**.9457**
		.9461	**.9508**	.9575	.9800			.9412	.9426	.9444	**.9472**
		.9988	.9985	.9991	.9992			.9418	.9434	.9453	**.9495**
.1	.3	.9225	.9078	.9084	.9289	.5	.5	.9191	.9189	.9133	.9233
		.9238	.9100	.9193	.9470			.9214	.9241	.9208	.9520
		.9230	.9076	.9070	.9302			.9214	.9241	.9207	.9349
		.9452	.9432	.9451	**.9544**			.9387	.9423	.9401	.9413
		.9433	.9416	.9423	**.9528**			.9390	.9428	.9409	.9428
		.9819	.9844	.9823	.9873			.9395	.9437	.9424	.9446
.1	.7	.9147	.9039	.9008	.9240	.9	.9	.9134	.9034	.8883	.9335
		.9251	.9558	.9564	.9661			.9134	.9177	.8997	**.9476**
		.9187	.9118	.9132	.9375			.9100	.8893	.8688	.9246
		.9360	.9387	.9395	**.9494**			.9422	**.9507**	**.9540**	.9606
		.9358	.9368	.9399	**.9479**			.9418	.9506	.9542	.9783
		.9445	.9452	**.9492**	.9587			.9974	.9988	.9981	.9989
.1	.9	.9041	.8793	.8658	.9163	.9	.01	.9053	.8755	.8606	.9017
		.9540	**.9496**	.9649	.9597			.9174	.9035	.8722	.9277
		.9123	.8893	.8922	.9267			.9170	.8922	.8704	.8951
		.9336	.9273	.9271	.9393			.9441	.9282	.9137	.8943
		.9379	.9311	.9330	.9564			.9400	.9254	.9134	.9050
		.9379	.9315	.9342	.9589			**.9468**	.9329	.9240	.9175
.3	.3	.9234	.9201	.9224	.9249	.05	.05	.9140	.8930	.9073	.9610
		.9248	.9229	.9264	.9381			.9137	.9066	.8939	**.9510**
		.9244	.9221	.9251	.9329			.9050	.8554	.8485	**.9458**
		.9411	.9447	.9435	**.9470**			**.9532**	.9659	.9849	.9974
		.9413	.9452	.9441	**.9479**			**.9525**	.9653	.9846	.9975
		.9537	.9629	.9652	.9657			.9999	.9999	.9999	1.000
.3	.5	.9211	.9191	.9464	.9322	.05	.01	.8757	.8132	.7896	.9117
		.9227	.9227	.9282	.9534			.8991	.8832	.8791	.9042
		.9226	.9225	.9253	.9414			.8716	.8011	.7977	.8825
		.9414	.9392	.9442	**.9482**			.9557	.9410	.9397	.9903
		.9419	.9405	.9439	**.9498**			.9232	.9005	.9001	.9832
		.9473	**.9464**	**.9510**	.9557			.9998	.9999	1.000	1.000

Table 10: $J_1 = 5$, $J_2 = 10$, $n_{ij} = 10$ $\forall_{i,j}$, $\alpha = .05$, equal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

			ρ						ρ		
p_1	p_2	.1	.3	.5	.9	p_1	p_2	.1	.3	.5	.9
.1	.1	.7925	.8357	.8669	.9159	.3	.7	.8192	.7791	.7839	.9069
		.9156	.9231	**.9467**	.9569			.9061	.9087	.8997	.9601
		.9006	.9046	.9311	**.9519**			.8387	.8501	.8338	.9109
		.8564	.8230	.7997	.9006			.9698	.9741	.9807	**.9493**
		.9092	.9001	.8971	**.9480**			.9799	.9949	.9839	.9791
		.9867	.9869	.9826	.9961			.9977	.9985	.9967	.9703
.1	.3	.8604	.8349	.8803	.9153	.5	.5	.8109	.8189	.8501	.8669
		.9134	.9207	.9406	.9597			.8878	.8903	.9129	**.9518**
		.9081	.8850	.9003	.9204			.8615	.8479	.8574	.9218
		.9499	.9097	.8917	.9208			.9983	.9881	.9728	.9554
		.9562	.9319	.9315	.9249			.9977	.9911	.9799	.9797
		.9887	.9830	.9667	.9627			.9991	.9921	.9855	.9802
.1	.7	.8319	.8329	.8399	.9007	.9	.9	.7561	.8191	.8607	.9227
		.9409	**.9492**	**.9483**	.9591			.9389	.9109	**.9480**	.9614
		.9029	.8937	.8691	.9203			.9004	.9061	.9301	.9779
		.9791	.9832	.9732	.9097			.9937	.9957	.9990	.9608
		.9982	.9889	.9792	.9401			.9967	.9949	.9991	.9891
		.9986	.9954	.9797	.9599			.9994	.9991	.9967	.9997
.1	.9	.7915	.7664	.7813	.9406	.9	.01	.7673	.7170	.7389	.8109
		.9498	.9502	.9567	.9596			.9119	**.9477**	.9250	**.9470**
		.8934	.8807	.8774	.8978			.9046	.9199	.8963	.8398
		.9809	.9607	.9217	.8796			.9719	.9709	.9109	.8701
		.9989	.9893	.9694	.9177			.9819	.9889	.9728	.8931
		.9987	.9963	.9838	.9300			.9803	.9839	.9953	.8833
.3	.3	.7914	.7799	.7682	.8891	.05	.05	.8697	.9009	.9319	.9554
		.8772	.8803	.9030	.9619			.9297	.9559	.9493	.9759
		.8391	.8228	.8367	.9338			.9109	.9061	.9409	.9664
		.9528	.9296	.8972	.9329			.8572	.8992	.8534	.8908
		.9891	.9547	.9496	.9644			.9335	.8979	.8807	.9792
		.9955	.9762	.9608	.9758			.9981	.9869	.9973	.9891
.3	.5	.7943	.7737	.8197	.9090	.05	.01	.6731	.6501	.6297	.9087
		.9089	.8991	.9096	**.9520**			**.9514**	.9687	.9597	.9819
		.8513	8531	.8537	.9324			.9711	.9338	.9680	.9857
		.9781	.9836	**.9456**	.9242			.9611	.9099	.8930	.9691
		.9936	.9963	.9859	.9703			.9819	.9717	**.9537**	.9919
		.9991	.9959	.9903	.9727			.9958	.9984	.9991	.9984

Confidence Levels when nominal level is .95

Table 11: $J_1 = 5$, $J_2 = 25$, $n_{ij} = 5$ $\forall_{i,j}$, $\alpha = .05$, equal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

