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EXPERT CONCEPTUALIZATIONS OF THE CONVERGENCE OF  
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YESTERDAY, TODAY, AND TOMORROW

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EXPERT CONCEPTUALIZATIONS OF THE CONVERGENCE OF  
TAYLOR SERIES  
YESTERDAY, TODAY, AND TOMORROW

A DISSERTATION APPROVED FOR THE  
DEPARTMENT OF MATHEMATICS

BY

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## ABSTRACT

Taylor series is a topic briefly covered in most university calculus sequences. In many cases it constitutes only one or two sections of a calculus textbook. With this limited exposure, what do calculus students really understand about the convergence of Taylor series? Do they think of Taylor series convergence as a sequence of converging polynomials? Do they think of convergence as a remainder going to zero? Do they think the Taylor series for sine really “equals” sine, or is it merely a good estimation for sine? Furthermore, how might experts respond to these questions?

This study reported qualitative research methods which utilized multiple phases of data collection consisting of questionnaires and interviews from expert and novice (undergraduate student) participant groups. In addition, this study utilized multiple layers of analysis incorporating methods such as Strauss and Corbin’s open coding and Sfard’s discourse analysis. Using Tall and Vinner’s notion of concept images, I analyzed and described the different ways in which both experts and novices conceptualized the convergence of Taylor series. The focus of this study was on identifying and categorizing particular types of conceptualizations and on how these conceptions may differ depending on the extent of a participant’s mathematical background. In so doing, commonalities and differences amongst the expert and novice participant groups emerged. In addition, a secondary focus was to gain insights into what might be influencing particular types of knowledge.

The main result from this study was found in the descriptions of thirteen different concept images that experts and novices employed concerning the convergence of Taylor series. Some of these images were used more than others by the different participant groups, and some images appeared to date back to the early years of calculus. Even though both groups employed a variety of images, on an individual level, experts were more prone to use a wider range of images that they efficiently and effectively employed as different situations prompted. The most notable difference between experts and novices was found in their graphical images of Taylor series convergence. Experts demonstrated little to no difficulties interpreting graphs of Taylor series, but the vast majority of novices were unable to correctly produce graphs related to Taylor series convergence. In several cases, novices appeared to be incorrectly applying previous knowledge of graphical properties of translating functions in an attempt to build their conceptions of Taylor polynomial graphs. This finding has implications for future research into the effects of graphical images, both dynamic and static, on student conceptions of the convergence of Taylor series.

## CHAPTER 1

### Introduction

Throughout the history of calculus many things have influenced its development into one of the greatest achievements of humankind. Few things have influenced calculus' development as much as Taylor series. Isaac Newton, Gottfried Leibniz, Brook Taylor, Leonhard Euler, Joseph Louis Lagrange, and Augustin-Louis Cauchy are just a few of the great mathematicians who made use of Taylor series in their calculus. Lagrange even went as far as to attempt to make Taylor series the fundamental building block upon which all of calculus was built (Burton, 2007; Grabiner, 1981). Over the years, people's understanding of the magnitude of the influence of Taylor series to the calculus has been lost. The advance of computers has helped to both uncover the importance of Taylor series estimation techniques and to subdue their significance. So, how do today's students - tomorrow's experts - comprehend the convergence of Taylor series? Do they understand how Taylor series may be used to estimate functions such as sine, cosine, arctangent and  $e^x$ ? Do they think about convergence of Taylor series graphically or algebraically? What does it mean to them when they see a function equaling its Taylor series? It was questions like these that provided the thrust for this research into how different people comprehend the convergence of Taylor series.

Taylor series are seen by many students for the first time within a university calculus sequence. In many traditional calculus textbooks, Taylor series may constitute

four, or fewer, sections that usually follow a quick foray into sequences and basic series results (Anton, Bivens, & Stephen, 2009; Hass, Weir, & Thomas, 2007; Larson, Edwards, & Hostetler, 2005; Stewart, 2008). Prior to the students' introduction to Taylor series, these texts first address the question of series convergence using basic tests such as the ratio and root tests.

Following this, students are introduced to either power series (Hass et al., 2007; Stewart, 2008) or Maclaurin and Taylor polynomials (Anton et al., 2009; Larson et al., 2005). Convergence issues are again discussed, but this time in the context of determining an interval of convergence. Later the geometric series is used to express a fraction as a power series, and the expansion for natural log quickly follows. After an introduction to the formulas for Taylor and Maclaurin series, convergence issues of Taylor series are brought up using a remainder formula. To answer convergence questions about Taylor series, a version of Lagrange's remainder formula may be used to produce upper bounds for the remainder. Stewart (2008) choose to use what he called Taylor's inequality while other textbooks, like Hass et al. (2007), might choose to show Taylor's formula in more generality. Using Lagrange's formula, students might see the proof of why  $\sin x$  and  $e^x$  are really equal to their Taylor series representations. Students may or may not see an example of a function that does not have a Taylor series expansion, Stewart (2008) hides this example in the exercises. After this introduction to Taylor series, application questions are posed and the Binomial series is quickly mentioned. Using Taylor series for estimation methods, and perhaps looking at some examples of how Taylor series are used in physics, might constitute the last time that students see Taylor series in their calculus classes.

With this limited exposure that calculus students have with Taylor series, what are they really retaining? To help answer this question, this study did not merely consider the understanding of students who have simply seen Taylor series for the first time, but it included students at varying points in their mathematical careers. Some students were a semester removed from the class in which they first saw Taylor series. Others were about to graduate with their Bachelor's degree. The novice participants in this study represented undergraduate students from calculus classes and advanced undergraduate mathematics classes. The students in the calculus classes had either recently covered Taylor series for the first time in their current classes or were a semester removed from seeing Taylor series. The students from the advanced undergraduate mathematics classes had been exposed to series results at least twice. This study will show that some of these students had a very elementary understanding and some had an understanding comparable to that of an expert. By looking at students with a wide range of mathematical experiences this study presented a better picture of student comprehension at different levels of mathematical maturity.

This study explored student understanding of the convergence of Taylor series, but it did not stop with students. What about expert comprehension? How do experts think about the convergence of Taylor series? What differences can be seen between expert understanding and novice understanding? What commonalities can be seen? If an instructional goal is to turn today's novices into tomorrow's experts, then instructors must also consider how today's experts think about the convergence of Taylor series. Understanding how experts think can give insight into steps potentially needed to help better prepare the experts of tomorrow. By considering expert conceptualizations, this

study became more than simply another study of student understanding of a concept from calculus. This study gave insights into both expert and novice conceptualizations of the convergence of Taylor series.

This study was not limited to merely looking at today's students and today's experts. Used by many early analysts, Taylor series has a very rich history in the development of calculus. Therefore, this study traced the history of Taylor series and how it was influenced by people, including James Gregory, Isaac Newton, Joseph Louis Lagrange, and Augustin-Louis Cauchy. Some of the writings of these great mathematicians addressing the concepts behind Taylor series have survived over the centuries. Therefore, historians are not only able to see what these great mathematicians have contributed to mathematics, but they can gain insights into how these great mathematicians may have thought about the mathematics with which they were working. Thus, this study also attempted to illuminate the understanding of yesterday's experts. How might the founders of calculus have understood Taylor series? In what ways did they use Taylor series? What questions did they ask about Taylor series? How does the understanding of yesterday's experts relate to the understanding of today's experts? What are the commonalities and differences between the different experts from different times? Therefore, this study attempted more than to simply compare and contrast the conceptualizations of different people from today, it brought together the understanding of yesterday's, today's, and tomorrow's experts. Thus, this was the study of expert conceptualizations of the convergence of Taylor series, yesterday, today, and tomorrow.

This chapter begins with an analysis of the complexities of the concept of convergence of Taylor series. This analysis revealed why people may struggle with

comprehending Taylor series by discussing multiple elements essential to Taylor series. How well individuals understand each of these intrinsic elements may positively or negatively influence their comprehension of Taylor series convergence. Following the discussion of the complexities of the convergence of Taylor series, the problem statement is presented, the significance of the problem is elaborated, and the research questions are detailed.

### *The Complexities of Taylor Series*

Taylor series brings together many ideas from calculus. They allow one to look at functions from a different viewpoint. Using this different viewpoint, one can differentiate and integrate the series as if it was an infinite polynomial and relate it back to the derivative and integral of the function from which the series was generated. For example, instead of defining sine using geometric methods,  $\sin x$  can be defined by its Taylor series. The series expansion for sine allows one to determine the derivatives and antiderivatives for sine. If derivatives and integrals can be defined using Taylor series, it is no wonder Lagrange attempted to make Taylor series the cornerstone of calculus. Unfortunately, Lagrange had not fully understood the issue of convergence that must be addressed when using Taylor series (Burton, 2007).

At the root of the concept of convergence of Taylor series is the limit concept. An individual's conception of the notion of limit may be their greatest ally or greatest enemy when learning about the convergence of Taylor series. On the flip side, learning about Taylor series convergence may aid individuals in confronting malformed notions of limit that had previously been relatively unchallenged. This confrontation could potentially cause one to build up better mental imagery of the concept of limit.



The limit concept is not the only concept influencing understanding of the convergence of Taylor series. Because Taylor series incorporates so many concepts from calculus, it may become very difficult for many students to fully comprehend what it means for a Taylor series to converge. Some of these concepts include: the variable concept, the function concept, and estimation techniques. It is the bringing together and the interaction between all the concepts related to Taylor series that adds extra layers of complexity. It is this complexity that may cause Taylor series to be difficult for many students to grasp. Therefore, students need to have a very firm foundation in these core calculus concepts prior to embarking on a study of Taylor series. If they don't have a firm foundation, they may be doomed for failure, or at the very least, have numerous challenges to overcome in order to develop a proper concept of the convergence of Taylor series.

#### *The Complexities of the Informal Limit Concept*

The complexity of the limit concept and the epistemological effects on student knowledge has been well documented over the last few decades (Cornu, 1991; Cottrill et al., 1996; Davis & Vinner, 1986; Monaghan, 1991; Oehrtman, 2002; Schwarzenberger & Tall, 1978; Tall & Vinner, 1981; Williams, 1991). Some of this understanding is affected by the very language that is used to describe limits (Davis & Vinner, 1986; Ferrini-Mundy & Graham, 1994; Lauten, Graham, & Ferrini-Mundy, 1994; Monaghan, 1991). In this study, students were sampled from both the University of Oklahoma and from a regional community college. These institutions use Stewart's (2008) calculus book and Hass et al.'s (2007) book, respectively. Both in Stewart's calculus book and in Hass et al.'s book, students are first exposed to an informal definition and later to the formal

definition of the limit. Consider the language used by Stewart (2008) when he informally defined the limit of a function for the first time.

We write

$$\lim_{x \rightarrow a} f(x) = L$$

and say “the limit of  $f(x)$ , as  $x$  approaches  $a$ , equals  $L$ ” if we can make the values of  $f(x)$  arbitrarily close to  $L$  (as close to  $L$  as we like) by taking  $x$  to be sufficiently close to  $a$  (on either side of  $a$ ) but not equal to  $a$  (Stewart, 2008, p. 66, quotes in original).

Notice how Stewart used the words “limit,” “approaches,” and “close,” even “arbitrarily” and “sufficiently close,” in his informal definition of the limit. In Hass et al., a similar informal definition appeared in which they use the words “limit,” “arbitrarily” and “sufficiently close,” and “approaches” (Hass et al., 2007, p. 62). “Approaches” and “close” have meaning, not just in calculus but in everyday language. Many times the everyday meanings can be in conflict with the formal mathematical meaning, which might lead a student to create metaphors that are inconsistent with the mathematical definition (Davis & Vinner, 1986). For example, “limit” in everyday language commonly refers to something that one can approach but not go over, consider the “speed limit” for example (Davis & Vinner, 1986, p. 299). Many times when one uses the word “approach,” implied within the context is that of “never reaching.” In addition, when one approaches something in the physical sense, one is moving toward that thing. Therefore, the use of the word “approach” may inadvertently direct students into a dynamic image of the limit concept. Stewart (2008) continued with this dynamic metaphor when, directly after the informal definition for the limit when he said the following:

Roughly speaking, this says that the values of  $f(x)$  tend to get closer and closer to the number  $L$  as  $x$  gets closer and closer to the number  $a$  (from either side of  $a$ ) but  $x \neq a$  (p. 66).

Not only does “closer and closer” imply movement, but what does “close” mean? When used in everyday language, “close” is a relative term. Hass et al. explained:

This definition is “informal” [referring to their informal definition of limit similar to Stewart’s seen above] because phrases like *arbitrarily close* and *sufficiently close* are imprecise; their meaning depends on the context. To a machinist manufacturing a piston, *close* may mean *within a few thousandths of an inch*. To an astronomer studying distant galaxies, *close* may mean *within a few thousand light-years*. (Hass et al., 2007, p. 62, italics in original)

So when is something “sufficiently close”?

### *The Complexities of the Formal Definition for Limit*

Understanding the formal  $\delta - \varepsilon$  definition is essential for students to advance in more formal analysis training. These definitions allow mathematicians to rigorously establish what “close” really means (at least what it really means for mathematicians). Yet, the formal  $\delta - \varepsilon$  definition can cause many problems for students. For example, in Stewart’s calculus textbook, the formal definition of the limit of a function is given by the following:

Let  $f$  be a function defined on some open interval that contains the number  $a$ , except possibly at  $a$  itself. Then we say that the **limit of  $f$  as  $x$  approaches  $a$  is  $L$** , and we write

$$\lim_{x \rightarrow a} f(x) = L$$

if for every number  $\varepsilon > 0$  there is a number  $\delta > 0$  such that

$$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \varepsilon .$$

(Stewart, 2008, p. 88, bold in original)

By again using the word “approaches,” the language that he used in the formal definition may help perpetuate dynamic images that students may have formed when learning the informal definition. In addition to continuing the dynamic metaphor, the use of the “for all” and “there exists” and the order in which they appear in the formal definition have been noted to cause cognitive obstacles to comprehending the formal definition (Alcock

& Simpson, 2004, 2005; Cottrill et al., 1996; Davis & Vinner, 1986; Dubinsky & Yiparaki, 2000; Fernandez, 2004; Tall & Vinner, 1981). Furthermore, the implication of the “if-then” statement must be understood before one can fully grasp the formal definition. Even the absolute values can cause problems. Based on my own personal experience, I have noticed that many students may read the “ $0 < |x - a| < \delta$ ” literally as the “absolute value of  $x$  minus  $a$ ,” and thus, not indicate a mental connection to distance. In addition, as Bergthold (1999) pointed out, the  $L$  in the formal definition can create difficulties because the definition does not give the student the candidate for  $L$ . Therefore, one may tend to rely on more informal techniques to find the candidate for  $L$ , thereby reinforcing the informal mental conceptions. Along these same lines, student understanding of the formal  $\delta - \epsilon$  definition has been related to their ability to reverse their thinking (Davis & Vinner, 1986; Kidron & Zehavi, 2002; Roh, 2005, 2008). When using the definition, one started with an epsilon, but when using informal techniques, it was the delta that was first manipulated. For example, consider Hass et al.’s (2007) explanation of the limit of a particular function found in Table 1.

Table 1  
*Calculating Numerical Limits*

The closer $x$ gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2		
Values of $x$ below and above 1	$f(x) = \frac{x^2 - 1}{x - 1} = x + 1,$	$x \neq 1$
0.9	1.9	
1.1	2.1	
0.99	1.99	
1.01	2.01	
0.999	1.999	
1.001	2.001	
0.999999	1.999999	
1.000001	2.000001	

\* From *University Calculus*, by Hass, J., Weir, M., and Thomas, G., 2007, p. 63, Cambridge, Massachusetts: MIT Press.

In Table 1, notice how Hass et al. used the “close” metaphor and the way they first note how  $x$  gets close to 1 followed by how  $y$  gets close to 2. In essence, Hass et al. reversed the roles of epsilon and delta in their informal explanation of this limit. Furthermore, Table 1 illustrates how the notions of limit oft rely upon tables and graphs (Bergthold, 1999). Unfortunately, tables and graphs may be badly constructed, and thus, lead students to incorrect conclusions. With informal metaphors relied upon so heavily, even by the experts like Stewart and Hass et al., it may contribute to many students struggling to even see a need for a formal  $\delta - \epsilon$  definition of limit (Cottrill et al., 1996).

### *The Complexities of the Limit Concept Applied to Taylor Series*

The previous paragraphs have discussed the limit concept in the context of real valued functions. Yet the limit concept of a real valued function is only indirectly related to the convergence of Taylor series. More directly related is the concept of the limit of sequences. In many cases, the same metaphors are used to describe the convergence of sequences as were used to describe the limits of functions (e.g., Oehrtman, 2002). These conceptions about the limit of functions discussed in the paragraphs above can carry over and affect conceptions of the convergence of sequences and eventually the concept of series convergence. This carry over may be perpetuated by the similarities between the  $\delta - \epsilon$  and  $\epsilon - N$  definitions for convergence and the use of real valued functions that equal given sequences on the set of nonnegative integers.

A layer of complexity is added when moving from questions about the convergence of sequences, to considering questions about series convergence. Based on my personal teaching experience, the fact that a sequence of positive termed partial sums can converge initially surprises many students. When looking at series convergence,

people have to differentiate between the sequence of partial sums and the sequence of terms from the series. Failure to make this distinction may have caused many novices and, as I will show in Chapter 4, some of yesterday's experts to conclude that term convergence to zero is a sufficient condition for series convergence. A valid sufficient condition for series convergence is that the sequence of remainders must converge to zero. I will show in Chapter 6 that the notion of remainder convergence to zero was sometimes incorrectly applied by participants. Textbooks like Stewart (2008) and Hass et al. (2007) use multiple sections to cover convergence tests like the integral, alternating series, ratio, and root tests but may use only one section to discuss Taylor series convergence. Thus, the definition for series convergence and the remainder method used to show convergence are sometimes overshadowed by the relative abundance of tests for basic series convergence. During this study I found that some students attempted convergence tests, like the ratio test, when asked about proving Taylor series convergence.

Another layer of complexity is added when moving from convergence of series to convergence of power series and Taylor series. Now the terms of the series are functions in terms of  $x$ , as are the partial sums. With the addition of the interval of convergence, the focus of questions about convergence change from "if" a series converges to "where" a series converges. Convergent Taylor series are not merely equal to isolated numbers; they are equal to functions whose domains are restricted to the interval on which the series converges.

The Lagrange formula for the remainder  $R_n(x)$  adds another layer of complexity. Consider Hass et al. (2007) on the next page:

If  $f$  has derivatives of all orders in an open interval  $I$  containing  $a$ , then for each positive integer  $n$  and for each  $x$  in  $I$ ,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \text{ for some } c \text{ between } a \text{ and } x \text{ (p. 560).}$$

The Lagrange remainder formula,  $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ , is used to

rigorously demonstrate convergence of Taylor series by providing a formula for the remainder. Thus, proving that this formula converges to zero is sufficient to prove that the remainder converges to zero, and hence, the Taylor series converges to its generating function. Understanding the Lagrange remainder formula is related to comprehending the role of the unknown “ $c$ ” in the formula. It has already been documented how the unknown “ $c$ ” adds to the complexity of Taylor series (Kidron, 2004). Not only do students have to interact with the variable  $x$ , but they also have to use this unknown “ $c$ ” in the  $(n+1)^{\text{st}}$  derivative of  $f$  to show that series converges for certain  $x$ . Being able to prove that a series converges, but not ever knowing exactly what the “ $c$ ” really is, causes students to question the validity of their results (Kidron, 2004).

To help eliminate this cognitive obstacle caused by the “ $c$ ,” both Stewart (2008) and Hass et al. (2007) introduced students to Taylor’s inequality:

$$R_n(x) \leq \frac{M}{(n+1)!}(x-a)^{n+1} \text{ where } M \text{ is an upper bound on the } (n+1)^{\text{st}} \text{ derivative of}$$

$f$ . This is still a complicated formula. Based on my experience, some novices may get bogged down in the process of finding the  $M$ . Even if they find the  $M$ , they may have difficulty recalling what they were going to do after they have

found it. This study will reveal that some do not see a need for the Lagrange remainder formula, perhaps because all the functions, or at least all but one function, that they see in their calculus classes equal their Taylor series expansions on their respective intervals of convergence. Assuming that the instructor emphasizes it, students might see the example of

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0, \\ 0 & x = 0 \end{cases},$$

which is  $C^\infty$  but is only equal to its Taylor series when  $x = 0$ . This one example may be overshadowed by an abundant supply of functions equaling their Taylor series on positive measure intervals of convergence. In addition, novice understanding of mathematical truth may cause students to completely disregard this isolated counterexample and therefore cause the counterexample to be ineffective in dislodging any strong held beliefs (Williams, 1991).

Interacting with notions of convergence of Taylor series are notions of variable and function. As the previous paragraph mentioned, students struggle with understanding the unknown variable “ $c$ .” But they can also struggle with understanding the role that  $x$  plays in the context of Taylor series. Poor notions of variable and function may facilitate an incomplete understanding of the concept of convergence of Taylor series. People with poor notions of variable and function might be able to manipulate Taylor series formulas to get correct answers, but have very little conceptual understanding of (White & Mitchelmore, 1996). In addition, people with poor notions of variable and function might fail to relate convergent power series to functions with domains equal to that of their corresponding intervals of convergence.



In addition to the notion of variable and function interacting with notions of Taylor series, one's concept of infinity can also influence the concept of convergence of Taylor series. Someone who views infinity as only a potentiality, not something that could be actually achieved, will have difficulty comprehending how limits can actually equal a definite object. Many times infinity is viewed as an unreachable process embedded in time (E. Fischbein, Tirosh, & Hess, 1979; Tall, 1980; Tirosh, 1991). Therefore, the idea that one can add up infinitely many things and achieve a sum in finite time goes against the potential notion of infinity (Cornu, 1991; E. Fischbein et al., 1979; Sierpiska, 1987; Tall & Vinner, 1981).

#### *The Complexities of Estimation Techniques Related to Taylor Series*

Following the introduction to Taylor series, both Stewart (2008) and Hess et al. (2007) address applications of Taylor series. Most notably they address the use of Taylor series to estimate functions like sine, cosine, arctangent,  $e^x$ , etc. All of the complexities of the convergence of Taylor series carry over when talking about estimation techniques using Taylor series. Even the Lagrange remainder term with its intricacies may be used to help answer estimation questions.

There are two common estimation questions that textbooks ask students to answer concerning Taylor series. (1) Students are asked to find an upper bound of the error associated with using a certain Taylor polynomial to approximate a function over a specified interval. (2) They are asked to find the appropriate Taylor polynomial that guarantees that the polynomial approximates the function within a given error over a specified interval. Both of these questions have distinct features that can cause different problems for students attempting their answers. They both rely on estimation theorems,

perhaps Lagrange's remainder formula or the alternating series estimation theorem. It is conceivable that a student needs to have a conceptual understanding of the relationship of the Taylor polynomials to the function that they are supposedly approximating to be able to fully comprehend these questions. Plus, to answer these questions, students must be able to reverse their focus. In (1), students were asked to find the error given a polynomial, but in (2), students were asked to find the polynomial given the error.

When addressing all of these changing variables, a student may have to rely upon a type of covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Oehrtman, Carlson, & Thompson, 2008).

We define *covariational reasoning* to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other (Carlson et al., 2002, p. 354, italics in original).

In the previous research, covariational reasoning was applied to student comprehension concerning functions. The two varying quantities were the  $x$  and the  $y$  from the domain and range, respectively. When  $x$  varies,  $y$  varies. In the context of Taylor series, many variables can change, and if a student has not developed a good sense of covariational reasoning then that student is posed to have a very difficult time comprehending how polynomial functions vary as  $n$  increases to approximate another function.

#### *Problem Statement*

The analyses above suggests that Taylor series is a very intricate system of overlapping concepts with added complexities over that of preceding concepts such as the limit and function concepts. The problem addressed in this study was to analyze and describe the different ways in which people, both novices and experts, conceptualize this complex topic, the convergence of Taylor series. The focus was on identifying and

categorizing particular types of conceptualizations and on how these conceptions may differ depending on the extent of a participant's mathematical background. A secondary focus was to gain insights into what might be influencing particular types of knowledge. These insights can provide an impetus for future research.

Since the concept of convergence of Taylor series is such an intricate system of overlapping conceptions with added complexities, analyzing a person's understanding of the convergence of Taylor series proved to be a very complicated endeavor. An instrument designed to access participant conceptualizations could not be limited to merely solving problems out of a calculus text like Stewart (2008) or Hass et al. (2007). These problems from the textbook might be solved by rote memorization of processes with little or no link to conceptual understanding (Hiebert & Carpenter, 1992; Vinner, 1997). Therefore, looking at a person's solutions to problems assigned from Stewart (2008) or Hass et al. (2007) would not give sufficient insight into that person's understanding of Taylor series. Thus, a combination of carefully created task-based questionnaires and interviews designed to elicit a person's conceptual understanding was used. Together these questionnaires and interviews provided a rich picture of expert and novice conceptions of this very complex topic of the convergence of Taylor series.

#### *Significance of the Problem*

While the understanding of the limit concept has been extensively researched (Cornu, 1991; Cottrill et al., 1996; Davis & Vinner, 1986; Monaghan, 1991; Schwarzenberger & Tall, 1978; Tall & Vinner, 1981; Williams, 1991), the concept of convergence specific to Taylor series is relatively unexplored territory. While Kidron and Zehavi (2002) specifically studied student understanding of Taylor series and the

Lagrange remainder theorem, most research that addresses Taylor series only does so when considering a broader topic such as limit or estimation (Alcock & Simpson, 2004, 2005; Kidron, 2004; Oehrtman, 2002). Even Kidron and Zehavi (2002) were primarily addressing the effect of animations on student understanding. Kidron and Zehavi (2002), Kidron (2004), and Oehrtman (2002) will be discussed in more detail in Chapter 2.

A precedent has been set for exploring and analyzing student understanding of the concept of limit using task-based questionnaires followed by interviews (Williams, 1991). Over the years, the information gained from questionnaires and interviews has successfully categorized and described student understanding using the idea of concept images as defined by Tall and Vinner (Tall & Vinner, 1981; Vinner, 1983, 1991).

We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures. (Tall & Vinner, 1981, p. 152, italics in original)

One of the reasons that concept image has lasted over the years is that it has proven to be a helpful theoretical framework for informing pedagogy (Mason, 2008). The role that concept image played in this study will be discussed in more detail in Chapter 2.

Most research using concept images considers the question of student understanding of a particular topic (e.g., R. C. Moore, 1994; Tall & Vinner, 1981; Vinner, 1983, 1991; Weber, Porter, & Housman, 2008). This study did not restrict its focus to just students. By analyzing the understanding of students at different stages in their mathematical careers and by investigating the understanding of experts, this study was able to get a very detailed picture of how the concept images of the convergence of

Taylor series might be “built up over the years through experiences of all kinds” (Tall & Vinner, 1981, p. 152).

The analysis and descriptions of concept images relied upon by experts and novices found within this study should provide a firm foundation for future research into this topic. Simply because little research has been conducted specific to the concept of convergence of Taylor series, this study can potentially make a substantial contribution to mathematics education research. Since an instructional goal is to turn novices into experts, seeing the differences between the two participant groups can yield pedagogical insights into how to better instruct novices so that they can become the next generation of experts.

#### *Research Questions*

1. *Images of the Convergence of Taylor Series.* When a function has a Taylor series expansion, in what ways do experts and novices think about convergence? What themes emerge from the expert and novice data? Williams (1991) categorized limit concept images as either “dynamic-theoretical,” “acting as a boundary,” “formal,” “unreachable,” or “acting as an approximation,” and “dynamic-practical” (p. 221). Do experts and novices view convergence of Taylor series using similar images? Do additional images appear? How are all the images characterized? How do they interact? What factors influence a good understanding of the convergence of Taylor series? What about estimation? Do experts and novices see how Taylor polynomials are useful for estimating functions?
2. *Graphical Images of the Convergence of Taylor Series.* When a function has a Taylor series expansion, in what ways do novices and experts relate the graph of the

- Taylor polynomials to the graph of the function? In particular, do they visually see the Taylor polynomials approximating the function on the interval of convergence, or is their concept of Taylor polynomial approximation completely disjoint from any visual references? Is re-centering included in their visual concept image?
3. *Making the Connection to Functions.* Do experts and novices relate Taylor series to functions? What are some ways in which experts and novices relate Taylor series to functions? For example, do they relate the interval of convergence to the domain of the series? Are there certain images that they use to reason about Taylor series that may be related to their function understanding?
  4. *Interval of Convergence (IOC) and Re-Centering.* Within the typical calculus sequence, the idea of the IOC is distinct to power series. Do experts and novices respond differently to problems with finite and infinite IOCs? Do they use re-centering as a tool for obtaining more accurate Taylor polynomial approximations for a particular  $x$ -value or range of  $x$ -values?
  5. *Various Other Effects.* Since Taylor series is influenced by several different mathematical concepts, the convergence of Taylor series can be expressed in many ways. For example, when discussing convergence one can use say that a single polynomial converges to the infinite series or that a sequence of Taylor polynomials converges to the infinite series (i.e. “ $1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$  converges to  $e^x$ ,” verses “ $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  converges to  $e^x$  where each  $T_n(x)$  is the corresponding  $n^{\text{th}}$  degree Taylor polynomial”). How might these different expressions effect participants’ conceptions? In Chapter 3 I will describe three

different possible effects and demonstrate how they were accounted for in more detail.

### *Chapter Abstracts*

This study attempted to dive into the minds of novices and experts to unpack the mental imagery, both visual and nonvisual, that they have concerning the convergence of Taylor series. Chapter 1 introduced literature that was relevant to the research questions. In Chapter 2, I examine some of this literature in more detail. In that chapter, I will discuss what research has been done relating to the convergence of Taylor series and examine how this research affected this study. In Chapter 3, I elaborate on the methodology employed in this study. Furthermore, Chapter 3 demonstrates how participants were selected and protected. It also presents a close look at the interview and questionnaire tasks and provides detail into how data was collected and analyzed.

Taylor series has greatly influenced the development of calculus and Chapter 4 contains a brief history of Taylor series. In that chapter, I will reveal how yesterday's experts used Taylor series and how they might have thought about the convergence of Taylor series. Chapters 5 and 6 contain both the results and discussion of the findings from this research study. Chapter 5 looks at how the current experts understood the convergence of Taylor series, while Chapter 6 considers today's novices, tomorrow's experts. I conclude this study in Chapter 7 by commenting on some of the commonalities and differences observed between novices and experts.

## CHAPTER 2

### Review of the Literature

In this chapter I will discuss prominent literature and see how that literature influenced the design of this study of expert and novice conceptualizations of convergence of Taylor series. As discussed in the previous chapter, there are many underlying complexities of Taylor series that influence an individual's understanding of its convergence. At their root, all of these complexities were related to the concept of limit. Therefore, I will begin this chapter with a brief literature review of the concept of limit. Since this study focuses on a specific application of the concept of limit, Taylor series, the thrust of this chapter will focus on only the main studies of limit that have incorporated Taylor series. Thus, following the section on the concept of limit, I will present an additional section on the concept of Taylor series. Following that section, I will conclude this chapter with a discussion of the theoretical framework that influenced the methods and procedures that were used to gather and analyze data for this study.

#### *The Concept of Limit*

The concept of limit has been extensively studied over the past three decades, and much of the literature has primarily focused on the concept of limit applied to the notion of functions (e.g., Lauten et al., 1994; Monaghan, 1991; Malgorzata Przenioslo, 2004; Williams, 1991). Until recently, almost all of the literature on the concept of limit concentrated on functions of one variable, but now, Fisher (2008) has developed three models of limit applied to multivariable functions. Another large chunk of limit literature



on the concept of limit has focused on sequences and series (e.g., Alcock & Simpson, 2004, 2005; Davis & Vinner, 1986; Mamona-Downs, 2001; M. Przenioslo, 2006; Roh, 2008; Tall & Vinner, 1981). Other pieces of literature have concentrated on applications of limits, such as derivatives (e.g., Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Hahkioniemi, 2006; Zandieh, 2000), continuity (e.g., Bezuidenhout, 2001; Tall & Vinner, 1981), and integration (e.g., Engelke & Sealey, 2009; Orton, 1983; Sealey, 2006; Sealey & Oehrtman, 2005, 2007). Bezuidenhout (2001) found that student understanding of some of these applications “rests largely upon isolated facts and procedures, and that their conceptual understanding of the relationships between these concepts is deficient” (p. 498, italics in original). In other words, students may tend to see all these concepts, such as continuity, differentiation, and integration, as disjoint entities that are not related to the notion of limit.

Other research has concentrated on instructional procedures to enhance student learning of the concept of limit (e.g., Cottrill et al., 1996; McDonald, Mathews, & Strobel, 2000; Oehrtman, 2004, 2008; Swinyard, 2009; Williams, 2001). Some have noted how the notion of infinity influences the conception of limit and vice versa (e.g., Cornu, 1991; Sierpinska, 1987; Tall, 2001). Still others have researched the role that technology, including animations, plays in student understanding of the limit concept (e.g., Cory, 2005, 2008; Kidron & Zehavi, 2002). Szydlik (2000) even showed how mathematical beliefs effect student conceptual understanding of the limit of function. In 1994, Gray and Tall coined the term procept to refer to an “amalgam of concept and process represented by the same symbol” (p. 121). Many of the studies already mentioned and some that I have yet to discuss, show that the notation used for limit,

whether  $\lim_{x \rightarrow \infty} f(x)$ ,  $\sum_{n=1}^{\infty} a_n$ , or  $\int_a^b f(x)dx$ , can evoke both a procedural conception, where limit is viewed as a process that may be performed in a dynamic fashion, and an object conception, where the resultant state of the limit is encapsulated in its entirety.

All of this literature mentioned above demonstrates how the concept of limit is a complicated notion that is not easily understood. Its various applications to continuity, derivatives, and integration add to the complexities of understanding this concept. Furthermore, not only is the concept of limit not easily understood by students, but as I learned, student understanding of this concept is not easily unpacked by researchers. Researchers, like Oehrtman, Roh, Sealey, and Swinyard continue to investigate the concept of limit applied to various applications from different epistemological and pedagogical theoretical perspectives so that the mathematical community may continue to grow in its understanding of how people comprehend this complex topic.

In the next section, I will focus this literature review on the concept of Taylor series. Instead of going into a detailed review of each of the aforementioned pieces of literature, I will refer to them as needed throughout this and the following chapters.

### *The Concept of Taylor Series*

Very little research has been conducted specific to student understanding of the concept of Taylor series. This was not only confirmed by the following literature review, it was also confirmed by other experts in the field (I. Kidron, personal communication, March 12, 2008; M. Oehrtman, personal communication, February 28, 2009; C. Rasmussen, personal communication, February 27, 2009). In most cases any study that addressed convergence of Taylor series did so vicariously while considering some broader topic, such as limits or function approximation techniques. This is certainly

understandable because of the intimate relationship between limits and function approximation with that of the convergence of Taylor series. To find previous work on student understanding of the convergence of Taylor series, I had to meticulously review literature on other topics and look for results specific to Taylor series. When researchers reported results concerning Taylor series, their reports may have only consisted of a paragraph while other researchers may have devoted entire sections of their reports to Taylor series. I found only one study that claimed to be entirely devoted to discovering student understanding of Taylor series convergence (Kidron & Zehavi, 2002).

One of the most notable studies of Taylor series was done by Kidron in 1999. A few years later, two papers followed: *The Role of Animation in Teaching the Limit Concept* (Kidron & Zehavi, 2002), and *Polynomial Approximation of Functions: Historical Perspective and New Tools* (Kidron, 2004). It is worth noting that the focus of these papers were limit and function approximation and not necessarily Taylor series. Even so, Taylor series and Taylor's formula both played major roles in this research. In addition to Kidron's research, Oehrtman (2002) and Alcock and Simpson (2004, 2005) have revealed interesting results regarding Taylor series while looking at other concepts. Within this section I will take a closer look at these studies.

*Kidron's Works.* In *The Role of Animation in Teaching the Limit Concept*, and *Polynomial Approximation of Functions: Historical Perspective and New Tools*, Kidron and Zehavi, and Kidron, respectively, reported on a qualitative study that was conducted for Kidron's dissertation (Kidron, 1999). Both papers described how animations were used to enhance student comprehension of approximation techniques. In particular, Kidron & Zehavi (2002) exclusively focused on student comprehension of Taylor series

convergence, and Kidron (2004) included additional explanations of student understanding of interpolation techniques. Both papers based their findings on data collected from four classes containing high school students, ages 16 and 17. In total 84 students participated in the study. In a computer laboratory classroom setting containing 20 computers, Kidron used dynamic graphs produced by a computer algebra system (CAS) as an instructional tool to illustrate convergence (Kidron, 2004; Kidron & Zehavi, 2002). Kidron desired to relate Taylor series convergence to a dynamic process and said that “the aim of using animation was to enable students to see the dynamic process in one picture” (2004, p. 304). Furthermore, students did not only manipulate the dynamic graphs, they programmed them as well. Kidron and Zehavi (2002) claimed that “the syntax of the animation explicitly makes clear *the way the pictures are related to each other*” (p. 210, italics in original).

Kidron taught these participants mathematics six hours each week where two hours were devoted to studying approximation techniques using the CAS. The other four hours were devoted to “standard subjects in Analysis, Algebra, and Trigonometry” (Kidron, 2004, p. 316). In the later publication, Kidron included a description of how she incorporated the historicity of the subject matter into her lessons. She elaborated:

I believe that the combination of dynamic graphics, algorithms, and historical perspective may lead to a more stimulating way of learning analysis by means of numerical processes. My research has focused on examining the extent to which this combination actually helped the students in the transition from their visual, intuitive interpretation of mathematical concepts to formal reasoning. (Kidron, 2004, p. 300)

To better describe how animations were used to enhance student comprehension of approximation, Kidron (2004) posed two questions: (1) “To what extent did [students] come to understand the formal statements of the mathematical theory?” and

(2) “As the students experimented with the CAS, did they develop visual intuitions that supported the theory?” (p. 317) To answer these questions in relation to Taylor series, students used the CAS Mathematica to illustrate the convergence of Taylor polynomials to particular generating functions, for example “ $f(x) = \frac{1}{1-x}$ ” (Kidron, 2004, p. 318) and “ $f(x) = \sin x \cos x$ ” (Kidron & Zehavi, 2002, p. 222).