			Confidence Levels when nominal level is .95								
			ρ						ρ		
p_1	p_2	.1	.3	.5	.9	p_1	p_2	.1	.3	.5	.9
.1	.1	.6402	.6154	.6843	.9031	.3	.7	.7882	.7689	.7925	.8777
		.9354	.9424	.9624	.9736			.8865	.9005	.8923	.9564
		.9201	.9163	.9283	.9624			.8202	.8212	.8339	.9067
		.8581	.8322	.8140	.8862			.9988	.9896	.9624	.9413
		.8871	.8821	.9090	.9769			.9987	.9897	.9634	**.9533**
		.9968	.9954	.9951	.9944			.9988	.9900	.9635	**.9543**
.1	.3	.7892	.8240	.8616	.9457	.5	.5	.8188	.8104	.8073	.8976
		.9332	.9509	.9648	.9804			.8949	.8872	.8998	.9576
		.9168	.9152	.9171	.9342			.8452	.8410	.8452	.9383
		.9577	.9259	.9010	.8974			.9942	.9792	.9586	.9725
		.9818	.9620	**.9470**	.9663			.9942	.9791	.9584	.9750
		.9994	.9967	.9917	.9884			.9954	.9814	.9609	.9768
.1	.7	.7840	.7928	.8251	.9153	.9	.9	.7097	.7443	.7086	.9110
		.9668	.9673	.9617	.9827			.9331	.9306	.9378	.9675
		.9159	.9096	.9155	.9214			.9158	.9129	.9225	.9623
		.9961	.9798	.9642	.8953			1.000	.9998	.9988	.9705
		.9992	.9918	.9856	.9604			.9997	1.000	.9996	.9936
		.9997	.9951	.9928	.9784			1.000	1.000	.9997	.9979
.1	.9	.7885	.7095	.7344	.9074	.9	.01	.6474	.7060	.6922	.8895
		.9622	.9585	.9753	.9896			.9561	**.9534**	.9448	.9669
		.9104	.8993	.9124	.9322			**.9483**	.9335	.9386	.9290
		.9886	.9655	**.9487**	.8776			.9993	.9912	.9719	.8626
		.9977	.9918	.9762	.9678			.9999	.9995	.9871	.9591
		.9982	.9920	.9803	.9789			.9998	.9997	.9883	.9734
.3	.3	.7931	.7889	.7413	.8765	.05	.05	.7868	.7700	.8645	.9525
		.8738	.8697	.8707	**.9495**			.9705	**.9500**	.9681	.9881
		.8560	.8079	.8088	.9139			**.9544**	.9366	.9438	.9679
		.9535	.9254	.8988	.9285			.8117	.8240	.7878	.8425
		.9505	.9227	.9025	.9403			**.9464**	.8988	.8856	.9783
		.9781	**.9503**	.9341	.9596			.9999	1.000	.9998	.9980
.3	.5	.7962	.7541	.7758	.8861	.05	.01	.6209	.6913	.5230	.8144
		.8790	.8879	.9010	**.9534**			.9632	.9599	.9776	.9993
		.8328	.8170	.8343	.9203			.9561	**.9496**	.9706	.9911
		.9862	.9565	.9323	.9427			.9174	.8703	.8556	.9560
		.9871	.9612	.9410	.9557			.9965	.9876	.9869	.9966
		.9901	.9710	**.9492**	.9610			1.000	.9999	.9999	.9990

Table 12: $J_1 = 5$, $J_2 = 25$, $n_{ij} = 10$ $\forall_{i,j}$, $\alpha = .05$, equal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

			Confidence Levels when nominal level is .95								
			ρ						ρ		
p_1	p_2	.1	.3	.5	.9	p_1	p_2	.1	.3	.5	.9
.1	.1	.7073	.8117	.8188	.8945	.3	.7	.7601	.7655	.7792	.8677
		.8913	.8921	.9008	.9179			.8905	.9039	.8999	.9234
		.8635	.8675	.8756	.9038			.8397	.8059	.8189	.9067
		.8694	.8977	.8922	.8979			.9791	.9794	.9624	.9429
		.8997	.9229	.9229	.9447			.9885	.9868	.9634	**.9495**
		.9843	.9900	.9925	.9980			.9918	.9934	.9635	.9559
.1	.3	.8401	.8755	.8497	.9317	.5	.5	.8019	.8308	.8587	.8792
		.9035	.9196	.9247	.9687			.8889	.8732	.9067	**.9495**
		.8882	.9040	.9001	.9031			.8515	.8097	.8621	.9186
		.9617	.9605	.8975	.8891			.9792	.9792	.9586	.9635
		.9837	.9729	.9271	.9561			.9842	.9798	**.9528**	.9704
		.9991	.9980	.9908	.9819			.9853	.9774	.9677	.9801
.1	.7	.7618	.7792	.8501	.9009	.9	.9	.7119	.7443	.7086	.9115
		.9389	.9297	.9068	.9791			.9591	.9306	.9377	.9675
		.9299	.9101	.8956	.9033			.9387	.9129	.9301	.9553
		.9949	.9635	.9589	.8959			.9840	.9949	.9967	.9722
		.9894	.9939	.9791	.9568			.9969	.9863	.9937	.9809
		.9925	.9839	.9829	.9803			.9907	.9961	.9988	.9990
.1	.9	.6985	.7995	.7501	.8896	.9	.01	.6961	.7158	.6792	.8681
		.9503	**.9501**	**.9497**	.9791			.9347	.9203	.9189	.9559
		.9093	.9004	.9031	.9387			**.9501**	.9097	.9228	.9299
		.9692	.9631	.9587	.8936			.9891	.9921	.9687	.8891
		.9877	.9941	.9752	.9646			.9919	.9985	.9844	.9601
		.9919	.9907	.9829	.9806			.9991	.9994	.9897	.9799
.3	.3	.7641	.7569	.7239	.8915	.05	.05	.7298	.7597	.8349	.9005
		.8908	.8539	.8967	.9235			.9665	.9419	.9597	.9697
		.8814	.8198	.8201	.9061			.9389	.9116	.9438	.9679
		.9701	.9331	.8981	.9087			.8317	.8391	.7911	.8655
		.9139	.9598	.9106	**.9519**			.9551	.9087	.8956	.9693
		.9897	.9197	.9449	**.9519**			.9991	.9947	.9977	.9936
.3	.5	.7662	.7791	.7301	.8997	.05	.01	.6509	.6897	.6140	.7944
		.8915	.8689	.8997	.9401			**.9511**	**.9468**	.9637	.9792
		.8038	.8217	.8654	.9331			.9303	.9197	.9597	.9661
		.9879	**.9511**	.9323	.9427			.9301	.8991	.8497	.9229
		.9809	**.9497**	.9687	.9347			.9879	.9871	.9879	.9801
		.9867	.9719	**.9499**	.9639			.9937	.9949	.9891	.9973

Table 13: $J_1 = J_2 = 5$, $n_{ij} = 5$ $\forall_{i,j}$, $\alpha = .05$, unequal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

		Confidence Levels when nominal level is .95									
		$\rho_1 \mid \rho_2$						$\rho_1 \mid \rho_2$			
p₁	p₂	.1 | .3	.1 | .5	.1 | .9	.3 | .9	p₁	p₂	.1 | .3	.1 | .5	.1 | .9	.3 | .9
.1	.1	.8744	.8905	.9465	.9406	.3	.7	.7607	7622	.8565	.8570
		.9410	.9402	.9558	.9733			.9019	.8873	.9542	.9589
		.9120	.9143	**.9503**	**.9486**			.8135	.8208	.9035	.8983
		.9688	.9863	.9933	.9928			.9028	.9031	.9214	.9202
		.9891	.9898	.9954	.9943			.9114	.9094	.9341	.9301
		.9963	.9860	.9918	.9979			.9144	.9128	.9423	.9376
.1	.3	.7264	.7269	.8794	.8688	.5	.5	.8027	.7918	.8613	.8585
		.8860	.8963	.9439	.9239			.9017	.8981	**.9462**	.9420
		.7927	.8021	.9026	.8957			.8326	.8307	.8957	.8969
		.9106	.8938	.9648	.9559			.9054	.8968	.9408	.9340
		.9043	.8843	.9578	.9445			.9071	.8999	**.9461**	.9397
		.9669	.9636	.9844	.9805			.9097	.9044	**.9504**	.9439
.1	.7	.7417	.7408	.8859	.8775	.9	.9	.8731	.8920	**.9467**	.9608
		.8886	.8634	**.9487**	**.9490**			**.9479**	.9623	.9705	.9702
		.7943	.7982	.8849	.8889			.9099	.9147	**.9533**	**.9462**
		.9116	.8996	.8993	.9101			.9841	.9860	.9933	.9921
		.9006	.8843	.9054	.9102			.9879	.9889	.9943	.9938
		.9191	.9041	.9252	.9310			.9962	.9951	.9985	.9982
.1	.9	.7535	.7562	.8847	.8812	.9	.01	.7646	.7891	.7600	.8861
		.9142	.9234	**.9524**	**.9528**			.8865	.8787	.9001	.9001
		.8488	.8570	.8529	.8795			.8534	.8564	.8468	.8505
		.8998	.8877	.7623	.8002			.7557	.8074	.7691	.7144
		.9341	.9286	.8525	.8767			.8386	.8192	.7707	.8103
		.9416	.9399	.8700	.8956			.8462	.8265	.7909	.8144
.3	.3	.7515	.7436	.8797	.8731	.05	.05	.9357	.9410	.9569	**.9468**
		.8858	.8795	.9372	.9345			.9547	.9625	.9715	.9668
		.8143	.8089	.8959	.8956			.9412	.9598	.9615	.9713
		.9192	.9052	**.9542**	**.9531**			.9871	.9981	.9983	.9979
		.9260	.9108	**.9525**	**.9539**			.9877	.9985	.9983	.9971
		.9422	.9311	.9715	.9663			.9899	.9995	.9969	.9996
.3	.5	.7909	.7876	.8695	.8581	.05	.01	.6989	.7593	.8777	.8695
		.9086	.9019	**.9457**	.9379			.9551	.9692	.9745	**.9515**
		.8271	.8281	.8972	.8955			.9419	.9568	.9679	.9631
		.9024	.8944	**.9462**	.9414			.9895	.9999	.9997	.9999
		.9051	.8957	**.9467**	.9415			.9899	.9991	.9998	1.000
		.9155	.9078	.9550	.9523			.9908	1.000	.9999	1.000

Table 14: $J_1 = J_2 = 10$, $n_{ij} = 5 \; \forall_{i,j}$, $\alpha = .05$, unequal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

Confidence Levels when nominal level is .95											
		$\rho_1 \mid \rho_2$						$\rho_1 \mid \rho_2$			
p_1	p_2	.1 | .3	.1 | .5	.1 | .9	.3 | .9	p_1	p_2	.1 | .3	.1 | .5	.1 | .9	.3 | .9
.1	.1	.8971	.8957	.9141	.9447	.3	.7	.8962	.8888	.8961	.8988
		.9202	.9255	**.9535**	**.9529**			**.9542**	**.9525**	**.9538**	.9658
		.9040	.9038	**.9496**	**.9503**			.9090	.9065	.9262	.9237
		.9685	.9682	.9855	.9910			.9275	.9228	.9319	.9326
		.9692	.9690	.9857	.9913			.9289	.9237	.9348	.9333
		.9976	.9980	.9978	.9972			.9321	.9276	.9391	.9369
.1	.3	.8818	.8777	.8894	.8975	.5	.5	.9006	.8956	.9048	.9034
		.9029	.9082	**.9534**	**.9543**			.9249	.9383	**.9543**	**.9530**
		.8869	.8917	.9288	.9274			.9124	.9147	.9302	.9274
		.9277	.9247	.9472	.9531			.9263	.9268	.9292	.9357
		.9294	.9156	.9338	.9418			.9269	.9277	.9304	.9380
		.9719	.9706	.9767	.9789			.9301	.9318	.9362	.9430
.1	.7	.8872	.8749	.8911	.9032	.9	.9	.8951	.8906	.9389	**.9561**
		.9551	**.9489**	.9691	.9693			.9190	.9193	.9665	.9660
		.9076	.9007	.9349	.9272			.9018	.8997	.9452	.9517
		.9264	.9244	.9374	.9368			.9655	.9657	.9865	.9905
		.9233	.9186	.9365	.9352			.9657	.9660	.9867	.9908
		.9350	.8348	.9546	.9544			.9971	.9971	.9981	.9982
.1	.9	.8524	.8502	.9313	.9314	.9	.01	.8957	.8200	.8640	.9009
		.9484	**.9458**	.9604	.9695			.9155	.8995	.9091	.8895
		.8943	.8923	.9343	.9171			.8934	.8732	.8546	.8153
		.9277	.9185	.9102	.9015			.8817	.8504	.8115	.7913
		.9296	.9266	.9363	.9346			.9111	.9004	.8577	.8498
		.9317	.9295	.9424	.9405			.9203	.9085	.8740	.8662
.3	.3	8984	8924	.9025	.9108	.05	.05	.9224	**.9498**	.8389	.9068
		.9100	.9162	.9482	.9537			**.9538**	**.9512**	.9285	.9840
		.9047	.9054	.9287	.9307			**.9491**	.9545	.9786	.9780
		.9299	.9236	.9346	**.9486**			.9936	.9948	.9985	.9984
		.9310	.9243	.9360	**.9471**			.9991	.9950	.9986	.9985
		.9524	**.9479**	.9568	.9646			.999	1.000	.9999	.9997
.3	.5	.9020	.8939	.9017	.8950	.05	.01	.8782	.8950	.9340	.9420
		.9479	**.9520**	**.9533**	**.9543**			.9539	.6518	.9779	.9688
		.9128	.9116	.9276	.9219			.9042	.8953	.9428	.9402
		.9294	.9230	.9271	.9332			.9988	.9996	.9999	.9997
		.9309	.9245	.9280	.9339			.9982	.9993	1.000	.9997
		.9387	.9339	.9401	**.9456**			.9999	1.000	1.000	1.000

Table 15: $J_1 = J_2 = 25$, $n_{ij} = 5$ $\forall_{i,j}$, $\alpha = .05$, unequal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