These researchers observed attributes of an approximation conceptualization of Taylor polynomials to a generating function. When discussing the relationship that Taylor polynomials played in approximating  $f(x) = \frac{1}{1-x}$ , Kidron (2004) quoted one student as having said that “the higher degree of the approximating polynomial, the bigger is the interval in which  $f(x)$  and the polynomial coincided” (p. 318). This student demonstrated his understanding that a Taylor polynomial progressively approximated the generating function better and better as the degree of the polynomial increased. I will later call the “interval in which  $f(x)$  and the polynomial coincided” as this student’s representation of an interval of accuracy.

In addition, the researchers observed convergence images of Taylor series on which the remainder’s convergence to zero was emphasized. When Kidron & Zehavi (2002) asked a student to explain what he meant when he referred to Taylor polynomial approximations getting “better” as the degree increases, they quoted the student as having said that “the error decreases” (p. 220). To illustrate the remainder’s convergence to zero, instead of looking at the difference between the generating function and Taylor polynomials, they used a three-dimensional dynamic graph of the Lagrange remainder

formula (simply written as  $R_n(x) = \frac{f^{n+1}(c)x^{n+1}}{(n+1)!}$  for some  $c$  between 0 and  $x$ ). The graphs were 3-D representations of estimations of the remainder function  $R_n(x)$  and they allowed students to pick out the “ $c$ ” that would result in an upper bound for the remainder function (Kidron, 2002). Even with this 3-D representation she observed that students had issues with the existence of the unknown “ $c$ ” in Taylor’s formula (Kidron, 2004, p. 328). In particular, students expressed concerns with how they could give answers to an approximation question using Taylor’s formula even though they might not know exactly what the number “ $c$ ” represented.

The researchers not only discovered that the CAS might spur students to ask better questions, but it encouraged students to investigate their own questions using the CAS. For example, students posed questions related to the interval of convergence that led them to centering the interval of convergence at some other number besides zero and allowed them to formulate a general Taylor series equation, not just another Maclaurin series equation. This supported earlier results for the use of a CAS as a discovery tool to help students get a “sense” of a problem before the development of formal theory (Tall, 2000). Unfortunately, Kidron and Zehavi (2002) noted that not all students were able to write some of the programs required to get the animations. Despite the CAS stimulating students to experimentally discover new results on their own, they concluded that students still had to understand certain problem contexts to be able to correctly interpret the CAS’ results. Students that had done the actual analysis work (on paper) more accurately interpreted the CAS’s results. In Kidron’s own words, “The results showed the students’ ability to use Mathematica’s numerical and graphical capabilities, but we observed that the ability to use these capabilities did not help much those students who

had not succeeded in performing the analytical reasoning” (Kidron, 2004, p. 328). In addition, Kidron and Zehavi (2002) observed that the dynamic graphs were a “source of trouble” in formulating the formal  $\varepsilon - N$  definition for limit. Students tended to reverse the order of  $\varepsilon - N$  because they remembered “the order in which they worked in the lab (beginning with domain and finding the error)” (p. 226). This problem with reversibility has also been observed in other literature (Davis & Vinner, 1986; Roh, 2005, 2008).

One of the main things that separated Kidron and Zehavi (2002) and Kidron (2004) from my own work was that they did not create a typology of conceptual understanding of convergence of Taylor series. Contained within their works are a few examples of different types of student understanding of convergence of Taylor series, but they tended to concentrate on explaining the effectiveness of their animations instead of elaborating on the details of the student’s conceptions. Therefore, both articles appeared to be explaining the effectiveness of an instructional technique more so than analyzing and describing student understanding. Furthermore, most of the tasks presented in their two papers were related to student understanding of particular animations which naturally biased their students to elicit certain conceptions. As the next chapter will reveal, during the initial portion of each interview I attempted to refrain from leading participants into certain types of responses to allow conceptions to more spontaneously appear. The spontaneity of the appearance of the conceptions allowed for me to make inferences about what conceptions my participants might have been primary relying upon when responding to Taylor series tasks. Only after several questions did interview tasks become more leading in an effort to better analyze and describe those conceptions that had already appeared. It is also important to note that their participants were working

with approximation techniques on the CAS prior to any formal understanding of Taylor series (Kidron, 2004, p. 327). The CAS appeared to be primarily used as an instructional tool for student discovery of concepts and key results, and their data was collected from students as they progressed through the learning process using the CAS. In contrast, my data was collected from individuals who already had exposure to Taylor series, some were even classified as Taylor series experts, and I was not following any particular student through the learning process.

*Oehrtman's Work.* In *Collapsing Dimensions, Physical Limitation, and Other Student Metaphors for Limit Concepts: An Instrumentalist Investigation into Calculus Students' Spontaneous Reasoning* (2002), Oehrtman characterized “students’ spontaneous language and patterns of reasoning about limits as they emerged in the process of learning” (p. 2). In his study Oehrtman collected data from 116 first year college calculus students using a series of open-ended interviews and writing assignments. To help characterize student conceptions he used the theory of conceptual metaphors and metaphorical reasoning which he attributed to Max Black (1962, 1977) and which has been used recently in mathematics education (e.g., Lakoff & Núñez, 2000). According to Oehrtman (2002), metaphorical reasoning, “characterizes the students’ connections between their spontaneous and scientific concepts through linguistic aspects of thought, that is, as revealed in the language, symbols and images, and even entire domains of nonmathematical experience used as signs to point to mathematical ideas that are used by the students” (pp. 6-7).

As one can tell from his title, Oehrtman (2002) was investigating far more than just Taylor series, he investigated student understanding of limit in general. Using the



theory of metaphors he defined metaphor clusters to “capture global patterns in students’ responses” and create categories of student conceptions of limits revolving around the metaphors that students utilized (Oehrtman, 2002, p. 103). Using this approach he was able to define five metaphor clusters: collapse, approximation, closeness, infinity as a number, and physical limitation.

In relation to Taylor series, Oehrtman posed a writing assignment involving the Maclaurin series for  $\sin x$ . Of the metaphors mentioned above, Oehrtman observed students using collapse, approximation, closeness, and infinity as a number metaphors in response to this writing assignment. He characterized students as using an approximation metaphor based on utterances of “words traditionally associated with approximation such as ‘estimate,’ ‘error,’ ‘accuracy,’ etc” (Oehrtman, 2002, p. 160). In an excerpt from one of his participants, the student alluded to the “power series for  $\sin x$  and  $\sin x$ ” as being “two different functions.” He went on to say that the “power series for  $\sin x$  will approximate a value infinitely close to the value of  $\sin x$ ,” that a remainder “can be calculated,” and that this remainder is “designed to show how much a power series deviates from the value of a function at a particular point” (for the full excerpt, see Oehrtman (2002), p. 166). These comments contained indicators of an interplay between conceptual images applied to Taylor series that I will later define as approximation, remainder, and pointwise. In addition, he found that of all the metaphors, the approximation metaphor appeared the most in response to the Taylor series writing assignment (p. 162).

According to Oehrtman (2002), a collapse metaphor for the changing quantity in the limiting process was characterized by “imagining a physical referent for the changing

quantity collapsing along one of its dimensions, yielding an object that was one dimension smaller” (p. 150). Oehrtman (2002) then produced examples of students using the collapse metaphor on different applications, such as cylindrical volumes of water, secant lines, and volumes of solids of revolution (pp. 150-159). For example, students used the collapse metaphor to explain how a secant line between two points collapsed to a tangent line through “‘two points’ at a single location” (Oehrtman, 2002, p. 153). Consider how the following student collapsed two points into one location in their explanation of graphical attributes of the derivative.

First given two points on a curve, its slope (or also known as the rate of change between the two points) can be found. Now if the distance between those two points were to begin to decrease (or as  $h \rightarrow 0$ , where  $h$  is  $\Delta x$ ) they would eventually be one in the same point. At this point, the slope depends only on one point rather than two and gives an instantaneous rate of change rather than an average.

(Oehrtman, 2002, pp. 154-155)

In relationship to Taylor series, Oehrtman (2002) observed students who used a remainder image that collapsed to zero as  $n$  went to infinity (p. 165). In essence, the error was a 2-D object, a function, that collapsed down to a 1-D object, namely zero, if  $n$  went to infinity.

Oehrtman (2002) characterized the close metaphor as indicating “spatial proximity or ‘closeness’ or ‘clustering’” (p. 171). He then gave examples related to continuity of functions where students discussed the closeness of two points on the  $x$ -axis necessitating the function’s corresponding outputs being close on the  $y$ -axis. In relation to derivatives, students might note that secant lines get close to the tangent line. In the context of Taylor series, Oehrtman (2002) alluded to one student who said that “the more polynomials we use to approximate the original function, the closer the polynomials will

wrap themselves around the original function” (p. 175). This student’s utterance of the word “closer” was the indicator for the close metaphor in the context of approximating the generating function. If the student’s reference to “more polynomials” was related to adding more terms to a Taylor polynomial, he could have been indicating what I will later call a dynamic partial sum conceptual image of Taylor series convergence.

Infinity as a number metaphor was related to students performing algebraic operations with infinity, plugging infinity into functions, or representing infinity as a point. (Oehrtman, 2002, pp. 180-184). Oehrtman (2002) found that students used this metaphor in relation to limits involving infinity. For example, he noted that students “typically treated infinity as a number” when attempting to explain the indeterminate form  $\frac{\infty}{\infty}$  related to L’Hopital’s Rule. In relation to Taylor series, Oehrtman (2002) quoted one student as saying, “If we were to use an infinitely large polynomial we could write the function as a polynomial literally” (p. 184). Oehrtman categorized this student’s comment as a reference to infinity as a number because the student alluded to viewing the Taylor series as an “infinitely large polynomial” that could be “literally” written. In so doing, the student was treating infinity as a number since the infinite amount of terms contained within the polynomial could be written down.

The main thing that separated Oehrtman (2002) from my own work was that he did not focus exclusively on the concept of limit applied to Taylor series. He created a typology, but his typology was for the broader topic of limits based on metaphorical reasoning. It was not clear if additional conceptions would be discovered upon a closer analysis of Taylor series, and as Chapters 5 and 6 will reveal, additional conceptions did emerge from my data. Later chapters will also reveal that some of Oehrtman’s students

(e.g., see the excerpt on p. 166) displayed many of the indicators for various conceptual categories that I later define, but Oehrtman did not detail some of the same features that I will discuss. For example, he did not separate remainder from approximation, but I found it necessary to separate approximation and remainder conceptions because participants did not always associate remainder with an approximation conception. By separating the two conceptions, this helped me to give a more detailed description of the conceptual understanding of Taylor series. In addition, although our methods of analysis were similar, I used metaphors as potential indicators of conceptual images and incorporated a detailed focus analysis that I will discuss in Chapter 3. Furthermore, I included much more than just first year college calculus students to provide a broader picture of conceptual understanding of Taylor series convergence amongst different levels of mathematical experience.

*Alcock and Simpson's Works.* In 2004 and 2005 Alcock and Simpson produced a pair of articles entitled *Convergence of Sequences and Series: Interactions Between Visual Reasoning and the Learner's Beliefs About Their Own Role*, and *Convergence of Sequences and Series 2: Interactions Between Nonvisual Reasoning and the Learner's Beliefs About Their Own Role*. As the titles indicate, Alcock and Simpson (2004, 2005) were investigating much more than just Taylor series. 18 student participants from a British university were selected from two first-term, first-year real analysis courses, one standard lecture, and one that incorporated group work (for a more detailed description of the latter, see Alcock and Simpson (2001)). In their study they distinguished between students who tended to produce visual representations (called visualizers) and students who tended to not introduce visual representations (called non-visualizers) based on

characteristics of responses to particular analysis questions. Questions were designed in such a way as to not encourage or discourage participants to draw or not. In their study, they found that both visualizers and non-visualizers could successfully respond to analysis questions. What caused students to produce correct results was linked to their sense of authority, their ability to link concepts together, and their understanding of the structure of mathematics. They found that visualizers with an internal sense of authority (those who judge the correctness of tasks based on an internal sense) who linked the visual to the algebraic concepts tended to respond successfully to analysis questions (Alcock & Simpson, 2004, p. 29). Successful non-visualizers demonstrated an internal sense of authority and sought to identify “higher-level structures of mathematics” (Alcock & Simpson, 2005, p. 97). On the other end, they found that students with external senses of authority (those who rely on teachers, textbooks, solution keys, etc. to judge the correctness of tasks instead of an internal sense) who did not link concepts together and who demonstrated a poor understanding of the structure of mathematics were more likely to give erroneous responses.

In their articles, Alcock and Simpson discussed one question directly concerning Taylor series that they posed to their students, “When does  $\sum \frac{(-x)^n}{n}$  converge?” (2004, p. 7). Within the articles themselves, Alcock and Simpson specifically discussed student responses to the Taylor series question on only a few occasions. They observed weak visualizers attempting to draw Taylor polynomial graphs and nonvisualizers attempting to build on previous familiar knowledge. A few of the student excerpts demonstrated a couple of the conceptual images that I will later discuss, but Alcock and Simpson did not detail these conceptions, nor was that their intent. In the few cases where Alcock and

Simpson specifically referenced student responses to the Taylor series task, they mostly did so to give an example of an erroneous response. This further illustrated student difficulties with the concept of Taylor series and the need for this study of the conceptualizations of the convergence of Taylor series.

### *Theoretical Framework*

As the previous section demonstrated, the concept of convergence in the context of Taylor series had not been previously studied in much detail. Therefore the goal of this study was to analyze and describe the different ways in which people conceptualize the convergence of Taylor series. The theoretical framework had to provide a proper means for achieving this goal. Two major articles influenced the theoretical framework: *Concept Image and Concept Definition in Mathematics with Particular Reference to Limit and Continuity* (1981) by Tall and Vinner, and *Models of Limit Held by College Calculus Students* (1991) by Williams. Within this section I will take a closer look at each of these frameworks and explain how they influenced this study.

*Tall & Vinner's Work.* Tall and Vinner's work of 1981, entitled *Concept Image and Concept Definition in Mathematics with Particular Reference to Limit and Continuity* is one of the single most influential works on student understanding of the concept of limit. What made this article so important is not that Tall and Vinner studied student understanding of limits and continuity, but that they introduced concept image and concept definition as a tool to aid in unraveling understanding. They defined an individual's concept image of a mathematical topic to consist of the "total cognitive structure that is associated with the concept" (Tall & Vinner, 1981, p. 152). Inclusive in the "total cognitive structure" were not just mental pictures, but all "associated properties

and process” (Tall & Vinner, 1981, p. 152). They went on to say that an individual’s concept image was “built up over the years,” and that it “need not be coherent at all times” during the development of the image (Tall & Vinner, 1981, p. 152). When images are not coherent, they may contain “cognitive conflict factors” but students may not be aware of such factors unless the conflicting parts of their image are evoked at the same time (Tall & Vinner, 1981, pp. 153-154).

In this same work, they also defined the notion of concept definition as follows:

We shall regard the *concept definition* to be a form of words used to specify that concept. It may be learnt by an individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole. It may also be a personal reconstruction by the student of a definition. (Tall & Vinner, 1981, p. 152)

While an individual’s concept image could include pictures, properties, and process, an individual’s concept definition was restricted to a “form or words.” Tall and Vinner (1981), then made a distinction between two different types of concept definitions: “personal concept definition” and “formal concept definition” (p. 152). The personal concept definition need not be similar to the formal definition, and therefore, can lead to misconceptions. In addition, a definition can create concept images that are much more often relied upon, and therefore cause a definition to be difficult to recall or even be forgotten (Tall & Vinner, 1981; Vinner, 1983, 1991). For the purpose of this study into the conceptualizations of convergence of Taylor series, I will focus my analysis on uncovering concept images and unpacking their characteristics.

The idea of concept image was spurred by previous work by Vinner on mental pictures (see Vinner, 1975). Vinner (1975) defined a person’s mental picture of a given concept to be “the set of all pictures that have even been associated with [the concept] in

[the person's] mind" (p. 339). In a later paper, Vinner went on to say that "the word 'picture' here was used in the broadest sense of the word and it included any visual representation of the concept (even symbols)" (Vinner, 1983, p. 293). When distinguishing the difference between mental picture and concept image, Vinner added that concept images additionally include the set of properties and processes associated with the concept (p. 293). Thus, concept image is a broader notion than mental picture. A person's mental picture is included in his or her concept image but their concept image is not limited to just a visual representation.

Using previous research, Tall and Vinner used the notion of limit and continuity to demonstrate how their notion of concept image was used to analyze and describe student understanding. For the notion of limit, the authors considered limits in the contexts of functions and sequences. They used verbal and written cues as indicators to help classify particular concept images. For example, they found that students often relied upon dynamic images when discussing limits. They used words like "get close to," "approaches," and "tends to" as indicators of a dynamic concept image (Tall & Vinner, 1981, p. 163). As an illustration with sequences they quote a student from a previous work by Schwarzenberger and Tall (1978), " $s_n \rightarrow s$  means  $s_n$  gets close to  $s$  as  $n$  gets large, but does not actually reach  $s$  until infinity" (as cited in Tall & Vinner, 1981, p. 159). They viewed the student's utterance of "gets close to" as an indicator of a dynamic concept image of how sequences converge. To further illustrate student dependence on dynamic imagery, Tall and Vinner asked students to write down a definition for  $\lim_{x \rightarrow a} f(x) = c$  (Tall & Vinner, 1981, p. 161). Of the seventy who responded, 18 (42%) attempted a formal static definition but only four were correct. Furthermore, 43 (61%)



responded with a dynamic definition and 27 of those were deemed correct (Tall & Vinner, 1981, p. 162). More on the dynamic concept image will be discussed later.

The use of conceptual images to unpack student understanding has not been restricted to the study of student understanding of limits. For example, it has been used in conjunction with trying to unpack student understanding of proof (e.g., R. C. Moore, 1994; Weber et al., 2008) and definitions (e.g., Vinner, 1991). Since the main goal of this study was to analyze and describe, and since concept image is tied to the “total cognitive structure” that includes all “mental pictures,” “properties,” and “processes” related to a given mathematical concept, concept image provided the bulk of my theoretical framework for analyzing and describing different conceptions of convergence of Taylor series.

*Williams’ Work.* In 1991, Williams took results from previous research on student understanding of limits and categorized the models that students used in his work entitled *Models of Limit Held by College Calculus Students*. Descriptions of the six models of limit that Williams considered can be found in Table 2. He divided his study into two phases, and in the first phase, 341 university students from two second-semester college calculus courses took a short questionnaire designed to inform Williams of the models that they were using. He allowed students to affirm any one or all of his six models of limit. Furthermore, he asked students to select which model best described the concept of limit. He then interviewed particular students based on the models that they indicated in their questionnaire.

It is interesting to note that in Oehrtman (2002), of his metaphors considered, the approximation metaphor was the metaphor that was used the most. 74 percent of

Table 2  
*Models of Limit*

Limit Model	Description
Dynamic-theoretical	“A limit describes how a function moves as $x$ moves toward a certain point.”
Acting as a boundary	“A limit is a number or point past which a function cannot go.”
Formal	“A limit is a number that the $y$ -values of a function can be made arbitrarily close to by restricting $x$ -values.”
Unreachable	“A limit is a number or point the function gets close to but never reaches.”
Acting as an approximation	“A limit is an approximation that can be made as accurate as you wish.”
Dynamic-practical	“A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.”

\* Adapted from Models of Limit, by Williams, S., 1991, p. 221, *Journal for Research in Mathematics Education*, 22(3), 219-236

Table 3  
*Limit Model Questionnaire Results*

Limit Model	Percentages of Student Responses Out of N = 341		
	True	False	Best
Dyanmic-theoretical	80	19	30
Acting as a boundary	33	67	3
Formal	66	31	19
Unreachable	70	30	36
Acting as an approximation	49	50	4
Dynamic-practical	43	57	5

\* From Models of Limit, by Williams, S., 1991, p. 223, *Journal for Research in Mathematics Education*, 22(3), 219-236

Oehrtman’s participants utilized an approximation image in response to a question concerning Taylor series and 70 percent utilized this image when responding to a question involving  $0.\bar{9} = 1$ . But according to Table 3, the approximation model of limit was frequently affirmed (49%) but it was only the fourth most affirmed of Williams’ models. In addition, a high number of participants rejected Williams’ approximation model (50%). Oehrtman (2002) showed that different situations can cause students to rely on different metaphors (images). Therefore, when comparing and contrasting

Oehrtman's and Williams' results it seems likely that some of Williams' participants may not have been currently aware of the situations in which they utilized an approximation image. In other words, they may not have been currently evoking an approximation conceptual image when responding to Williams' questionnaire even though they may utilize an approximation image under certain conditions.

Imbedded in several of the descriptions for Williams' models of limit was dynamic language that can be indicative of a dynamic concept image. In Williams' descriptions, the dynamic language can be seen in more than just his models that received the word "dynamic" in their title. For example, the unreachable model contained a reference to "getting close to" and acting as a boundary model contained the word "go." Dynamic aspects of understanding limits have been observed by other researchers (e.g., Cornu, 1991; Cottrill et al., 1996; Davis & Vinner, 1986; Oehrtman, 2002; Tall & Vinner, 1981). Indeed, a dynamic image of the concept of limit has been shown to be popular amongst students. Recall that in Tall and Vinner's work from 1981, over 60 percent of the students in their study appeared to utilize a dynamic concept image when attempting to write a definition for  $\lim_{x \rightarrow a} f(x) = c$ . In addition, Table 3 shows that the dynamic-theoretical model was the most often affirmed model with 80 percent of Williams' participants indicating true. Furthermore, the dynamic-theoretical was also the least rejected model (19%).

Oehrtman (2002) warned that even though individuals may use language that is commonly associated with movement it does not mean that they were literally imaging actual movement when answering questions pertaining to limit. Therefore, just because someone may use dynamic words, like "approaches," it does not necessarily mean that

they are viewing a function as literally moving. They might be simply using this language because it was what was commonly used throughout their learning process. Therefore, today's researchers must be careful to not over interpret utterances that initially appear to be dynamic because in some cases they may not be.

Sometimes the dynamic image includes the idea of attainment or the lack of attainment of some limiting value, and Williams (1991) referred to these notions as dynamic-practical and unreachable. When discussing why a function never reached its limit, Williams (1991) quoted a student as having said the following:

As  $x$  approaches 2,  $f(x)$  approaches 3. What I mean by never reaches it is because you can go 2.9, 2.99, 2.999, but you can never really reach 3. (p. 226)

Table 3 shows that out of a Williams' sample of 341 calculus students, 70 percent indicated that a limit was unreachable and 36 percent indicated that unreachable was the best way to think about limits. Upon a closer inspection of Table 3, one might find it surprising that the unreachable model was the model most often affirmed as best. Perhaps even more surprising was that of the students who indicated having a more formal view of limit, 82 percent also indicated a dynamic image and 65 percent indicated that they still accepted unreachable as a valid way of thinking about the limit concept (Williams, 1991, p. 225). Even among the students who thought that the formal view was best, 71 percent indicated a dynamic image and nearly 50 percent still indicated that limit as unreachable was acceptable (Williams, 1991, p. 225). Therefore, even after apparently accepting a more formal definition for limit, students may still tend to rely upon other notions for limit. In addition to Williams (1991), other researchers have observed and studied student conceptions about the attainment or lack of attainment of

limits and its relevance to student understanding of limit (e.g., Cornu, 1991; Davis & Vinner, 1986; Mamona-Downs, 2001; Schwarzenberger & Tall, 1978; Sierpiska, 1987; Szydlik, 2000; Tall, 1980).

Tall (2001) claimed that the notion of limit has “built-in cognitive obstacles” related to the “idea of ‘getting closer and closer’” (p. 234). He asserted that the “idea of ‘getting closer and closer’ carries an embodied sense of ‘near there, but not quite,’” and thus, propagating the notion of the unattainable limit (p. 234). Cornu (1991) defined epistemological obstacles as a difficulty encountered by students in the learning process “which occur because of the nature of the mathematical concepts themselves” (p. 158). He went on to credit Gaston Bachelard with the definition and said that epistemological obstacles may be “unavoidable” and are found “in the historical development” of concepts (Cornu, 1991, p. 158). In Chapter 4, I will elaborate on the historical debate over the attainment of the limiting value, and because of its history, Cornu (1991) classified this debate as an epistemological obstacle and concluded that the debate is “still alive in our students” (p. 162). In Chapter 6, I will demonstrate that the dynamic reachable and unreachable limit images relative to Taylor series are “alive” in the minds of today’s students.

In relationship to sequences and series, Tall and Vinner (1981) elaborated on why some students concluded that  $0.\overline{9} \neq 1$ . To demonstrate this, the authors used previous work by Schwarzenberger and Tall (1978) and quoted students as having said the following about  $0.\overline{9}$  :

Student A: “It is just less than one, because the difference between it and one is infinitely small” (Tall & Vinner, 1981, p. 159).

Student B: “Just less than one, because even at infinity the number though close to one is still not technically one” (Tall & Vinner, 1981, p. 159).

Even when students concluded that there was no difference between  $0.\bar{9}$  and 1, they can still used dynamic language to describe the limit.

Student C: “[ $0.\bar{9}$  and 1 are] the same, for at infinity it [ $0.\bar{9}$ ] comes so close to one it can be considered the same” (Schwarzenberger & Tall, 1978).

Later, in a different context, of the students who had participated in the research referenced by Tall and Vinner (1981), of those students who had viewed  $0.\bar{9}$  to be different than 1, most changed their minds and concluded that  $0.\bar{9} = 1$  (p. 159). Tall and Vinner reported that the students were experiencing cognitive conflict but from the context it is not clear if the students had experienced actual conflict or if they had chosen to rely upon a different image to answer the problem.

The reliance of students on the concept of limit as unreachable may be related to their notions of infinity (Sierpinska, 1987). In Schwarzenberger and Tall (1978), the authors list off five notions of infinity that are commonly adhered to by students.

1. “Infinity is a concept invented in order to give an endpoint to the real numbers, beyond which there is no more real numbers.”
  2. “A symbol to represent the unreachable.”
  3. “The biggest possible number that exists.”
  4. “A number which does not exist, but is the largest value for any number to have.”
  5. “The idea of a last number in a never ending chain of numbers.”
- (p. 46)

In particular, the “unreachableness” of infinity exemplifies the notion of potential infinity, a process that goes on without end. For example, Tall and Vinner (1981) reported that some students could not overcome  $0.\bar{9} \neq 1$  because they related it to a

process that was conducted in time. When infinity is viewed as a process that goes on without end, it is referred to as potential infinity (Tirosh, 1991). In contrast, when infinity is viewed as achievable, it is referred to as actual infinity (Tirosh, 1991). When someone views infinity as potential, this view can influence the idea that limits are unreachable, and the “unreachableness” of the limit can be grounded in finite realities (Monaghan, 2001; Sierpinska, 1987; Tall, 2001). Consider the following comments from Sierpinska (1987):

The limit of a sequence is what that sequence is infinitely approaching without ever reaching it; the impossibility of reaching the limit is implied by the impossibility of running through infinity in a finite time; in particular, the number 0.999... is an infinite sequence which is being constructed in time; it is a number that tends to 1 without ever reaching it (p. 385).

Many studies have shown that students tend to view infinity as a process that cannot be completed in finite time (e.g., E. Fischbein et al., 1979; E. Fischbein, Tirosh, & Melamed, 1981; Tall, 1980). Because these intuitions are strongly adhered to finite realities, one’s intuitions of infinity may remain relatively unchanged and unaffected by instruction (E. Fischbein et al., 1979). Fortunately, Fischbein, Tirosh, and Hess go on to note that “the concept of infinity may develop itself by the instructional process, while the intuitions of infinity may remain unchanged, starting with age 12” (E. Fischbein et al., 1979, p. 33). That is, a student’s concept image of infinity may contain formal images but the student’s intuition of infinity will continue to be that of potentiality. Because of this procedural view of infinity, when encountering limit problems, the focus of student attention may fall on the “infinite ongoing process rather than the finite limit value,” and in so doing it is only natural to have a dynamic unreachable view of the limit concept (Tall, 2001, p. 233). Lakoff and Núñez (2000) claimed that the conception of actual infinity is based on

a potential infinity conception given a “metaphorical completion” (Lakoff & Núñez, 2000, p. 158). They called this “metaphorical completion” of an infinite process the “Basic Metaphor for Infinity” and concluded that this basic metaphor is the “basis of our understanding of infinity in virtually all mathematical domains” (Lakoff & Núñez, 2000, p. 8).

The dynamic-theoretical, unreachable, acting as approximation, and dynamic-practical models of limit influenced the creation of a questionnaire that I used to gain insights into student conceptions of the convergence of Taylor series. I used the questionnaire to select participants for interviews based on their questionnaire responses to tasks adapted from the four limit models mentioned above. Selecting interview participants based on questionnaire responses was the same procedure that Williams (1991) utilized. Therefore, Williams’ work affected the framework and methodology behind this study. It should be noted that I was not the first to utilize adaptations of Williams’ models and his method of using responses to a limit model questionnaire to select participants for interviews (e.g., Fisher, 2008; Lauten et al., 1994; Oehrtman, 2002).

In the next chapter I will not only explain how I used adapted versions of some of Williams’ questionnaire tasks to uncover and unpack student conceptions of limit applied to Taylor series, but I will also explain how I allowed for additional images to emerge and how I accounted for the interactions between these images.



## CHAPTER 3

### Methodology

Since student understanding of Taylor series has not been extensively researched in the past, this study incorporated an exploratory design. I classified participants into two main categories, novices and experts, based on their experience with Taylor series. I used qualitative methods spread out over three phases to elicit participant understanding of the convergence of Taylor series. The three phases are:

Phase 1: A focus group and individual interviews conducted with experts in Taylor series;

Phase 2: Questionnaires given to novices in Taylor series; and

Phase 3: Interviews conducted with selected novices.

The focus group session, interviews, and questionnaires were designed to bring out how participants think about convergence of Taylor series. The focus group and interviews consisted of a series of tasks in which the participant went through a handout that asked questions about their understanding of Taylor series. The handout served as an interview guide and it provided a place for participants to write to help them answer questions. Tasks on the handout included short answer or true/false type questions. Tasks on the questionnaires included short answer, multiple choice, or true/false type questions. The questionnaires were given only to novices. The primary data for this study consisted of visual and oral comments made during the focus group and interviews, and written work in response to the interview handout and questionnaire. The data were

analyzed for occurrences of various concept images that participants used to describe the convergence of Taylor series.

### *Participants*

To be a participant in this study an individual had to have had some previous experience with Taylor series. Because of the varying levels of participant exposure with Taylor series, for example some student participants may have only seen Taylor series in one class while faculty participants may have taught Taylor series multiple times, it was necessary to take into account the amount of contact that each individual participant may have had with Taylor series. Based on their amount of experience with Taylor series, participants were divided into two main groups: experts and novices.

*Experts.* Only experts participated during the first phase of the research. Experts were graduate students or faculty in the Department of Mathematics at the University of Oklahoma (OU). Most experts were chosen based on previous teaching and research experience with Taylor series. Some experts neither had teaching experience with Taylor series nor used series in their research endeavors. These experts were chosen because of previous extensive exposure, such as found in graduate level analysis classes, to Taylor series and/or series. Therefore, because individual expert familiarity with Taylor series varied greatly, experts were classified into two categories: experienced experts and capable experts.

Experienced expert participants (EEPs) were graduate students and faculty who have taught Taylor series or who have used series in their research. Therefore, many EEPs were very conversant and knowledgeable about Taylor series because of their exposure to and use of series inside the classroom and/or in their own research. Capable

expert participants (CEPs) might also be very conversant and knowledgeable about Taylor series but they had not yet taught series nor had they used methods that involve series in their own research. Even so, CEPs were well qualified to teach a class containing Taylor series even though they had not yet done so.

It is important to note that the two expert participant categories were created prior to analyzing the data. These categories are only intended to inform the reader of each participant's amount and type of exposure to Taylor series and series in general. They are not intended to be an indicator of an individual participant's type or level of understanding. It was possible for a CEP to demonstrate understanding similar to that of an EEP and vice versa. The same will apply when novices are discussed later.

In total, 16 experts participated in this study. Eleven of the 16 were graduate students, while the remaining five were faculty. Table 4 contains the demographics of the expert participants based on their level of experience with Taylor series.

Table 4  
*Expert demographics by Level of Experience with Taylor Series*

Expert classification	EEP	CEP
Number of participants	9	7

Data were first collected from participants by using a moderated focus group format in which I was the moderator. Of the 11 graduate student participants in this study, 7 graduate students participated in a graduate student focus group. Those who participated in the focus group also agreed to participate as needed in two follow-up interviews. Since the interviews were task-based, a second interview was not necessary if the participant completed all tasks during the first interview. Two graduate students were contacted for further interviews because they were identified as experienced experts

with seemingly dissimilar views of the convergence of Taylor series. Since the main purpose of the study is to analyze and describe the different ways in which people, both novices and experts, understand the convergence of Taylor series, it was not necessary to interview several participants who demonstrated similar patterns of thought during the focus group. People demonstrating analogous concept images would have similar responses during individual interviews and therefore, interviewing additional people with analogous imagery would add little to the description gained from a previous interview.

Not all graduate students were able to participate in the focus group. Plus the focus group size needed to remain small to allow for optimal involvement from each focus group participant. Therefore, individual, task-based, face-to-face interviews were conducted with experienced experts who did not participate in the focus group. Expert participants who did not participate in the focus group agreed to participate in no more than three interviews, and since these were task-based interviews, not all interviews were needed if the participant completed all tasks in a prior interview. In most cases, all participants completed all tasks in just two interviews. It is also worth noting that the interview protocol is nearly identical to the focus group protocol since all of the material from the focus group handout is contained in the interview handout. Plus, the interview handout contains additional questions near the end of the handout. The two graduate student participants from the focus group, who were later interviewed, completed these additional questions. Therefore, all experts who participated in interviews responded to the exact same set of questions to create consistency and aid in comparisons.

*Novices.* Novices were undergraduate students from OU and from a regional community college (RCC). Novices were sampled from classes in which they had no

more than one earlier exposure to Taylor series, for example, a calculus course. On the other end, novices were also sampled from undergraduate analysis classes in which they may have been seeing series for the third, fourth, or more time. Therefore, the mathematical experience of novices might range from only having calculus classes to having experience with discrete math, linear algebra, ordinary and partial differential equations, and even real analysis and numerical methods. Novices only participated in Phase 2 and Phase 3 of the research. Many novice participants who volunteered for Phase 2 did not volunteer for Phase 3, hence the potential number of novice interviews decreased considerably when compared to the number of participants who volunteered to complete questionnaires. Fortunately, having many student interviews was not necessary. Novices who volunteered to participate in Phase 3 of the research were contacted as needed depending on their responses to tasks presented during Phase 2. For the same reasons that it was not necessary to interview multiple experts demonstrating analogous concept images, it was not necessary to interview multiple students who had similar questionnaire responses.

Students in classes such as Calculus 3 and 4, Ordinary and Partial Differential Equations, Introduction to Real Analysis 1 and 2, Numerical Methods, and Numerical Analysis 1 and 2 at OU as well as Calculus 2 and 3, and Differential Equations at RCC were targeted because they included students who had previous experience with Taylor series. Even though these classes were targeted, undergraduate student participation was not solicited from all sections. Class time had to be used by the principal investigator to solicit student participation. In some cases there was not sufficient class time available for me to solicit participation. Some classes were not scheduled during the semester in

which this study was being conducted. In the end, participation was solicited from Calculus 3 and 4, Introduction to Real Analysis I, and Numerical Analysis I at OU and from Calculus 2 and 3 at RCC.

In all classes, the instructors chose to give a grade incentive equivalent to a small percentage of each student's individual overall grade, such as one quiz grade or one homework grade. Following OU's Institutional Review Board (IRB) requirements, these grade incentives were small so that it would not significantly affect a student's overall course grade; for example two instructors choose to give their students a grade incentive equivalent to 1.6 percent of the overall course grade. More on the IRB requirements and grade incentives will be discussed in the next section.

In total, 292 students agreed to participate in Phase 2 of the study by signing an informed consent form (ICF). Table 5 contains the class demographics of the novices who indicated that they were willing to participant. If a novice participant was concurrently taking two classes, instead of removing that participant or counting that participant in both classes, the participant is only counted in the rightmost of the two columns corresponding to the two classes that the participant was taking at the time of the questionnaire. For example, if a participant was enrolled currently in both Calculus 4 and Introduction to Real Analysis I, that participant was only counted as a student in Introduction to Real Analysis I because Introduction to Real Analysis I corresponds to the rightmost of the two columns corresponding to the two classes that the participant was taking. Since there were very few participants concurrently enrolled in two classes, if participants were identified as taking two classes, then it might be remotely possible for instructors to identify the participants. Therefore, by identifying participants with only

one class allows for the identity of the concurrently enrolled participants to remain undetectable. Furthermore, the rightmost class is more indicative of a participant's level of experience with Taylor series. This will become clear after the discussion of the division of the novice group into two subgroups representative of their level of exposure to Taylor series.

Table 5  
*Potential Novice Demographics by Class*

Class (Location)	Calc. 2 (RCC)	Calc. 3 (OU)	Calc. 3 (RCC)	Calc. 4 (OU)	Introduction to Real Analysis I (OU)	Numerical Analysis I (OU)
Number of participants	16	126	13	85	28	24

The Calculus 2 and Calculus 3 classes at RCC were four hour courses that used Hass, Weir, and Thomas (2007). The Calculus 3 and Calculus 4 classes at OU were three hour courses that used Stewart (2008). The real analysis students used the 3<sup>rd</sup> edition of *Introduction to Real Analysis* by Bartle and Sherbert (2000), and the numerical analysis students used the 8<sup>th</sup> edition of *Numerical Analysis* by Burden and Faires (2005).

Of the 292 novices who indicated that they were willing to participate in the study, 131 participated by completing the questionnaire in time for this study. Table 6 contains the class demographics of the novices that actually participated in the study. Again, if a participant was concurrently enrolled in two classes, that participant is only counted in the rightmost of the two columns corresponding to the two classes that the participant was taking.

Table 6  
*Actual Novice Participant Demographics by Class*

Class (Location)	Calc. 2 (RCC)	Calc. 3 (OU)	Calc. 3 (RCC)	Calc. 4 (OU)	Introduction to Real Analysis I (OU)	Numerical Analysis I (OU)
Number of participants	13	40	13	37	16	12

The novice participants from RCC and 12 Calculus 3 students from OU took the questionnaire during the 2008 summer semester. The remaining novice participants took the questionnaire during the 2008 fall semester. The novice participants from Calculus 2 at RCC and Calculus 3 at OU had just completed lessons on Taylor series. This was their first exposure to Taylor series in calculus. The novice participants from Calculus 3 at RCC and Calculus 4 at OU had completed the previous calculus class containing Taylor series but had not completed another class that typically contains series. Therefore, the Calculus 3 participants from RCC and the Calculus 4 participants from OU were at least one semester removed from studying Taylor series in a classroom setting, and in some cases, the participants were more than a summer removed. The participants from Introduction to Real Analysis had recently completed Chapter 3 of Bartle and Sherbert (2000, pp. 52-95) which has sections covering topics ranging from the limits of sequences, to the Bolzano-Weierstrass Theorem, to the Cauchy Criterion, and to infinite series. The numerical analysis participants had just finished specifically discussing how Taylor series could be used for numerical approximation methods.