			Confidence Levels when nominal level is .95								
		$\rho_1 \mid \rho_2$						$\rho_1 \mid \rho_2$			
p_1	p_2	.1 | .3	.1 | .5	.1 | .9	.3 | .9	p_1	p_2	.1 | .3	.1 | .5	.1 | .9	.3 | .9
.1	.1	.9120	.9060	.9365	.9374	.3	.7	.9106	.9177	.9228	.9270
		.9227	.9144	.9374	.9409			.9429	.9580	.9716	.9706
		.9177	.9056	.9280	.9322			.9174	.9262	.9355	.9377
		.9473	.9384	.9734	.9655			.9429	.9384	.9412	.9418
		.9473	.9384	.9736	.9655			.9425	.9392	.9431	.9430
		.9983	.9979	.9982	.9985			.9436	.9402	.9446	.9454
.1	.3	.9165	.9129	.9196	.9250	.5	.5	.9228	.9221	.9300	.9284
		.9228	.9222	.9307	.9367			.9271	.9283	.9406	.9371
		.9201	.9203	.9259	.9341			.9270	.9283	.9395	.9368
		.9489	.9399	**.9516**	**.9495**			.9427	.9405	**.9467**	.9410
		.9484	.9355	**.9457**	.9422			.9427	.9409	**.9477**	.9424
		.9799	.9796	.9844	.9842			.9440	.9423	**.9493**	.9454
.1	.7	.9141	.9130	.9130	.9205	.9	.9	.9222	.9071	.9414	.9342
		.9621	.9618	.9697	.9676			.9249	.9162	.9418	.9405
		.9213	.9243	.9296	.9299			.9194	.9082	.9320	.9311
		.9389	.9367	.9441	**.9460**			**.9506**	.9396	.9719	.9646
		.9390	.9352	.9416	**.9461**			**.9506**	.9396	.9718	.9646
		.9469	**.9476**	**.9524**	**.9527**			.9990	.9983	.9988	.9988
.1	.9	.9004	.9034	.9150	.9100	.9	.01	.9213	.9326	.8939	.9170
		.9539	**.9542**	**.9543**	.9712			.9218	.9153	**.9531**	.9060
		.9150	.9161	.9308	.9342			.9210	.9124	.8914	.8966
		.9329	.9281	.9339	.9342			.9379	.9275	.9004	.8933
		.9359	.9333	**.9465**	**.9504**			.9382	.9358	.9182	.9226
		.9366	.9346	**.9469**	**.9523**			**.9469**	.9435	.9260	.9314
.3	.3	.9251	.9188	.9250	.9264	.05	.05	.9285	.9274	.9664	.9467
		.9285	.9251	.9333	.9355			.9307	.9270	.9656	.9590
		.9279	.9239	.9315	.9347			.9160	.9101	**.9492**	**.9458**
		.9439	.9390	.9431	.9436			.9640	.9625	.9950	.9903
		.9446	.9395	.9440	.9447			.9639	.9625	.9951	.9902
		.9620	.9603	.9621	.9641			.9999	.9999	.9991	1.000
.3	.5	.9201	.9240	.9237	.9283	.05	.01	.8928	.9087	.9285	.9442
		.9254	.9310	.9657	.9622			.9205	.9212	.8916	.9198
		.9253	.9304	.9340	.9372			.8940	.8917	.8418	.8735
		.9369	.9420	.9421	.9447			.9797	.9880	.9736	.9937
		.9373	.9422	.9420	**.9457**			**.9462**	.9628	.9643	.9844
		.9435	**.9490**	**.9494**	**.9535**			.9998	1.000	.9989	1.000

Table 16: $J_1 = J_2 = 5$, $n_{ij} = 10$ $\forall_{i,j}$, $\alpha = .05$, unequal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

		$\rho_1 \mid \rho_2$						$\rho_1 \mid \rho_2$			
p_1	p_2	.1 | .3	.1 | .5	.1 | .9	.3 | .9	p_1	p_2	.1 | .3	.1 | .5	.1 | .9	.3 | .9
.1	.1	.7552	.7020	.8655	.8688	.3	.7	.7684	.7471	.8612	.8587
		.8681	.8612	.9252	.9329			.8804	.8613	.9404	.9305
		.8320	.8185	.9060	.9136			.8048	.7948	.8547	.8741
		.9512	.9560	.9752	.9795			.8828	.8813	.9015	.8991
		.9517	.9564	.9772	.9812			.8865	.8816	.8997	.9005
		.9956	.9932	.9972	.9983			.8901	.8892	.9023	.9123
.1	.3	.7568	.7216	.8465	.8397	.5	.5	.7836	.7604	.8594	.8987
		.8790	.8797	.8888	.9021			.9137	.8991	.9401	.9301
		.8068	.7926	.8679	.8624			.8597	.8408	.8919	.8919
		.8773	.8625	.9205	.9152			.9097	.8918	.9307	**.9497**
		.8605	.8500	.9096	.9048			.9118	.8939	**.9459**	**.9501**
		.9392	.9221	.9420	.9348			.9058	.9201	.9387	**.9467**
.1	.7	.7468	.7116	.8408	.8604	.9	.9	.8524	.8387	.9187	.9437
		.8669	.8352	.9157	.9284			.9224	**.9529**	.9439	.9606
		.8085	.8099	.8657	.8805			.9099	.9147	.9533	.9462
		.8725	.8548	.8955	.8989			.9791	.9597	.9679	.9893
		.8568	.8348	.8900	.8960			.9879	.9619	.9833	.9904
		.8756	.8556	.9084	.9120			.9963	.9929	.9879	.9867
.1	.9	.7140	.7104	.8627	.8648	.9	.01	.7559	.7918	.7403	.8791
		.9048	.9205	.9595	**.9528**			.8861	.8950	.9134	.9197
		.8440	.8574	.9053	.9040			.8097	.8516	.8419	.8753
		.9181	.8932	.8588	.8711			.7987	.8297	.7937	.8359
		.9204	.9092	.8884	.9012			.8504	.8397	.8671	.8179
		.9240	.9152	.9140	.9256			.8499	.8198	.8674	.8597
.3	.3	.7556	.7324	.8536	.8476	.05	.05	.9007	.9190	.9237	.9168
		.8725	.8353	.9068	.9008			**.9541**	**.9532**	.9660	.9619
		.8020	.7824	.8856	.8761			.9439	.9319	**.9468**	.9716
		.8784	.8540	.9144	.9093			.9705	.9881	.9883	.9729
		.8797	.8564	.9165	.9104			.9698	.9697	.9814	.9839
		.9141	.8947	.9293	.9273			.9853	.9875	.9989	.9916
.3	.5	.7860	.7856	.8804	.8804	.05	.01	.6922	.7993	.8650	.8885
		.8741	.8800	.9392	.9425			.9441	.9329	.9549	.9394
		.8169	.8181	.9240	.9211			.9439	.9591	.9547	.9539
		.8828	.8753	.9249	.9273			.9895	.9934	.9957	.9991
		.8833	.8752	.9273	.9312			.9891	.9921	.9874	.9891
		.8972	.8849	.9424	.9412			.9878	.9990	.9913	.9968

Table 17: $J_1 = J_2 = 10$, $n_{ij} = 10$ $\forall_{i,j}$, $\alpha = .05$, unequal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