Because the exposure of novices to series varied considerably, the novice group is divided into two subgroups: new novice participants (NNPs) and the mature novice participants (MNPs). Table 7 contains the demographics of the novice participants based on their level of experience with Taylor series.



Table 7  
*Novice Demographics by Level of Experience with Taylor Series*

Novice classification	NNP	MNP
Number of participants	103	28

The 103 students who had recently completed studies on Taylor series are reported together as NNPs; these are students from Calculus 2, Calculus 3 and Calculus 4. These new novices represent students who most likely have had only one exposure to series, this exposure occurring in either their current calculus class or the previous calculus class. Since this study is not attempting to make institutional comparisons, all NNPs are considered together whether they are from OU or RCC. The most mathematically mature novices were grouped together as MNPs: these are students from Introduction to Real Analysis I and Numerical Analysis I at OU. These 28 students had studied series in at least one previous calculus class and had seen series in their current class. Therefore, all MNPs had seen series results in class at least twice while the NNPs had seen series results most likely only once. If a participant was concurrently enrolled in classes that would lead to a classification of both NNP and MNP, then that participant was only classified as a MNP. This classification scheme is consistent with the scheme used to report the novice demographics by class and it is consistent with MNP description of students who had seen series results in at least two classes. Furthermore, as with the experts, these classifications are only intended to inform the reader of each participant's amount and type of exposure to Taylor series and series in general. These categories are not intended to be an indicator of an individual participant's type or level of understanding.

Of the 131 novices who participated in Phase 2, 41 initially indicated that they were willing to participate in the follow-up interviews of Phase 3. Of the 41 who indicated that they were willing, several were asked to participate in Phase 3 based on their responses to the questionnaire. Participants were asked to participate in phase 3 based on how they thought about the convergence of Taylor series as indicated by their answers to questionnaire tasks from Phase 2. Since one of the purposes of this study was to list and explain the differences in how people think about the convergence of Taylor series, it was not necessary to interview several participants who had demonstrated similar patterns of thought on the questionnaire. In addition, participants who indicated that they were willing to participate in Phase 3 were given the option to decline any invitation for an interview. In the end, 8 novices participated in the interviews of Phase 3. These novices responded to many of the exact same interview tasks that were posed to the experts during expert interviews. This helped to create consistency and aid in expert verses novice comparisons. More on the interview tasks will be discussed later in this chapter.

#### *Protection of Human Subjects*

The use of human subjects as participants in this research was approved by the IRB of the University of Oklahoma for using both students and faculty (see Appendix A for IRB Approval Forms). The protection of confidentiality began with the distribution and the collection of the Informed Consent Forms (ICFs). At the beginning of Phase 1, experts were privately solicited outside of a classroom setting. By soliciting experts privately, it should have helped the experts to not feel unduly influenced by their peers to accept or decline the invitation to participate in this study. Expert participants were also

assured verbally and on the ICF that participation was completely voluntary and that they could withdraw from the study at any time.

At the beginning of Phase 2, participation from all novices was solicited within a classroom setting by myself. The ICF was distributed to all members of the particular targeted class and was then used as a guide when I was explaining the research. After I answered all of the student questions about the research, those who wanted to participate were instructed to sign the consent form. All students in each class were then instructed to return the consent form signed or not, thus preventing their instructor and their peers from knowing who consented to participate in this study. As discussed in the pervious section, all instructors decided to give a small grade incentive for students who completed the questionnaire. For those who did not wish to participate or who were unable to participate in this study, there was an option to take the questionnaire for the exact same grade incentive but their questionnaire was not included in the data for this study. Having this alternate method for increasing their grade should have helped to alleviate students from feeling unduly pressured to participate. Participants and non-participants were given an opportunity to acquire the questionnaire directly from myself, circumventing their instructor and, therefore, keeping their participation or lack thereof undisclosed to their instructor. At the end of each semester, each instructor received a list of all of their students who took the questionnaire. This list did not distinguish students that were allowing the questionnaire to be used for data from those who were not. Therefore, based on this list, the instructor was able to credit a student's grade without knowing if that student was a research participant. During the administration of the ICFs, students were assured that their instructor would not be able to distinguish participants

from non-participants and of the steps mentioned above that would assure their instructor's blindness. Student participants were assured verbally and on the ICF that this study was completely voluntary and that they could withdraw from this study at any time without penalty to their grade. See Appendix A for the ICFs.

At the beginning of Phase 2, novice participants were given the opportunity to indicate on the ICF if they were willing to be considered for individual interviews. They were verbally informed that any grade incentive only applied to the questionnaire and not to the interviews and that the interviews were voluntary. Of the novice participants who indicated that they were willing to be considered for interviews, at the beginning of Phase 3 some of these participants were individually contacted via e-mail soliciting interview participants. Participants contacted for interviews were allowed to decline any invitation to participate in Phase 3. Instructors of interview participants were never informed in any way about their student's participation in interviews.

*Focus Group and Interview Data.* Data from the focus group and interviews consisted of both audio and video records and written responses to tasks presented during the focus group and interviews. All identifiable information was removed from transcripts of interviews and any written responses. Each expert was assigned a two digit numerical code and novices were assigned a three digit numerical code. The numerical code is the only identifying code that linked a participant's transcript to his/her written responses.

*Questionnaire Data.* Data from questionnaire consisted only of written responses to tasks that were presented in the questionnaire. As with the focus group and interview data, all identifiable information was removed from the questionnaires. Names were

replaced by a three digit numerical code. This code was the only link between questionnaire and interview data. All data, including audio and video records, transcripts, written responses, and questionnaires, have been kept secure and will be kept secure in case other studies may benefit from this data.

*Pseudonyms.* Pseudonyms have been used for reporting, including in this dissertation, presentations, and subsequent papers. Due to the small number of female participants, all gender references that may have identified a participant's gender as female has been replaced by either gender neutral or male identifiers. This includes the use of only gender neutral or male pseudonyms.

#### *Phase 1 Expert Focus Group / Interview Tasks*

Seven graduate students in the Department of Mathematics at the University of Oklahoma participated in one 60 minute focus group designed to elicit their understanding of and help expose them to how other experts understand the convergence of Taylor series. Two of these graduate students were classified as EEPs and the other five as CEPs. The focus group marked the first time that data were collected from participants during the course of this study.

Eleven expert participants from the Department of Mathematics at the University of Oklahoma participated in no more than three 60 minute, face-to-face, task-based interviews designed to elicit their understanding of the convergence of Taylor series. Two of these participants also participated in the focus group, and all eleven completed the exact same tasks. In total, 16 experts participated in either a focus group, focus group and interviews, or interviews.

The data collected from the focus group consisted of both audio and video recordings and written responses to tasks presented in a handout (see Appendix B for the focus group protocol and handout). The handout given during the focus group consisted of 35 questions that represented a portion of a larger handout that would be given during face-to-face interviews. The complete handout contained an additional seven tasks that were not in the handout given during the focus group. Of the 42 total tasks in the handout that a participant would see during interviews, the type of tasks presented falls into three categories:

Identification and background information: 3 questions

Short answer type questions: 26 questions

True / false type questions: 13 questions

For the purpose of analysis, the short answer and the true / false type questions can be classified as either high inference type or low inference type questions.

*High Inference Type Questions.* The high inference type questions were designed to elicit response from each individual participant without excessively leading a participant into using one mental image over another and therefore causing the participant to respond in a certain way. The high inference type questions were open form in the sense that the “respondents can make any response they wish” (Gall, Gall, & Borg, 2007, p. 234). The open form allowed me to make more reliable inferences about a given participant’s dominant mental imagery concerning the convergence of Taylor series. For example, if a participant relied upon approximation language (Oehrtman, 2002) throughout the high inference tasks, I would infer that an approximation image might be primary in that participant’s conceptual imagery concerning the convergence of Taylor

series. This inference would be deduced with a higher level of reliability and accuracy since the deduction was based on the participant's response to a high inference task as opposed to the participant's response to a low inference task. If another participant, relied upon an unreachable model (Williams, 1991) throughout the high inference tasks, I would infer that an unreachable conceptual image might dominate that participant's conceptual imagery. I believed that the high inference type questions presented during the interviews could potentially draw out a very wide range of responses from within the expert group. Chapter 5 on Today's Experts will demonstrate the breadth of the themes that emerged from the interviews. Table 8 lists the high inference type questions from the interview tasks.

In the following tables of focus group / interview tasks, question numbers indicate the intended order for questions to be posed. Questions appearing on the same row in the table appeared together on the same page of the handout. For example, a participant would see questions one and two at the same time when responding to each question. In addition, during the interview I might ask participants to elaborate on certain topics brought up by the participant during responses. After responding to all questions on each page, participants were asked if they wanted to add anything to their answers before moving to the next page. For the full expert interview protocol and handout, see Appendix B.

*Low Inference Type Questions.* Following the high inference type questions were more specific questions designed to elicit responses that might not have otherwise appeared when answering the open form questions. The tradeoff is that the specific questions might be more leading, and therefore, potentially bias participant responses.

Table 8

*Expert Interview High Inference Type Questions for Convergence Imagery*

Convergence imagery high inference type question
1. What are Taylor series?
2. Why are Taylor series studied in calculus?
3. What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$ when $x$ is any real number?”
4. What is meant by the “ $(-1,1)$ ” in, “ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ when $x$ is in the interval $(-1,1)$ ?”
6. What is meant by the word “prove” if you were asked to, “Prove that sine is equal to its Taylor series.”
7. What are the steps in proving that $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$ ?
Convergence imagery high inference type question: Excluding approximation imagery
8. How can we estimate sine by using its Taylor series?
9. What is meant by the “approximation” symbol in “ $\sin(x) \approx x - \frac{x^3}{3!}$ = a Taylor polynomial for sine when $x$ is near 0?”
10. What is meant by the “near” in “ $\sin(x) \approx x - \frac{x^3}{3!}$ = a Taylor polynomial for sine when $x$ is near 0?”
11. How can we get a better approximation for sine than using $x - \frac{x^3}{3!}$ ?

These more specific questions are referred to as low inference type questions because I cannot make reliable inferences about the dominance of one type of imagery over another within an individual participant. A participant completed high inference type questions prior to being biased by low inference type questions. Typical examples of high inference type questions include short answer, while typical examples of low inference tasks are true / false type questions that force the participant to respond in a certain way. See Table 9 for the low inference true / false type questions.

True / false type questions are closed form in the sense that “the question permits only prespecified responses” (Gall et al., 2007, p. 234). In addition to permitting participants to select “True” or “False”, participants were also given an option to selected “DN” for “Don’t kNow”. Depending on time restrictions during interviews, most



individual interview participants were encouraged to think out loud as they were answering the true / false type questions.

Table 9  
*Expert Interview Low Inference True / False Type Questions*

Convergence imagery low inference type question*
19. $\overline{.9} = 1$ . (Note $\overline{.9}$ means .999999 with 9's continuing to repeat).
20. Every bounded increasing sequence is convergent.
21. Every bounded decreasing sequence is convergent.
22. A convergent series of continuous functions is continuous.
23. Every function can be represented as a Taylor series. That is, every function equals its Taylor series.
24. The function producing a given Taylor series is unique. That is, if $f(x)$ produces Taylor series $T(x)$ , then no other function $g(x)$ can produce the same Taylor series $T(x)$ .
25. The Taylor series for a given function is unique. That is, $f(x)$ cannot have two different Taylor series representations.
26. If $f(x) = T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$ on the interval $[0,1]$ , then on $(0,1)$
$\frac{d}{dx}(f(x)) = \frac{d}{dx} \left( \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \right) \text{ and } \int_0^1 f(x) dx = \int_0^1 \left( \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \right) dx .$
27. A series $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n \rightarrow \infty} a_n = 0$ .
28. If $a_m + a_{m+1} + a_{m+2} + \dots \rightarrow 0$ as $m \rightarrow \infty$ , then $\sum_{n=1}^{\infty} a_n$ converges.
29. If $a_m + a_{m+1} + a_{m+2} + \dots + a_{k-1} + a_k \rightarrow 0$ as $m, k \rightarrow \infty$ , then $\sum_{n=1}^{\infty} a_n$ converges.
30. Since $\lim_{m \rightarrow \infty} \left( \frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \frac{x^{m+2}}{(m+2)!} + \dots \right) = 0$ for all real numbers $x$ , $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all real numbers $x$ .
31. When $C$ is some constant number $\sqrt{C^2 + \pi C + \sin \phi} \approx \sqrt{C^2 + \pi C + 1 - \frac{1}{6} \phi^3}$ when $\phi$ is near zero. (Note: $\approx$ means approximately)

\* True / False / DN appeared next to each questions (DN for Don't kNow). Participants were instructed to circle either True, False, or DN.

True / false type questions are not the only low inference type questions contained within the interview tasks. Some short answer type questions are considered low inference type questions because they are closed form, for example, Task 5 in Table 10. Other short answer type questions are considered low inference type because they use language that could bias a participant's response. For example, Task 12 in Table 10, uses the words "approximate," "Taylor polynomial," and "error." During the analysis of the data, these three words, when used by a participant, were indicators that a participant may be relying on three related but different concept images. Since these words are contained within the question itself, the participant has already been biased to use this language. Therefore, I cannot conclude that the images related to each word are primary in the participant's conceptions because the participant did not evoke the image independently of the question. Instead, I can only gain insight into images that may or may not have been evoked prior to this question. Table 10 contains the list of low inference type questions for the focus group and individual interviews that are not true / false type questions.

In addition to questions designed to illuminate expert understanding of the convergence of Taylor series, questions concerning expert opinion of how students understand this topic were posed. These are low inference type questions in the sense that the questions are not designed to illuminate expert understanding of the topic of convergence of Taylor series, but illuminate how experts perceive their students to understand this topic. Therefore, responses need not represent how experts understand this topic. Table 11 contains the list of these questions. In some cases, questions 13 – 18 were posed after the true / false type questions.

Table 10

*Other Expert Interview Low Inference Type Questions for Convergence Imagery*

Convergence imagery low inference type question
5. Find the exact sum of the series $\sum_{n=0}^{\infty} \frac{5^n}{n!} = 1 + \frac{5}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \dots$ .
12. What is the difference between the following questions. a. What is the maximum error in using a given $n^{\text{th}}$ degree Taylor polynomial to approximate $f(x)$ ? Verses b. How large does $n$ need to be to guarantee that the $n^{\text{th}}$ degree Taylor polynomial is within 0.1 of $f(x)$ ?
32. Now I am going to show you an image. a. What are you seeing when I move the Top slider? b. What are you seeing when I move the Bottom slider?
33. For part a) and part b), consider the graph below: a) Estimate the error in using the Taylor polynomial to approximate $\sin\left(\frac{\pi}{4}\right)$ . b) For what values of $x$ is the Taylor polynomial within 0.1 for $\sin x$ .
34. What is the maximum error in using $1 + \frac{x}{1!}$ to approximate $e^x$ on $(0,2)$ ? Justify your steps?
35. How large does $n$ need to be to guarantee that the $n^{\text{th}}$ degree Taylor polynomial $1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$ is within 0.001 of $e^x$ when $x$ is in the interval $(0,2)$ ? Justify your steps?
36. Is there a way to decrease the $n$ you got on problem 35?
37. Use the image with sliders to answer part a) and part b). Please move the sliders to help get your answer. a. How large does $n$ have to be to get the Taylor polynomial function within 0.1 of $\sin x$ when $x$ is in the interval $(-4,4)$ ? What is the $n$ doing to the Taylor polynomial formula? b. Take the $n$ value that you got from part a. What happens to the values of $x$ that guarantee that the Taylor polynomial function is within 0.1 of $\sin x$ when you move the “a” slider? What is the happening to the Taylor polynomial formula?
38. What are Taylor series and why are they studied in Calculus?
39. What other comments would you like to provide that you believe would inform the topic of convergence of Taylor series and student understanding.

\* All graphs and dynamic images are not contained in this table. Graphs and screen shots of dynamic images can be found in the corresponding interview protocol found in Appendix B.

Table 11

*Expert Interview Low Inference Type Questions Concerning Expert Opinion about Instruction and Student Understanding*

Convergence imagery low inference type question
13. What problems would you include in a lesson plan for convergence of Taylor series? Why would you include these problems?
14. A student asks you, "Why does sine equal its Taylor series?" How would you answer the student's question? Why?
15. If there was one thing that students should take from Taylor series, what would that be?
16. What do you perceive to be the biggest problems that students have with Taylor series?
17. What mistakes do you commonly see students make when working with Taylor series? Why do you think they make these mistakes?
18. What other comments would you like to provide that you believe would inform the topic of convergence of Taylor series and student understanding.

*Why Experts First?* The major goal of this study is to produce and explain a list of mental images as indicated by metaphors that people use when communicating about the convergence of Taylor series. Experts were chosen to go first during this study because of their experience with Taylor series. According to Hiebert and Carpenter (1992) individuals with good understanding of a particular mathematical topic should have a complex system of internal and external relevant representations that are joined together by numerous strong connections. In summarizing the work of Schoenfeld (1985, 1987a, 1987b), Lester (1994) compares and contrasts "good" and "poor" problem solvers.

1. Good problem solvers know more than poor problem solvers and what they know, they know differently-their knowledge is well connected and composed of rich schemata.
2. Good problem solvers tend to focus their attention on structural features of problems, poor problem solvers on surface features.
3. Good problem solvers are more aware than poor problem solvers of their strengths and weaknesses as problem solvers.
4. Good problem solvers are better than poor problem solvers at monitoring and regulating their problem-solving efforts.
5. Good problem solvers tend to be more concerned than poor problem solvers about obtaining "elegant" solutions to problems. (p. 665)

In a more recent study, Carlson and Bloom (2005) studied the problem solving abilities of eight research mathematicians and four Ph.D. candidates who eventually completed

their degrees. Had these participants been participants in this study, they would have been classified as experts. They found that while working on mathematical problems, their experts accessed “conceptual knowledge, facts, and algorithms” (p. 68). In addition, “The efficiency and effectiveness of their actions appeared to be strongly influenced by their fluency in accessing a wide repertoire of heuristics, algorithms, and computational approaches” (p. 68). Furthermore, Carlson and Bloom (2005) noted their experts monitored their thought process and products regularly when working mathematical problems.

Therefore, Taylor series experts should have a more robust mental imagery of Taylor series, have more numerous images, be able to switch efficiently between different images, and be more cognizant of how they think about Taylor series and of how others think about Taylor series. Conversely, novices should have a very simple mental image (if a mental image existed at all). When presented with nonstandard problems, novices might be unable to mentally transition, and they may not be very cognizant of how they think about Taylor series, much less how others think about Taylor series (Hiebert & Carpenter, 1992; Lester, 1994).

The data from the focus group and the expert interviews were used to help produce the questionnaire that was given to the novices. Two months elapsed between the time the first interview was conducted with an expert and the time the first questionnaire was administered to a novice. Much of that time was used going over the data and creating a stronger questionnaire. Because experts demonstrated a variety of mental images, I was able to design the questionnaire to account for mental images that I would not have initially considered. Therefore, having the experts participate first in this

study allowed for the creation of an improved questionnaire that would help better illuminate the understanding of novices. Serving in this capacity, the experts were key informers because they had “special knowledge or perceptions that would not otherwise be available to the researcher” (Gall et al., 2007, p. 243).

### *Phase 2 Novice Questionnaire Tasks*

131 students from OU and from RCC voluntarily completed questionnaires (see Appendix C for the complete questionnaire). The questionnaires were administered in a classroom setting monitored by myself. I was there to answer questions that participants had while taking the questionnaire and to ensure that participants followed directions.

The questionnaires consisted of 33 questions that fell into four categories:

Identification and background information: 4 questions

Short answer type questions: 15 questions, some with multiple parts.

Multiple choice type questions: 6 questions.

True / false type questions: 12 questions.

Because of the length and content of the questionnaire, each questionnaire took 45 to 75 minutes for a novice participant to complete. In most cases, 60 minutes was sufficient for a participant to complete the questionnaire.

*The Order of the Questions.* Questionnaire tasks were ordered from short answer to multiple choice to true / false type questions. As with the expert interview tasks, the questionnaire tasks were arranged in this order to create a higher degree of accuracy when making inferences about participant thinking when reporting the results.

Participant responses to the short answer questions could vary widely because the short answer questions tended to be open form questions that were less leading than the closed

form multiple choice and true / false type questions. The open form questions allowed for a higher degree of accuracy when I was making inferences about a participant's conceptual imagery. On the other hand, multiple choice and true / false type questions restrict participant responses and in so doing are more leading. The multiple choice and true / false type questions are designed to elicit responses that might not have otherwise appeared when answering the short answer type questions.

In some cases, the ordering within the short answer type questions goes from general to specific. For example, participants were asked to draw Taylor polynomials prior to seeing graphs of Taylor polynomials presented in conjunction with a later question. Therefore, a participant who correctly responded to the later question but were unable to draw a Taylor polynomial graph in the previous question might not have a very well developed primary concept image that contains a complete visual understanding of Taylor polynomials. Even though their understanding might not be complete, the participant that correctly responds to the later question but not to the first is demonstrating some visual understanding of Taylor polynomials. While a participant who is unable to respond to either question may have little or no understanding of Taylor polynomial graphs. On the other hand, had these questions been reversed, the graphs would be presented with the first question and it would then influence a participant's ability to draw Taylor polynomial graphs.

*Accounting for Effects.* Because of the complexities of Taylor series, there are many factors that can influence a person's understanding. This study attempts to account for these factors within the questionnaire by asking very similar tasks multiple times. Each time the question is posed, it incorporates a different combination of the factors to

aid in determining the factor's effect on participant understanding. This approach helped to reduce false conclusions about an individual's understanding. Some of the factors that the questionnaire attempted to take into account include: the interval of convergence (IOC), different notations used for series and partial sums, the presence or lack of a known function generating the series, and the convergence to a function.

For the interval of convergence, the questionnaire repeated similar tasks concerning series with finite intervals of convergence (finite in measure) and series with infinite intervals of convergence (infinite in measure). For example, as discussed in Chapter 2, Steven R. Williams (1991) classified student views of the limit concept into six categories; dynamic-theoretical, acting as boundary, formal, unreachable, acting as an approximation, and dynamic-practical. Four of these categories emerged from the expert interview data: dynamic-theoretical, unreachable, acting as an approximation, and dynamic-practical. In the context of Taylor series, I call these dynamic, dynamic unreachable, approximation, and dynamic reachable, respectively. In addition another theme emerged that I refer to as "exact." I will discuss the description of each of these themes later in this chapter.

Table 12 contains the question matrix used to analyze the questionnaire tasks corresponding to four of the five themes mentioned above. For example, "19 k" refers to part "k" of task "19" in the questionnaire. 19 k incorporates a dynamic reachable image of convergence of Taylor series in which the IOC is infinite. 16 d incorporates a nearly identical question that references a dynamic reachable image of convergence of Taylor series in which the IOC is finite. Therefore, a participant who affirmed either 19 k or 16 d, but not the other, may be indicating an effect caused by the interval of convergence.



For the actual questions see the questionnaire in Appendix C. A merely dynamic image was excluded from the questionnaire because the expert interview data indicated that the dynamic image was usually accompanied with the idea of attainment or lack of attainment. Since the majority of these themes are a direct adaption of the models of limit used by Williams (1991), I will refer to all five of these themes as existing themes which are indicative of the corresponding existing images defined later in this chapter.

Table 12  
*Questionnaire Matrix for Existing Images Accounting for Different Intervals of Convergence (IOCs)*

Interval of Convergence	Existing Images			
	Dynamic Reachable	Dynamic Unreachable	Approximation	Exact
Finite IOC	16 d	16 e	16 f	16 g
Infinite IOC	19 k	19 l	19 m	19 n

The questionnaire made a distinction between different notations that are used for series and partial sums. For example, when referring to series convergence, the convergence can be represented as a sequence, “ $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  converges as  $n$  goes to infinity.” Convergence can be referred to using the  $n^{\text{th}}$  Taylor polynomial, “ $T_n(x)$  converges as  $n$  goes to infinity.” The first notation will be referred to as a sequence of partial sums and the later notation will be referred to as the  $n^{\text{th}}$  partial sum notation.

In addition to accounting for the notational effect, the questionnaire attempted to account for the effect of the presence of a generating function specifically equated to a Taylor series. A function  $f(x)$  generates a Taylor series if the Taylor series for  $f(x)$  is equal to

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

for some given center  $a$ . In other words, a function is a generating function for a given series if it generates the given series. To account for the effect of the presence of a stated generating function explicitly equated to the Taylor series, questions were asked about series that had a stated generating function and then asked again with another series in which no generating function was specifically revealed to the participant. For example, problem 19 states that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \text{for all numbers } x,$$

while problem 18 states that

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \text{ converges for all numbers } x.$$

Problem 18 never equates the given series to a specific generating function, but problem 19 explicitly states that the generating function equals the given Taylor series.

Furthermore, the equated series effect was considered. This effect considers the effects of “convergence to” a series versus “convergence” without a series explicated stated was considered. For example, to account for the effect of convergence to a series,

in problem 18 participants could choose from 18 h, “ $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!}$

converges as  $n$  goes to infinity,” or 18 i, “ $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!}$  converges to

$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$  as  $n$  goes to infinity.” Therefore, in response to problem 18, participants

could choose between simply affirming that the Taylor polynomial simply “converges” as  $n$  goes to infinite, affirming that the Taylor polynomial “converges to” a given series, or both. When a generating function was stated, the series was replaced by the

generating function. For example, in problem 19, a participant could affirm that the given Taylor polynomial “converges to  $e^x$  as  $n$  goes to infinity.”

Tables 13 and 14 contain the questionnaire matrices used to analyze the questionnaire tasks corresponding to the notational, stated generating function, and equated series effects. In addition, the effect of the interval of convergence was also considered. Therefore, when considering all four factors, there are  $2^4 = 16$  different possibilities. This approach helped to reduce dubious conclusions about participant conceptions by accounting for multiple factors that might influence each participant’s conceptualization of the convergence of Taylor series. Chapter 5 on Tomorrow’s Experts will reveal how Tables 13 and 14 were used to analyze and describe novice understanding of the convergence of Taylor series.

Table 13

*3D Questionnaire Matrix for Notational Effect with Finite IOC*

	Equated Series Effect	Notational Effect	
		$n^{\text{th}}$ Partial Sum Notation	Sequence of Partial Sums Notation
Stated equated generating function (series is replaced by generating function)			
	“Converges to” series	16 i	16 k
	“Converges”	16 h	16 j
No stated equated generating function			
	“Converges to” series	17 d	17 j
	“Converges”	17 c	17 i

Table 14

*3D Questionnaire Matrix for Notational Effect with Infinite IOC*

	Equated Series Effect	Notational Effect	
		$n^{\text{th}}$ Partial Sum Notation	Sequence of Partial Sums Notation
Stated equated generating function (series is replaced by generating function)			
	“Converges to” series	19 g	19 i
	“Converges”	19 f	19 h
No stated equated generating function			
	“Converges to” series	18 i	18 k
	“Converges”	18 h	18 j

*Consistency Among Questions.* Similar language was used throughout the questionnaire. When a question involved the notion of the limit applied to a series, I would use the word “converge.” I only deviated from using the word “converge” when attempting to uncover differing participant views of convergence (see questions 16 d-g and 19 k-n in Appendix C). I consistently used the words “n goes to infinity” and consistently placed “n goes to infinity” at the end of statements when appropriate. This would be in contrast to using “n approaches infinity” or the symbolic notation of “ $n \rightarrow \infty$ ” and placing it at varying points within different statements. Since students sometime struggle with interpreting mathematical notation, the use of symbolic notation was avoided when possible because I did not want the notation preventing students from answering questions and ultimately causing problems with the interpretation of their data. The order of multiple choice selections was consistent between different numbered tasks whenever possible. For example, the multiple choice selection about graphs being “identical” followed the selection about graphs being “similar” for tasks 18 and 20 (see Appendix C). Also, all multiple choice questions ended with the exact same three statements to choose from. In addition, many questions were presented from the perspective of a student. These questions typically began with, “A student correctly writes the following...” All multiple choice questions referenced a hypothetical student who had correctly written a mathematical result that should then be considered by the participant. “Correctly” was underlined, emphasizing the validity of the student’s mathematical statements even though the student making the statement was fictitious. Therefore the participant should not be in doubt of any mathematical results obtained by

the imaginary student. Following the statement of each correct mathematical result, all multiple choice questions asked the participant to do two things:

1. Read ALL responses on the next page before circling.
2. Circle ALL responses that you can correctly conclude based off of what the student had written.

Following these directions, the corresponding multiple choice selections followed on the adjacent page. The consistency of the language employed, the avoidance of symbolic notation, and the uniformity of the tasks helped to contribute to a more readable and reliable questionnaire.

Because of the length of the questionnaire, I am not showing all the questionnaire tasks at this time. For the complete questionnaire see Appendix C. I will discuss novice responses to the questionnaire in Chapter 6.

### *Phase 3 Novice Interview Tasks*

Eight undergraduate students from OU participated in no more than two 60 minute, face-to-face, task-based interviews designed to elicit novice understanding of the convergence of Taylor series. Of the eight novice interview participants, five were classified as NNPs and three as MNPs. The data collected from these interviews consisted of both audio and video recording and written responses to tasks presented in a handout (see the novice interview protocol and handout in Appendix C).

The handout given during the interview consisted of 13 written questions. These 13 questions included all the high inference type questions found in the expert interview tasks (see Tasks 1 – 4 and Task 6 – 11 in Table 8). This consistency between expert and

novice interviews allowed for expert and novice comparisons between the high inference type questions. Table 15 is a list of the questions found in the novice interview handout.

Table 15  
*Novice Interview Tasks*

Convergence imagery high inference type question
1. What are Taylor series? 2. Why are Taylor series studied in calculus?
3. What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$ when $x$ is any real number?”
4. What is meant by the “(-1,1)” in, “ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ when $x$ is in the interval (-1,1)”
6. What is meant by the word “prove” if you were asked to, “Prove that sine is equal to its Taylor series.”
7. What are the steps in proving that $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$ ?
Convergence imagery high inference type question: Excluding approximation imagery
8. How can we estimate sine by using its Taylor series?
9. What is meant by the “approximation” symbol in “ $\sin(x) \approx x - \frac{x^3}{3!}$ = a Taylor polynomial for sine when $x$ is near 0?”
10. What is meant by the “near” in “ $\sin(x) \approx x - \frac{x^3}{3!}$ = a Taylor polynomial for sine when $x$ is near 0?”
11. How can we get a better approximation for sine than using $x - \frac{x^3}{3!}$ ?
Convergence imagery low inference type question
12. What is the difference between the following questions. a. What is the maximum error in using a given $n^{\text{th}}$ degree Taylor polynomial to approximate $f(x)$ ? Verses b. How large does $n$ need to be to guarantee that the $n^{\text{th}}$ degree Taylor polynomial is within 0.1 of $f(x)$ ?
<i>Taylor’s Inequality states that if there exists a number <math>M</math> such that <math> f^{(n+1)}(x)  \leq M</math> for all <math>x</math> in an interval, then the remainder function <math>R_n(x)</math> of the Taylor series for <math>f(x)</math> at <math>x = a</math> satisfies the inequality <math>R_n(x) \leq \frac{M}{(n+1)!} x-a ^{n+1}</math> for all <math>x</math> in the interval.</i>
13. How is Taylor’s Inequality useful?
14. Elaborate on each statement in problem 19 from the questionnaire. [Problem 19 is repeated for the novice in the handout so that the novice doesn’t have to look it up in the questionnaire.]

In addition to the thirteen questions found on the novice interview handout, each participant was asked to go over selected tasks from the questionnaire. These additional questions were selected for each individual participant based on consistency or inconsistency indicated by that participant's questionnaire responses. For example, if a participant indicated an approximation image in one task but later failed to indicate an approximation image in another similar task, the participant was asked to elaborate on why they responded in this way. In most cases, novices did not initially see their previous questionnaire responses so as not to bias their current responses during their interview. In some cases, I revealed previous responses because I was prompted by the participant to do so. Participants were also informed that they were not being asked to go over selected questionnaire tasks because they incorrectly responded to the tasks but because I desired to understand their thinking when responding to the selected tasks. By helping participants understand that they were not going over questionnaire tasks because they were incorrect should have helped to reduce participant temptation to switch their answers because they assumed that they missed a task.

Fewer questions were needed for the novice interview than for the expert interview because some of the tasks presented to experts during interviews were presented to novices on the questionnaire. For example, many of the true / false questions presented to the experts during interviews were on the questionnaire. Therefore, it was not necessary to repeat the same true / false question during the novice interview. Similar versions of some of the graphs that the experts saw during their interviews appeared on the questionnaire. Also, expert interview tasks pertaining to

instruction (see Table 11) did not apply for novices, and were therefore removed from their interview tasks.

### *Data Preparation*

*Demographics.* Background data were collected on experts and novices to help determine the level of experience and exposure to convergence of Taylor series. The data were used to create the distinctions between experienced expert participants (EEPs), capable expert participants (CEPs), new novice participants (NNPs), and mature novice participants (MNPs). For experts, this included how long that they had taught college courses containing sequences. For novices, this included a list of mathematics classes that they had taken.

*Transcriptions.* Interview tapes were transcribed as accurately as feasible. This transcription not only included verbal interaction with participants, but also gestures indicative of Taylor series comprehensions. According to the *Merriam-Webster Online Dictionary*, gesture is “a movement usually of the body or limbs that expresses or emphasizes an idea, sentiment, or attitude” (gesture, 2009). Relevant gestures are contained within brackets [...]. Consider the excerpt below where EEP Lewis discusses the remainder associated with a Taylor polynomial and notice how Lewis’ movements have meaning related to what he is discussing.

#### *Excerpt 1*

EEP LEWIS: You sorta gotta see the function [holds up both hands together and moves them apart forming a concave up parabola] and you gotta see the approximating function [moves hands apart along a line in the air] and, and, and see where they’re furthest apart [brings both hands together at his fingers and thumbs and moves fingers away from thumbs forming a ‘C’ shape with both hands as he moves his hands apart].



Lewis' physical movements indicated his internal visualization of a function as a parabola, of a linear Taylor polynomial to the parabola, and of the difference between the two as a 'C' shape that gets larger as he moved away from where the linear Taylor polynomial touched the parabola. Without the detail of Lewis' gestures one would not understand the depths of his thoughts.

Researchers (Hiebert & Carpenter, 1992; Núñez, 2006; Sfard, 2000, 2001; Tabaghi & Sinclair, 2009) affirm that an individual's communication is an indicator of their internal representations and processes. In Sfard's words, "what happens in a conversation... is indicative of what might be taking place in the 'individual heads'" (2000, p. 298). In a later article, Sfard goes on to say the following:

*Thinking may be conceptualized as a case of communication, that is, as one's communication with oneself... The word communication is used here in a very broad sense and is not confined to interactions mediated by language... Speech is no longer considered as a mere 'window to the mind' – as an activity secondary to thinking and coming just to 'express' a ready-made thought. Although there is still room for the talk about thought and speech as two different things, these two 'things' are to be understood as inseparable aspects of basically one and the same phenomenon, with none of them being prior to the other. (Sfard, 2001, pp. 26-27, italics in original).*

Either the spoken word provides one with a "window into the mind" or is an instance of thought itself. In either case, the analysis of verbal transcripts gives a better understanding of participant thoughts, and therefore sheds light on conceptual images that a participant may have. According to Hiebert and Carpenter (1992), "the way in which a student deals with or generates an external representation reveals something of how the student has represented that information internally" (p. 66).

Since communication is more than just "interactions mediated by language" and communication provides one with at least a "window into the mind," everything that goes

into communication must be analyzed, including gestures (Sfard, 2001). According to Tabaghi and Sinclair (2009), “Mathematicians use *gestures* and metaphors to express their thinking about concepts” (p. 9, italics added). According to Núñez (2006), “Gestures show that the fundamental dynamic contents involving infinite sequences, limits, continuity, and so on, are in fact *constitutive* of the inferential organization of these ideas” (p. 178, italics in original). Furthermore, gestures show that the “dynamism involved in these ideas have full psychological and cognitive reality” (Núñez, 2006, p. 178). Physical movements have been recently transcribed by other researchers (Oehrtman, 2008; Rasmussen & Ruan, 2008; Zandieh, Larson, & Nunley, 2008) attempting to better illuminate student understanding and pedagogy in various areas of undergraduate mathematics education.

*Handouts.* The handouts given to participants during interviews not only provided consistency amongst interview tasks by providing a guide, they also provided a place where participants could transcribe their thoughts either symbolically, graphically, or in written word. When elaborating on a particular task in the handout, a participant might write on the handout to collect their thoughts or clarify what they were saying. In many instances the hand written remarks on the handout came in the form of a formula or of a graph. If a participant sketched or wrote while making comments that are contained in an excerpt, a copy of that participant’s dictations will either be contained in the excerpt or will follow the excerpt. Together, the participant’s sketches, writings, verbalizations, and physical movements supplied a much richer picture of the participant’s internal representation and processes than verbal transcripts alone.

## *Data Analysis*

I will conclude this chapter with a discussion of the methods that I used to analyze the data. In this section, I will demonstrate how the focus group, interview, and questionnaire data from the experts and novices were analyzed in an effort to illuminate expert and novice conceptions about the convergence of Taylor series. The analysis of the data consisted of four different layers. The four layers are the following:

Layer 1: The focus group / interview data was coded for existing themes that were an adaptation of an existing framework that was previously created by Williams (1991);

Layer 2: The focus group / interview data was coded for emerging themes. I used an open coding procedure to allow for new themes to emerge from the data in addition to those themes contingent upon Williams (1991);

Layer 3: The questionnaire was analyzed for the themes from Layer 1 and Layer 2. Furthermore, the questionnaire was used to investigate possible indicators and influencing factors for particular themes; and

Layer 4: A detailed focus analysis of the high inference type interviews tasks was used to help demonstrate the differences between experts and novices.

See Figure 1 for the overall design of this study showing the three phases of data collection and the four layers of analysis. I will begin this section with a discussion about why qualitative methods were used in this study. Following this discussion, I will elaborate on each of the four layers of the data analysis.

*Why a Qualitative Study?* As Chapter 2 revealed, very little educational research has been conducted concerning Taylor series, and even when research contained Taylor

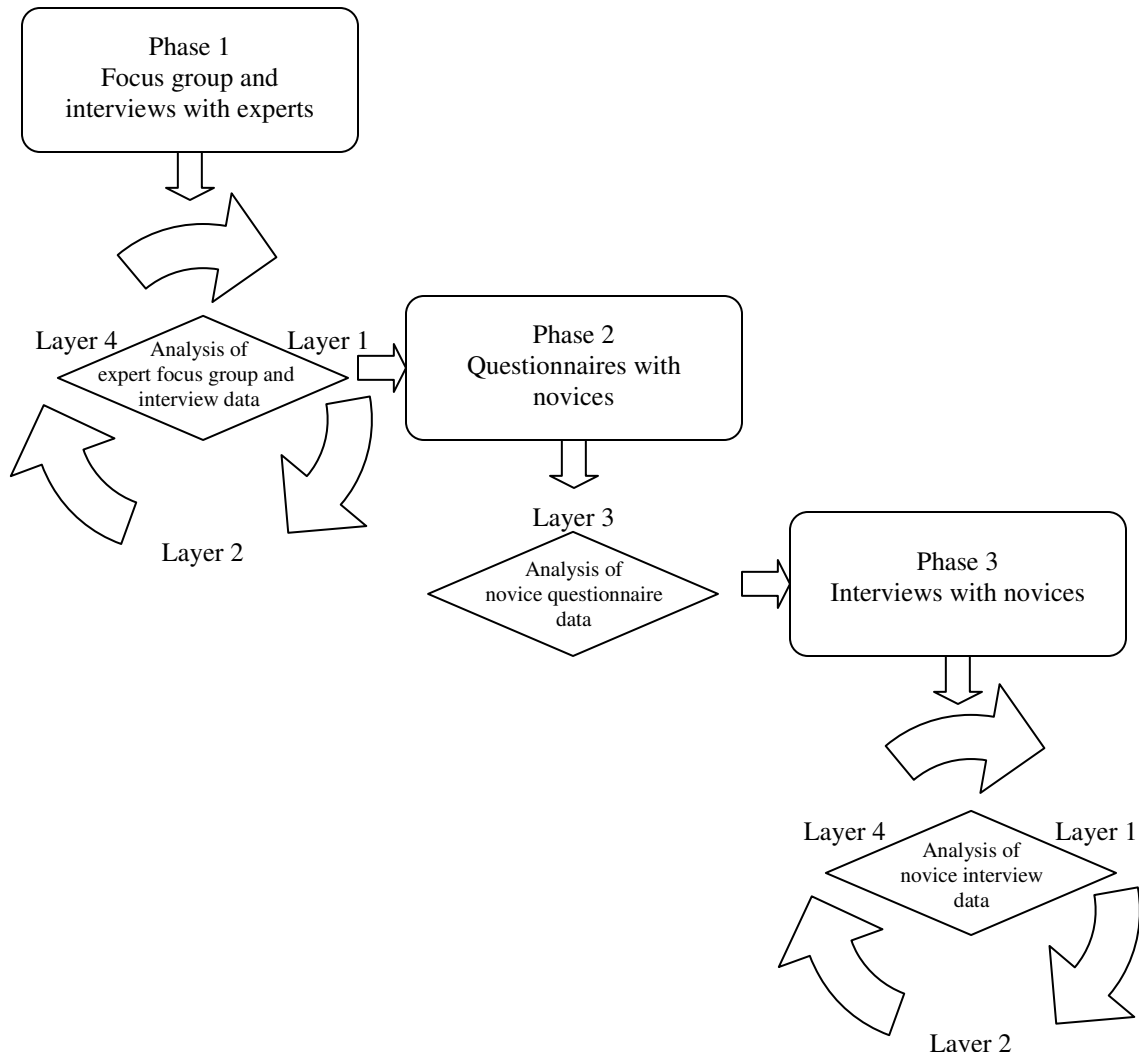


Figure 1. Overall Study Design

series, it alluded to it only as a means of demonstrating an example of what the authors were really researching: for example, limit (Oehrtman, 2002), and approximation techniques (Kidron, 2003). According to Strauss and Corbin (1990), “Qualitative methods can be used to uncover and understand what lies behind any phenomenon about which little is yet known” (p. 19). Therefore, since little was yet known through research methods about how one understands the convergence of Taylor series, I choose to create

a study designed to analyze and describe the understanding of this topic. Hence, an exploratory qualitative study was a natural choice.

Since a qualitative design was used for the study, qualitative methods for analyzing the data were considered. The data were coded using a combination of both grounded theory and focal analysis (Sfard, 2000, 2001; Strauss & Corbin, 1990). Using grounded theory and focal analysis allowed for themes to emerge from the data and informed me of conceptual images in addition to those images modified from Williams' (1991) framework. More on the coding that was contingent upon Williams' (1991) framework will be discussed in the following section.

*Layer 1: Coding the Focus Group / Interviews for Existing Images.* As discussed in Chapter 2, Williams (1991) classified student views of the limit concept into six categories. Following Williams' framework, expert interview tasks were primarily designed to elicit participant views of the notion of limit in the context of convergence of Taylor series. Instead of six categories, five categories were considered. Table 16 describes these five categories.

Table 16  
*Existing Concept Images of the Convergence of Taylor Series*

Image	Description
Dynamic	The Taylor polynomials move toward the generating function within the interval of convergence.
Dynamic unreachable	The Taylor polynomials get close to but never reach the generating function.
Dynamic reachable	The Taylor polynomials get closer and closer to the generating function until the function is reached.
Approximation	The Taylor polynomials approximate the generating function and can be made as accurate as you wish.
Exact	The Taylor series is identical to the function.

\* Adapted from Models of Limit, by Williams, S., 1991, p. 221, *Journal for Research in Mathematics Education*, 22(3), 219-236

Some of William's (1991) categories were excluded because of lack of relevance to Taylor series. For example, limit as bound was excluded. Under William's (1991) description, limit acting as a boundary would not make much sense for Taylor series, because the graphs of the approximating Taylor polynomials frequently pass through the generating function's graph. Exactness was added because of the use of equality when ascribing that the generating function is equal to its Taylor series, and because functions like  $\sin x$ ,  $\cos x$ , and  $e^x$  can be defined by their Taylor series expansions. I believed that participants would demonstrate these images modified from Williams' (1991) images of limit because, mathematically, the concept of limit is the foundation upon which the concept of convergence of Taylor series is rigorously built.

The initial coding of the expert focus group / interview data consisted of direct reading of the transcripts along with corresponding interview handouts. During these readings, I attempted to identify participant statements related to one of these five concept image categories. Keywords such as those found in the statement descriptions were essential in categorizing statements. Once statements were identified as relating to one of the five categories, a more in-depth exploration of each statement ensued. Each statement was coded to analyze and describe its characteristics and properties. To this end, I noted the language used, sketches and writings on the handouts, physical movements, and images relied upon. Some of the coding used during this time could be described as a type of axial coding in the sense that it focused on describing a category in terms of the "conditions that give rise to it; the *context* (its specific set of properties) in which it is embedded; the action / interactional *strategies* by which it is handled,

managed, carried out; and the *consequences* of those strategies” (Strauss & Corbin, 1990, p. 97, italics in original).

*Layer 2: Coding the Focus Group / Interviews for Emerging Images.* The initial coding of the data was contingent upon a predetermined framework. A predetermined framework does not allow for additional concept image categories to emerge from the data. Therefore, additional coding of the transcripts was necessary.

For this additional coding, I read the transcripts and corresponding interview handouts several times. During each reading, I marked the problem context, language usage, and relevant gestures. Preference was not given to the themes contingent upon Williams’ (1991) work so that other themes could be allowed to emerge. During this stage of the coding the goal was to build a framework rather than simply test a framework. A type of open coding was used to create a new list of conceptual images directly related to the understanding of the convergence of Taylor series. According to Strauss and Corbin, “Open coding is the part of the analysis that pertains specifically to the naming and categorizing of phenomena through close examination of the data” (Strauss & Corbin, 1990, p. 62). Furthermore, axial coding was also used. Additional concept images were initially loosely defined and became more tightly defined after reanalysis. The emergent categories from the expert and novice interviews will be revealed and discussed in detail in Chapters 5 and 6, respectively.

*Layer 3: Analyzing the Questionnaires.* The short answer type tasks contained within the questionnaire were coded similar to the coding for the interviews. The questionnaire not only provided additional insight into the understanding of interview participants, but also insight into the understanding of the participants who were not a

part of the interviews. The purpose of the initial analysis of the questionnaires was to identify and classify participant concept images.

Even though the qualitative study was the thrust of the research, a few quantitative elements appeared in this study. The number of participants and the questionnaire tasks provided data that could be analyzed using some statistical procedures. Cross-tabs ("Cross tabulation," 2007) and cluster analysis ("Hierarchical cluster analysis," 2007; Johnson & Wichern, 2002) on the some of the questionnaire data were implemented to determine interactions between tasks and differences between participant groups. Even though these quantitative elements appeared, because of the goal of this study, to analyze and describe understanding, the quantitative elements were used to help create a better qualitative description of participant conceptualizations. Therefore, since the quantitative elements were used for qualitative reasons, I conclude that this study was by far primarily a qualitative study and not a mixed method study.

The analysis of the questionnaire was not merely limited to my interpretation of the data. During the novice participant interviews, novices were asked to interpret the meaning behind selected questions. Gathering this information increased the dependability of my interpretation of the meaning behind questionnaire responses. This part of the data proved vital because in some cases the questionnaire solicited unexpected information. The results and discussion from the questionnaire data will be discussed in Chapter 6.

*Layer 4: Detailed Focus Analysis of the Interviews.* I used the emergent categories to code the high inference type questions that led to an indicator of a participant's primary images, and I used keywords as clues for interpreting the data



(Sfard, 2000). It was during this analysis phase that the interview data were analyzed for the pronounced, attended, and intended focus (Sfard, 2000, 2001). Sfard defines the pronounced focus as the “expression used by [the participant] to identify the object of her or his attention” (2001, p. 34). This “expression” can reveal itself as verbal, written, in the form of physical movements, etc. The keywords used as clues to interpret the data represented a large portion of a participant’s pronounced focus while engaged in a mathematical discourse (Sfard, 2000, 2001). Sfard defines a participant’s attended focus as what a participant is “attending to – looking at, listening to, etc.” (Sfard, 2001, p. 34). During interviews, a participant’s attended focus should be driven by the current question. In other words, participants were attending to listening to or looking at a given question.

The intended focus is the [participant’s] interpretation of the pronounced and attended foci: It is the whole cluster of experiences evoked by these other focal components [pronounced and attended] as well as all the statements he or she would be able to make on the entity in question, even if they have not appeared in the present exchange (Sfard, 2000, p. 304).

According to Sfard (2000, 2001) a type of duality exists between the attended focus and the intended focus. The attended focus can give rise to the intended focus and the intended focus can guide the attended focus. In addition, what is pronounced “gives rise to a primary intended focus clear enough to dictate a rather well-defined attended focus” (Sfard, 2000, p. 314). The intended focus is mostly tacit, even so, because of its intimate ties to the pronounced and attended foci, observing the pronounced and attended foci allowed me to make more dependable inferences about the intended focus of individual participants during interview tasks (Sfard, 2000, 2001).

I used a type of focus analysis to create a focus diagram for each participant. From each diagram one can rapidly determine the focus of the participant as the participant moved through the high inference type questions. Each diagram tracks the changes in focus for each participant during the interview process. In addition, the focus diagrams can illustrate the primary focus of individual participants. If a participant consistently relied upon a particular image throughout the high inference type questions, then that is an indicator that that image is the primary focus in that participant's mind. Hence, the diagram tells the "story" of each participant as the participant interacted with the mathematical content posed during the interview. Besides looking at individual diagrams, grouping the diagrams together, such as experts and novices, are an aid in making comparisons between groups. Therefore, the diagrams not only tell the "story" of the individual participants, but taken together, the diagrams are instrumental in telling the "story" of experts and novice understanding of the convergence of Taylor series. The following chapters will introduce the focus diagrams and the story will start to unfold.

### *Chapter Summary*

Participants in this study were divided into two main groups, experts and novices, based on each participant's amount of experience with Taylor series. Experts were classified as graduate students or faculty from The University of Oklahoma (OU) and novices were classified as undergraduate students from either OU or a regional community college (RCC). In total, 16 experts and 131 novices participated in this study. Experts and novices were each divided into two subgroups, experienced expert participants (EEP) and capable expert participants (CEP), and new novice participants (NNP) and mature novice participants (MNP), respectively. These subgroups were

created to help further identify a participant's level of exposure to Taylor series. Using these subcategories NNPs had the least amount of exposure to Taylor series, while EEPs had the most amount of exposure.

There were three phases of data collection. Phase 1 consisted of focus group and individual interview data collected from experts. Phase 2 consisted of questionnaire data collected from novices. Phase 3 consisted of individual interview data collected from novices. During each phase of data collection, tasks were arranged from high inference type questions to low inference type questions when applicable. This format allowed me to make more reliable inferences about a given participant's primary mental imagery and probe deeper into their understanding. Experts went first because their mental capabilities (Carlson & Bloom, 2005; Hiebert & Carpenter, 1992; Lester, 1994) proved useful for me to create a better questionnaire to use during Phase 2 of the study. Within Phase 2, there was consistency amongst the questionnaire questions to help create a more readable and reliable questionnaire. In addition, effects, such as the interval of convergence and differing notations, were accounted for within the questionnaire to help determine their influence on the conceptions of Taylor series. In Phase 3, many tasks presented to experts during interviews were repeated during interviews with novices. This allowed for better comparison between experts and novices.

Data for this study consisted of background data, transcripts, and handouts. The background data enabled me to differentiate between the different categories of participants. Transcriptions of relevant video and audio tapes included both verbal interactions and gestures related to Taylor series. The handouts were a place where participants could sketch or write down their thoughts during interviews.

The analysis of the data came in four layers. In the first layer, the focus group / interview data were coded for existing themes adapted from Williams (1991). The second layer consisted of an open coding scheme applied to the focus group / interview data. During this coding, additional themes could emerge distinct from those themes related to Williams (1991). The third layer consisted of an analysis of the questionnaire data for both the existing and emerging themes. The fourth layer of analysis was a focus analysis of the common tasks between the experts and novices. The fourth layer of analysis used focus analysis to help identify similarities and differences between experts and novices. Focus diagrams were created for each participant to help tell the “story” of how experts and novices interacted with particular Taylor series tasks.

## CHAPTER 4

### Yesterday's Experts A Discussion of the History of Taylor Series

#### *Excerpt 2*

Task 1: What are Taylor series?

MNP PHILIP: Well Taylor series are uh, I'm not sure, I can't remember exactly how they were uh, uh, where they originated from, how they were uh, determined to begin with, who came up with it. I guess it was a guy named Taylor that came up with it.

Did a guy named Taylor come up with Taylor series or are Taylor series merely named after him? If a guy named Taylor did not come up with Taylor series, who did? Why did someone first use Taylor series? How did they conceptualize convergence in the context of Taylor series? Did Taylor series influence the development of calculus? These were some of the questions that provided the impetus for this chapter, and in some cases the answers proved to be more allusive than anticipated. Even so, in this chapter I will attempt to provide answers to these questions.

When considering how people comprehend topics within mathematics it is important to consider how the conceptions were developed and to investigate how they may have changed over time. In many cases this can be a very beneficial tool that aids instruction by giving insights into the development of student conceptions that mirror the historical development of particular mathematical topics (Juter, 2006; Kidron, 2003). Therefore, in this chapter I will use historical evidence to comment on some of the ways in which experts throughout history may have conceptualized the convergence of Taylor series. I will demonstrate how people like, Isaac Newton, Brook Taylor, James Gregory,

Joseph Louis Lagrange, and Augustin-Louis Cauchy along with other analysts their time, used Taylor series in various contexts. I will show some of the questions that they asked concerning Taylor series, and perhaps more importantly, I will remark on some of the questions that they didn't ask. I will show how Lagrange attempted to counter criticisms from a Bishop by using Taylor series as the very cornerstone of calculus, and how Cauchy influenced conceptualizations of the convergence of Taylor series by introducing rigor. I will conclude this chapter with insights into how these founders of calculus might have understood the convergence of Taylor series and how that understanding relates to individuals today.

### *Setting the Stage*

In 1685, James II became King of England after the death of King Charles II. Louis XIV was King of France and a mentally failing Charles II of Spain was ruling his struggling country. Ten of the thirteen colonies in the Americas were organized. Names like René Descartes, Baruck Spinoza, John Locke, and David Hume were influencing what is today called the Age of Reason. This age was characterized by a philosophy that either believed that knowledge came through the senses, the Empiricist, or that knowledge came through reason alone, the Rationalist. Isaac Newton was at the University of Cambridge as the Lucasian chair, one of the highest honors that could be bestowed upon a scientist. Newton was currently working on the famous *Principia Mathematica* which was to be published in two years in its complete Latin version (Burton, 2007, p. 387). In the previous year, Gottfried Wilhelm Leibniz had published some of his results on differential calculus which would be followed the next year with a

paper on integral calculus. It was in this year, 1685, that a boy was born by the name of Brook Taylor.

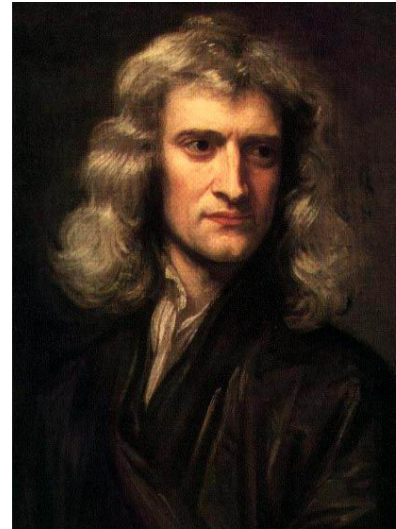
### *Isaac Newton*

More than four decades earlier, the year was 1642 and Isaac Newton was born on Christmas Day near the university in which he would someday be one of the most influential mathematicians to ever live. Newton was born during the Thirty Years' War involving most of Europe and during the English Civil War which was a power struggle that pitted the monarchy against the

parliament. The Thirty Years' War ended six years later in 1648 and the civil war concluded the next year when the monarchy was removed from power after King Charles I was executed and Oliver Cromwell was

established as Lord Protector of England. After a twelve year hiatus from the rule of a king, the monarchy regained control after Cromwell's death and the son of Charles I, Charles II, was crowned King of England in 1661. By this time, Isaac Newton was 18 years old. This period of political unrest and war witnessed by a young Newton was soon to be followed by a period of disease and death.

At intermittent points throughout history, the Bubonic plague hit parts of Europe. From 1665 to 1666 the plague struck England, killing up to a fourth of London's population (Burton, 2007). Newton was admitted to Cambridge a few months after Charles II was crowned King of England, and in 1665 he received his degree. Shortly



*Figure 2.* Isaac Newton  
\*From Isaac Newton. (2009, July 29). In *Wikipedia, The Free Encyclopedia*. Retrieved July 29, 2009, from [http://en.wikipedia.org/w/index.php?title=Isaac\\_Newton&oldid=304814916](http://en.wikipedia.org/w/index.php?title=Isaac_Newton&oldid=304814916)

thereafter, Cambridge was closed during the time of the Bubonic plague and Newton, then 22 years old, was forced to live in isolation at home. It was during this seemingly bleak period that the secluded Newton discovered differential calculus, separated the visual spectrum, and discovered the law of gravitation (Burton, 2007, pp. 392-393). In 1666, the Great Fire of London started in a bakery and lasted over a period of four days. The fire destroyed the vast majority of homes in London and left about half of the city in ashes. After the fire, the plague ended and Newton returned to Cambridge in 1667 with his newfound discoveries.

### *Brook Taylor*

Moving forward to the year 1715, George I had just become King of Great Britain, succeeding Queen Anne who had successfully united Scotland and England and in so doing established the United Kingdom of Great Britain. The year before, Britain won a thirteen year war with France and Spain. As a result of the win, the British Empire gained territories from France and Spain, and made its mark in the history books as one of the greatest empires in history. In Spain, King Philip V ruled, and in France, Louis XIV died after a fifty-four year rule and left his throne to his great-grandson, Louis XV. In the Americas, the United Kingdom of Great Britain controlled twelve of the thirteen colonies. Furthermore, the Age of Enlightenment had begun. In this age, Voltaire published his *Elements de la philosophie de Newton* in which he attempted to



*Figure 3.* Brook Taylor  
\*From Brook Taylor. (2009, July 29). In *The MacTutor History of Mathematics Archive*. Retrieved July 29, 2009 from <http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Taylor.html>



demonstrate that there was order in the universe and that this order could be described using the rules of mathematics (Burton, 2007). Others joined Voltaire and attempted to use rules similar to mathematical laws in an effort to describe the “whole range of human experience” (Burton, 2007, p. 520). As “the most influential work of the eighteenth century,” the *Encyclopédie* which praised the scientific method, would also be published during this age (Burton, 2007, p. 522). In addition, the flying shuttle and the Spinning Jenny were to be soon invented, necessary ingredients for the upcoming industrial revolution which begun around 1770.

It was in this year, 1715, at the age of 30, that Brook Taylor published *Methodus Incrementorum Directa et Inversa*. This would be the first publication in which a general formula for Taylor series was given, a formula that Taylor discovered at least three years prior in 1712 (Burton, 2007; Jahnke, 2003; Struik, 1969). Because he was influenced by Newton’s calculus, the notation that Taylor used was very different from the notation that is commonly used today. In Corollary II of Proposition VII of Theorem III from Taylor’s *Methodus Incrementorum*, the classical Taylor series was written as the following:

When  $z$  flows uniformly into  $z + v$ ,  $x$  becomes

$$x + \dot{x} \frac{v}{1.\dot{z}} + \ddot{x} \frac{v^2}{1.2.\dot{z}^2} + \ddot{\ddot{x}} \frac{v^3}{1.2.3.\dot{z}^3} \&c$$

(as cited by Struik, 1969, p. 332)

As this excerpt from Taylor’s *Methodus Incrementorum* demonstrated, Taylor’s notation may be difficult to read by some modern mathematicians not familiar with Newton’s fluxion and evanescent quantity notation. In today’s modern notation this translated into

$$f(x+h) = f(x) + \frac{f'(x)h}{1!} + \frac{f''(x)h^2}{2!} + \frac{f'''(x)h^3}{3!} + \dots$$

for a general function  $f$  (Calinger, 1995, p. 465; Struik, 1969).

Taylor entered Cambridge in 1701, and received a Bachelor of Laws in 1709 and a Doctor of Laws in 1714. Prior to receiving his Doctor of Laws, in 1712 Taylor was elected to the Royal Society of London, a society for science in which Newton had been president since 1703, and in 1714, Taylor became the secretary of this society (Calinger, 1995; O'Connor & Robertson, 2000a). Even though Taylor was born into a wealthy family who frequently had visits from artists and musicians, Taylor was not kept from illness and tragedy (Anton, 1992). In 1718, Taylor stepped down from being secretary of the Royal Society of London due to ill health and maybe from lack of interest in the secretary's responsibilities (Calinger, 1995; O'Connor & Robertson, 2000a). Without his father's approval, Taylor married in 1721 causing a rift between him and his father that would not be mended until two years later after the death of his wife during childbirth (Anton, 1992; Calinger, 1995). The child was also lost. In 1725, Taylor married again but this time with his father's approval. His father died in 1729 and the next year Taylor lost his second wife in childbirth, fortunately this child, a girl he named Elizabeth, survived (Calinger, 1995). A year later, Taylor lost his battle with his ill health and died.

Before Taylor's health began to fail, Taylor was able to write on a wide range of subjects. For the purposes of this study, I will concentrate on his mathematics. Taylor studied mathematics under John Machin and John Keill (Calinger, 1995; O'Connor & Robertson, 2000a). John Keill, discipled by David Gregory and Newton himself, was an outspoken defendant of Newton's discovery of the calculus and an outspoken adversary of Leibniz's discovery (Ball, 2001; Boyer, 1991; Burton, 2007; O'Connor & Robertson, 2005b). In fact, it was John Keill's abrasive accusations that Leibniz plagiarized

Newton's work that fueled the calculus discovery dispute and eventually led to the Royal Society of London's formation of a special committee in 1712 to settle the calculus controversy (Ball, 2001; Boyer, 1991; Burton, 2007). Brook Taylor, and John Keill were both on this committee that, after a month, concluded that their current Royal Society president, Newton, was the "first inventor" of calculus and failed to vindicate Leibniz from plagiarizing (Burton, 2007; Calinger, 1995; O'Connor & Robertson, 2003).

Besides John Keill, the other already mentioned significant mathematical influence on Brook Taylor was John Machin. Together with Taylor and Keill, Machin was also a member of the Royal Society and was on the committee that "settled" the calculus controversy (O'Connor & Robertson, 2003). In addition, it was Machin who took Taylor's place in 1718 as secretary of the Royal Society and remained in that position for nearly thirty years (O'Connor & Robertson, 2003). As a mathematician, John Machin developed a series that converged to pi, and in 1706 he used this series to calculate pi correct to 100 decimal places (Burton, 2007; Eves & Eves, 1992; O'Connor & Robertson, 2003).

Three years prior to his *Methodus Incremento*, Brook Taylor wrote a letter to Machin on the 26 of July in 1712. It was in this letter that one first sees Taylor's Theorem (O'Connor & Robertson, 2000a). In the letter Taylor credited the idea to coffeehouse conversations with Machin about Isaac Newton's works on Kepler's problem on planetary motion and about Edmond Halley's work with roots of polynomials (Anton, 1992; O'Connor & Robertson, 2003). But was Taylor the first to come up with formula for series that now bears his name?

### *Don't Forget James Gregory*

John Keill, one of the primary mathematical influences of Brook Taylor, studied under David Gregory, the nephew of James Gregory (O'Connor & Robertson, 2005a, 2005b). Over forty years before Brook Taylor's discovery of Taylor Series, James Gregory uses specific "Taylor" series in his works dating back to 1667. Eight years after his initial use of these specific Taylor series, James Gregory become ill and died prior to his 37<sup>th</sup> birthday. Had Gregory lived longer, the respected mathematics historian Dirk J. Struik



*Figure 4. James Gregory*  
\*From James Gregory. (2009, July 31). In *The MacTutor History of Mathematics Archive*. Retrieved July 31, 2009 from <http://www.gap-system.org/~history/Mathematicians/Gregory.html>

believed that Gregory "might have ranked with Newton and Leibniz as an inventor of calculus" (Struik, 1967, p. 114). For example, Gregory had related the derivate to the integral via inverses in his *Geometriae* of 1668, which unfortunately remained virtually unnoticed at the time (Kline, 1972). The mathematics historian Carl Boyer echoed Struik's comments about Gregory when he said that "[Gregory] might have anticipated Newton in the invention of the calculus, for virtually all the fundamental elements were known to him by the end of 1668" (Boyer, 1991, p. 386).

Much of what is known about Gregory's mathematical ideas and discoveries were buried in letters and notes effectively keeping some of his work out of public view until 1939 (Dehn & Hellinger, 1943; Kline, 1972). In Gregory's *appendicula ad veram circuli et hyperbolae quadratam* of 1668 there is strong indication that he knew the first few terms of the Taylor series for certain trigonometric and hyperbolic functions (Dehn &

Hellinger, 1943). Other historians credit Gregory with the discovery of the Taylor expansions for sine and cosine in the year prior, 1667, in his *Vera circuli et hyperbolae quadratura* (Ball, 2001). Still others credit Isaac Newton with the discovery of the expansions for sine and cosine in *De Analysisi* of 1669 (Kline, 1972). Even Leibniz may have independently discovered series expansions for sine and cosine in 1673 (Kline, 1972). If Gregory had discovered the Taylor expansions for sine and cosine in 1667, it would predate Newton's discovery by two years and Leibniz's by six. In addition to the discovery of series expansions for certain trigonometric and hyperbolic functions, in 1670 Gregory discovered the binomial series (Struik, 1969). Gregory most likely did this without knowledge of Newton's prior discovery in 1666 because Newton's discovery was unpublished and remained relatively concealed for a decade (Boyer, 1991; Dehn & Hellinger, 1943; Struik, 1969). In a letter dated the 15 February 1671, addressed to a publisher friend and librarian of the Royal Society, John Collins, Gregory listed series expansions up to five or six terms for the following seven functions found in Table 17.

Table 17  
*Seven Nonstandard Functions Expanded by James Gregory*

arctan $x$	$\tan x$	$\sec x$	$\log \sec x$
$\log \tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$	$\text{Arc sec}(\sqrt{2}e^x)$	$2 \arctan\left(\tanh \frac{x}{2}\right)$	

\*As cited in Certain Mathematical Achievements of James Gregory by Dehn, M., & Hellinger, E., 1943, p. 149, *The American Mathematical Monthly*, 50 (3), 149-163.

Further indicating Gregory's genius was his consideration given to a series expansion's remainder, something that would not be extensively considered by mathematicians again for over a century. In a letter dated the 9 April 1672, James

Gregory stated the first three terms of the Taylor series for hyperbolic sine with the addition of an estimated remainder term (Dehn & Hellinger, 1943). He then stated that one can get a better approximation of the hyperbolic sine by using more terms, and he indicated the corresponding estimated remainder term associated with using more terms (Dehn & Hellinger, 1943). Gregory's use of the remainder was indicative of what I will later call a remainder concept image in Chapter 5. Furthermore, many historians would argue that Gregory was the first to distinguish between convergent and divergent series (Ball, 2001; Boyer, 1991; Eves & Eves, 1992).

Gregory and Newton were not the only mathematicians to use series expansions prior to Taylor's publication in 1715. As noted previously, Leibniz used series expansions in his calculus. In 1668, using a method from Gregory, German mathematician Nicolaus Mercator produced a series for  $\log(1+x)$  which Newton had already known previously but had failed to publish prior (Burton, 2007; Kline, 1972). There is evidence that another mathematician by the name of Johann Bernoulli had known of a type of Taylor theorem prior to Taylor's discovery in 1712 (Calinger, 1995; Jahnke, 2003). For example, two decades prior to Taylor's publication, in Johann Bernoulli's *Acta eruditorum* of 1694 appeared the following formula:

$$\text{int. } n \, dz = nz - \frac{1}{2} z z \frac{dn}{dz} + \frac{1}{1 \cdot 2 \cdot 3} \frac{z^3 ddn}{dz^2} - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \frac{z^4 dddn}{dz^3} + \text{etc.}$$

where "int." was the symbol used by Bernoulli to represent the integral and "n" represented a function in terms of  $z$  (Jahnke, 2003, p. 111). Because Bernoulli's formula from 1694 was equivalent to Taylor's theorem published in 1715, some historians credit Bernoulli with the discovery of Taylor's theorem (Calinger, 1995). Yet there is little evidence to suggest that Johann Bernoulli realized that he was in possession of a variant

of Taylor's theorem. For example, because of the way in which it was presented by Bernoulli in this integral form, it turned out to not be as usable as Taylor's version (Jahnke, 2003, p. 112). Had he possessed Taylor's version of the formula he might have chosen to use it instead of his unwieldy integral version.

Gregory and Newton were not the first mathematicians to know about series expansions. Unknown to Gregory and Newton, the Indian mathematician Madhava of Sangamagramma had successfully produced the first few terms of the Taylor expansions for sine, cosine, and arctangent (O'Connor & Robertson, 2000b). Although none of Madhava's original work still exists, his students credited Madhava with accomplishing this feat around the year 1400, over 250 years prior to Gregory's and Newton's works (O'Connor & Robertson, 2000b)! Even if it was Madhava's students, and not Madhava himself, who had discovered these series expansions, a discovery by his students would still predate Gregory's and Newton's discovery by a century (O'Connor & Robertson, 2000b)!

Even though James Gregory applied Taylor series to several specific functions, was James Gregory in possession of the formula for Taylor series applied to generic functions? Since James Gregory was able to list off series expansions for several different functions one might believe that Gregory did indeed know Taylor's formula applied to generic functions. Because of this fact, some historians credit Gregory with the first discovery of Taylor's formula applied to generic functions (Boyer, 1991; Kline, 1972; Struik, 1967). Yet some historians believe that there is little indication that Gregory understood Taylor's theorem applied to generic functions (Dehn & Hellinger, 1943). Although James Gregory found series expansions for several functions, Gregory

was not using series expansions for the sake of studying series expansions. Although mathematicians such as Gregory and Newton extensively used series expansions prior to Brook Taylor's 1715 *Methods Incrementorum Directa et Inversa*, they used the expansions for very specific functions, such as for sine and cosine, to solve very specific problems. Even then, they were only concerned with using the first few terms of the series because it would ensure their desired accuracy (Grabiner, 1981). Much of this was the result of applications being the driving force behind discovering series expansions. For example, where one sees Gregory use the series expansions for sine and cosine in his *Vera circuli et hyperbolae quadratura*, Gregory was working with problems concerning the area of the circle and hyperbola (Ball, 2001). For Isaac Newton, the first few terms of series expansions were a tool to help solve problems in physics and astronomy. Therefore, these mathematicians were concerned with the applications of Taylor series more than the study of Taylor series themselves. Not until Brook Taylor's 1715 theorem does one see Taylor's theorem stated for a general function  $f$ . Even after this discovery, most mathematicians continued to look at Taylor series expansions for only specific functions where only the first few terms of the expansions were used for estimations.

From Gregory to Newton to Taylor, all the mathematicians of this time had only partial understanding of Taylor's series. Gregory and Newton both worked with series expansions for specific functions. Although Gregory estimated the remainder term, he had not come up with the remainder formula associated with Taylor's Theorem. Indeed many people may find it surprising that the remainder formula was not discovered by Brook Taylor. It was not until the turn of the century that another mathematician came up with the full version of Taylor's Theorem with an additional remainder term. Brook



Taylor would have had little motivation to discover a remainder formula because mathematicians of his day believed that all functions could be represented by series expansions (Grabiner, 1981; Jahnke, 2003). Therefore, the evidence suggests that Gregory, Newton, and Taylor all believed that every function could be viewed as an infinite polynomial. Why would mathematicians of this time period think otherwise? After all, the specific functions that they were working with could be represented with series expansions.

*Enter a Bishop*

George Berkley was Irish Bishop and theologian. He was also an admirable mathematician and a scathing critic. In particular, his accurate criticisms of Newton's calculus helped spur the mathematics community to consider rigor in areas where it had previously validated results using merely empirical tests. One of his most notable attacks was *The analyst: or a discourse addressed to an infidel mathematician* published in 1734 (Struik, 1969). In it he attacked Newton's notion of limits and derivatives using fluxions.



*Figure 5. George Berkeley*  
 \*From George Berkeley. (2009, July 31). In *The MacTutor History of Mathematics Archive*. Retrieved July 31, 2009 from <http://www.gap-system.org/~history/Mathematicians/Berkeley.html>

*And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them ghosts of departed quantities?* (as cited in Struik, 1969, p. 338)

Berkley did not question Newton's conclusions, only Newton's method (Boman, 2008). Newton's use of fluxions and empirical tests for validation of calculus results certainly put Bishop Berkley's critical comments on solid ground and seemingly put calculus in

sinking sand. It is important to note that Berkley was not specifically attacking Taylor series but the very foundations of Newton's calculus. Yet, this seemingly negative event would cause the mathematics community to respond and in the end, for the first time, put calculus on solid ground and usher in the age of rigor. Some of the first major attempts to put calculus on solid ground employed Taylor series.

Colin Maclaurin, most known today for the specific Taylor series centered at zero that bears his name, was amongst the first to respond to Bishop Berkley's critics of Newton's calculus. Eight years after Berkley's *Analyst*, Maclaurin's *Treatise on Fluxions* of 1742 was a noteworthy attempt to make calculus rigorous using geometrical methods similar to what ancient Greeks had used (Boyer, 1991; Burton, 2007; Calinger, 1995; Jahnke, 2003). Despite a valiant effort, Maclaurin's attempt failed to refute Berkley's critiques, partly because of the very geometrical methods that he had intentionally employed (Boyer, 1991; Calinger, 1995). Even though the *Treatise on Fluxions* failed to refute Berkley, it contained the integral test for series and, although not in modern notation, a method involving differentiation for formulating the Maclaurin series that is still used today in many calculus books to introduce students to Taylor series (Boyer, 1991; Burton, 2007; Calinger, 1995; Stewart, 2008; Struik, 1969).

#### *Enter Joseph Louis Lagrange*

Along with Leonhard Euler, Joseph Louis Lagrange was one of the greatest mathematicians of the 18<sup>th</sup> century. Lagrange was able to influence almost all areas of mathematics and in many areas this influence was significant. Born in Turin Italy in 1736, he became a professor in geometry in Turin at the age of 18 (Burton, 2007). At the request of Frederick the Great, Lagrange moved to Berlin to take Euler's place at the

Berlin Academy in 1766. Lagrange stayed in Prussia at the Berlin Academy for twenty years, until the death of Frederick the Great in 1787. Following Frederick's death, King Louis XVI of France offered Lagrange an opportunity to settle in Paris in the Louvre and become a French citizen (Ball, 2001; Burton, 2007). Lagrange accepted King Louis XVI invitation, and turned down similar invitations from Spain and Naples (Ball, 2001). Under Napoleon, in 1797 Lagrange was appointed as a professor in the Ecole Polytechnique, which would soon become a fertile breeding ground for many future mathematicians (Ball, 2001; Burton, 2007). In 1808 Lagrange was recognized with the Legion of Honour and Count of the Empire awards (Anton, 1992; O'Connor & Robertson, 1999). A few years later, Lagrange died in Paris on the 10 April 1813 and was buried in the Panthéon with honors (Anton, 1992; Ball, 2001).

During his lifetime, Lagrange was a favorite of leaders, from Frederick the Great, to King Louis XVI, to Napoleon Bonaparte. Also during his lifetime, multiple revolutions forever changed the course of history. Oversees the American Revolution liberated America from the control of Great Britain. Ten years after the end of America's war with Great Britain, Lagrange was in France during the French Revolution in which King Louis XVI was tried and beheaded for treason in January 1793 (Burton, 2007). Shortly thereafter, Napoleon rose to power and crowned himself Emperor in 1804.