		Confidence Levels when nominal level is .95									
		$\rho_1 \mid \rho_2$						$\rho_1 \mid \rho_2$			
p₁	p₂	.1 | .3	.1 | .5	.1 | .9	.3 | .9	p₁	p₂	.1 | .3	.1 | .5	.1 | .9	.3 | .9
.1	.1	.8707	.8354	.8926	.9170	.3	.7	.8882	.8801	.8881	.8935
		.8992	.8797	.9288	.9217			.9435	**.9454**	.9589	.9622
		.8584	.8401	.9076	.9067			.8960	.8983	.9220	.9174
		.9353	.9247	.9589	.9721			.9205	.9163	.9257	.9305
		.9353	.9275	.9592	.9728			.9229	.9155	.9294	.9353
		.9849	.9956	.9970	.9979			.9251	.9194	.9350	.9386
.1	.3	.8823	.8660	.8836	.8868	.5	.5	.8862	.8821	.8891	.8936
		.9000	.8885	.9384	**.9467**			.8995	.9121	**.9519**	.9548
		.8851	.8789	.9219	.9170			.8980	.9005	.9181	.9184
		.9299	.9177	.9415	.9442			.9255	.9186	.9186	.9290
		.9348	.9079	.9266	.9387			.9265	.9197	.9208	.9311
		.9672	.9592	.9705	.9752			.9288	.9228	.9283	.9355
.1	.7	.8796	.8692	.8779	.8788	.9	.9	.8753	.8407	.8876	.9177
		.9455	.9400	.9630	.9614			.8815	.8605	.9319	.9363
		.8960	.8937	.9234	.9180			.8644	.8410	.9060	.9055
		.9180	.9113	.9322	.9250			.9396	.9223	**.9533**	.9832
		.9167	.9063	.9311	.9364			.9397	.9226	.9540	.9836
		.9265	.9245	**.9499**	**.9535**			.9957	.9950	.9969	.9981
.1	.9	.8342	.8155	.8942	.8999	.9	.01	.8851	.8980	.8633	.8918
		.9307	.9083	**.9538**	.9593			.9012	.9304	.9437	.9105
		.8663	.8545	.9114	.8984			.8785	.8863	.8825	.8158
		.9020	.8890	.9047	.8957			.9062	.8942	.7916	.8577
		.9099	.8957	.9233	.9207			.9194	.9113	.8772	.8621
		.9110	.8990	.9301	.9269			.9285	.9191	.8832	.8693
.3	.3	.8972	.8793	.8918	.8886	.05	.05	.8991	.8912	.9227	.9596
		.9054	.8933	.9328	.9339			.9127	.9198	.9573	.9659
		.9026	.8896	.9202	.9091			.8843	.8877	.9445	.9422
		9236	.9115	.9276	.9309			.9781	.9781	.9904	.9968
		.9248	.9124	.9286	.9328			.9781	.9871	.9905	.9969
		.9477	.9360	**.9504**	**.9532**			.9999	.9994	1.000	.9999
.3	.5	.8895	.8827	.8865	.8973	.05	.01	.8329	.8535	.9432	.9208
		.9205	.9374	.9629	.9639			.8641	.8761	.9345	.9435
		.9021	.9050	.9202	.9211			.8204	.8266	.8554	.8091
		.9245	9247	.9197	.9315			.9697	.9856	.9998	.9999
		.9256	.9257	.9192	.9332			.9603	.9747	.9928	.9890
		.9332	.9358	.9332	.9431			.9991	1.000	1.000	.9993

Table 18: $J_1 = J_2 = 25$, $n_{ij} = 10$ $\forall_{i,j}$, $\alpha = .05$, unequal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

		$\rho_1 \mid \rho_2$						$\rho_1 \mid \rho_2$			
p_1	p_2	.1|.3	.1|.5	.1|.9	.3|.9	p_1	p_2	.1|.3	.1|.5	.1|.9	.3|.9
.1	.1	.9004	.8774	.9079	.9134	.3	.7	.9057	.9124	.9191	.9166
		.9023	.8988	.9322	.9191			.9237	.9413	.9691	.9673
		.8966	.8733	.9222	.9116			.9126	.9229	.9354	.9331
		.9393	.9231	.9361	**.9529**			.9424	.9367	.9385	.9425
		.9392	.9230	.9363	**.9529**			.9435	.9371	.9417	.9429
		.9973	.9956	.9935	.9984			.9446	.9386	.9433	.9448
.1	.3	.9138	.9145	.9139	.9194	.5	.5	.9218	.9183	.9278	.9266
		.9164	.9224	.9298	.9326			.9260	.9256	.9368	.9370
		.9153	.9204	.9274	.9303			.9258	.9256	.9368	.9369
		.9393	.9438	.9443	**.9500**			.9397	.9394	.9421	.9443
		.9356	.9402	.9365	.9429			.9397	.9397	.9433	.9451
		.9783	.9822	.9819	.9831			.9405	.9415	.9455	.9471
.1	.7	.9045	.9027	.9164	.9079	.9	.9	.9000	.8812	.9110	.9177
		.9508	.9571	.9634	.9650			.9007	.8872	.9244	.9210
		.9118	.9206	.9325	.9246			.8949	.8805	.9187	.9131
		.9360	.9395	.9439	.9387			.9394	.9247	.9401	**.9540**
		.9352	.9378	.9390	.9363			.9392	.9246	.9400	**.9540**
		.9453	**.9481**	**.9514**	**.9483**			.9979	.9962	.9939	.9983
.1	.9	.8896	.8653	.9036	.9080	.9	.01	.9042	.9117	.9428	.9444
		.9531	**.9459**	**.9537**	.9642			.9350	.9029	.9026	.8805
		.9020	.8894	.9187	.9242			.9140	.8995	.8874	.8700
		.9302	.9209	.9220	.9337			.9391	.9291	.9132	.9007
		.9334	.9247	.9324	**.9458**			.9381	.9332	.9379	.9237
		.9341	.9261	.9347	**.9486**			.9453	.9391	.9433	.9322
.3	.3	.9211	.9082	.9213	.9205	.05	.05	.8846	.8609	.9101	.9303
		.9257	.9161	.9347	.9310			.9052	.8844	.9207	.9216
		.9243	.9148	.9336	.9294			.8820	.8601	.9096	.9062
		.9418	.9347	.9391	.9402			.9437	.9244	.9578	.9808
		.9425	.9354	.9399	.9414			.9431	.9239	.9578	.9805
		.9640	.9581	.9614	.9614			.9999	1.000	.9997	1.000
.3	.5	.9199	.9167	.9245	.9226	.05	.01	.8901	.9072	.9467	.9278
		.9239	.9251	.9432	**.9513**			.9077	.9021	.9331	.9005
		.9239	.9250	.9375	.9337			.8782	.8739	.8926	.8532
		.9403	.9364	.9409	.9445			.9745	.9814	.9952	.9904
		.9398	.9375	.9409	.9442			.9504	.9629	.9901	.9797
		.9460	.9442	**.9493**	**.9522**			.9991	.9998	1.000	1.000