*Figure 6. Joseph-Louis Lagrange*

\*From Joseph-Louis Lagrange. (2009, July 31). In *The MacTutor History of Mathematics Archive*. Retrieved July 31, 2009 from <http://www.gap-system.org/~history/Mathematicians/Lagrange.html>

Lagrange did not live to see Napoleon's defeat and the brother of Louis XVI reclaim the throne of France in 1814.

So, what was Lagrange's contribution to Taylor series? Taylor's Theorem with the remainder term. In fact, Lagrange was the first to write Taylor's Formula in its modern notation of

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

where  $c$  is some number between  $x$  and  $a$  (Grabiner, 1981). The remainder term to Taylor's Theorem appeared for the first time in his 1797 book *Théorie des fonctions alaytiques* (Burton, 2007; Grabiner, 1981; Jahnke, 2003).

Grabiner (1981) went on to conclude that it was d'Alembert who influenced Lagrange to find the remainder when nearly three decades earlier, d'Alembert asked, "What are the bounds on the error made in approximating the sum of the infinite [binomial] series by the sum of a finite number of its terms?" (p. 61) D'Alembert also asked, "Under what circumstances do the successive terms decrease?" (Grabiner, 1981, p. 61) D'Alembert went on to answer his own questions using, for the first time, the ratio test (Grabiner, 1981). According to Grabiner (1981), d'Alembert's paper of 1768 and a paper by Lagrange, written at about the same time, were the first instances when d'Alembert's two questions were considered together. Unfortunately for d'Alembert and other mathematicians of the 18<sup>th</sup> century, the question of convergence of series was identical to the question of successively decreasing terms (Grabiner, 1981, p. 61). Therefore, for d'Alembert, the second question above was the same as asking, "Under what circumstances does the series converge?" Even though d'Alembert's sufficient condition for series convergence was in error, asking these questions was a necessary step

to come up with the full version of Taylor's theorem and eventually rigorously show that certain functions definitely have Taylor series expansions.

Recall that Bishop Berkley had criticized Newton's calculus for not having a rigorous foundation. Mathematicians began to respond and, like Colin Maclaurin, Lagrange began to take rigor seriously (Grabiner, 1981). This was clearly evidenced in that while at the Berlin Academy, Lagrange acknowledged Bishop Berkley's critiques through the Berlin Academy's prize competition when he asked all of academia to find better foundations for calculus (Grabiner, 1981). This was one of many steps that would help move calculus into an age of rigor.

One small step toward a better foundation for calculus occurred with a seemingly insignificant notational change that would forever alter the emphasis of derivatives. As noted previously, Brook Taylor's notation for Taylor series did not resemble the notation used today since it relied on Newton's fluxion notation (see Struik 1969 for an example of Taylor's work on series). Colin Maclaurin's notation also made use of Newton's fluxion notation even though Maclaurin had attempted to cut ties with infinitesimals and moments (see Struik 1969 for an example of Maclaurin's work on series) (Jahnke, 2003). Lagrange moved away from Newton's cumbersome fluxion notation in his highly influential work *Théorie des fonctions alaytiques* of 1797, the same book that contained Lagrange's remainder theorem for Taylor series expansions. To establish algebraic-like rigor in the calculus, instead of using infinitesimals and fluxions as had been done before, Lagrange used series expansions to define the first derivative as the coefficient of the linear term in the Taylor series (Burton, 2007; Grabiner, 1981). By doing this he was able to quickly define higher order derivatives by looking at the corresponding

coefficients of higher order terms in Taylor series (Burton, 2007; Grabiner, 1981). To denote the derivatives in the Taylor series Lagrange used the prime notation, which is still used today to denote derivatives (Burton, 2007; Grabiner, 1981; Jahnke, 2003). Lagrange was the first to use the prime notation, and it appeared as early as 1772 in another paper in which he shared his idea of attempting use algebra as the cornerstone of calculus (Grabiner, 1981). The prime notation helped move the focus of mathematicians away from seeing derivatives merely as ratios of infinitesimals and more to seeing derivatives as functions (Grabiner, 1981). *Théorie des fonctions alaytiques* was an attempt by Lagrange, not necessarily to introduce new results, but to reduce calculus to algebra by moving it away from infinitesimals and put it on algebra's already firm foundation, therefore seemingly countering Bishop Berkley's critiques of calculus (Burton, 2007; Grabiner, 1981; Jahnke, 2003). By using Taylor series instead of limits as the foundational element, Lagrange was in effect declaring the supremacy of Taylor series in the calculus.

Lagrange's fatal flaw was that he had not well addressed the question of convergence when working on his *Théorie des fonctions analytiques*. As is now known, not every continuous, infinitely differentiable function has a Taylor series expansion. Like mathematicians of his time, Lagrange believed that every function worthy of consideration in calculus was equal to its Taylor series except possibly at particular isolated values of  $x$  (Grabiner, 1981; Jahnke, 2003). Therefore, Lagrange did not seriously consider the question of convergence. As noted previously, d'Alembert had considered the question of convergence but had erroneously reduced this question to the question of successively decreasing terms. Before d'Alembert and almost fifty years

prior to Lagrange's *Théorie des fonctions alaytiques*, Euler considered the question of the convergence of Taylor expansions to their generating functions in his *Introductio in analysin infinittorum* of 1748 (Grabiner, 1981; Jahnke, 2003). Euler, in his *Introductio*, showed that the functions that were commonly used at that time could be represented as infinite series (Grabiner, 1981). Like many other analyst of the 18<sup>th</sup> century Lagrange had little reason to consider the question of convergence of Taylor series because it was believed that every important function effectively equaled its Taylor series expansion. Therefore Lagrange never actually used his remainder formula to show that series converged (Kline, 1972). Lagrange only used his remainder formula to determine the remainder associated with using a given Taylor polynomial (Kline, 1972).

#### *Enter Cauchy*

Newton and Leibniz may have discovered calculus but it was Cauchy who effectively laid a firm foundation and silenced the long lasting criticisms of Bishop George Berkley. How did Cauchy do this? Cauchy did it with a collection of small changes that together constituted one monumental shift in how to approach calculus.

Augustin Louis Cauchy was born in Paris on 21 August 1789 and died on 23 May 1857 in Sceaux (near Paris). After attending school at the Ecole Polytechnique, the school in which Lagrange taught, Cauchy pursued a career in engineering, but after publishing a series of memoirs, Cauchy was contacted by



*Figure 7.* Augustin Louis Cauchy

\*From Augustin Louis Cauchy. (2009, July 31). In *The MacTutor History of Mathematics Archive*. Retrieved July 31, 2009 from <http://www.gap-system.org/~history/Mathematicians/Cauchy.html>

Lagrange and Laplace and persuaded to pursue a career in science and mathematics instead (Burton, 2007; Calinger, 1995). In 1813 he accepted an instructorship at the Ecole Polytechnique and was full professor by 1816 (Burton, 2007). In 1814 Napoleon was defeated and Louis XVIII reclaimed the throne for the Bourbon monarchy. Because Cauchy was an avid supporter of the Bourbon monarchy, Cauchy went into exile soon after Charles X abdicated the throne, and Louis-Phillipe King of the French, a non-Bourbon, took control of France as a result of the French Revolution of 1830 (Burton, 2007). During this period, Cauchy spent time at the University in Turin and in Prague tutoring the grandson of Charles X (Burton, 2007). Cauchy returned to France in 1838 but was not allowed to be a professor at Ecole Polytechnique. It took another French revolution that ended with the removal of Louis-Phillipe in 1848 before Cauchy would have a professorship at any university in France (Burton, 2007).

By the time of Euler, many 18<sup>th</sup> century analyst were aware that they were using series results “whose legitimacy they could not really prove” (Jahnke, 2003, p. 119). Newton, Liebnez, Taylor, and Maclaurin were well aware of divergent series throughout the 18<sup>th</sup> century. For example, in 1689, Jakob and Johann Bernoulli had shown that the harmonic series was divergent (Jahnke, 2003, p. 118). But what did divergence mean for 18<sup>th</sup> century analysts? Many analysts of this time treated divergent series similar to convergent series. For example, when using the geometric series, it was know that  $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ , and so when  $x = 2$ , an 18<sup>th</sup> century analyst might write that  $-1 = 1 + 2 + 4 + 8 + \dots$  (Burton, 2007; Jahnke, 2003). Some questioned this practice.



Guido Grandi used the geometric series  $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$  in his *Quadratura*

*Circuli et Hyperbolae* of 1703 to show that  $\frac{1}{2} = 0$  by demonstrating that

$$\frac{1}{2} = \frac{1}{1+1} = 1 - 1 + 1 - 1 + \dots = (1-1) + (1-1) + \dots = 0 + 0 + \dots = 0$$

(Burton, 2007; Jahnke, 2003). Guido went on to proclaim that because of this result it proved that the world could be created out of nothing (Burton, 2007; Jahnke, 2003).

Liebniz then noted that depending on how one grouped the terms in the series expansion,

$$(1-1) + (1-1) + (1-1) + \dots \text{ verses } 1 + (-1+1) + (-1+1) + (-1+1) + \dots,$$

one might get the sum to be zero or one might get the sum to be 1, and therefore the sum should be the average of the two results, 1/2 (Jahnke, 2003). Another mathematician by the name of Varignon proposed that the series had no sum (Jahnke, 2003). According to Morris Kline (Kline, 1972), in 1768, d'Alembert had reservations about how divergent series were being used, but his reservations were disregarded by 18<sup>th</sup> century analysts.

When the next century started, another mathematician by the name of Sylvestre François Lacroix would express similar concerns (Grabiner, 1981).

It is important to note that analysts of the 18<sup>th</sup> century knew about divergent series but could not decide on how to handle divergent series. As was demonstrated in the previous paragraph, many mathematicians, like Liebniz and Guido, treated divergent series like convergent series. For them, a divergent series had a sum just like a convergent series even though the actual “sum” might be in debate. So for 18<sup>th</sup> century analysts, like Newton, Liebniz, Taylor, and Maclaurin, the question of convergence wasn't much of an issue because the question of divergence, although sometimes

debated, had not yet become much of an issue. Even though Euler addressed the issue of convergence in the middle of the 18<sup>th</sup> century, his conclusion that every commonly used function had an expansion may have led mathematicians, like Lagrange, to dismiss the issue and use tools, like the Lagrange remainder theorem, to merely estimate errors for given partial sums and not prove that certain functions actually have Taylor series expansions. Convergence became an issue when Cauchy forcefully refused to accept that divergent series have sums (Grabiner, 1981; Lutzen, 2003).

In Cauchy's *Cours d'analyse de l'École Royale Polytechnique* of 1821, Cauchy gave his definitions for the limit and for the convergence and divergence of series.

When the values successively attributed to the same variable approach a fixed value indefinitely, in such a way as to end up by differing from it as little as one could wish, this last value is called the *limit* of all the others. (as cited in Lutzen, 2003, p. 158, italics in original)

A series is an indefinite sequence of quantities

$$u_0, u_1, u_2, u_3, \text{ etc...}$$

which succeed each other according to a fixed law. These quantities themselves are the different *terms* of the series considered. Let

$$s_n = u_0 + u_1 + u_2 + \cdots + u_{n-1}$$

be the sum of the first  $n$  terms, where  $n$  is an arbitrary integer. If the sum  $s_n$  approaches a certain limit  $S$  for increasing values of  $n$ , then the series is said to be *convergent*, and the limit in question is called the sum of the series. On the contrary, if the sum  $s_n$  approaches no fixed limit when  $n$  increases indefinitely, the series is divergent and has no sum. (as cited in Lutzen, 2003, p. 159, italics in original)

Also in this work, the *Cours d'analyse de l'École Royale Polytechnique*, Cauchy defined infinitesimals and continuous functions, and discussed the radius of convergence of a series (Birkhoff, 1973; Lutzen, 2003). Two years later, in his *Résumé des leçons données à l'école royale polytechnique sur le calcul infinitesimal* of 1823, Cauchy defined the derivative and the integral (Birkhoff, 1973; Lutzen, 2003). In particular the

derivative was defined as the limit of the difference quotient and incorporated Lagrange's prime notation (Birkhoff, 1973; Lutzen, 2003).

It is clear that Cauchy borrowed the ideas for his definitions from other mathematicians (Lutzen, 2003). For example, according to Lutzen (2003), in 1755 Euler had a similar definition for limit, Euler also had used the convergence of partial sums to answer questions about series convergence. Prior to 1795, many notable mathematicians throughout the 18<sup>th</sup> century, like d'Alembert, had insisted that a variable cannot surpass its limit. According to Grabiner (1981), in 1795, almost 30 years prior to Cauchy's *Cours d'analyse*, Simon l'Huilier made a separate definition for limits that was particular for alternating series. He needed this particular definition because alternating series can surpass their limit, and he wanted to discuss the convergence of alternating series separately from positively termed series. Fifteen years later, in 1810, Sylvestre François Lacroix, the mathematician who had filled Lagrange's Chair at the Ecole Polytechnique, joined Huilier and abandoned the idea that a variable could not surpass its limit (Grabiner, 1981). Lacroix went on to say that a variable quantity can approach its limit "as closely as desired" (Grabiner, 1981, p. 81). Notably missing from Lacroix's definition was any notion that the limit cannot be surpassed by the variable. Later it was Lacroix who defined the derivative as the limit of the difference quotient prior to Cauchy's definition of the derivative in 1823 (Grabiner, 1981, p. 81). Lacroix would go on to greatly influence the definitions that Cauchy used for his calculus (Grabiner, 1981).

It is also worth noting that in Cauchy's definitions, the deltas, epsilons, and  $N$ 's were missing. For example, in modern notation one might write the definition of the limit of a function  $f$  as the following:

The limit as  $x$  approaches  $a$  of a function  $f$  equals  $L$ , if for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that whenever  $|x - a| < \delta$ ,  $|f(x) - L| < \varepsilon$ .

This definition would not become known until Weierstrass starting teaching at the University of Berlin in 1859 (Bagni, 2005; Burton, 2007; Kline, 1972). Quoting Kline (1972), Bagni (2005) noted that, “Weierstrass tried to avoid intuition ... and did not like the sentence *a variable approaches a limit* because it would suggest ideas of time and motion” (p. 460, italics in original). As noted by Lutzen (2003), today one may have a hard time separating  $f(x) \rightarrow L$  from  $x \rightarrow a$  because they are considered in tandem when proving that the limit of a function exists. Yet, Cauchy’s definition of limit allowed one to look only at what it meant for  $x \rightarrow a$ . Even though Cauchy’s definition of limit allowed one to consider  $f(x) \rightarrow L$  and  $x \rightarrow a$  separately, Cauchy clearly understood how to use his definition applied to the statements in tandem (Grabiner, 1981; Lutzen, 2003). So what was Cauchy’s contribution to calculus? Lutzen (2003) and Grabiner (1981) are quick to point out that to answer this question one cannot look at each of Cauchy’s calculus elements separately. One must look at all of Cauchy’s calculus as a whole and see all of his subtle changes taken together which were designed to make calculus more rigorous.

If one just looked at Cauchy’s definitions and not his proofs, one would see how he borrowed much of the components of his definitions from other sources. One might even conclude that he had assembled together the best possible definitions up to that point in time. One might also notice Cauchy’s attention to detail. For example, although Lacroix had used the limit of the difference quotient as a definition for derivatives,

Cauchy added to Lacroix's definition and said that the derivative exists when the limit exists (Grabiner, 1981; Lutzen, 2003, p. 159). To the definition of a convergent series, Cauchy added the definition of a divergent series and specifically stated that a divergent series "has no sum" (Lutzen, 2003, p. 159). If one looked only at Cauchy's definitions, his use of deltas, epsilons, and  $N$ 's contained within his proofs would not be seen. For example, the first time that both deltas and epsilons appeared in a limit proof can be found in one of Cauchy's proofs concerning derivatives in 1823 (Grabiner, 1981, p. 115). Because of his use of deltas, epsilons, and  $N$ 's within his proofs, one quickly sees that Cauchy's notions of limit, convergence, and derivative were much more advanced than his contemporaries (Grabiner, 1981; Lutzen, 2003). Cauchy's new method for convergence proofs opened up the door for questions about existence and nonexistence of derivatives, and convergence and divergence of series, and his method allowed for these questions to be rigorously answered.

Over two decades earlier, in 1797, Lagrange's method of defining derivatives as the coefficient of the terms in the Taylor series had serious problems. Contemporaries of the time had criticized Lagrange's method as being circular (Burton, 2007; Jahnke, 2003). As Burton notes, "the [Taylor] series is constructed from the derivatives, not the other way around" (Burton, 2007, p. 606). Lagrange's method avoided limits, a necessity of convergence. Unknown to Lagrange at the time was that not every infinitely differentiable continuous function has a Taylor series expansion (Burton, 2007; Grabiner, 1981). Since every differentiable function does not have a Taylor series expansion, using Lagrange's method to define a function's derivative was going to cause problems for special classes of functions. In addition, Lagrange had not known that given a Taylor

series expansion for a function  $f$ , there might exist another function completely distinct from  $f$  that could have the exact same Taylor series expansion (Burton, 2007). Plus, coming up with Taylor series expansion for a given function need not be an easy task. Even with these flaws, Lagrange's calculus remained influential for almost two decades (Jahnke, 2003).

Now equipped with his own definition of series convergence, and maybe even more important, his strict adherence to his definition for divergence and his method of proof, Cauchy was able to effectively distinguish between convergent and divergent series. By incorporating his definition of divergence as having "no sum," there was now little reason to consider those  $x$ 's for which a power series diverged. In response to Guido Grandi's statement that

$$\frac{1}{2} = \frac{1}{1+1} = 1 - 1 + 1 - 1 + \dots = (1-1) + (1-1) + \dots = 0 + 0 + \dots = 0,$$

Cauchy was able to rigorously show that Grandi's conclusion had no meaning because the Taylor series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

was divergent when  $|x| \geq 1$  (Burton, 2007). Therefore, unlike many of his predecessors, Cauchy seriously considered the question of convergence of series.

Cauchy was the first to notice that Taylor series expansions were not uniquely determined by only one function (Burton, 2007; Lutzen, 2003). In his *Résumé des leçons données à l'école royale polytechnique sur le calcul infinitesimal* of 1823, Cauchy considered the Taylor series centered at zero for the function defined by

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

which is  $C^\infty$  (i.e. a continuous function with continuous derivatives of all orders) (Burton, 2007; Lutzen, 2003). All of the derivatives of  $f$  at  $x=0$  are identically equal to zero for all  $x$ . Therefore, the Taylor series centered at  $x=0$  is convergent for all  $x$  and is identically equal to zero for all  $x$ . Since  $e^{-1/x^2}$  does not equal zero for all  $x \neq 0$ , Cauchy found a Taylor series that is convergent everywhere but only equal to its continuous, infinitely differentiable, generating function at one point. In addition, the function defined by  $f$  and the function that is identically equal to zero both generate the same Taylor series. Therefore, every function need not be equal to its Taylor series on its interval of convergence, and every Taylor series need not be uniquely determined by only one generating function. Cauchy was able to point out both of these facts with this one example (Burton, 2007; Lutzen, 2003). Thus, mathematicians could not continue to merely assume that functions automatically equaled their Taylor series expansions.

So how did Cauchy go about proving that functions equal their Taylor series expansions? Recall that in 1797, Lagrange came up with his remainder formula for Taylor series. Yet, Lagrange never used his remainder formula to address the question of convergence. In Lagrange's defense, Lagrange had not seen Cauchy's definitions of convergence and divergence nor had Lagrange seen Cauchy's example with  $e^{-1/x^2}$ . Cauchy, now equipped with his own definitions for convergence and divergence, and equipped with Lagrange's remainder formula, was the first who could rigorously prove when Taylor series converge to their generating functions (Kline, 1972).

### *Insights Into How These Experts Might Have Understood Taylor Series*

Taylor series was not a topic that developed overnight. Nor was it developed by one person. Discovered by James Gregory, used by Isaac Newton, and generalized by Brook Taylor, it quickly became a vital tool for analysts of the 18<sup>th</sup> century. In the beginning Taylor series would never have constituted only two sections of a calculus textbook, nay too be seen again within its pages. It was too fundamental in solving various calculus problems to be totally isolated, and was an essential tool used by the early analyst to estimate functions using finite polynomials. It allowed mathematicians, like Newton, to simplify complicated equations originating from physics, astronomy, and other areas of science. An analyst not recognizing the importance of Taylor series would have been unheard of in the 18<sup>th</sup> century because an analyst of that time without some conceptual understanding of Taylor series would have been ineffective.

Even though an understanding of Taylor series was a requirement for 18<sup>th</sup> century analyst, this understanding need not be rigorous. By today's standards, this understanding might even be called very rudimentary. Analysts of the 18<sup>th</sup> century may have intuitively understood how to use Taylor series, but they didn't fully understand all of the intricate details of Taylor series. Part of this lack of understanding originated from a lack of interest in the question of convergence. Analysts were aware that they were using series results that they couldn't prove, but why would they have cared? The results they were using were working in the applications that they were most concerned about. When Euler addressed the question of convergence in 1748, Euler only propagated the indifference to the question of convergence when he showed that all the commonly used functions of the time had Taylor series representations. Twenty years later in 1768,



d'Alembert started asking about error bounds and convergence, but unfortunately for d'Alembert, his intuition about convergence was in error. Influenced by d'Alembert, Lagrange found the remainder term bearing his name and started addressing error bound questions. But Lagrange did not fully address the question of convergence. Even though Lagrange based all of his calculus on Taylor series in his *Théorie des fonctions analytiques* of 1797, Lagrange based his proofs off of a more intuitive reasoning than rigorous demonstrations (Burton, 2007). Not until the time of Cauchy in the 19<sup>th</sup> century did the understanding of Taylor series move from an intuitive understanding to a rigorous understanding.

Much of what was causing the proliferation of intuitive arguments over rigorous demonstrations was the definition of limit in the 18<sup>th</sup> century. The language describing limits paralleled the language of motion that incorporated the idea of “closeness.” Much of the limit concepts investigated by Williams (1991) seemed to be present in the minds of 18<sup>th</sup> century analysts. In Newton's *Principia*, when he was discussing the limit of secant lines, one can see Newton use language indicative of a dynamic-practical image imbedded in time.

Quantities, and also ratios of quantities, which in any finite time constantly tend to equality, and which tend to equality, and which before the end of that time approach so close to one another that their difference is less than any given quantity, become ultimately equal (as cited in Guicciardini, 2003, p. 82).

At this moment it appeared that Newton was conceiving of the limit as something that can move through time and “ultimately” be reached. Following his statement, a very intuitive “proof” appeared.

If you deny this, let them become ultimately unequal, and let their ultimate difference be  $D$ . Then they cannot approach so close to equality that their difference is less than the given difference  $D$ , contrary to the hypothesis (as cited in Guicciardini, 2003, p. 82).

Referencing Struik (1986), Kaput (1994) pointed out that dynamic images were essential to the development of Newton's calculus. Consider the following quote from Newton's *De quadratura curvarum* of 1676:

I consider mathematical quantities in this place not as consisting of very small parts, but as described by a continued motion. Lines [curves] are described, and thereby generated, not by the apposition of parts but by the continued motion of points. (as cited in Struik, 1969, p. 303)

Newton's use of dynamic language clearly influenced how many mathematicians of that time period talked about limits. For example, in the 19<sup>th</sup> century, Lacroix, who influenced Cauchy's definition of limit, said, "The application of limits is made by means of principles whose truth rests only on the possibility of proving that a variable quantity can approach its limit as closely as desired" (Grabiner, 1981, p. 85). Lacroix demonstrated a dynamic image of limit that allowed for the limit to be reached. As the previous chapters have already noted, this dynamic metaphor continues to be used in textbooks like Stewart (2008) and Hass et al. (2007).

In addition to demonstrating a dynamic image, Colin Maclaurin seemed to be calling on an unreachable image when discussing the derivative as a ratio of differences.

There is nothing to hinder us from knowing what was the ratio of those increments at any term of the time while they had a real existence, how this ratio varied, and to what limit it approached, while the increments were continually diminished (as cited in Grabiner, 1981, p. 85).

So Maclaurin appeared to be saying that the limit of the ratios didn't have a "real existence," and implicit in his statement was that only the ratios over the diminishing small increments approaching the limit existed. Similarly, in his *Encyclopédie*

d'Alembert was clearly viewing the notion of limit dynamically as a number that was unreachable when he said, "The tangent is the limit of the secants; they approach closer and closer to it without ever reaching it" (as cited in Burton, 2007, p. 604). Gregory of Saint Vincent in his *Opus geometricum* of 1647 demonstrated an unreachable conception when he said the following:

The conclusion of a progression is the end of the series that the considered progression does not reach, although it is indefinitely lengthened; it can approach such value as close as it is possible. (as cited in Bagni, 2005, p. 459) which was adapted from Kline (1972, p. 437).

The limit as bound image appeared in d'Alembert's mental imagery concerning limits when in the *Encyclopédie* he also said, "One magnitude is said to be the limit of another magnitude when the second may approach the first within any given magnitude however small, though the first magnitude may never exceed the magnitude it approaches" (as cited in Burton, 2007, p. 603). Lagrange's exclusive use of his remainder term to calculate error given a Taylor polynomial was evidence that Lagrange viewed limit as approximation in the context of Taylor series.

Even the metaphors that Cauchy relied upon in the 19<sup>th</sup> century for his definitions had a dynamic element. He used the words "successively," "decrease," "increases," and "approach" in his definitions for limit, infinitesimal, and convergence of series (Lutzen, 2003). Referencing Lakatos (1978), Monaghan (2001) believed that Lakatos proved that "Cauchy's limit ideas were dynamic" (2001, p. 253). After a further investigation into Cauchy work one can quickly see that if Cauchy had a dynamic image, he must have also been in possession of very formal static images that he called upon when producing proofs. This was clearly evident by the appearance of deltas, epsilons, and  $N$ 's in his

proofs. Thus, it can be argued that Cauchy appeared to be effectively switching between different images depending on the context in which he was working.

Some of these mental images might have led many 18<sup>th</sup> century analyst astray and therefore prolong the full development of Taylor series. For example, some believed that every function was equal to its Taylor series for every  $x$ . Therefore, some mathematicians used Taylor series to conclude that  $\frac{1}{2} = 0$ . Others thought that Taylor series were uniquely determined by only one function. Still others, like d'Alembert, demonstrated a limit as bound mental image when dealing with series convergence, and thus assumed that the terms of the series going to zero was sufficient to prove series convergence (Grabiner, 1981). In many cases, it took specific counterexamples to show some problems that these images can cause. Cauchy used the interval of convergence and  $e^{-1/x^2}$  as counterexamples to the first two, and l'Huilier used convergent alternating series to dispel the notion of limit as bound (Grabiner, 1981).

One might think that I am being unfair to the analyst of the 18<sup>th</sup> century by pointing out what appeared to be mental flaws. After all, the limit concept was still developing. Had 18<sup>th</sup> century analysts had Cauchy's definition, the mental images that they relied upon might have changed. They may have used images that were more static and less dynamic. They may have produced the exact same proofs that Cauchy produced using the exact same method. Indeed this is true. But because of the world in which one lives and the words in which one uses, people bring in preconceived notions about limits prior to any introduction to the formal definition of limit (Davis & Vinner, 1986; Monaghan, 1991; Sierpinska, 1987). Furthermore, they may hold onto these preconceived notions well after introduction to the formal definition. Therefore, had they

had Cauchy's definitions, some of these experts of the 18<sup>th</sup> century may have still held on to their old notions. As Cauchy appears to have done, they may have incorporated their old notions with their new notions and used them at different times depending on the context in which they were working. Plus, these mental flaws of the 18<sup>th</sup> century analyst may not be mental flaws, but illustrations of necessary building blocks upon which an individual can build more effective and complete mental images. Cauchy's calculus was built off of the calculus done before his time, and even though Cauchy had a very formal static mental image for the limit, he did not exclusively rely on the formal image. The dynamic language in which he used in his definitions is indication that Cauchy had a dynamic image to complement his formal image. Before Cauchy, the dynamic image was used by analyst like Newton, Maclaurin, and Lacroix, but for Cauchy, the dynamic image had to bow to the definition of limit. The proof technique in which Cauchy used is evidence that the static formal definition was primary for Cauchy since any dynamic image utilized by Cauchy had to eventually succumb to the formal image.

Even though Cauchy effectively introduced the mathematical world to a rigorous calculus, and in so doing avoided some of the oversights of his predecessors, one should not think Cauchy as so clever as to be incapable of any error. It took mathematicians, like Gauss, Bolzano, Fourier, Weierstrass, and Abel to continue what Cauchy had started and usher in the next century of calculus. One of the most notable mistakes in Cauchy's calculus, was his assumption found in the *Cours d'analyse* that every convergent series of continuous functions must have a continuous sum (Burton, 2007; Lutzen, 2003). It took another five years after the *Course d'analyse* for Niels Henrik Abel to point out that

Cauchy's theorem was not applicable to every function (Burton, 2007; Lutzen, 2003).

Abel demonstrated that the trigonometric series

$$\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x) + \dots$$

violated Cauchy's theorem about continuous sums because the sum is not continuous at every odd multiple of  $\pi$ ,  $x = (2n+1)\pi$  (Burton, 2007; Lutzen, 2003). More importantly,

Abel was able to rigorously show this because of Cauchy's previous work with convergence and divergence (Lutzen, 2003).

### *Informing Pedagogy*

When students struggle with Taylor series, instructors would be wise to keep history in mind. The struggles that students have are no different than the struggles that the very founders of calculus encountered centuries ago. The things that they don't see are the things that the founders didn't see. The questions that they don't ask are the same questions the founders didn't ask. The way that students think is the way the founders thought. The language that they use is the language that the founders used. They bring in preconceived notions just like the founders of calculus had. Just like the founders of calculus, today's students will have to overcome mental obstacles. They will have to make their preconceived notions bow to formal definitions. They may need to be equipped with counterexamples, like Cauchy used for one of Lagrange's misconceptions and Abel used for one of Cauchy's misconceptions. It took over a century for calculus to develop into the calculus of Cauchy, students need not think that what they are learning will come easy, and instructors need not give up on their students as they represent the experts of the future. It is the responsibility of instructors, to meet their students where

they are and teach them to think in the best way possible so that they can begin to ask the right questions and start looking like experts.

In the next chapters I will give insights into how today's and tomorrow's experts conceptualize the convergence of Taylor series.

## CHAPTER 5

### Today's Experts Results & Discussion of How Experts Understand Taylor Series

In this chapter I will analyze and describe the different ways in which experts conceptualized the convergence of Taylor series. In Chapter 3, experts were defined as graduate students or faculty from the Department of Mathematics at the University of Oklahoma. Experts were divided into two groups, experienced expert participants (EEPs) and capable expert participants (CEPs), based on their level of exposure to the concept of convergence of series. I will begin this chapter by demonstrating how the focus group and interview data from the experts, both EEPs and CEPs, were analyzed. The analysis of the data from the experts incorporated three of the four layers of data analysis as discussed in Chapter 3. The expert analysis incorporated Layer 1 where focus group and interview data were coded for existing themes adapted from Williams (1991), Layer 2 where focus group and interview data were coded for emerging themes, and Layer 4 where a detailed focus analysis of the high inference tasks interview results was performed. Only Layer 3 of the analysis was absent for the experts because the experts did not participate in the questionnaires. This chapter is divided into five main sections. The first three sections are based on the three layers of analysis. The chapter concludes with a potpourri section that discusses a verity of interesting results that didn't fit into the previous sections, and a conclusion section that brings together some of the main results from within the chapter.



### *Layer 1: Existing Images*

In this section I will discuss the adaptation of the existing framework created by Williams (1991) and demonstrate how this framework was expanded upon when analyzing the expert data. As discussed in Chapter 2, Williams (1991) classified student models of the limit concept into six categories: limit as bound, limit as approximation, limit as unreachable, limit as dynamic-theoretical, limit as dynamic-practical, and formal view (see Table 2 in Chapter 2). During Layer 1 analysis, focus group / interview data were coded for five existing themes, each indicative of a corresponding existing image. Chapter 3 explained how four of Williams' limit models were adapted for Taylor series: dynamic, dynamic unreachable, dynamic reachable, and approximation. A fifth category, that I called exact, was also included in the existing image categories. See Table 16 in Chapter 3 for a description of all five existing image categories. The following subsections describe how each existing image materialized from the expert data.

*Dynamic.* The dynamic image was characterized by metaphors involving motion. The dynamic language is expressed in words like, “gets close to,” “tends to,” “approaches,” “goes to,” and “moves.” In addition to Williams (1991), motion imagery has been noted by other researchers studying various topics in calculus (e.g., Lauten et al., 1994; Monk, 1992; Sierpinska, 1987; Tall, 1992; Tall & Vinner, 1981; Thompson, 1994). Some researchers indicate that the conceptualization of types of dynamic images is fundamental for the understanding of calculus (Carlson et al., 2002; Cottrill et al., 1996; Monk, 1992; Tall, 1992; Thompson, 1994). Indeed, the previous chapter noted how dynamic images were essential in Newton's development of the calculus. The proliferation of dynamic images may be propagated by the very language that is used

when discussing limits (Cornu, 1991; Davis & Vinner, 1986; Monaghan, 1991; Schwarzenberger & Tall, 1978). Even so, Oehrtman (2002) pointed out that just because a student uses dynamic language it does not necessarily indicate that they are thinking of something as literally moving. Therefore, I must be careful and not over interpret participant comments.

Because of the complexities of Taylor series, language potentially indicative of dynamic images was used for Taylor series in many different contexts, such as for Taylor polynomials, remainders, and intervals of convergence. When talking about Taylor polynomials, the dynamic image might materialize in a context of adding more terms to the Taylor polynomial. Consider EEP LOGAN's response to the high inference Task 8.

*Excerpt 3*

Task 8: How can we estimate sine by using its Taylor series?

EEP LOGAN: Sin  $x$  is actually equal to its Taylor series.

I: Okay.

EEP LOGAN: And therefore we can estimate sine by just truncating Taylor series, by using [a] certain number of terms in the Taylor series.

Because we know that the more terms we take in the Taylor series, the closer they are to the value of the sin  $x$ .

LOGAN noticed the difference between Taylor series and Taylor polynomials when he said, "sin  $x$  is actually equal to its Taylor series." He went on to say that "we can estimate sine by just truncating Taylor series, by using [a] certain number of terms in the Taylor series." For LOGAN, the series "equaled" the generating function, it did not estimate the generating function, the Taylor polynomials estimated the generating function. A possible dynamic image is indicated by LOGAN's use of the word "closer." For LOGAN, "closer" was associated with "taking" Taylor polynomials with "more terms." As "more terms" were "taken," the "closer" the Taylor polynomials came to the generating function.

EEP LOGAN was not the only expert to use the idea of getting “closer” in the context of considering Taylor polynomials with more terms. Consider EEP JAMES’ excerpt below:

*Excerpt 4*

I: What does it mean to converge?

EEP JAMES: Well-

I: For cosine of  $x$  to equal this?

EEP JAMES: Well, as you put more and more terms, you're getting closer and closer to the right value.

For JAMES, it appeared that the generating function was estimated by a single dynamic polynomial to which additional terms were added. With each addition, the polynomial became a better and better approximation that got “closer and closer” to the generating function. In total, out of the nine experts who answered the high inference tasks during individual interviews, six used the “closer” language when discussing partial sums. In fact, the “closer” language was primarily associated with the idea of a dynamic polynomial converging to a given generating function.

“Closer” was not the only word indicative of a potential dynamic image associated with Taylor series convergence.

*Excerpt 5*

I: [If] you had to approximate sine of 103 radians, how would you go about doing that?

EEP MARSHAL: Well, you probably wouldn't do it with a Taylor series. Okay. Uh, because uh, uh, you would have to move the Taylor series out, increase so many terms in the Taylor series.

Here, to get a good approximation of sine of 103 radians, the Taylor polynomial is verbalized as “moving” when MARSHAL said, “You would have to move the Taylor series out.” MARSHAL quickly clarified the meaning of “move the Taylor series out” when he immediately alluded to “increasing” the “terms” of the series. Later, when he

discussed how to approximate 103 radians using Taylor series, EEP MARSHAL made the distinction between Taylor series and Maclaurin series. He went on to reference movement in relationship to the center of the series.

*Excerpt 6*

EEP MARSHAL: Yeah, you move it. Yeah, you say Taylor series, we are not talking about Maclaurin series. So a Taylor series, we-we want to look at, you can move it around [taking right hand and waving it far from participant's body] at a different point a, okay. And so you choose that a close to 103.

The dynamic elements of Taylor series are not restricted to merely one dimension.

MARSHAL's reference to moving the series by both adding more terms and by re-centering the series is illustrative of the multidimensional dynamic aspects of the convergence of Taylor series. One can add more terms to a Taylor polynomial or one can move the center, both are valid ways to increase the accuracy of an approximation depending on the context of the situation.

As illustrated in the previous chapter, the conceptualization of the convergence of Taylor series is twofold. A proper conception of convergence comes with a proper conception of divergence. If someone has one but not the other, then the validity of that person holding any proper conception is justly in dispute. The following excerpt indicated that EEP JAMES may have conceived of convergences of Taylor series as a consistent "tending" toward a value since JAMES classified divergence as an "erratic" movement that does not "tend" toward a specific value.

*Excerpt 7*

I: What, for you, does it mean to diverge?

EEP JAMES: Well, the sum behaves erratically; it never tends to a value or jumps between two values or more.

“Goes to” language is nearly inextricably linked to the notion of limit in any context, Taylor series not excluded. Variations of “n goes to infinity” is not only contained within all transcripts from this study, but it is prevalent within most. Try holding a conversation about the convergence of Taylor series without uttering the words, “n goes to infinity.” One of the more interesting uses of the “goes to” language occurred in the context of remainders related to Taylor polynomials. Seven of the nine experts used “goes to” language in the context of remainders “going to” zero. Consider MARSHAL’s and DYLAN’s response to Task 7 in Excerpt 8.

*Excerpt 8*

Task 7: What are the steps in proving that

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots?$$

EEP MARSHAL: So, it uh, the steps typically uh, involve looking at the difference between what you think it converges to and the n<sup>th</sup> partial sum and then showing that that error goes to zero.

EEP DYLAN: Um, you gave me—you gave me the first step because you wrote down the Taylor series. And to prove that these are equal, all I have to do is observe that the sine function has all its derivatives, well orders, and the remainder term goes to zero.

Note that DYLAN demonstrated his “expertness” by noting that the generating function must have derivatives of all orders for the Taylor series to exist for that given generating function. In Excerpt 9, another experienced expert participant, EEP DEAN, demonstrated his “expertness” when he made reference to Taylor’s Inequality (a modification of the Lagrange Remainder Formula found in Stewart, 2008, p. 773) and the sandwich theorem.

*Excerpt 9*

Task 6: What is meant by the word “prove” if you were asked to, “Prove that sine is equal to its Taylor series.”

EEP DEAN: Um, now if you want to show that sequence of Taylor polynomials converge, analogous to that would be to show that the remainders go to zero [downward motion with right hand], or converge to zero [brief downward motion with right hand]. Um, and Taylor Inequality is a way to bound [holds up both hands as if holding something between], to put a boundary on the absolute value of uh, the remainder. So remainder has a certain form, put an absolute value [makes absolute value bars in the air with both hands] on it, there is a way to prove um, that that is going to be less than or equal to something. Now once you do that, what you want to show is, if you, if you want to use sandwich theorem, or absolute values [makes absolute value marks in the air with both index fingers] of sandwich theorem, if you want show that the absolute values of something go to zero [downward motion with right hand], you, you wanta, first of all it is always-always greater than or equal to zero. So it's always sandwiched with zero on one side. And now you want to find an upper bound [holds right hand up in the air above head] for it, make sure that the sequence of upper bounds go to zero too [downward motion right hand]. And the, you have a sequence sandwiched between two sequences going to zero [downward motions with both hands], in absolute value [absolute value marks in the air with both index fingers], which means that it goes to zero [downward motion with right hand centered in middle of body] too. And if you have sequence of remainders going to zero [left hand downward motion], that means the sequence of the Taylor polynomials are convergent [downward motions with left hand].

Although DEAN had difficulty articulating his understanding, there are several important aspects of his understanding worthy of mentioning. First, he understood that Taylor’s Inequality yields a “sequence of upper bounds.” As opposed to believing that Taylor’s Inequality yields the actual error, DEAN knew that Taylor Inequality produces an overestimate for the error. Second, he recognized the relationship that the sandwich theorem has with Taylor’s Inequality. He indicated that this relationship is characterized by the remainders being sandwiched between zero and the “sequence of upper bounds” given by Taylor’s Inequality. Thirdly, DEAN knew that to prove that the sequence of Taylor polynomials were convergent, it was sufficient to prove that the sequence of

remainders go to zero and that this is accomplished by showing that the “sequence of upper bounds” go to zero. This interaction between Taylor polynomials and corresponding remainders and the role theorems play in proving convergence of Taylor series is characteristic of what many experts might call expert understanding of the convergence of Taylor series. Under this assumption, clearly, EEP DEAN demonstrated expert understanding of convergence in the context of Taylor series. More on this excerpt, including DEAN’s gestures, will be discussed in the next section.

*Dynamic Reachable and Unreachable.* Sometimes the dynamic image also includes the idea of attainment or the lack of attainment of some limiting value. As discussed in Chapter 2, several researchers have observed and studied student conceptions about the attainment or lack of attainment of limits and its relevance to student understanding of limit (Cornu, 1991; Davis & Vinner, 1986; Mamona-Downs, 2001; Sierpiska, 1987; Szydlik, 2000; Tall, 1980; Williams, 1991). Cornu (1991) classified the historical debate over the attainment of a limiting value as an epistemological obstacle that is “still alive in our students” (p. 162). The next chapter will demonstrate that the dynamic reachable and unreachable limit images relative to Taylor series are still “alive” in the minds of students. But what about in the minds of today’s experts?

In the context of Taylor series, the dynamic reachable image was typically characterized by metaphors involving movement in which a sequence of Taylor polynomials eventually reached a given generating function. With this image comes the idea of ultimate completion. Past research has linked this image of limit with an actual view of infinity (Sierpiska, 1987). An indicator of a reachable image was seen in EEP

JAMES' excerpt below when JAMES viewed the series as converging "right on" cosine.

In addition, he emphasized the convergence "on" cosine with his gesture.

*Excerpt 10*

Task 3: What is meant by the "=" in " $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$ " when  $x$  is any real number?"

EEP JAMES: Well, I guess it's meant that the, the series on the right hand side converges to the value on the left hand side as we take the limit of the partial sums to be, uh, as we include more and more terms... I mean it converges right on [quick downward motion with right hand] the value of cosine.

In the context of Taylor series, the dynamic unreachable image was typically characterized by metaphors involving movement in which a sequence of Taylor polynomials got close to but never reached a given generating function. As demonstrated in the previous chapter, historical experts in mathematics, such as Colin Maclaurin, D'Alembert, and Gregory of Saint Vincent, have demonstrated the dynamic unreachable image. As discussed in Chapter 2, some researchers have classified the dynamic unreachable image as seemingly unavoidable (Cornu, 1991; Davis & Vinner, 1986; Tall, 2001) and may be related to a potential (or procedural) view of infinity (Fischbein, Tirosh, & Melamed, 1981; Schwarzenberger & Tall, 1978; Sierpinska, 1987; Tall, 1980; Tall & Vinner, 1981).

The dynamic unreachable image appeared in small amounts within the data from the expert group. In particular, of all the experts, two experts demonstrated what might be considered a dynamic unreachable image by indicating that reaching the completion of the Taylor series was unnecessary under certain conditions. For both experts, the lack of need to reach the generating function was in the context of approximation.



For EEP JAMES, it is worth noting that he did not hold relentlessly to an unreachable image, but for “practical purposes” one would only extend the Taylor polynomial out far enough to get “enough precision.”

*Excerpt 11*

Task 3: What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$ ”

when  $x$  is any real number?”

I: What does it mean to converge?

EEP JAMES: Well-

I: For  $\cos x$  to equal this [referring to cosine’s Maclaurin series]?

EEP JAMES: Well, as you put more and more terms, you’re getting closer and closer to the right value.

I: Okay.

EEP JAMES: Simply, simply explained that’s what it is.

I: Okay.

EEP JAMES: And for practical purposes actually that’s what you want, I mean you add more and more terms and at some point to get uh, you get enough precision, you stop. So, so for practical people infinite series are very finite sums, I mean you never add infinitely many terms, like when you, when you write a program you always put some upper, you never add infinitely many terms. So, yeah, yeah so, I mean, I’m not good at philosophy about what equal sign is here.

I: Yeah, I just want a-

EEP JAMES: -But for me just equality between numbers. And infinite sum, well you can philosophize a lot about infinite sums, but eventually this goes to a number. It never reaches it maybe. But uh, if you know what convergence means, then, then you write cosine of  $\pi$  over two is zero, and so this sum is going to converge to zero.

As you can see from Excerpt 11, the lack of need for extending the Taylor polynomial out indefinitely was grounded in the context of approximation. For JAMES, “practical people” viewed infinite series as very “finite sums” that were created by adding “more and more terms” until one achieves “enough precision.” Furthermore, JAMES later goes on to say that the series “never reaches it maybe.” It appears that the debate between reachable versus unreachable was going on in JAMES’ mind within this very excerpt. He went on to create a number to plug into cosine,  $\pi/2$ , and concludes that

“if you know what convergence means” then the sum created by plugging in  $\pi/2$  is “going to converge to zero.” This debate becomes more visible when one notes that it was JAMES who demonstrated a reachable image in Excerpt 10. Furthermore, JAMES’ reachable image was in response to the same task in which he demonstrated his unreachable image, and Excerpt 11 occurred within about fifteen seconds from Excerpt 10.

Now I will consider the capable expert participant (CEP) WALLACE. Like the experienced expert participant (EEP) JAMES, CEP WALLACE demonstrated an unreachable image in response to the same task. Like JAMES, WALLACE brought up the unreachable image in the context of approximation. In addition, WALLACE attached the unreachable image to a notion of infinity.

*Excerpt 12*

Task 3: What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$ ”

when  $x$  is any real number?”

CEP WALLACE: Um, it’s meant by, it is a real close approximation, it is not necessarily equal to that. But uh, this is an infinite expansion. There’re infinitely many elements here. So it is simply, when we look at the summation part,  $m$  from zero to infinity we have that equal part. So it takes us to the knowledge of infinity.

Following this excerpt WALLACE went on to bring up other properties of Taylor series, such as the radius of convergence. After this, the next excerpts reveal why WALLACE seemed to struggle between an unreachable and reachable image.

*Excerpt 13*

CEP WALLACE: It can not be really the same. I’m not sure really about that though, right now. [pause] But we use it as equal, as it is close to that. Because there is limit involved in this. And limit means uh, for infinitely many numbers, I mean this is definitely equal but uh, there might be error side in that. If you write it as uh, let’s see,  $T_n$  of  $x$  and  $R_n$  of  $x$  where  $R$  is the uh, um,  $n$  plus one through infinity may elements and if you look at the limit, it takes us to limit knowledge. So limit as  $n$

approaches infinity of  $R_n$  of  $x$  is equal to zero will tell us that this is precise equal. Um, so it is depending on the limit knowledge. If limit is precisely zero, then we just say that it is equal. So this is precise equal, but again, what does limit tell us?

In the first sentence, WALLACE relied on an unreachable image and referred to the Taylor series as not being “really the same” as the generating function. Then the debate between reachable verses unreachable ensued after a short pause. He appeared to make a distinction between two different types of knowledge, one that he repeatedly referred to as “limit knowledge.” He concluded that if one can show that the limit of the remainder is equal to zero, then what is meant by the equal sign in Task 3 is “precise equal.” The use of “precise” seemed indicative of a reachable image. For WALLACE, this reachable image seemed to only be linked to his “limit knowledge.” In the next excerpt, WALLACE goes on to contrast “limit knowledge” with a “geometric way” of thinking about convergence.

*Excerpt 14*

CEP WALLACE: Limit knowledge is sort of included in here and I think that this is precisely equal but there is no uh, geometric way of saying this and I like uh, looking at things geometrically. Algebraically they are equal because of the knowledge that we have.

When asked about what he meant by “geometric,” WALLACE clarified that geometric referred to looking “at the curves.” So what might cause one type of knowledge to yield a reachable image, and another type of knowledge to yield an unreachable image?

WALLACE explained:

*Excerpt 15*

CEP WALLACE: When we go along these elements [referring to the terms of the Taylor series] here, [pointing to the series equated to cosine in problem 4], one, then two, then three, just include three elements, then include four elements, what we are seeing is a curve that gets really close and close to the curve cosine  $x$ . But when we insert infinitely many elements, how does it match with the  $\cos x$  function? So that, that is the

first question that came to my mind when I saw this question, and I like looking at things geometrically. As there are infinitely many elements [circling the “...” in problem 3], it looks like you cannot really see it in the way that you can see. But when I look at it algebraically it’s definitely equal.

When WALLACE first saw the problem, he thought about it visually. He thought about it dynamically in the context of adding more terms to a Taylor polynomial that progressively becomes a better approximation to the generating function. This type of image appeared to be imbedded in time, “then three, just include three elements, then four elements.” Being able to insert infinitely many elements moved him to considering a different type of knowledge, “limit knowledge.” For WALLACE limit knowledge was related to an “algebraic” understanding as opposed to a “geometric” understanding. When considering the question “algebraically,” WALLACE was able conclude that the series was “definitely equal” to the generating function.

WALLACE was not the only expert to make a distinction between two different ways of thinking about the limit of partial sums. In response to the same task, CEP KELLEN admits to “having a hard time” with answering the question. KELLEN then went on to say the following:

*Excerpt 16*

CEP KELLEN: Well um, I’m, I’m thinking it’s more like equality in the sense of a limit. Um, that since, since this is a[n] infinite sum, and unless  $x$  is equal to zero, this is going to be an infinite sum. That um, when you find an infinite sum, then we’re really talking about limits instead of strict equalities.

For KELLEN it appears that there are two types of equalities; “strict equalities” and those associated with limits. In the interview discussion following this excerpt, KELLEN never clearly associated either of the two types of equalities with reachable or unreachable images.

One could argue that both JAMES' and WALLACE's imagery concerning the attainability of the limit is very labile. This may be due to them relying on their notions of infinity. It has already been shown by other researchers that individual conceptions of infinity easily change depending on the current problem context (E. Fischbein et al., 1979; E. Fischbein et al., 1981). If JAMES and WALLACE were relying on their notions of infinity, then these findings would be consistent with previous research. Even if they were not relying on a notion of infinity, the capacity to switch between images depending upon the problem context found within various calculus problems has been observed in students (Ferrini-Mundy & Graham, 1994; Lauten et al., 1994). JAMES and WALLACE have demonstrated that this switching ability can persist within the experts and may be even more cultivated. In the end, these experts, both JAMES and WALLACE have used this ability in an apparently efficient and effective manner. Furthermore, the use of these reachable and unreachable images as needed appears to have caused little worry. Consider JAMES insightful announcement in Excerpt 17:

*Excerpt 17*

EEP JAMES: Over the years I've had no psychological trouble [chuckle] looking at infinite series and putting an equal sign.

Perhaps a little more pronounced than JAMES' and WALLANCE's likely reliance on their notion of infinity, was their dependence on approximation images that seemed to influence their use of unreachable images. JAMES said that when "you get enough precision, you stop," and WALLACE called the equal sign "a real close approximation." It is important to note how dominate the approximation imagery was in each expert's mind, because both were responding to the question, "What does it mean to converge?" Neither were responding to "How do we estimate sine by using its Taylor

series?” For JAMES, because the approximation imagery was so primary in his understanding, it was superfluous for “practical people” to consider the completion of the Taylor series. In the next subsection I will discuss the approximation image relative to the convergence of Taylor series in more detail.

*Approximation.* The approximation image is characterized by a focus on being able to make the Taylor polynomials as accurate as you wish. Previously, in Excerpt 11, EEP JAMES demonstrated an approximation image when he alluded to adding more terms to a Taylor polynomial and stopping once you get “enough precision.” In addition to Williams’ (1991) study, approximation imagery has been observed and studied by other researchers (e.g., Kidron, 2003; Lauten et al., 1994; Oehrtman, 2002). As indicated in the next excerpts, the approximation properties of Taylor polynomials are one of the main reasons that Taylor series are studied in calculus.

*Excerpt 18*

Task 2: Why are Taylor series studied in calculus?

EEP CLARK: The first thing you learn in calculus is that functions can be approximated, general functions can be approximated by linear functions, that's what you do when you learn what the derivative is. Uh, but then you start to generalize that, you come up with more sophisticated tools for approximating functions where you don't just use linear approximations but quadratic and approximations by higher order polynomials.

*Excerpt 19*

Task 2: Why are Taylor series studied in calculus?

EEP LOGAN: Well, because uh we use the Taylor series to approximate functions, and uh there are many applications. For example, comes to mind uh, linear, quadratic approximations, uh computations of integrals, approximately, uh, and then later in differential equations. We use uh, Taylor series to approximate solutions of differential equations as it was done many years ago by Newton and others.

Both CLARK and LOGAN reference the approximation properties of Taylor series as being one of the main reasons that Taylor series are studied in calculus.

CLARK and LOGAN were not the only experts to clearly reference the approximation properties of Taylor series in response to Task 2. Six of the nine experts individually interviewed clearly referenced an approximation image of Taylor series when responding to this task. Of the three who were not clear, one said that “Taylor series is a way of writing a function down in polynomial form,” and another said that Taylor series is “a very useful tool for studying functions locally.” In addition, seven of nine experts demonstrated an unbiased approximation image at some point during the high inference tasks. At some point within the high inference tasks, all nine demonstrated elements of an approximation image to some degree or another. This type of dominance of the approximation image for convergence in relationship to Taylor series was also observed by Oehrtman (2002).

*Excerpt 20*

Task 3: What is meant by the “=” in  $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$

when  $x$  is any real number?

EEP DYLAN: So what you mean here is that no matter what  $x$  value you plug in, if you want to know the cosine of uh, pi divided by 100, something, maybe we shouldn't take pi, because we are going to have to approximate pi. If you want the cosine of uh, 37, or whatever, measured in radians, um, all you have to do is plug that number into the formula on the right and take it out sufficiently far and you will get an actual decimal approximation for that number.

Consider this expert’s response to Task 9, “What is meant by the ‘near’ in ‘

$\sin(x) \approx x - \frac{x^3}{3!}$  = a Taylor polynomial for sine when  $x$  is near 0?’”

*Excerpt 21*

EEP DEAN: Near would be something much smaller than one, pretty close to zero. So when  $\sin x$  is near zero, I would say zero point zero one close, or zero point one close, even though that kinda sounds like a big number because it's one tenth of uh the value, the maximal value of sine. So I would say when I say something near something or something

approximately close to something in way it's, part of it is kinda common sense, near [holds up left hand and with pointer finger and thumb extended close together] as opposed to [holds out both hands away from body] not so near. Apart of it's uh, in the context of the maximal and the minimum values that function is taking. So because sine is not taking very high values, then I would suppose that  $x$  being near zero means, well, zero point one close or zero point zero one close. That close. Like much smaller than the maximum value of sine. Even though really you could say well but  $x$ , you know, sine is taking values supposedly on the  $y$ -axes if think about it that way and  $x$  is not taking its values,  $x$  is like a different variable taking values on a different axes.

This expert demonstrated great detail in his approximation imagery. For example, closeness was actually assigned numerical values. These numerical values also appear dependent upon the range of the function. Thus, for a function with a smaller or larger range than sine, “near” might take on different numerical values.

In addition, EEP DEAN has reasoned very well about two varying quantities that take on values from two different axes. For this expert, the first varying quantity was sine which was related to the varying values along the  $y$ -axes. The other varying quantity was  $x$ . DEAN explicitly stated that  $x$  is “not taking” sine’s values but  $x$  was a “different variable taking values on a different axes.” This expert’s ability to reason about sine and  $x$  in this way demonstrated an apparently well developed ability to apply covariational reasoning in the context of Taylor series. In Chapter 7, I will show an example of a novice who struggled to answer Tasks 9 and 10 due to possible difficulties with covariational reasoning.

*Exact.* The exact image is characterized by the idea of the series being identical to the generating function or that the generating function is defined by the series. What differentiates the exact image from the dynamic reachable image is the lack of the dynamic element. The series and the generating function are simply seen as one and the



same thing. In a couple of cases this image materialized when experts viewed particular Taylor series as “defining” the corresponding generating function. In such cases, the Taylor series was seen as generating the generating function, not vice versa. Consider EEP CLARK’s and EEP LOGAN’s responses to Task 3 in Excerpt 22.

*Excerpt 22*

Task 3: What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$ ” when  $x$  is any real number?”

EEP CLARK: “In order for that statement [referring to Task 4] to have content, uh, you need to define what the function is on the left hand side. Actually, sometimes the cosine function is defined as the series on the right hand side. In which case the equal would mean this is the definition of the  $\cos x$ .”

EEP LOGAN: This makes me think. Uh. [pause] Well, what is  $\cos x$  actually?  $\cos x$  is, a proper way to define cosine is through defini-, through, uh Taylor series. That is, through this series. So this actually, definition, I would say.

For CLARK and LOGAN,  $\cos x$  could be defined using its series expansion, and therefore, the series expansion for  $\cos x$  and  $\cos x$  are one and the same thing. In other words, depending on how one views the definition of cosine, there may be no distinction between the Taylor series and the generating function, and so the Taylor series and the generating function are identical. Notably missing from both expert comments was any reference to adding more terms to a Taylor polynomial until the generating function was reached. Therefore, they were not relying on a dynamic reachable image, simply that the series is the generating function, an exact image.

*Excerpt 23*

Task 4: What is meant by the “(-1,1)” in, “ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ ” when  $x$  is in the interval  $(-1,1)$ ?”

EEP LOGAN: Well, the way it is written, it says that the left side, this function one over one minus  $x$  is equal to the right side. Once again the

right side is just a series, so it's a limit of partial sums and so for every  $x$  between minus one and one, strictly between minus, strictly bigger than minus one, strictly less than one, uh limit in the right side, that is this sum of this series is equal to the value of the function in the left side. So if you plug in particular  $x$  in the left and compute limit in the right, it will be the same.

In LOGAN's reponse to the next task, the exact image showed up again. Near the end of Excerpt 23 he said, "This sum of this series is equal to the value of the function in the left side." It appeared that for LOGAN, the sum was viewed as the completion of the series and that the sum and the generating function were the same thing since they are equal. Then, in reference to what happens when a "particular  $x$ " is plugged into both sides of the equation, he reiterated that the sum and the generating function "will be the same" for that particular  $x$ .

While responding to Task 3, the same question that prompted the exact image from EEP CLARK and EEP LOGAN, EEP DEAN brought up the word "converge." I then asked DEAN to "elaborate on what it means to converge." Consider DEAN's response in Excerpt 24:

*Excerpt 24*

EEP DEAN: So for, for a series to converge means that the sum is a real number. So in this case, when I say series converge um, for a particular, let me put it this way, for a particular  $x$ , well if you just write the series for any  $x$  whatsoever it's just, it's a function, but for a particular  $x$  it becomes a series uh, which has uh, entries real numbers and then the question can become whither the sum actually makes sense in the sense of whither it's a real number or not. If it's a real number then it's, the series converges.

The exact image seemed to permeate this excerpt. For DEAN, series convergence meant that the "sum is a real number." DEAN made a distinction between a "series" and a "series for any  $x$ ." For DEAN it appeared that a "series" was devoid of any varying  $x$ . A "series" was a series consisting of merely numbers, such as those series found in sections

two through seven in Chapter 12 of Stewart (2008, pp. 723-258). A “series for any  $x$ ,” “it’s a function.” Note DEAN’s use of the word “is,” as if to say that the “series for any  $x$ ” and the function were one and the same thing. DEAN then went on to elaborate on what he meant by a “series.” He explained that for “particular  $x$ ” a series is convergent if “it’s a real number.” Therefore, for particular  $x$ , the series and the real number are the same thing. This exact imagery for Taylor series convergence demonstrated by DEAN was clothed in a pointwise explanation.

### *Layer 2: Emerging Images*

In this section I will describe how additional themes emerged in the expert interview data concerning the convergence of Taylor series. The previous excerpts have already indicated that there is much more to understanding of the convergence of Taylor series than just five types of concept images. The framework of Williams (1991) provided a backdrop for an initial investigation of the expert interviews. But a quick analysis of the preceding excerpts will quickly reveal that there are addition themes that are influencing a person’s understanding of convergence in the context of Taylor series.

For the second layer of the data analysis, I reread the transcripts using an open coding scheme (Strauss & Corbin, 1990) and loosely categorized emergent themes related to determining the convergence of Taylor series. This approach eventually led to the images that appear in Table 18. Within an individual’s conceptual understanding, some of these images might have been primary while other images were more secluded and only appeared when answering certain tasks. Yet, all experts demonstrated most of these images to one degree or another, and in many cases these emergent images interacted closely with the existing images found in the first layer of analysis.

Table 18

*Emergent Concept Images of the Convergence of Taylor Series*

Image	Description
Pointwise	Taylor series convergence was viewed as a point-by-point convergence to the generating function evaluated at each corresponding point. Here the focus was on plugging in single points, and not on an interval of points.
Sequence of Partial Sums	Taylor series convergence was viewed as a limit of the sequence of Taylor polynomials converging to the generating function. Here the focus is on a sequence of polynomials and not on an individual Taylor polynomial.
Dynamic Partial Sum	Taylor series convergence is viewed in the context of a single dynamic polynomial on which terms are added one by one. As this process continues the polynomial converges to the generating function. Here the focus is on an individual Taylor polynomial converging to the generating function.
Remainder	Taylor series convergence is viewed as a limit of remainders. If the limit of remainders is zero, then the series converges to the generating function. Here the focus is on difference between the Taylor polynomials and the generating function.
Termwise*	Taylor series convergence is viewed as a limit of the terms of the Taylor series. Here the focus is on noticing if the individual terms of the series go to zero.

\*Termwise emerged while considering the historical experts found in Chapter 4 and it clearly appeared in the novice group found in Chapter 6.

In the remainder of this section I will discuss each of these images in more detail. I will also describe some ways in which these five images interacted with the five existing images described in the previous section.

*Pointwise.* In the context of functions, a pointwise conceptualization has already been characterized by other researchers (e.g., Dubinsky & Harel, 1992; Ferrini-Mundy & Graham, 1994; Monk, 1992; Oehrtman et al., 2008). Monk (1992) says that “some students do seem to conceive of the information in a function as made up of more or less isolated values, or of input-output pairs” (p. 183). Dubinsky and Harel (1992) mentioned a pointwise element to what they called an action conception of function. According to Dubinsky and Harel (1992),

Such a conception of function would involve, for example, the ability to plug numbers into an algebraic expression and calculate. It is a static conception in that the subject will tend to think about it one step at a time (e.g., one evaluation of an expression). (p. 85)

What I have called a pointwise image of Taylor series convergence is ontologically similar. For a participant to display the pointwise image for convergence of Taylor series, their focus must be on point-by-point convergence to a generating function. Unlike Monk (1992), to be included as a pointwise image I did not require participants to fully believe that Taylor polynomials and what they convergence to are made up of isolated values of input-output pairs. For pointwise image inclusion, I merely required participants to indicate some reliance upon isolated values when describing Taylor series convergence.

A typical pointwise type image related to the convergence of Taylor series is given by EEP CLARK in Excerpt 25. In this Excerpt note how CLARK referred to “a given value of  $x$ .” At this moment CLARK was explaining Taylor series convergence by using one idealized point from the interval of convergence.

*Excerpt 25*

I: What do you mean by converge?

EEP CLARK: Uh, the, uh, I understand that to mean, uh if you take the sum of the, uh, if you take a given value of  $x$  and look at the terms on the right hand side going up to the  $n^{\text{th}}$  power of  $x$ , then, uh, the limit of that as  $n$  goes to infinity is equal to the left hand side. And then that gets in to the definition of what is the limit of a sequence, I guess, huh.

Previous excerpts have also revealed a pointwise image in relation to existing themes. In Excerpt 23, EEP LOGAN referred to plugging in “a particular  $x$ ” in the context of an exact image. Also in the context of an exact image, in Excerpt 24 EEP DEAN also referenced a “particular  $x$ ” and explained that a “series for any  $x$ ” is different than “series.” In Excerpt 20, EEP DYLAN used actual values,  $\frac{\pi}{100}$ ,  $\pi$ , and 37 radians,

in the context of the approximation image. In response to the same task DYLAN continued to reference a pointwise explanation as seen in the following excerpt. This time DYLAN referred to how trigonometric tables might be built using Taylor series.

*Excerpt 26*

Task 3: What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$ ”

when  $x$  is any real number?”

EEP DYLAN: Most of them [referring to students] took, at some point in high school, a trig course or they've seen these tables for uh, trig values and I know they wonder where in world these numbers come from. And it comes from using the Taylor series and plugging in the  $x$  appropriately, and you get an actual decimal approximation, as good as you want for the value.

Not all indicators of a pointwise image needed to be in the context of convergence, divergence also appeared alongside pointwise images. In the following excerpt, MARSHAL plugged in the endpoints of the interval of convergence. By plugging in the endpoints, MARSHAL indicated that he viewed a pointwise image as helpful in interpreting divergence. It also should be noted that in this case, MARSHAL came to the wrong conclusion.

*Excerpt 27*

Task 4: What is meant by the “ $(-1,1)$ ” in, “ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ ”

when  $x$  is in the interval  $(-1,1)$ ?”

I: And uh, what happens outside this interval?

EEP MARSHAL: Well, uh, in this particular one it diverges. You can show divergence. The interesting points are negative one and one. At one it diverges. At negative one it will converge.

*Sequence of Partial Sums.* The sequence of partial sums image of Taylor series convergence was characterized by a focus on the limit of the sequence of Taylor polynomials. In contrast to the dynamic partial sum image, which I will discuss in more detail later, the focus of the sequence of partial sums image was on a sequence of

polynomials and not on an individual Taylor polynomial. In the context of series convergence, this image is the most in line with the formal definition of series convergence as defined by Cauchy. In Excerpt 28, EEP DEAN exemplified the typical sequence of partial sums image as seen in the expert participant group.

*Excerpt 28*

I: Can you elaborate on what it means to converge?

EEP DEAN: I, I guess I could say, um, for a series to converge it means uh, for a sequence of partial sums to converge. Uh, that's the definition of convergence of series in terms of convergence of sequences.

As you can see, when discussing what it meant to converge, DEAN's focus appeared to fixate on the sequence aspect of convergence of partial sums. At this moment, for DEAN, series convergence was intimately tied to sequence convergence. In his own words, "for a series to converge it means uh, for a sequence of partial sums to converge." To reiterate, he even noted that "that's the definition of convergence of series." Later, DEAN alluded to the "sequence of partial sums" in the context of divergence.

*Excerpt 29*

I: What does it mean to diverge?

EEP DEAN: Um, for a series it means, um, that, well, in, in, as far as our sum is concerned, either, well, I would say a sum is infinity in this case, but see in, in, in sequences it's a little more complicated. Um, the limit of the sequence of partial sums need not exist at all.

One of the questions during the interviews that elicited a sequence of partial sums image from some experts was Task 6, "What is meant by the word 'prove' if you were asked to, 'Prove that sine is equal to its Taylor series.'" EEP MARSHAL was one of the experts who responded to Task 6 with a sequence of partial sums image. Consider MARSHAL's response to this task as found in Excerpt 30.

*Excerpt 30*

EEP MARSHAL: You have to know what it means for, what a, for a series to converge and that actually means that the sequence of  $n^{\text{th}}$  partial sums, okay, defined by the series, okay. So you take the sum  $n$  terms, the sum of first  $n$  plus one and so forth, and define a sequence. That sequence converges to something, okay. To prove that that series converges means that you're proving that the sequence of  $n^{\text{th}}$  partial sums converge, okay.

In this excerpt MARSHAL indicated that series convergence really “means” sequence of  $n^{\text{th}}$  partial sums convergence. He specified the partial sums to the  $n^{\text{th}}$  term, to the  $(n + 1)^{\text{st}}$  term, etc. He then restated that “to prove that that series converges means that you’re proving that the sequence of  $n^{\text{th}}$  partial sums converge.” For EEP MARSHAL it seemed that the sequence of partial sums image is intimately related to the method for proving series convergence using sequences.

*Dynamic Partial Sum.* In contrast to the sequence of partial sums image, the dynamic partial sum image had a dynamic component associated to adding more terms to a single Taylor polynomial. The sequence of partial sums image could materialize as an image involving several or all Taylor polynomials at once. Yet the dynamic partial sum image contained a single polynomial on which more terms were added one-by-one. Once a term has been added to create the new polynomial, the image of the old polynomial is no longer evoked. In a sense, the old polynomial has become (or is at least a part of) the new polynomial. Another term can then be added to the new Taylor polynomial and the process can iterate indefinitely. Note CLARK’s comment about taking “more and more terms” found in Excerpt 31.

Excerpt 31 is a typical example of the dynamic partial sum image. CLARK’s image of Taylor series was dynamic in that the “take more and more” feature of his utterance was naturally imbedded in time. Once one takes more, one has to take more



*Excerpt 31*

Task 3: What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$ ” when  $x$  is any real number?”

EEP CLARK: The series on the right converges as you take more and more terms to the function which has been defined previously in a different way on the left.

again, and take more again ad infinitum. In this instance, his image of convergence was linked to this ability to take more and more terms. Technically, one would have to “take” terms out of the series, and add them to a Taylor polynomial which converges to the generating function.

Many examples of the dynamic partial sum image can be found in several of the previous excerpts. Other typically examples of the dynamic partial sum image can be found in EEP JAMES in Excerpts 4 and 10. In Excerpt 4, the emergent dynamic partial sum image was an example of the existing dynamic image. In this excerpt, JAMES “put more and more terms,” and understood that as one did this, the Taylor polynomials got “closer and closer.” In Excerpt 10, JAMES alluded to the dynamic partial sum image in proximity to the dynamic reachable image. He noted that as “we include more and more terms” the growing Taylor polynomial “converges right on the value of cosine.” In Excerpt 3, EEP MARSHAL uses the dynamic partial sum image in response to an approximation question concerning approximating the value of sine at a specific point. In fact the dynamic partial sum image is common amongst all individually interviewed experts relative to varying types of approximation questions. Consider Excerpt 32:

*Excerpt 32*

Task 11: How can we get a better approximation for sine than using

$$x - \frac{x^3}{3!} ?$$

CEP WALLACE: Uh, just use as many uh, Taylor series expansion elements as we can and that will give a better approximation because if we just add one more element, that will be a better approximation than this element, then this [pointing to the Taylor polynomial in problem 11] polynomial. And if we add other term, it will be like that and as we go [moves right hand to the right] along the uh, Taylor series expansion of sine, we'll always get a better approximation when we add another term.

All nine experts who responded to this task during individual interviews responded in very similar ways. WALLACE clearly indicated that he understood a relationship between adding more terms to a Taylor polynomial and the approximation properties of the polynomial, "If we just add one more element [term], that will be a better approximation." He went on to state that adding "another term" would "always" yield a better approximation.

In some cases the dynamic partial sum image was not verbalized, but gestured. Excerpts 33 and 34 contain instances of a type of dynamic partial sum image that materialized in physical movements. In both cases, KELLEN and WALLACE were responding to interview Task 8, "How can we estimate sine by using its Taylor series?"

*Excerpt 33*

I: What if we wanted to estimate sine at 103 radians?

CEP KELLEN: So put in 103 for  $x$ , go until, until you feel like stopping or use your beautiful TI83 [laughing] and just choose a really big number to sum it up to.

I: So go until you feel like stopping. What are you, what, what determines when you stop?

CEP KELLEN: Right, so, how accurate you want your answer to be. So sometimes you just want a ball park figure and sometimes you really need to be within one one-thousandth or one ten-thousandth. And depending on how accurate you need your answer to be that will determine how many [moving right hand to the right with chopping motions] terms you actually have to find the sum of.

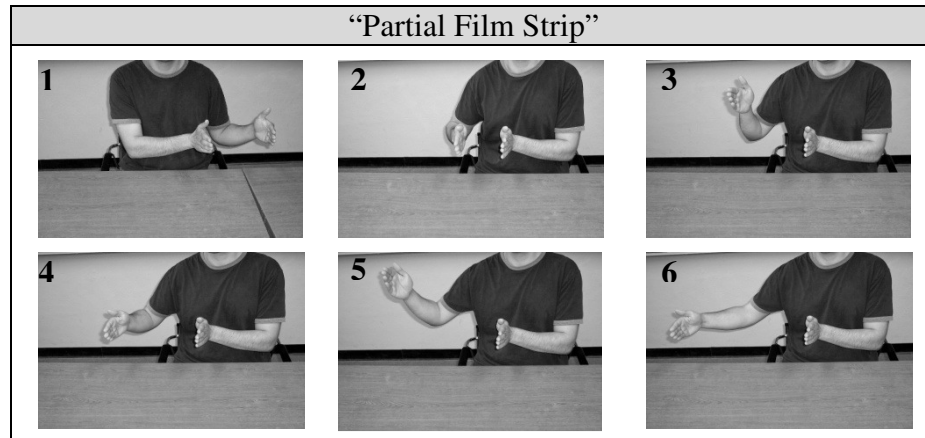
*Excerpt 34*

CEP WALLACE: Estimating sine by using its Taylor series can be uh, with the particular parts [holds both hands up as if holding something between] that we take from Taylor series expansion. For example [now moves right hand away from left hand by making chopping motions], first two terms, first three terms, first four terms, and we can look at the uh estimation by going that way. And we can pick up for example, ten values, of course we cannot take infinitely many values because it's the Taylor series expansion, but we can at the uh, we can estimate sine by just looking at specific uh [pause], how should I say, specific parts of Taylor series.

The dynamic partial sum image was internally visually assessable to both KELLEN and WALLACE, and it appeared in the form of gestures. In response to how to approximate sine at 103 radians, KELLEN responded by saying, "Depending on how you need your answer to be that will determine how many [moving right hand to the right with chopping motions] terms you actually have to find the sum of. Adding more terms to a Taylor polynomial appeared in his chopping motions as he moved his right hand to the right. Each chop suggested another term being added to the Taylor polynomial. A similar gesture appeared when WALLACE provided his insights into how to estimate sine using Taylor series. A reproduction of some of WALLACE's physical movements can be found in Figure 8. At first, when WALLACE talked about the "particular parts" of Taylor series, he held up his hands as if he was holding the first term of the Taylor series (see picture 2 in Figure 8). When WALLACE went on to talk about the "first two terms, first three terms, first four terms," he moved his right hand toward the right while making chopping motion (see pictures 3-6 in Figure 8). After each chop, it was as if he was holding the next higher order Taylor polynomial (see pictures 4 and 6 in Figure 8).

KELLEN and WALLACE were not the only experts to use similar gestures when expressing their thoughts about partial sums and the role that they play relative to Taylor

series. For example, during remarks about the usefulness of Taylor series applied to solving ordinary differential equations EPP JAMES included, “You just don't just keep the first term [holds up right hand] you keep the quadratic [moves hand one movement



*Figure 8.* Reproduced\* Generating Function and Chopping Motion Gestures

Note. Picture 1 is a typical gesture representative of a generating function. Pictures 2, 4, and 6 are typical gestures representing Taylor polynomials of increasing orders. Pictures 3 and 5 are the apexes of the chopping motions separating the gestured Taylor polynomials.

\*All “Partial Film Strips” contain pictures that are not pictures of actual participants, but reproduced images based on participant physical movements using myself as the subject contained within each picture.

to the right] or you keep the cubic [moves hand one movement more to the right] or you do other tricks.” Contained within JAMES’ comments are physical movements representative of adding more terms to a Taylor polynomial. JAMES’ movements were similar to the right hand movements depicted in pictures 2, 4, and 6 in Figure 8.

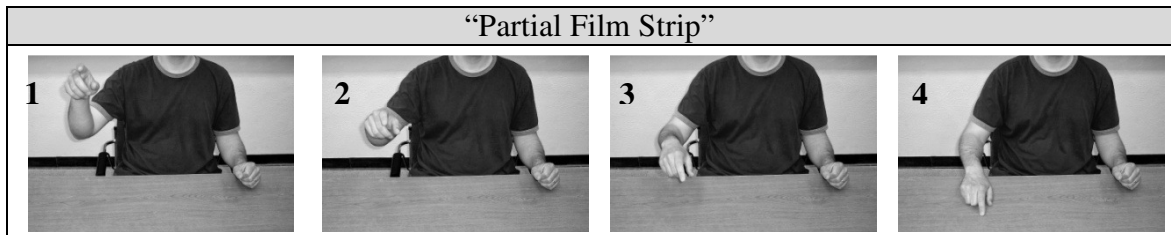
*Remainder.* The remainder image was characterized by a focus on the difference between the Taylor polynomials and the generating function. If this difference went to zero as the degree of the Taylor polynomials increased, then the sequence of Taylor polynomials converged to zero. To various amounts the remainder image has already appeared in several excerpts. In Excerpt 1, EEP LEWIS verbalized and gestured a type of remainder image in the context of approximation. LEWIS referred to seeing the

generating function and the approximating polynomial and seeing where “they’re furthest apart.” In the context of proving that a Taylor series converges to a generating function, both EEP MARSHAL and EEP DYLAN alluded to a remainder image in Excerpt 6. MARSHAL said that one of the “typical” steps for proving that sine’s Maclaurin series was equal to sine was to show that the “error goes to zero.” Within that excerpt, MARSHAL was very clear that he understood “error” as the “difference” between the generating function and a given Taylor polynomial. For DYLAN, showing that “the remainder term goes to zero” and that the sine function is infinitely differentiable was “all” that was necessary to prove that the Taylor series was equal to its generating function.