Table 19: $J_1 = 5$, $J_2 = 10$, $n_{ij} = 5$ $\forall_{i,j}$, $\alpha = .05$, unequal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

		|	|	|	|			|	|	|	|
		$\rho_1 \mid \rho_2$						$\rho_1 \mid \rho_2$			
p₁	p₂	.1 | .3	.1 | .5	.1 | .9	.3 | .9	p₁	p₂	.1 | .3	.1 | .5	.1 | .9	.3 | .9
.1	.1	.8335	.8788	.9209	.9250	.3	.7	.8259	.8283	.8571	.8332
		.9279	.9376	.9636	.9659			.9335	.9219	.9436	.9343
		.9047	.9157	**.9516**	**.9490**			.8605	.8643	.8942	.8800
		.8040	.8195	.8688	.8631			.9587	.9942	.9827	.9726
		.8931	.8963	.9339	.9559			.9697	.9977	.9881	.9797
		.9983	.9965	.9956	.9951			.9998	.9986	.9911	.9828
.1	.3	.8690	.8740	.9001	.9032	.5	.5	.8449	.8544	.8705	.8513
		.9237	.9238	.9600	.9609			.9125	.9283	.9439	.9364
		.8938	.8907	.9240	.9228			.8654	.8761	.8937	.8801
		.9384	.8994	.9451	.9413			.9973	.9964	.9900	.9784
		.9311	.9120	**.9519**	**.9482**			.9984	.9979	.9917	.9825
		.9830	.9568	.9631	.9617			.9991	.9984	.9924	.9841
.1	.7	.8412	.8519	.8764	.8754	.9	.9	.8342	.8801	.9394	.9399
		.9489	**.9499**	.9660	.9713			.9329	.9383	.9822	.9817
		.8913	.9006	.9253	.9204			.9119	.9098	**.9490**	**.9530**
		.9911	.9831	.9662	**.9506**			1.000	.9997	.9998	.9999
		.9943	.9876	.9721	.9596			.9997	1.000	1.000	1.000
		.9951	.9887	.9769	.9608			1.000	1.000	.9999	1.000
.1	.9	.7856	.8016	.9222	.9021	.9	.01	.7431	.7826	.9103	.9006
		.9422	**.9466**	.9747	.9773			.9175	.9073	.8945	.8969
		.8863	.8978	.9112	.9048			.9172	.9070	.8541	.8564
		.9706	.9574	.9191	.9025			.9876	.9779	**.9459**	.8891
		.9872	.9812	.9664	.9570			.9991	.9997	.9989	.9917
		.9899	.9847	.9747	.9651			.9997	1.000	.9992	.9931
.3	.3	.8131	.8212	.8567	.8253	.05	.05	.9195	.9096	.9088	.9620
		.8830	.8913	.9293	.9142			.9643	.9671	.9855	.9890
		.8522	.8566	.8912	.8703			.9627	.9659	.9715	.9825
		.9664	**.9494**	**.9505**	.9252			.8153	.8660	.8783	.8717
		.9838	.9764	.9720	.9568			.9138	.9201	.9745	.9662
		.9941	.9919	.9881	.9752			.9998	.9999	.9988	.9995
.3	.5	.8423	.8353	.8640	.8409	.05	.01	.6411	.5988	.8685	.8649
		.9144	.9217	.9426	.9419			.9710	.9808	.9857	.9968
		.8563	.8675	.8653	.8764			.9664	.9799	.9760	.9875
		.9912	.9834	.9711	.9599			.9360	.9102	.9286	.9345
		.9973	.9921	.9764	.9689			.9564	.9317	**.9519**	.9601
		.9990	.9969	.9827	.9749			.9999	.9999	1.000	.9998

Confidence Levels when nominal level is .95

Table 20: $J_1 = 5$, $J_2 = 10$, $n_{ij} = 10$ $\forall_{i,j}$, $\alpha = .05$, unequal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

		\multicolumn{4}{c}{$\rho_1 \mid \rho_2$}			\multicolumn{4}{c}{$\rho_1 \mid \rho_2$}
p₁	p₂	.1 | .3	.1 | .5	.1 | .9	.3 | .9	p₁	p₂	.1 | .3	.1 | .5	.1 | .9	.3 | .9
.1	.1	.8097	.8891	.9049	.9197	.3	.7	.8039	.8426	.9004	.8732
		.9217	.9408	.9449	**.9529**			.9156	.9305	.9358	.9111
		.8945	.9097	.9267	.9397			.8905	.8547	.8785	.8904
		.8197	.8401	.8891	.8931			.9847	.9872	.9914	.9836
		.8883	.8964	.9108	.9572			.9891	.9907	.9801	.9714
		.9794	.9897	.9956	.9847			.9937	.9896	.9884	.9697
.1	.3	.8412	.8397	.9088	.9132	.5	.5	.8056	.8697	.8335	.8744
		.9304	.9115	.9397	.9406			.9068	.9354	**.9541**	.9168
		.8789	.8882	.9315	.9197			.8854	.8487	.8952	.8701
		.9114	.8992	.9335	**.9501**			.9897	.9832	.9845	.9644
		.9291	.9138	.9419	.9551			.9901	.9697	.9814	.9687
		.9820	.9398	.9551	.9697			.9951	.9891	.9794	.9831
.1	.7	.8298	.8497	.8357	.8733	.9	.9	.8497	.8860	.9057	.9322
		.9301	.9222	.9522	.9715			.9187	.9401	.9687	.9698
		.8887	.8853	.9239	.9206			.9057	.9187	.9367	**.9492**
		.9872	.9687	.9397	**.9492**			.9881	.9797	.9802	.9887
		.9913	.9836	.9804	**.9519**			.9797	.9854	.9714	.9765
		.9867	.9901	.9711	.9622			.9927	.9967	.9924	.9895
.1	.9	.7844	.8168	.9247	.9091	.9	.01	.6980	.7452	.9087	.9005
		.9437	.9308	**.9489**	.9793			.9254	.9157	.9047	.9152
		.8792	.8933	.9003	.9048			.9136	.8875	.8537	.8764
		.9635	.9397	.9187	.9029			.9768	.9605	**.9466**	.8991
		.9771	.9791	**.9501**	**.9536**			.9751	.9802	.9889	.9677
		.9813	.9833	.9614	.9568			.9967	.9956	.9992	.9931
.3	.3	.8221	.8143	.8332	.8915	.05	.05	.8795	.9114	.9044	.9367
		.8903	.8884	.9193	.9202			**.9544**	.9254	.9766	.9632
		.8511	.8357	.9102	.8936			.9637	.9659	.9735	.9825
		.9594	**.9463**	.9444	.9301			.8687	.8587	.8744	.8681
		.9774	.9739	.9797	**.9515**			.9223	.9201	.9685	.9409
		.9891	.9839	.9723	.9803			.9918	.9979	.9991	.9939
.3	.5	.8342	.8442	.8750	.8690	.05	.01	.6977	.6087	.7958	.8857
		.9057	.9269	.9325	.9502			.9635	.9754	**.9544**	.9788
		.8618	.8897	.8435	.8803			**.9456**	.9698	.9587	.9724
		.9815	.9874	.9657	.9387			.9540	.9305	.9198	.9287
		.9849	.9867	.9763	.9701			.9601	.9442	.9657	.9722
		.9981	.9895	.9838	.9711			.9924	.9839	.9885	.9915