EEP MARSHAL and EEP DYLAN were not the only experts who alluded to a remainder image in relation to proving that a Taylor series was equal to its generating function. Tasks 6 and 7 both relate to how one proves Taylor series convergence, and in response to these questions, all experienced experts referenced a remainder image during individual interviews. Only the capable expert participants did not reference a remainder image during individual interviews in response to Tasks 6 and 7. Therefore, the responses to these two questions separated the capable expert participants from experienced expert participants. It is important to note that since the sample size is so small, it is unclear if this trend would continue with a larger sample. For these participants, this result seemed to indicate that for the capable experts, the remainder image might be a pseudo image (Vinner, 1997) or at the very least, not readily accessible. The lack of ease of accessibility could be due to a lack of use, which is one of the defining characteristics of capable experts. It should be noted that each CEP individually

interviewed referred to a remainder image, but these references were limited and did not occur during Tasks 6 or 7.

In Excerpt 9, EEP DEAN used a multifaceted remainder image intricately interwoven with other concepts. The paragraph following Excerpt 9 explained some of DEAN's conception of remainder and its relationship to Taylor polynomials and associated theorems. Also present in Excerpt 9, were gestures from DEAN. In relationship to his remainder image, a downward motion gesture is prevalent throughout Excerpt 9 in proximity to comments about the "remainder going to zero" (see Figure 9). For example, at the very beginning of his comment about how to prove that a Taylor series is equal to a generating function, DEAN said that "if you want to show that sequence of Taylor polynomials converge, analogous to that would be to show that the remainders go to zero [downward motion with right hand], or converge to zero [brief downward motion with right hand]." Contained within this brief comment were two references to remainder convergence to zero, the later reference was simply a restatement of the first. In both cases, movements similar to those depicted in Figure 9 were intertwined with convergence to zero. For EEP DEAN it was as if zero was depicted as the tabletop. Therefore, convergence to zero would consist of a downward movement to the table. These movements in Excerpt 9 gave insight into how DEAN may tend to visualize convergence to zero, and in particular, remainder convergence to zero.



*Figure 9.* Reproduced\* Downward Motion Gesture

\*All "Partial Film Strips" contain pictures that are not pictures of actual participants, but reproduced images based on participant physical movements using myself as the subject contained within each picture.

*Termwise.* The termwise concept image was characterized by the notion that convergence was determined by merely considering the limit of the individual terms contained within the series. A participant with a termwise conception of convergence might indicate that a Taylor series converges because the limit of the terms of the series was zero. Of course, by today's definitions for series convergence, the limit of the terms being zero is a necessary but not sufficient condition for series convergence. Just because the limit is zero, the series need not converge. The key to this lack of sufficiency is in the use of Cauchy's definition for series convergence and the rejection of d'Alembert's definition for series convergence (see Chapter 4).

The previous chapter revealed that yesterday's experts argued about termwise convergence to zero being a sufficient condition for series convergence prior to the acceptance of Cauchy definition for convergence. Therefore the termwise image of convergence was held by and used by yesterday's experts, but what about today's experts? At no point during the high inference interview tasks, did any expert clearly demonstrate a termwise conception of convergence. Yet, during low inference type tasks, some experts displayed indicators of a termwise image.

During the focus group / interview, all expert participants were asked to determine the validity of Task 27, "A series  $\sum_{n=1}^{\infty} a_n$  converges if  $\lim_{n \rightarrow \infty} a_n = 0$ ." Out of all sixteen experts, only twelve affirmed that it was a false statement. Of the four who said that the statement was true, three were CEPs and one was an EEP. Two of the true responses were made during individual interviews, while the other two were made during the focus group. The two participants who responded true to this task during interviews were CEP WALLACE and EEP MARSHAL. Since these experts missed what many

would consider as a relative easy question concerning series convergence, it might bring WALLACE's and MARSHAL's qualifications as Taylor series experts into question. First of all, it should be noted that both participants moved relatively quickly through this task as they were moving through all the true / false type tasks. WALLACE spent 27 seconds in total attending to this task while MARSHAL spent 51 seconds. As many people sometimes do, they simply may not have paid much attention to the details of the question and they moved on too quickly without fully thinking about the problem. Secondly, for MARSHAL, during the questions following Task 27, it appears that MARSHAL momentarily fixated on alternating series. If that fixation had occurred during Task 27, termwise convergence to zero is a partial indicator for alternating series convergence. Unfortunately, from my data it is not clear if MARSHAL was thinking about alternating series at the same moment he was responding to Task 27. Much of the 51 seconds that MARSHAL took to answer the problem was spent in silence as he contemplated the proper response. Rest assured, MARSHAL was an experienced expert who was well versed in Taylor series, as all experienced experts were. Therefore, even the best of experts can make small oversights.

The typical answers to Task 27 are provided by GRIFFIN and DEAN in Excerpt 35. Out of the nine experts who participated exclusively in individual interviews, five recalled a specific counterexample to Task 27. In all cases, that counterexample was the harmonic series. Not only were the experts able to recall the apparently prototypical counterexample, but most of them who recalled the harmonic series did so with relative ease (for example see EEP DEAN in Excerpt 35).



*Excerpt 35*

EEP GRIFFIN: Okay. That's, that's false of course. So, you know, this is the, you know is the converse of the divergence theorem or the  $n^{\text{th}}$  term divergence test or the divergence test, whatever you want to call it. Is that true? You know, and of course and the, the classic example is one over  $n$  which diverges by the integral test, by a clever sum you can show that it diverges. So it's uh, even though the terms go to zero.

EEP DEAN: A series  $a_n$  converges if the limit of the sequence of terms is zero. No. Because um, [writes “ $\sum \frac{1}{n} = \infty$ ”] harmonic series is divergent. So, false.

In Excerpt 25, EEP CLARK said, “if you take a given value of  $x$  and look at the terms on the right hand side going up to the  $n^{\text{th}}$  power of  $x$ , then uh, the limit of that as  $n$  goes to infinity is equal to the left hand side” (emphasis added). In some cases, the experts would use the word “terms” interchangeably with different mathematical objects. For CLARK, it appeared that he was not referring to the terms of the series, but that “terms” referenced actual Taylor polynomials viewed as terms in a sequence. In the same excerpt, CLARK immediately went on to say, “And then that gets into the definition of what is the limit of a sequence.” Participants who used the word “term” but not in reference to the terms of the series were not indicating what I called a termwise image of convergence. The next chapter will reveal that the termwise conception of convergence was more pervasive in the novice participant group.

*Layer 4: Focus Analysis*

The previous sections described the five existing images (see Table 16 in Chapter 3) and the five emerging images (see Table 18) of convergence in the context of Taylor series. Many of the preceding excerpts have indicated a complex interplay between these images. This intricate interaction is abundantly represented in the following excerpt from

EEP DYLAN's transcript. During this excerpt the approximation, pointwise, sequence of partial sums, dynamic partial sum, and remainder images all appeared at various levels.

*Excerpt 36*

EEP DYLAN: So let's say I want to use the series to approximate one. Now, okay, granted I know ahead of time what the answer is but uh, well if I take it out, my first step a half. A half is not a very good approximation for one. If I put my second term, a half plus a fourth, then I'm closer to one, and if I take it another step, I'm even closer. Well, this sequence of partial sums works in the same way. If I look at my first partial sum it's probably, it's probably not a good approximation at all to the sine of  $x$ . If I take my second partial sum, it's a little closer. It may not be a lot closer. The, if I take my third partial sum, I'm getting closer still. I know that my sequence of partial sums converges and so, I want the difference between my  $n^{\text{th}}$  partial sum and the sine of  $x$  to be quite small. So the way I do this is by choosing  $n$  to be large.

I: Okay.

EEP DYLAN: And I can make my  $s_n$  arbitrarily close to sine of  $x$  by choosing  $n$  sufficiently large and by using my remainder theorem to see how close I'm coming.

Now that the existing and emerging images have been defined in the above sections, I will take a closer look at the interactions between some of the images. A focus analysis, as described in Chapter 3, revealed some of the interactions between the emergent images. The focus analysis was only performed on the nine experts who exclusively participated in individual interviews because there was not sufficient data from the focus group to perform a focus analysis that would reveal any one individual's tendencies. Furthermore, only the high inference task responses were coded since the low inference tasks were more deceptive in revealing a participant's tendency (see Table 8 for the list of the interview high-inference type tasks). From the focus analysis I created focus diagrams which track each participant through the high inference tasks and visually point out a participant's primary focus. In this section, I will walk through the focus analysis coding used to create a portion of a focus diagram for one of the expert

participants. After that I will reveal other focus diagrams from some experts and comment on tendencies that they seemed to reveal. Before one begins this section, it should be noted that in most cases, tallies are relatively low, and therefore, numbers and percentages are only intended to represent possible expert tendencies. Even so, by the end of this section, one will see that the data appeared to yield some solid insights into expert understanding of the convergence of Taylor series.

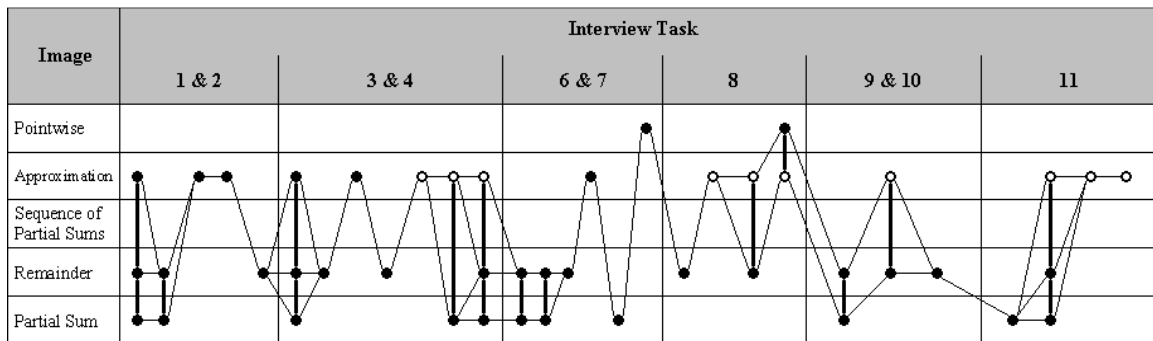


Figure 10. EEP GRIFFIN's Focus Analysis Diagram

Figure 10 contains the focus analysis diagram for EEP GRIFFIN who typified many of the experts' responses. Each row of the diagram corresponds to a particular image, and each column corresponds to an interview task or tasks. When an image was indicated within the transcript, a circle was placed in the corresponding cell. For example, an indicator of a remainder image appeared from GRIFFIN during Task 11, and a circle was placed in the cell corresponding to the remainder image row and the Task 11 column. If a participant was prompted to discuss a certain image, the circle is not filled in, but if the image appeared spontaneously, the circle was filled in. For example, open circles always appear in the approximation row corresponding to Tasks 8 – 11 because these tasks utilized approximation language within each question, therefore prompting participants to use the same language. When image indicators appeared simultaneously, they were connected using a thick vertical line. I defined images that appear on the same

vertical line as occurring during the same instance of focus. For example, indicators of the approximation, remainder, and partial sum images appeared within the same instance of focus during GRIFFIN's response to Task 11. Shifts in focus, as indicated by the participant's pronounced focus, constituted a rightward movement in the diagram to a new instance of focus. Therefore, in Task 11, GRIFFIN's focus shifted between four instances of focus: it began as partial sum, then it moved to a ternary focus on approximation, remainder, and partial sum, then to a singular focus on approximation, and finally to approximation again. Since two singular approximation images appeared side-by-side in Task 11, between the two approximation images, GRIFFIN's focus shifted to something that was not directly related to any of the images represented by the diagram. Therefore, as one moves from left to right across the diagram, indicated images appear in the order in which they appeared during the interview. These images were connected using thin lines representative of the progression through instances of focus relative to each of the interview tasks. The creation of this diagram will become clear upon a closer analysis of a portion of GRIFFIN's transcripts below.

The focus analysis that I used, concentrated on a participant's pronounced focus (i.e. what the participant literally said). The circles in the focus analysis diagrams indicate a literal utterance or utterances indicative of images. The interview task columns of the focus analysis diagram are analogous to a rudimentary attended focus because they indicate what interview task or tasks that participants were attending to as they saw them in the interview handout. Two tasks may appear in one column because both tasks appeared on the same page of the interview handout, and therefore, both tasks could be influencing a participant's response to either. All responses taken together can give

insight into a participant's primary focus. Using the focus analysis diagram, I defined a primary focus as a focus that is pronounced in all columns except one. Therefore, GRIFFIN's had a ternary primary focus that included approximation, remainder, and partial sum.

To better explain the focus diagram, I will start by using interview Tasks 1 and 2 and Table 19. The left column of Table 19 contains EEP GRIFFIN's response to high inference Tasks 1 and 2, and the right column contains potential images indicated by the corresponding excerpt. In Excerpt (1) of Table 19, GRIFFIN noted that "functions can be approximated by a finite polynomial and a remainder term." This language indicated that approximation, remainder, and partial sum images might have been evoked simultaneously (i.e. in the same instance of focus) by GRIFFIN during this utterance. Therefore, in one-to-one correspondence to the utterance in the transcript, "Approximation / Remainder / Partial Sum" appears in the right column. All images that appeared in the same instance of focus are always separated by a backslash "/" in the right column. Following this, GRIFFIN's pronounced focus appeared to move to properties of the generating function, "derivatives have to be continuous." Then, as indicated by his utterance, GRIFFIN's focus appeared to come back to the emergent images of remainder and partial sum simultaneously. In this case, both the partial sum and the remainder images had dynamic elements, "as the polynomial [waves right hand left to right] increases in size the remainder [two chops followed by waving right hand toward the right] decreases and goes to zero." In Excerpt (3) of Table 19, he moved to a pronounced focus on approximation in the context of defining generating functions using Taylor series. Between Excerpt (3) and Excerpt (8), he commented on various topics,

such as properties of polynomials in general, complex numbers, and Taylor series. None of these topics were specifically related to Taylor series convergence and thus, no corresponding image is indicated in the right column. In Excerpt (8), he again indicated a potential approximation image when he mentioned “Taylor theorem and approximation.” Following that, a remainder image appeared in the utterance of “error.” In this case, the remainder image was brought up in the context of “error approximations.” “Approximations” is not indicative of an approximation image in this instance because it was the error that was being approximated and not a generating function. Therefore, a singular focus on remainder and not a remainder / approximation focus was indicated in the right column of Table 19. He finished out these tasks by discussing Euler’s formula and particular math courses. The images represented in the right column of Table 19 were used to create the column corresponding to interview Tasks 1 and 2 found in Figure 10.

Table 20 contains the focus coding scheme for interview Tasks 3 and 4. While attending to Task 3, GRIFFIN demonstrated indicators for a ternary image of approximation, remainder, and partial sum, followed by a singular focus on remainder. Following these two instances of focus, GIFFIN then moved to Task 4 and indicated an approximation and remainder image in Excerpt (2) of Table 20. In Excerpt (5), I prompted EEP GRIFFIN to “elaborate” on the approximation language that he had used in the pervious excerpts. Since GRIFFIN had now been biased to use approximation language, I marked the appearance of approximation image indicators in the right column using a strikethrough which corresponds to an unfilled in circle in the focus diagram. This allowed me to track GRIFFIN’s continued use of approximation images and be

Table 19

*EEP GRIFFIN's Focus Coding for High Inference Interview Tasks 1 & 2*

Excerpt	Potential Image Indicated
<p>(1) EEP: [Attending to Task 1] And uh to me Taylor series are uh, uh series that arise from Taylor's theorem which is a generalization of the mean value theorem where it says that functions can be approximated by a finite polynomial and a remainder term whose coefficients depend on the derivative on an interval. And the function has to have nice properties of a uh, the derivatives have to be continuous up to a point and uh. What makes the Taylor series interesting is when they're, for most, for nice functions the remainder is something that as the polynomial [waves right hand left to right] increases in size the remainder [two chops followed by waving right hand toward the right] decreases and goes to zero.</p> <p>(2) I: Okay.</p> <p>(3) EEP: And so that approximation of a function actually can be used to define a function, and that's why they're studied in calculus. It gives us ways to define functions which seems sort of uh, distant as polynomials. Polynomials are things that we can get our hand, hands on and things we can use complex number in, you know, huh-huh. And that applies to other areas of mathematics.</p> <p>(4) I: Okay. Anything else you want to add?</p> <p>(5) EEP: That's just the basic overview. I don't know what sort of details...</p> <p>(6) I: That's fine.</p> <p>(7) EEP: I can write Taylor's theorem on the board or huh-huh, I don't think that's necessary.</p> <p>(8) EEP: [Attending to Task 2] Uh, they're studied in calculus. Uh. I mean oh. They're studied in calculus because they wanta in-, I guess they want to introduce the, the idea of uh Taylor theorem and approximations and also they want to introduce uh, the Taylor expansions for uh exponential, sine and cosine. I guess it's debatable whether how necessary it is for engineers to know Taylor series, but it's certainly, it falls into the concept idea of error approximations and like that. I don't know it's necessary for doing the, because it's usually introduced in the middle of calculus, you know, I don't think it's, I don't remember it being necessary for multivariate calculus, but you know, it's a good idea, you know. And certainly when you, if you're going to uh brush over uh Euler's theorem relate <math>e</math> to the <math>i</math> theta to <math>\sin</math>, <math>i \sin</math> theta plus cosine theta which you uh, you do in differential equations and it's, it's nice to have seen that, the Taylor expansions.</p> <p>(9) I: Okay</p> <p>(10) EEP: You know, and I-I think it's also nice just to have that introduction to analysis. The idea, reinforce the idea of what is convergence and things like that.</p> <p>(11) I: Uh-huh.</p> <p>(12) EEP: Okay.</p>	<p>Approximation / Remainder / Partial Sum</p> <p>Remainder / Partial Sum</p> <p>Approximation</p> <p>Approximation</p> <p>Remainder</p>

Table 20

*EEP GRIFFIN's Focus Coding for High Inference Interview Tasks 3 & 4*

Excerpt	Potential Image Indicated
<p>(1) EEP: [Attending to Task 3] Sure. So uh, you have cosine uh, because of what I said, it's that the uh, the series uh, your function on a nice interval can be approximated by a polynomial plus a remainder and uh, what you do is uh, it's basically a equality of the [holds up left hand] function equals the [holds up right hand] polynomial plus the [holds left hand steady while moving right hand on movement to the right] remainder and you can take a limit of both sides of that equality with respect to the uh, the length of the polynomial of the function is independent of that so that stays the same. The uh, the sum you uh, you told me the sum is the series that you have written down here and things are nice, then the remainder goes to zero. So, I mean uh, they're equal in sense of that [pointing to the problem] the limit of this sum at any x is the value of cosine of x via the magical Taylor theorem. And taking limits and letting the remainder go to zero.</p> <p>(2) EEP: [Attending to Task 4] Uh, on this the interval I uh, like you said that uh, like, like for Taylor's theorem to apply use certain conditions and for these conditions, in order to have that sum be a good approximation, right, you actually have to have it within this interval. It's uh, it's a geometric series and if it's outside that interval you don't have, you don't have convergence of the sum, it blows up and so the remainder doesn't go to zero it's uh, I don't even know, let's see would the, would the Taylor theorem even apply for that? I don't know that it would outside of there.</p> <p>(3) I: Uh, on number [3] you uh-</p> <p>(4) EEP: I'd have to think about that. Like I, I have to, like I don't really remember the statement of Taylor's theorem precisely. It might just be that you have a remainder that doesn't go to zero or might be that it doesn't apply. I'd, I'd have to look at that.</p> <p>(5) I: On number [3] you brought up the word approximate, and could you elaborate on what... You elaborated some...-</p> <p>(6) EEP: Sure um, you know and, and, for the, for the calculus student, like the approximation is that you, you look at the cosine [takes right hand and moves it up and down from left to right] it's up and down up and down it's uh periodic function that uh, you know. It, it, it can be approximated by a polynomial [takes right hand and moves it up and down from left to right] that goes up and down a finite number of times. Well if you extend the uh, degree of the polynomial you know, it will keep [takes right hand and moves it up and down from left to right] going up and down more and more and it'll uh kinda follow the graph of the cosine function on [takes both hands, palms facing each other, and moves them away from each other] bigger interval. And uh the further you get out from uh, what's called the center of the uh, of the Taylor series, here's it's the Maclaurin so the center's zero, the further you go out [quickly moves palms facing each other, and moves them away from each other] the uh, the more, the higher [takes right hand and moves it up and down from left to right] the degree of the polynomial to get, you know, all [moves right hand</p>	<p>Approximation / Remainder / Partial Sum</p> <p>Remainder</p> <p>Approximation</p> <p>Remainder</p> <p>Approximation</p> <p>Approximation / Partial Sum</p>



Table 20 Continued

*EEP GRIFFIN's Focus Coding for High Inference Interview Tasks 3 & 4*

Excerpt	Potential Image Indicated
<p>up and down far to the right of his body] the humps in to, to follow the cosine function. Uh. Were you looking for something?</p> <p>(7) I: No, I just wanted to make sure that I was understanding what you were understanding?</p> <p>(8) EEP: Sure.</p> <p>(9) I: It's not about your students it is about how you understand.</p> <p>(10) EEP: Yeah, well I mean uh, like what I talked about was sorta a uh, a visual understanding of what the graph looks like but for me it's just that uh, it's uh, like the approximation is in terms of the, the algebraic sum, you know. Cosine <math>x</math> equals the sum from one to <math>n</math> of the Taylor series plus <math>R_{sub\ n}</math>, the remainder, you know, and as that remainder gets small then the sum gets closer to cosine.</p> <p>(11) I: Okay.</p>	<p>Approximation / Remainder / Partial Sum</p>

aware that this usage may have been influenced by the question posed. One can see the appearance of potentially biased approximation images in Excerpts (6) and (10) of Table 20.

Not only did EEP GRIFFIN have a primary focus consisting of three images, the remainder image appeared unbiased in all columns in his focus diagram. That is, the remainder image was indicated by at least one filled in circle within each column of the focus analysis diagram for GRIFFIN. Furthermore, the approximation image appeared in all columns in his focus diagram. Only the partial sum image of his primary focus failed to appear in all columns. Notably absent from GRIFFIN's responses to the high inference interview tasks was any clear reference to a sequence of partial sums image. In addition, the pointwise image was only indicated in two columns. Therefore, it appeared that GRIFFIN relied upon pointwise imagery infrequently and sequence of partial sums imagery even less.

The focus diagram in Figure 10 demonstrated that GRIFFIN sometimes evoked multiple images in the same instance of focus. 45 percent (5 of 11) of the instances containing partial sum were linked to approximation imagery. In 38 percent (6 of 16) of the approximation or remainder instances, approximation and remainder were linked together. Most notably for GRIFFIN was that he repeatedly linked partial sum images to remainder images. Remainder was linked to partial sum in 50 percent (8 of 16) of the instances containing remainder, and 73 percent (8 of 11) of the partial sum instances contained linkages to remainder. EEP GRIFFIN was not the only expert to have a large percentage of partial sum instances linked to remainder imagery, both EEP MARSHAL and EEP LOGAN consistently linked partial sum to remainder, 70 percent (7 of 10) and 50 percent (5 of 10) respectively (see Figure 12 contained within this chapter and Figure 41 in Appendix D).

In some cases, images may come linked to other images and in other cases, images may tend to stand alone. Both EEP MARSHAL and EEP LOGAN never clearly linked their sequence imagery to any other images, whereas EEP DEAN's (see Figure 38 in Appendix D) remainder image was always linked to another image.

An analysis of the all focus diagrams, the expert participant group linked images in 51 percent (89 of 175) of all instances of focus. EEP DEAN linked images in 64 percent (9 of 14) of his instances of focus, and in contrast, 36 percent (9 of 25) of EEP JAMES' instances of focus contained linked images. In addition to the linkages that have already been mentioned, EEP DYLAN (Figure 39 in Appendix D) tended to link pointwise instances to approximation images (7 of 9), and both CEP WALLACE (Figure 11) and EEP DEAN tended to link approximation instances to partial sum images (5 of 7).

EEP GRIFFIN was not the only expert to demonstrate a ternary primary focus consisting of approximation, remainder, and partial sum images, EEP DYLAN and EEP JAMES both demonstrated the same primary focus (see Figures 39 and 40 in Appendix D). EEP LOGAN nearly demonstrated the same primary focus but was missing one instance for each of the three images (see Figure 41 in Appendix D). Figure 11 shows that CEP WALLACE demonstrated a binary primary focus consisting of approximation and partial sum images. EEP CLARK relied heavily on the same two images but was missing one instance for each of the two images to be consider as approximation / partial sum primary (see Figure 37 in Appendix D). Figure 12 demonstrates that EEP MARSHAL's primary focus consisted of the remainder and partial sum images.

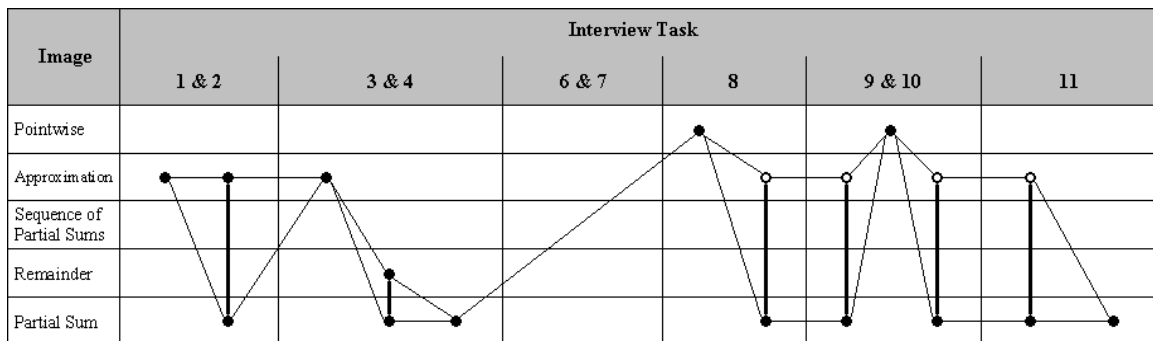


Figure 11. CEP WALLACE's Focus Analysis Diagram

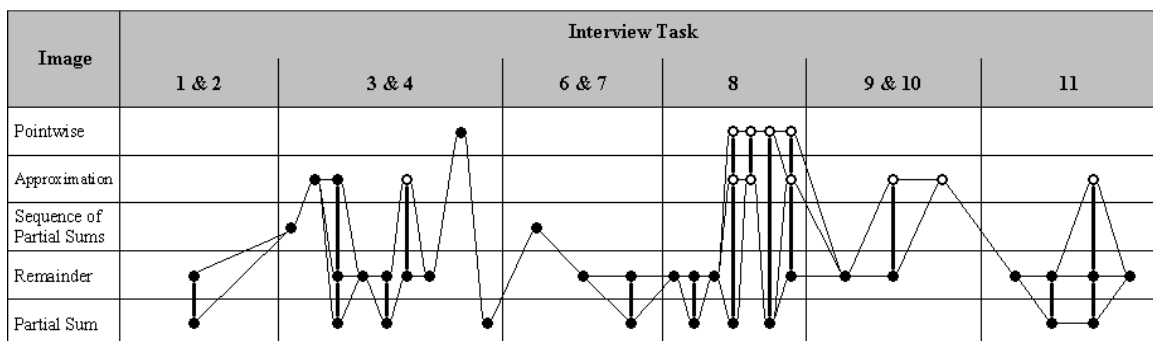


Figure 12. EEP MARSHAL's Focus Analysis Diagram

Table 21 revealed the number of experts who spontaneously indicated images in response to a particular interview task or tasks. For example, only one expert indicated a

pointwise image during their response to Tasks 1 and 2. While the focus analysis diagrams helped reveal the tendencies of individual participants, this table helped to disclose the tendencies of the expert group. This table revealed that the remainder and partial sum images were relied upon fairly heavily by the expert group, while the sequence of partial sums image was not utilized by every expert. It should be noted that if I allowed for prompted images to be tallied, the focus diagrams revealed that the approximation image was also heavily relied upon.

Table 21  
*Tallies of Experts Who Demonstrated Spontaneous\* Images during Responses to the High Inference Interview Tasks*

Image	Interview Tasks					
	1 & 2	3 & 4	6 & 7	8	9 & 10	11
Pointwise	1	6	1	6	4	2
Approximation	6	5	1	NA	NA	NA
Sequence of Partial Sums	0	4	2	0	0	0
Remainder	2	5	7	8	8	5
Partial Sum	4	6	6	8	6	9

\* As opposed to images that appeared because participants were prompted to discuss the image. Tasks 8 – 11 contained approximation language and therefore, any participant reference to an approximation image was prompted and therefore is not applicable (NA) to tally.

Almost as if by default, all nine individually interviewed expert participants partaking of the high inference tasks utilized a partial sum image in response to Task 11. Thus, adding more terms to a Taylor polynomial typified expert responses to, “How can we get a better approximation for sine than using  $x - \frac{x^3}{3!}$ ?” Seven experts responded with a remainder image during Tasks 6 and 7. A closer analysis of the data revealed that all

interviewed EEPs employed a remainder image in response to these tasks related to proving that a Taylor series was equal to its generating function. The remainder image then persisted within the EEP group throughout Tasks 8 – 10. Perhaps equally impressive is the lack of use of the sequence of partial sums image. Only four experts (all EEPs) used this sequence image in response to tasks seeking explanations for the equal sign and the interval of convergence related to particular Taylor series. Two of these experts continued to use the sequence of partial sums image to response to Tasks 6 and 7, but no other expert utilized this image and no other tasks evoked the image.

In summary, the focus analysis has reinforced that no two experts think alike when discussing the convergence of Taylor series. Experts appeared to be very robust in their thinking. That is, they were able to think about Taylor series in many different ways. They were able to move between different images and utilize multiple images as needed. Just because an expert tended to use one image didn't mean that that expert was unable to use another image. Of the images considered in the focus analysis, the approximation, remainder, and partial sum images were predominately used by the experts, whereas the sequence of partial sums image was used infrequently. As this section has demonstrated, the focus diagrams tell the story of each participant as they moved through the high inference interview tasks. Taken together, they begin to tell the story of expert conception of the convergence of Taylor series. The next section will tell some of the more interesting side stories.

#### *Expert Potpourri*

In this section I will discuss an assortment of interesting results that emerged from the expert interviews. Some of the topics addressed in this section did not fit into any of

the previous themes addressed in the preceding sections. I will begin this section by discussing experts and visualization, approximation and function replacement, and the interval of convergence and re-centering. Following those discussions, I will then elaborate on two additional emergent themes; remainder as tail image and Cauchy image. In the case of the images addressed in this section, these images emerged from the data but did not directly appear in the data for the high inference tasks for the vast majority of experts. Furthermore, as the next chapter will reveal, these images were likewise infrequent within the novice group. In relation to Taylor series, these images appeared to be secondary images that were infrequently, if at all, evoked by participants on an as needed basis. Even though the images' appearances may be brief and restricted to only a very few participants in only a few responses, these images are still that worthy of mentioning.

*Experts and Visualizing.* All experts who participated in individual interviews at various levels demonstrated an ability to visualize Taylor series convergence. When presented with problems that were more algebraic in nature, some experts attempted to draw graphs. When presented with problems that were more graphical in nature, some experts attempted to write formulas. Some of these tendencies of individual experts may have been indicators of visualizers and non-visualizers as described by Alcock and Simpson (2004, 2005). Even though describing visualizers and non-visualizers was not the primary focus of this study, enough information was gathered from the data to allow me to make some comments on this subject relative to Taylor series.

Some experts appeared to have demonstrated a tendency to visualize while other experts demonstrated a tendency toward non-visualization. Even though some experts

demonstrated these tendencies, all experts understood and could interpret graphs of Taylor polynomials. For example, when presented with the dynamic graphs of Taylor polynomials (see Task 32 in Appendix B) all experts were able to correctly identify the approximating Taylor polynomials and the effect of re-centering the polynomials. All experts were able to correctly answer approximation and remainder type questions associated with Taylor series when presented with a graph (see Tasks 33 and 37 in Appendix B). Experts might respond differently to some of these questions. For example, when asked to estimate the error in using the Taylor polynomial to approximate  $\sin\left(\frac{\pi}{4}\right)$ , of the nine experienced expert participants, eight calculated the straight vertical line distance between the two graphs while one reported the error as a percent change from one graph to the other. In either case, all nine EEPs gave correct responses. For the experts it appeared that their tendencies in visualization or non-visualization had little or no effect on their ability to respond to Taylor series problems.

*Approximation and Function Replacement.* Task 31 of the low inference true / false tasks (see Table 9 in Chapter 3) provided some interesting insight into expert understanding of the use of Taylor series in the context of function replacement. In Task 31, expert participants were asked to affirm an approximation as being true or false. In this approximation “equation”  $\sin \phi$  had been incorrectly replaced by what appeared to be the third order Taylor polynomial,  $1 - \frac{1}{6}\phi^3$ . The actual third order Taylor polynomial using sine’s Maclaurin series is  $\phi - \frac{1}{6}\phi^3$ .

Taylor polynomials are frequently used in other disciplines, such as physics, to simplify complicated equations by replacing parts of the equations with Taylor

polynomials. For example, Einstein’s theory of special relativity evolves a Taylor series, the binomial series to be precise. Under certain conditions, the higher order terms of the binomial series are unnecessary and can be approximated using a first order Taylor polynomial. By removing the unnecessary higher order terms, it significantly simplifies the equation. For a more detailed description see Stewart (2008, pp. 789-790).

Out of all the experts who responded to Task 31, six responded true, six responded false, and four said that they didn’t know the correct response. Out of the nine experience experts, three responded true, five false, and one said that they did not know because they were unsure of sine’s Maclaurin series. Based on interview data, experts responded true because they were incorrectly relating the first two terms of sine’s Maclaurin series with  $1 - \frac{1}{6}\phi^3$  and not noticing that the first term should be replaced with a  $\phi$ . Those who responded false did so because they recognized that the “Taylor polynomial” was incorrect. Two experts thought that I might had made a misprint in the interview handout, one of whom went on to comment that if the first term was replaced by  $\phi$ , then the correct response would be true. Influencing the validity of this statement, experts brought up the unknown “C” and the continuity of the square root. In addition, they brought up that the validity of the statement depended on what approximation meant and on what near meant. Only one expert clearly demonstrated a type of pointwise image by plugging in zero and noting that “the left hand side and right hand side are going to be totally different.”

All interviewed experts, no matter what response they made, attempted to associate  $1 - \frac{1}{6}\phi^3$  with  $\sin\phi$ . One CEP even changed their “don’t know” response to true



once they learned of the possible association of  $1 - \frac{1}{6}\phi^3$  to the Maclaurin series of sine.

Some experts just had recall problems with the correct formula for the cubic Taylor polynomial for sine. As with other expert mistakes that I have presented, this seemed not so much of an indicator of an error in understanding as much as an indicator of expert oversight. More important than the expert oversights themselves, all experts indicated that they understand some of the approximation properties of Taylor polynomials and the role that they can play in function replacement. In the next chapter, this type of expert understanding will stand in contrast to some novice understanding as indicated by select novice responses to the questionnaire.

*The Interval of Convergence and Re-Centering?* In the context of Taylor series convergence, the interval of convergence is the collection of  $x$  on which the series converges. At various points during the individual interviews all experts alluded to the interval of convergence when discussing Taylor series convergence. Furthermore, when responding to the high inference Task 4 (see Table 8 in Chapter 3), all experts interviewed for Task 4 demonstrated that they not only understood what was happening inside the interval, but they also understood that the series was divergent outside the interval of convergence. Therefore, the data indicated that most experts have a very clear conception of the interval of convergence and the role that it plays in association to Taylor series convergence.

When discussing the role that Taylor polynomials play in approximating, another image of a type of interval might appear. Consider a portion of EEP GRIFFIN's response to Task 3 in Excerpt 37.

*Excerpt 37*

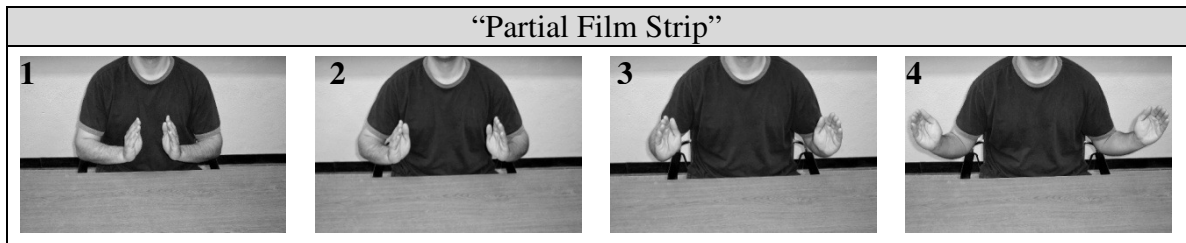
Task 3: What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$ ”

when  $x$  is any real number?”

EEP GRIFFIN: Well if you extend the uh, degree of the polynomial you know, it will keep [takes right hand and moves it up and down from left to right] going up and down more and more and it'll uh kinda follow the graph of the cosine function on [takes both hands, palms facing each other, and moves them away from each other] bigger interval.

In Excerpt 37, GRIFFIN discussed an interval, but it is not the interval of convergence.

Instead this interval seemed to depict a region on which the approximating Taylor polynomial was accurate to some predetermine level based on how well the Taylor polynomial followed the graph of the generating function. In this case, it was GRIFFIN's gesture that gave insight into how he visualized this dynamic interval of accuracy (see Figure 13).



*Figure 13.* Reproduced\* Growing Interval Gesture

\*All “Partial Film Strips” contain pictures that are not pictures of actual participants, but reproduced images based on participant physical movements using myself as the subject contained within each picture.