Confidence Levels when nominal level is .95

Table 21: $J_1 = 5$, $J_2 = 25$, $n_{ij} = 10 \ \forall_{i,j}$, $\alpha = .05$, unequal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

		$\rho_1 \mid \rho_2$						$\rho_1 \mid \rho_2$			
p_1	p_2	.1 | .3	.1 | .5	.1 | .9	.3 | .9	p_1	p_2	.1 | .3	.1 | .5	.1 | .9	.3 | .9
.1	.1	.7154	.6987	.8457	.8825	.3	.7	.7684	8056	.8319	.7990
		.9245	.9364	.9455	.9411			.9005	.9087	.9133	.8840
		.9058	.9221	.9112	.9028			.8791	.8447	.8459	.8126
		.8856	.8456	.9002	.8956			.9864	.9878	.9758	.9566
		.8756	.9011	.9354	.9333			.9865	.9922	.9787	.9478
		.9845	.9914	.9992	.9963			.9902	.9931	.9802	.9565
.1	.3	.8474	.8952	.8732	.9014	.5	.5	.8303	.8109	.8265	.8426
		.9154	.9256	.9668	**.9458**			.8755	.9057	.9009	.9125
		.9054	.9151	.9238	.9123			.8653	.8789	.8852	.8353
		.9296	.9333	.9302	.9254			.9804	.9856	.9884	.9717
		.9502	.9552	.9392	.9546			.9955	.9905	.9882	.9716
		.9933	.9865	.9834	.9918			.9937	.9929	.9940	.9750
.1	.7	.8444	.8504	.8733	.8698	.9	.9	.8560	.8511	.8297	.7159
		.9457	.9367	.9588	.9699			.9318	**.9528**	.9405	**.9495**
		.9064	.9096	.9308	.9002			.9167	.9402	.9432	.9319
		.9906	.9702	.9552	.9388			.9877	.9892	.9972	.9915
		.9923	.9833	.9661	.9405			.9997	1.000	.9991	1.000
		.9915	.9897	.9559	.9605			.9990	.9968	.9999	1.000
.1	.9	.7498	.7351	.8118	.8156	.9	.01	.6894	.7158	.8057	.8820
		.9521	.9449	.9692	.9705			**.9496**	.9377	.9055	.9123
		.9111	.8952	.9132	.9244			.9401	.9323	.9106	.9025
		.9702	.9611	.9385	.9305			.9912	.9723	.9677	.9845
		.9598	.9934	.9753	**.9457**			.9969	.9955	.9991	.9958
		.9866	.9838	.9711	.9744			.9997	.9990	.9971	.9938
.3	.3	.7522	.8081	.8598	.7980	.05	.05	.7992	.8256	.9014	.9332
		.8657	.8864	.9004	.8967			.9556	**.9501**	.9619	**.9504**
		.8425	.8534	.8635	.8109			.9363	.9109	.9717	**.9518**
		.9552	**.9498**	.9661	.9442			.8921	.8897	.9005	.8952
		.9622	.9701	.9676	.9402			.9335	.9248	.9582	.9499
		.9821	.9805	.9749	.9754			.9719	.9856	.9918	.9861
.3	.5	.8057	.8954	.8657	.8144	.05	.01	.6695	.6874	.7018	.5887
		.9001	.9231	.9154	.9111			.9612	.9569	.9610	.9591
		.8854	.8608	.8562	.8678			.9159	.9358	**.9501**	.9555
		.9687	.9715	.9633	.9401			.9280	.9390	.9669	.9618
		.9826	.9831	.9885	.9568			.9818	.9836	.9907	.9922
		.9793	.9790	.9687	.9493			.9969	.9946	.9991	.9987

Table 22: $J_1 = 50$, $J_2 = 50$, $n_{ij} = 10$ $\forall_{i,j}$, $\alpha = .05$, equal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

Confidence Levels when nominal level is .95											
		ρ						ρ			
p_1	p_2	.1	.3	.5	.9	p_1	p_2	.1	.3	.5	.9
.1	.1	.9067	.8942	.9065	.9116	.3	.7	.9267	.9257	.9312	.9302
		.9090	.9113	.9289	**.9456**			.9305	.9308	.9403	.9406
		.9055	.8962	.9247	.9173			.9305	.9307	.9403	.9369
		.9428	.9340	.9305	.9399			**.9457**	**.9464**	.9443	**.9466**
		.9428	.9342	.9304	.9396			.9448	**.9467**	.9438	**.9470**
		.9978	.9965	.9942	.9979			.9451	**.9468**	.9455	**.9488**
.1	.3	.9239	.9176	.9271	**.9506**	.5	.5	.9331	.9329	.9332	.9348
		.9467	.9239	.9354	**.9457**			.9345	.9362	.9400	**.9482**
		.9362	.9230	.9344	**.9461**			.9432	.9362	.9399	.9402
		.9458	**.9466**	**.9473**	.9505			.9435	**.9466**	.9544	**.9477**
		.9471	.9399	.9382	**.9468**			.9437	**.9469**	.9459	**.9483**
		.9814	.9839	.9827	.9861			.9510	**.9475**	**.9469**	**.9496**
.1	.7	.9216	.9185	.9292	.9234	.9	.9	.9110	.8961	.9089	.9128
		.9246	**.9477**	**.9542**	.9662			.9120	.8984	.9279	.9245
		.9235	.9272	**.9467**	.9321			.9082	.8944	.9216	.9180
		.9474	**.9459**	**.9479**	.9464			**.9459**	.9391	.9326	.9422
		.9461	.9443	**.9487**	**.9483**			**.9459**	.9390	.9324	.9422
		.9487	**.9524**	**.9542**	**.9523**			.9982	.9972	.9951	.9982
.1	.9	.9292	.9007	.8866	.9122	.9	.01	.9072	.9110	.9141	**.9458**
		.9662	**.9482**	**.9483**	.9541			.9181	.9185	**.9543**	.9096
		.9487	.9116	.9014	.9310			.9180	.9153	.9014	.9403
		.9459	.9399	.9350	.9354			.9451	**.9464**	.9441	**.9476**
		.9477	.9410	.9372	.9406			.9436	.9452	**.9528**	**.9530**
		.9542	**.9462**	.9380	.9410			.9487	.9498	.9575	**.9531**
.3	.3	.9279	.9240	.9320	**.9475**	.05	.05	.8894	.8641	.9064	.9108
		.9315	.9265	.9391	**.9464**			.9006	.8916	.9205	.9179
		.9310	.9260	.9381	.9421			.8845	.8634	.9138	.9066
		.9442	.9421	.9435	**.9483**			.9417	.9278	.9408	.9586
		.9442	.9423	.9440	**.9488**			.9412	.9274	.9406	.9585
		.9633	.9605	.9655	.9667			.9998	.9997	.9999	1.000
.3	.5	.9336	.9329	.9364	.9342	.05	.01	.9160	.9218	**.9543**	.9199
		.9467	.9351	**.9480**	.9404			.9137	.9137	**.9460**	.8911
		.9366	.9350	**.9459**	.9402			.9026	.8986	.9049	.8433
		.9436	**.9462**	**.9486**	.9481			.9630	.9720	.9937	.9895
		.9439	**.9459**	**.9473**	.9480			**.9519**	.9631	.9897	.9734
		.9495	**.9509**	**.9540**	**.9543**			1.000	1.000	.9999	.9992

Table 23: $J_1 = 150$, $J_2 = 150$, $n_{ij} = 15 \quad \forall_{i,j}$, $\alpha = .05$, equal correlation

Row 1 – Wald interval (usual); Row 2 – Jeffreys-Perks interval; Row 3 – Haldane interval; Row 4 – extended Newcombe interval; Row 5 – extended Beal interval; Row 6 – extended Peskun interval

Confidence Levels when nominal level is .95											
		ρ						ρ			
p_1	p_2	.1	.3	.5	.9	p_1	p_2	.1	.3	.5	.9
.1	.1	.9408	.9052	.9010	**.9456**	.3	.7	**.9480**	.9392	.9392	.9400
		.9458	**.9268**	.9015	**.9456**			.9392	**.9484**	**.9464**	**.9462**
		.9345	.9254	.8971	.9392			.9380	.9400	.9414	.9432
		.9552	**.9508**	**.9500**	.9540			.9452	**.9464**	**.9516**	**.9496**
		.9552	**.9508**	**.9500**	.9536			**.9464**	**.9456**	**.9508**	**.9480**
		.9988	.9996	.9999	.9984			**.9468**	**.9456**	**.9517**	**.9480**
.1	.3	.9388	.9244	.9352	.9336	.5	.5	.9468	**.9472**	.9408	**.9512**
		.9376	.9252	.9376	.9348			**.9472**	**.9472**	**.9460**	**.9524**
		.9372	.9256	.9314	.9340			**.9464**	**.9462**	.9422	**.9526**
		.9520	**.9456**	**.9464**	**.9476**			**.9496**	**.9520**	.9448	**.9532**
		.9520	**.9462**	**.9460**	**.9468**			**.9496**	**.9524**	**.9458**	**.9532**
		.9872	.9836	.9892	.9840			**.9498**	**.9523**	.9452	**.9536**
.1	.7	.9396	.9308	.9196	.9408	.9	.9	.9260	.9005	.9051	.9280
		.9480	.9372	.9352	**.9468**			.9244	.9087	.9112	.9316
		.9372	.9276	.9252	**.9478**			.9204	.9005	.9007	.9280
		.9476	.9448	.9448	**.9504**			**.9500**	.9400	**.9476**	**.9536**
		.9480	**.9460**	**.9460**	**.9524**			**.9500**	.9400	**.9476**	.9566
		.9536	**.9552**	.9440	.9608			.9988	.9984	.9988	.9999
.1	.9	.9454	.8928	.9540	.9208	.9	.01	.9224	.9012	.8980	.9280
		.9352	.9264	.9244	.9652			.9256	.9122	.9004	**.9536**
		.9348	.9064	.9264	.9308			.9236	.9022	.8997	.9420
		.9532	**.9516**	.9560	**.9540**			**.9516**	.9444	**.9492**	**.9484**
		.9504	**.9496**	**.9536**	**.9544**			**.9504**	.9404	**.9456**	**.9472**
		.9504	**.9500**	**.9536**	**.9544**			**.9540**	.9452	**.9496**	**.9512**
.3	.3	**.9470**	.9364	.9344	.9344	.05	.05	.9196	.9015	.9304	.9116
		.9478	**.9480**	.9340	**.9460**			.9136	.9112	**.9508**	**.9470**
		.9462	.9372	.9344	**.9495**			.9128	.9006	.9292	.9118
		.9460	**.9460**	**.9498**	.9412			**.9528**	**.9520**	**.9544**	**.9468**
		.9460	**.9460**	**.9484**	.9412			.9258	**.9520**	**.9536**	**.9490**
		.9636	.9672	.9680	.9620			.9999	.9997	.9991	1.000
.3	.5	**.9544**	.9380	**.9496**	**.9472**	.05	.01	.9374	.8905	.8576	.9076
		.9526	**.9476**	**.9516**	**.9512**			**.9509**	.9087	.9152	**.9468**
		.9506	.9372	**.9516**	**.9508**			.9392	.9011	.9053	.9144
		.9564	**.9492**	**.9532**	**.9536**			**.9544**	.9476	.9600	.9676
		.9560	**.9496**	**.9532**	**.9532**			**.9537**	.9448	**.9544**	.9492
		.9609	.9548	.9584	.9564			1.000	.9999	.9992	1.000

VITA

Timothy J. Maharry

Candidate for the Degree of

Doctor of Philosophy

Thesis: PROPORTION DIFFERENCES USING THE BETA-BINOMIAL
DISTRIBUTION

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Biographical:

Education:

Received a Bachelor of Arts degree in Mathematics from Hastings College,
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ABSTRACT

Name: Timothy J. Maharry   Date of Degree: December, 2006

Institution: Oklahoma State University   Location: Stillwater, Oklahoma

Title of Study: PROPORTION DIFFERENCES USING THE BETA-BINOMIAL DISTRIBUTION

Pages in Study: 137   Candidate for the Degree of Doctor of Philosophy

Major Field: Statistics

Scope and Method of Study: When estimating the difference between two proportions with overdispersion due to correlation within the trials, the usual asymptotic confidence interval based on the maximum likelihood estimators generally has lower than desired coverage rates for small sample sizes. Consequently, the purpose of this study is to construct confidence intervals in this setting that exhibit near nominal coverage even for small sample sizes. The beta-binomial model is one possible way to model correlated 0-1 data. This model is used to develop two new intervals, referred to as the Haldane and Jeffreys-Perks intervals. The paper then compares these two new intervals with four existing competitors and evaluates their performance via simulations.

Findings and Conclusions: The usual asymptotic interval based on the maximum likelihood estimators is discouraged for cases when the sample sizes are small or the proportions are close to zero or one. The Haldane interval is an improvement over the usual interval but still has many cases with less than desireable coverage probability. The Jeffreys-Perks interval provides significant improvement over the usual interval as do the existing intervals referred to as the extended Beal, extended Newcombe, and extended Peskun intervals. In particular, the Jeffreys-Perks interval is generally the best choice in terms of coverage probability for cases where the difference between the proportions is large.   In specific cases when the two proportions are equal, or close to equal, the extended Newcombe and extended Beal generally have the best results. In many other cases, the extended Newcombe, extended Beal, and the Jeffreys-Perks intervals provide very similar results.

Advisor's Approval: ___   Dr. Melinda H. McCann_____ _____