As EEP GRIFFIN depicted, the interval of accuracy can grow in size as more terms are added to an approximating Taylor polynomial. This growth depicted the graph of the Taylor polynomial “following” the graph of the generating function on a progressively larger interval. Implicit in GRIFFIN's comments is that outside of the growing intervals of accuracy were shrinking intervals on which the Taylor polynomials were not “following” the graph of the generating function. When viewed through the

lens of intervals of accuracy, the interval of convergence is the completion of the limit of or union of all intervals of accuracy.

The interval of accuracy could also shrink to produce a smaller error between the generating function and the approximating Taylor polynomial. Consider EEP JAMES' response to high inference Task 11 in Excerpt 38. At first JAMES answered the question with the prototypical response related to a dynamic partial sum image of convergence. After I asked him for another method, the first response that he gave was related to shrinking the interval of accuracy.

*Excerpt 38*

Task 11: How can we get a better approximation for sine than using

$$x - \frac{x^3}{3!} ?$$

EEP JAMES: By adding more terms, one, two, three terms more and see what's happening.

I: Okay. So if our goal now is to get as good approximation as possible, is there anything else you should consider besides adding more terms.

EEP JAMES: Well also shrinking interval where x can be.

Intimately related to the interval of convergence is the center of the series. Most experts verbally uttered how they understood how individual Taylor polynomial approximations get progressively better the closer  $x$  is to the center. Experts also demonstrated that they understood how Taylor series could be re-centered as a method for using Taylor polynomials of lower degree to approximate generating functions. The notion of re-centering Taylor polynomials did not immediately appear in some of the expert data. Some experts appeared less eager to allude to re-centering the Taylor series than others. For example, consider expert responses to high inference Task 8, "How can we estimate sine by using its Taylor series?" Seven experts did not allude to re-centering the series, but instead most used merely a dynamic partial sum image. If a re-centering

response did not surface, I would typically ask the experts, “What if we wanted to estimate sine at 103 radians?” If no re-centering image appeared, I might ask, “Is there anything else you want to consider when answering this question?” Even after these prompts designed to potentially elicit a re-centering image of Taylor series, four experts, one CEP and three EEPs, still did not clearly allude to re-centering as a viable approach.

I am not saying that the approximation imagery related to Taylor series for these four experts were devoid of re-centering, but it was not evoked during their responses to Task 8. In some cases, even if the notion of re-centering was used by an expert in response to Task 8, that notion was rarely called upon in response to other tasks. Therefore, it appeared that for some experts, perhaps for many experts, the image of Taylor polynomial approximation does not contain a notion of re-centering the polynomial as a primary attribute of the approximation image. This may be due to the influence of Maclaurin series and its use in exercises and applications in current curriculum (for example see Hass et al., 2007; Stewart, 2008). It should also be noted that although the interview tasks were presented in the context of Taylor series, the vast majority of series written in the interview handout were written in Maclaurin series form. Even so, some participants did rely upon re-centering as a tool for approximating. For some participants I would conclude that re-centering was foremost in their minds, while for others, re-centering appeared to be buried behind other concepts.

*Remainder as Tail Image.* Another type of remainder image that should be noted came up during expert responses to the low inference true / false Tasks 28 and 30 (see Table 9 in Chapter 3). Both Tasks 28 and 30 involved statements claiming that the limits of “tails” of series equaling zero were sufficient conditions for series convergence. In

Task 30, for example, participants were asked to affirm that the limit of the “tail” of a Taylor series equaling zero was a sufficient condition for Taylor series convergence to a generating function, the Maclaurin series for  $e^x$  to be precise. Eleven experts affirmed that Task 30 was true, three indicated false, and two didn’t know how to respond. Of the nine experienced experts, six said that Task 30 was true, two affirmed false, and one indicated that he didn’t know the correct response. Later, EEP LEWIS changed his true response to false, and when explaining his reason for doing so, he referenced partial sums in Excerpt 39.

*Excerpt 39*

EEP LEWIS: The convergence of the partial sums, converge to e to the x. So it's not because this [referring to Task 30] is true that that's e to the x. It's because the convergence of the partial sums.

Of the four EEPs that eventually did not affirm true, they all expressed concern with not having enough information to conclude that the statement was true. EEP

DYLAN said that  $\lim_{m \rightarrow \infty} \frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \frac{x^{m+2}}{(m+2)!} + \dots = 0$  was “not the reason” but “part of

the reason” that  $e^x$  equals its Maclaurin series. DYLAN also noted that the converse of

the statement was “certainly true.” In reference to  $\frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \frac{x^{m+2}}{(m+2)!} + \dots$ , EEP

COOKE proclaimed that “the limit of the error should be equal to zero, but that’s not the

error.” Another EEP who did not affirm true, associated  $\frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \frac{x^{m+2}}{(m+2)!} + \dots$

with the remainder but said that their was a “lot more stuff” associated with proving

convergence to  $e^x$ , such as noting that  $e^x$  has “all its derivatives” of all orders.

Half of the EEPs who responded true associated  $\frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \frac{x^{m+2}}{(m+2)!} + \dots$

with the remainder but did not mention “more stuff,” or at least did not see the “more stuff” as sufficient reasons for marking the statement as false. Some experts associated

Task 30 with Task 28. In Task 28, the  $a_m + a_{m+1} + a_{m+2} + \dots$  corresponds to the error

between the partial sum  $a_1 + a_2 + a_3 + \dots + a_{m-1}$  and  $\sum_{n=1}^{\infty} a_n$ , and showing that

$a_m + a_{m+1} + a_{m+2} + \dots$  converges to zero as  $m \rightarrow \infty$  is sufficient for proving that the series

converges. But this is not an indicator of what the series converges to, merely that the

series converges. In contrast, Task 30 claimed that the series not only converged but that

the series converged to  $e^x$ . When explaining why he said false to Task 30 but true to

Task 29, EEP DYLAN observed that 30 said “a lot more than that the series converges.”

He went on to say that not only did Task 30 make claims about convergence but that the

series “converges to” something, namely  $e^x$ . In contrast, when explaining why he said

false, EEP JAMES does not note the difference between 28 and 30

*Excerpt 40*

EEP JAMES: Yes. That's true and that's a particular case of question [turns the page back]. Which question is it? [28]. So [Task 28] again we stated the remainders, I mean the, what remains after we truncate [does one chopping motion with his right hand] the first terms of the Taylor series, so if this thing [holds hands out close together to the far right of his body as if holding something between] goes to zero then the Taylor series converges.

Task 30 should not be viewed as an indicator of expert mistakes but as a partial indicator of how much some experts are willing to assume when verifying the validity of a statement. As Task 30 demonstrated, some cases experts are willing to gloss over a little detail in certain situations.

The remainder as tail image that Tasks 28 and 30 seemed to evoke in expert participants is related to the image of remainder as the difference between the generating function and given Taylor polynomials, but it is different. Here the remainder was viewed as the “tail” of the series and not directly as a difference. One of the indicators of a conceptual image appeared in the use of gestures. In Excerpt 40, one can see evidence of EEP JAMES’ visual representation of this image of “remainder as a tail” when he held out his hands to the far right of his body as if holding the tail of the series between. JAMES was not the only expert to have similar gestures concerning the tail of the series when he responded to these two tasks. This type of remainder image was not clearly verifiable within expert responses to the high inference tasks but seemed limited to Tasks 28 and 30. Because of this apparent restricted use of the remainder as tail image, it appeared that this image was not primary in the minds of the experts in the context of Taylor series convergence.

*Cauchy Image.* A discussion about expert understanding of convergence of series wouldn’t be complete without commenting about Cauchy convergence. The experts were posed one question specifically designed to illicit a Cauchy image of convergence. For the purpose of this study, I will define the Cauchy image of Taylor series convergence as an image of the Cauchy criterion of convergence applied to series. According to Wade (2000), the Cauchy criterion for convergence of series is given by the following:

Let  $\{a_k\}$  be a real sequence. Then the infinite series  $\sum_{k=1}^{\infty} a_k$  converges if and only if for every  $\varepsilon > 0$  there is an  $N \in \mathbb{N}$  such that

$$m > n \geq N \text{ implies } \left| \sum_{k=n}^m a_k \right| < \varepsilon .$$

(p. 155)

Consider Excerpt 41 where EEP MARSHAL responded to the true / false Task 29.

*Excerpt 41*

Task 29: If  $a_m + a_{m+1} + a_{m+2} + \dots + a_{k-1} + a_k \rightarrow 0$  as  $m, k \rightarrow \infty$ , then  $\sum_{n=1}^{\infty} a_n$  converges.

EEP MARSHAL: What you're talking about here is if it's Cauchy. If the sequence of  $n^{\text{th}}$  partial sums is Cauchy, then the sequence converges.

Not all experts were as quick as MARSHAL to impart a Cauchy criterion for convergence onto Task 29. Of the 9 experts who participated in only interviews, 3 experts verbally associated Task 29 with Cauchy convergence. Of the 16 experts who answered this question, five experts affirmed that Task 29 was true, four of five were EEPs. Consider EEP JAMES' comments in Excerpt 42:

*Excerpt 42*

EEP JAMES: So  $m$  and  $k$  are both going to infinity but we don't know how.

I: Well, I will say that  $k$  is a bigger number than  $m$ .

EEP JAMES: Right, right, but that's it, you don't say more than that, I mean...

I: Just-

EEP JAMES: You know, so that this partial sum contains something right?

I: Yeah, it contains something.

EEP JAMES: Yeah. No, it's false. It's false. I mean this kinda, that's like question [27], and it'd be false for the same reason, so. Here we even allow more kinda leeway. So if question [27] says false, then this is even more false.

JAMES associated true / false Task 29 and with true / false Task 27, "A series

$\sum_{n=1}^{\infty} a_n$  converges if  $\lim_{n \rightarrow \infty} a_n = 0$ ." As indicated in the above excerpt, JAMES had

previously responded false to 27, and his false response to 27 influenced his false response to 29. By making this association, it appeared that JAMES may have viewed the distance between  $m$  and  $k$  in 29 as being fixed. Then the sum

$a_m + a_{m+1} + a_{m+2} + \dots + a_{k-1} + a_k$  would always contain a fixed number of elements. If



this sum always contained a fixed number of elements, then its convergence to zero would not be a sufficient condition for series convergence, consider the harmonic series for example. This view of  $m$  and  $k$  would lead an expert, like JAMES, to conclude that the correct response to Task 29 was false. In contrast, if someone viewed  $m$  and  $k$  as infinitely variable, this view might lead experts, like MARSHAL, to conclude that the correct response to Task 29 was true based on the Cauchy criterion applied to series.

Since Task 29 did not fully explain the roles of  $m$  and  $k$ , some experts could reasonably argue that Task 29 was not well posed. I would agree with these experts. But in the context of attempting to illicit various enlightening responses from experts, Task 29 proved to be very well posed. In addition, even though 29 was not specific to Taylor series, it was preceded by several Taylor series questions and thus, influenced by the concept of Taylor series. In this context, it appeared that for many expert participants, any image of Cauchy convergence was completely lacking, not evoked, or at least not primary in their mind. In the context of the high inference tasks, only EEP DYLAN briefly referred to the “difference between” partial sums as “getting smaller and smaller” during his response to Task 3. Since his focus was on the difference between Taylor polynomials and not the difference between the generating function and a Taylor polynomial, this could be an indication of a type of Cauchy image of convergence for DYLAN. But the brevity of the statement and lack of any reappearance in any transcript, not just DYLAN’s transcript, seemed to indicate that the Cauchy image is not relied upon by the experts when considering the convergence of Taylor series. It was certainly not a primary image in the minds of the experts when considering Taylor series convergence.

## *Conclusions*

This chapter has analyzed and described the different ways in which experts, EEPs and CEPs, conceptualized the convergence of Taylor series. At the beginning of the chapter I discussed the two categories of conceptual images found within the expert data: existing and emergent images. A focus analysis was completed to give a better indication of how some of the images interacted with the high inference tasks presented during individual interviews. Images contained within the two categories were not the only images to emerge from within the expert data. Additional images that appeared less influential were also addressed in a potpourri section that contained other interesting results.

The existing images were categorized as dynamic, dynamic unreachable, dynamic reachable, approximation, and exact (see Table 16 in Chapter 3 for a description of all five existing image categories). All dynamic images were typically expressed by words such as “gets close to,” “tends to,” “approaches,” “goes to,” and “moves.” In contrast, the exact image was usually devoid of any dynamic utterances. The dynamism indicated by experts was usually in the context of a Taylor polynomial approximating a generating function or the remainder function going to zero. Some experts did rely upon a dynamic unreachable type of image under certain conditions, and in most cases, when the unreachable image appeared it was linked to the notion of Taylor polynomial approximation. Furthermore, some experts discussed a type of dualism that existed between “limit knowledge” and another type of knowledge sometimes tacitly implied. These two types of knowledge appeared to be related to the actual and potential views of infinity. Moreover, experts seemed to be able to move between these two modes of

thinking with relative ease. For the majority of experts, the most relied upon existing image of convergence in the context of Taylor series was the approximation image, and several experts even commented about how this image was one of the main reasons, if not the main reason, that Taylor series are studied in calculus. In addition, when talking about Taylor polynomial approximation, some experts depicted a growing interval of accuracy.

The emerging images were categorized as pointwise, sequence of partial sums, dynamic partial sum, remainder, and termwise (see Table 18 for a description of all five emerging images). The pointwise image became explicit when participants discussed convergence by plugging in individual  $x$ -values. In some cases, participants used the pointwise image as a way of communicating divergence. The sequence of partial sums was arguably the image that appeared to be the most in line with Cauchy's definition of series convergence, but the remainder image was the image most relied upon when discussing how to prove convergence. The remainder and the dynamic partial sum images appeared to be the most often used emergent images throughout the entire interview. In some cases the dynamic partial sum image materialized in the form of a gesture involving chopping motions representative of adding more terms to a Taylor polynomial (see Figure 8). Additional emergent images used by today's experts were the termwise, remainder as tail, and the Cauchy image, but these images were infrequently relied upon.

The focus analysis demonstrated how experts thought in many different ways. Some experts may have focused primarily on approximation and partial sum images while others may have focused on remainder and partial sum images. Still others may

have used a pointwise image several times while some more heavily relied upon the sequence of partial sums image. Even though they may each think about Taylor series convergence in a variety of ways, this variation was not a hindrance. Most experts seemed to be able to move with ease between the different modes of thinking and use particular images when the situations called for them. Most notably, in the context of proof and getting better approximations, all experienced experts, almost as if by default, used remainder and dynamic partial sum images, respectively. Therefore, experts appeared efficient and effective in their thought processes concerning the convergence of Taylor series.

Furthermore, experts were able to visually understand and reason using the graphs of Taylor polynomials and generating functions. Also they were able to quickly recall prototypical examples, such as the harmonic series, and use these examples in effective ways. In addition, experts demonstrated an understanding of the interval of convergence and re-centering the series in relationship to Taylor series convergence. However, re-centering the series was a notion that was infrequently called upon during the interviews for some experts.

Although experts were usually efficient and effective in their thinking they were not beyond mistake. For this group of experts, it appeared that most expert mistakes could be characterized as either oversights, omissions, or misspeaks. Many of the oversights occurred during the true / false type tasks. In many cases oversights may have resulted because participants moved through questions too quickly without fully paying attention to the details of each task. Omissions occurred throughout the interviews, and in many cases a participant may have fixated on a particular aspect of a task or a

particular image, and failed to consider, or at least verbalize, all options. Some apparent errors were simply due to participants misspeaking. For example, a participant might say “Taylor series” when he meant to say “Taylor polynomial.”

The next chapter will reveal how these images interacted with the novice participant group. Instead of spending time to redefine and describe the images, I will spend more time pointing out differences in how the novices incorporated the images into their mental schema of Taylor series convergence. I will also be able to discuss all four layers of analysis since the novices engaged in an additional questionnaire not taken by the experts. Following the chapter describing the novices, I will conclude this study with a chapter that highlights some of the commonalities and differences between the expert and novice participant groups. In that chapter I will comment on some potential strategies for instruction and reveal several directions for future research.

## CHAPTER 6

### Tomorrow's Experts Results & Discussion of How Novices Understand Taylor Series

In this chapter I will analyze and describe the different ways in which novices conceptualized the convergence of Taylor series. In Chapter 3, novices were defined as undergraduate students from The University of Oklahoma (OU) and from a regional community college (RCC). Novices were divided into two groups, new novice participants (NNPs) and mature novice participants (MNPs) based on their amount of exposure to Taylor series. I will begin this chapter with a brief discussion of how the interview data from the novices were analyzed. Unlike the analysis of the experts, the analysis of the novices consisted of all four layers of analysis as described in Chapter 3. The novice analysis incorporated Layers 1 and 2 where interview data were coded for existing images adapted from Williams (1991) and coded for emerging images like those found in Tables 16 and 18. In addition, Layer 3 analysis consisted of a similar coding applied to questionnaire data as well as some descriptive statistics using cross-tabs and cluster analysis. As with the experts, Layer 4 analysis was conducted to provide a more detailed picture of each participant's response to the high inference type questions. This chapter is divided into four main sections. The first two sections describe the four levels of analysis, and the third presents some conclusions about the novice participant group.

### *Layers 1, 2, & 4: Existing & Emerging Images during Novice Interviews*

In this section I will discuss both the existing and emerging images as they appeared during individual interviews from the novice participant group (see Tables 16 and 18 for a description of each image). Since these images have already been discussed in detail in the previous chapter concerning expert conceptions, and since there were only minor changes in most of the descriptions of their appearances within the novice participant group, I will not explain each of the ten images in detail. I will present a few typical images as they appeared during novice interviews and then use the majority of this section highlighting some similarities and differences between the novices and experts. In an effort to highlight these similarities and differences, results from the fourth layer of analysis will be included as needed. For a detailed description of Layers 1, 2, and 4 analysis of the novice interview data see Chapter 3.

Most of what I called the existing images from the first layer of analysis were modified from Williams (1991), and I labeled them as dynamic, dynamic unreachable, dynamic reachable, approximation, and exact (see Table 16). The second layer of analysis was conducted to discover and describe additional emergent images. The data for the novices indicated similar emergent images to those images that emerged from the expert data. In Chapter 5, I labeled these concept images as pointwise, sequence of partial sums, dynamic partial sum, remainder, and termwise (see Table 18). The following three excerpts present a few of the typical themes for some of the existing and emergent images as they appeared during interviews with novices.

*Excerpt 43*

Task 2: Why are Taylor series studied in calculus?

NNP STEVE: And they're studied in calculus because there's a lot of things like sine and cosine, e to the x, that cannot be really approximated, well they can be approximated, but can't be found as a real number. They, you keep just [moves right hand from left to right making frequent small chopping motions]-

I: Uh-huh.

NNP STEVE: -go out as far as you want, as, to get as accurate as you want. And that's just the way to get closer and closer to what you need.

*Excerpt 44*

Task 3: What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$ ” when  $x$  is any real number?”

NNP JORDAN: If you were to plug in  $x$ , the, the first part is the function cosine [holds both hands to left with palms facing each other as if holding something between], and the second part [holds both hands to right with palms facing each other as if holding something between] is the function as defined by its [moves right hand away from left] infinite Taylor series. So, to add all those up [hands near and then moves right hand away from left making frequent small chopping motions] for any given  $x$ , would equal [both hands to left] cosine of  $x$ .

*Excerpt 45*

Task 6: What is meant by the word “prove” if you were asked to, “Prove that sine is equal to its Taylor series.”

MNP PHILIP: Um. In order to prove that the sine is equal to its Taylor series, you'd have to prove that the error between sine and its Taylor series is equal to zero.

In Excerpt 43, in his response to “Why are Taylor series studied in calculus?”

NNP STEVE displayed an approximation image and a dynamic partial sum image.

Indicators of an approximation image appeared, not only when STEVE uttered the word “approximated,” but it more clearly appeared when he noted that one can get “as accurate as you want” when one takes the partial sum “out.” Adding more terms to a Taylor polynomial was implied when he indicated that one could take the partial sum “out as far as you want.” Together with the gesture in which he made frequent small chopping motions (see Figure 8 in Chapter 5), taking the partial sum “out as far as you want” was



indicative of a dynamic partial sum image. It should also be noted that by linking this question to approximation, NNP STEVE was responding in a way that many expert participants responded.

In Excerpt 44, one again sees how gestures are also used in conjunction with a dynamic partial sum image. In this case, NNP JORDAN's reference to plugging in an  $x$  was an indicator of a pointwise image at the beginning of the excerpt. Therefore, the dynamic partial sum image was directly linked to a pointwise image. When JORDAN added "all those up," referring to adding terms to a partial sum as indicated by his chopping motion gesture (see Figure 8 in Chapter 5), he was doing so with a particular  $x$  plugged in. In addition, in the context of adding "all those up for any given  $x$ ," NNP JORDAN exhibited a dynamic reachable image when he noted that the resultant would "equal" the generating function. Also note that for JORDAN, a visual representation of the Taylor series was more than just some static object as indicated by his physical movement of his right hand away from his left. In contrast, the generating function was gestured as something that he held in its entirety to the left of his body. But the infinite Taylor series consisted of more than what could simply be held to the right of his body.

In Excerpt 45, MNP PHILIP used a remainder image when he uttered the "error between sine and its Taylor series is equal to zero." In many cases the remainder was seen as "going to zero" in a dynamic sense, but MNP PHILIP did not use this dynamic language for his response to high inference Task 6. In contrast, NNP JORDAN said that "you have to prove that the remainder goes to zero" when he responded to the same question. In both cases, these novices were demonstrating responses that all the experienced experts gave.

These three excerpts have presented some typical indicators of existing and emerging images as they appeared in the novice participant group during interviews. As one can see, these utterances and gestures were not unlike the experts. Therefore I will digress from a full explanation of each image and refer the reader to the previous chapter for a more complete account of each of the existing and emerging images. I will spend the rest of the section illuminating a few differences in usage of some of the images between the experts and novice participant group. Before I continue, it should be noted that for the novices, some of the questionnaires analysis preceded Layers 1, 2, and 4 of analysis. Therefore, some excerpts presented in this section represent novice participant interview responses to previously completed questionnaire tasks.

*Dynamic Reachable and Unreachable.* Consider the following excerpt from NNP STEVE:

*Excerpt 46*

I: So you brought up this word converges, can you elaborate on what that word means?

NNP STEVE: Well, lets like, um, like e to the x, and if you keep pushing more, more pieces of the, the series in [moving right hand to the right tapping the table], you get closer and closer to some number but you never really reach it unless [you] go to infinity number of figures.

A noticeable difference between some experts and novices appeared in the form of the dynamic reachable and unreachable image, in particular some novices seemed to rely heavily on dynamic unreachable images. In Excerpt 46, it first appeared clear that NNP STEVE used a dynamic unreachable image when thinking about Taylor series convergence when he uttered, “but you never really reach it.” But then he added, “unless [you] go to infinity number of figures.” In the addition, he appeared to be applying a rudimentary contrast between potential and actual views of infinity applied to Taylor

series convergence. If an infinite amount of “pieces” of the series was allowed (actual infinity), then the “number” was achieved. But if an infinite amount of “pieces” of the series was not allowed, then the “number” could only be approximated (potential infinity). Unlike the experts, NNP STEVE did not go into detail about two different kinds of knowledge (see CEPs WALLACE and KELLEN in Excerpts 12 – 16), but he seemed to be aware of two different ways of thinking about series convergence, either reachable or unreachable. In the end, NNP STEVE appeared to rely more on the dynamic unreachable image. In response to a later question STEVE said that “the series goes on and on and on, we can't get a specific number out of it, we just can get close.”

As NNP STEVE seemed to alight on an unreachable image, so did MNP PHILIP. When asked why he affirmed exactness (see Questionnaire Tasks 16 g and 19 n in Appendix C) but not dynamic reachable (see Questionnaire Tasks 16 d and 19 k in Appendix C) in the questionnaire, MNP PHILIP responded by saying the following:

*Excerpt 47*

MNP PHILIP: [The dynamic reachable question] is asking the wrong question basically... You wouldn't ask if it reaches a certain value because then you're considering uh, there to not be an infinite number of terms... Since you cannot actually write out an infinite amount of terms.

Notice that PHILIP made a distinction between “exact” and “reachable.” For him, the idea of reachable seemed to be imbedded in the idea of time, there isn't enough time to “actually write out an infinite amount of terms.” Implicit in this response was that the “exact” image was not imbedded in time because it was not associated with actually writing out an infinite amount of terms.

For both NNP STEVE and MNP PHILIP, the dynamic unreachable image seemed to permeate deep into their conceptions of the convergence of Taylor series. Moreover,

both STEVE and PHILIP seemed representative of many of the novice participants. Out of the eight interviewed novice participants, seven alluded to an unreachable image at some point during their responses to the high inference interview tasks. The next section will reveal that unreachable images were indicated by the majority of the novices (see Table 24). Cornu (1991) said that the debate between reachable and unreachable limits was “alive in our students” (p. 162). I would add, “And it appears that the unreachable image is winning when Taylor series is concerned.”

As noted in the previous chapter, the dynamic unreachable image has been observed by other researchers. In particular, while referencing Fischbein, Tirosh, and Melemed (1981), Fischbein (2001) pointed out how the usual answer that students gave concerning the convergence of the infinite series  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$  was that the sum of the series “tends to 1” (p. 320). Students were choosing this answer over the options between the sum which is “infinite” or the sum “equals 1” (p. 320). In fact, Fischbein (2001) claims that “very few students claimed that the sum was equal to 1” (p. 320). Therefore, it appears that the unreachable image is winning when series, in general, are concerned. Therefore, it is very possible that the use of the unreachable image concerning Taylor series is imputed from an unreachable image of series. Furthermore, the unreachable image of series may be attributed to an unreachable image of sequences which is ascribed to an unreachable image of limit. More research is needed before these questions can be fully answered.

*Approximation.* Sometimes, what novices viewed as Taylor polynomial approximation did not fully align with what was actually going on. Consider NNP

STEVE's response when he was asked to elaborate on his response to Questionnaire Task 6 (see Appendix C). His response can be found in Excerpt 48.

*Excerpt 48*

Questionnaire Task 6: Using the graph of  $\sin x$  below, on the same axis sketch two different Taylor polynomials for sine.

NNP STEVE: I would think that the, well, depending on the accuracy, that the sine function would probably get close [while drawing] and the further out you go in the uh, the uh [moves right hand left to right], more figures to it you get a more definite, not error prone squiggly line here.

I: Okay.

NNP STEVE: Or, I don't know, it might be something like this [drawing again], and that's not quite right. So you add up more to get closer to the actual line.

STEVE appeared to be relying on approximation imagery when he uttered “depending on the accuracy.” Also, it appeared that he was using a type of dynamic polynomial image when he alluded to “the further you go in the uh, the uh [moves right hand left to right].” The left to right motion with his right hand seemed to indicate that he was thinking about adding more terms to a Taylor polynomial. He then went on to say “more figures to it” as if to clarify that he is adding more terms to a Taylor polynomial. It is very important to note that while NNP STEVE was talking, he was also producing the graph found in Figure 14. One's understanding of his conception of how a Taylor polynomial visually approximates the generating function is not entire without seeing his actual graph. A quick glance at Figure 14 indicates that this novice did not possess a fully correct approximation image.

Without seeing what he drew at particular moments during the interview, the interpretation of his comments and his graphs remain partially unclear. In Figure 15, I reproduced the graphs based on how he sketched the curves within Excerpt 48. In the production of Figure 14, NNP STEVE first produced a graph very similar to the top

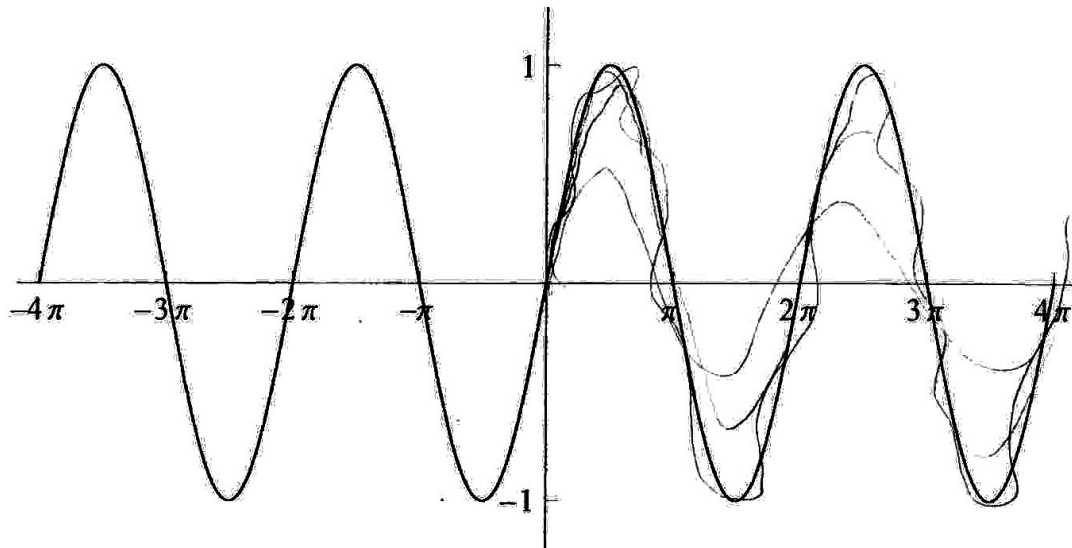


Figure 14. NNP STEVE's Graph of Approximating Taylor Polynomials

graph found in Figure 15 prior to his comments about a “not error prone squiggly line here.” As one can tell from the top graph in Figure 15, he was far from producing an appropriate graph for an approximating Taylor polynomial. The next two graphs in Figure 15 are reproductions of what he produced after uttering, “Or, I don’t know, it might be something like this.” After he said that, he first produced the middle graph found in Figure 15. This approximating polynomial seemed to oscillate with the same period as the sine curve but with less amplitude. After producing the middle graph, he drew another oscillating approximating polynomial with higher amplitude as seen in the bottom graph of Figure 15. While he attempted to make sense of the approximating properties of Taylor polynomials, this novice appeared to visually conceptualize

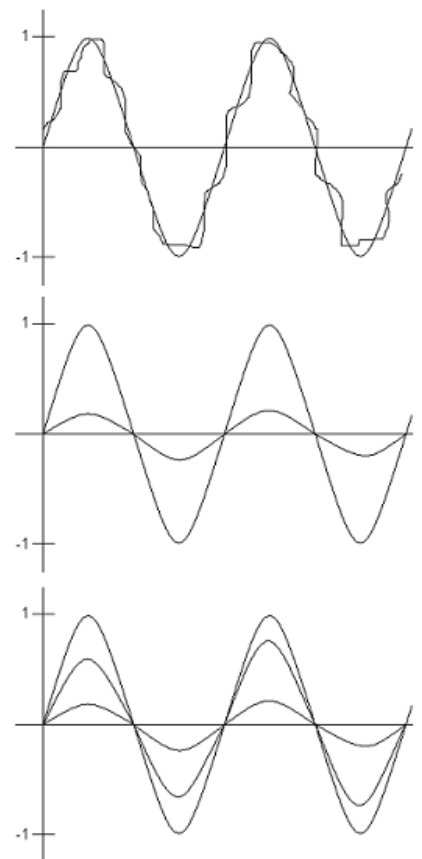


Figure 15. Reproduction of NNP STEVE's Graphs of Taylor Polynomials

Taylor polynomial convergence as sine waves with amplitudes increasing to the actual amplitude of the original sine function. It should be noted that this conceptualization of convergence seems restricted to the context of oscillating generating functions. Had the graph of what the participant was trying to approximate been something that didn't oscillate, I can only speculate that he might have had a different approach. It is also worth noting that NNP STEVE had enough scruples to detect that both ways of graphing the Taylor polynomials were inaccurate, but he remained unable to produce the correct method for graphing Taylor polynomials.

*Sequence of Partial Sums and Dynamic Partial Sum.* During an interview with MNP PHILIP, we discussed his responses to questionnaire questions six and seven concerning his graphs of Taylor polynomials and his graph of the Taylor series (see Appendix C). In response to graphing two Taylor polynomials, PHILIP produced the graphs found in Figure 16.

It is worth noting that not only did MNP PHILIP attempt to produce correct graphs of Taylor polynomials, he also attempted to produce the formulas for the Taylor polynomials relative to the Maclaurin series for sine. Furthermore, instead of drawing a linear and a cubic, he chose to draw a cubic and a quintic. Considering that he was not allowed to use a calculator, the graphs were markedly accurate, especially the cubic graph that he labeled " $f(x)$ ." The quintic graph, labeled " $g(x)$ " was not as true to the graph of the actual approximating quintic Taylor polynomial as " $f(x)$ " was true to the cubic. Even so, the graphs of " $f(x)$ " and " $g(x)$ " indicated his understanding of how Taylor polynomials approximate the generating function on progressively larger intervals as one adds more terms. The graphs also indicated that he understood that an  $n^{\text{th}}$  degree

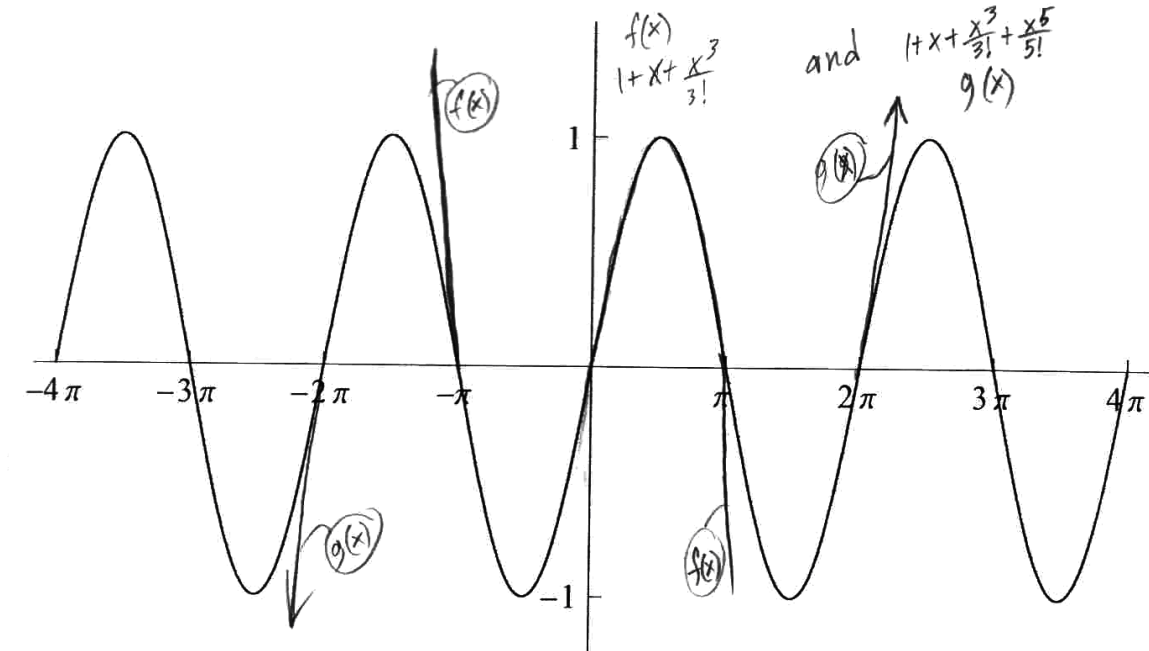


Figure 16. MNP PHILIP's Graph of Taylor Polynomials

polynomial can have at most  $n-1$  local extremum. PHILIP made a mistake in the formulas by not communicating the alternating nature of sine's Maclaurin series using negative signs in appropriate places. Other than that, his formulas for the Taylor polynomials were correct.

Considering the detail that MNP PHILIP put into his graphs of the Taylor polynomials in Figure 16, one might be tempted to conclude that this novice has demonstrated expert understanding of Taylor series. Now consider his response to being asked to graph the Taylor series for sine as shown in Figure 17.



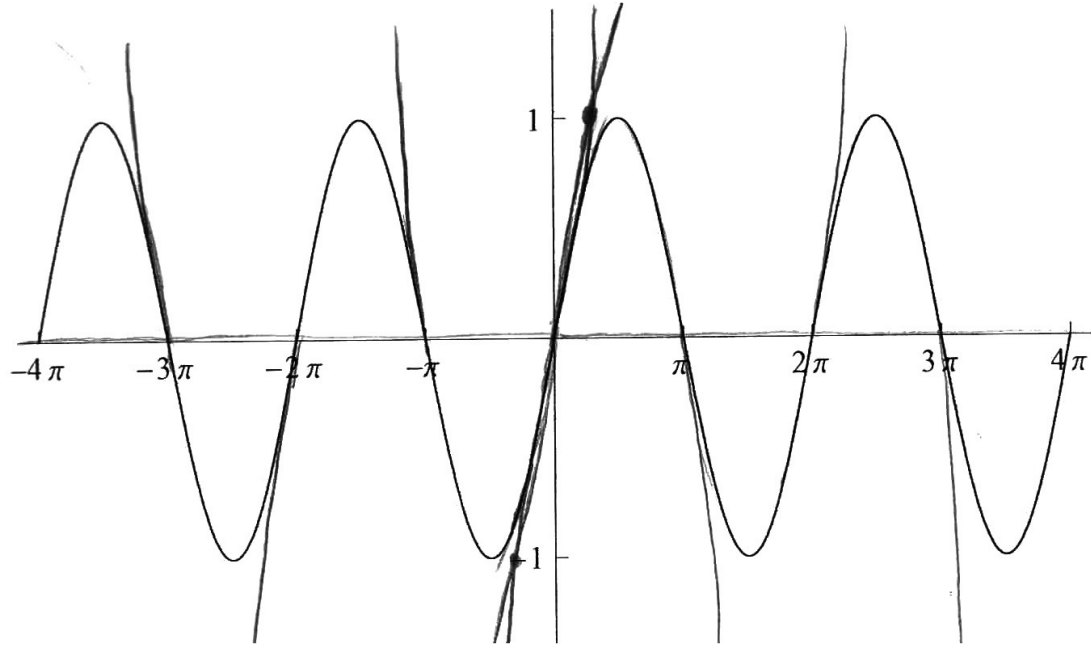


Figure 17. MNP PHILIP's Graph of a Taylor Series

Instead of tracing over the graph of sine, MNP PHILIP graphed at least 4 approximating Taylor polynomials. From the graph it is unclear if he viewed the Taylor series as the completion of all the Taylor polynomials, or if he viewed the Taylor series as the set of all Taylor polynomials. During the interview with this participant I attempted to approach this subject and learn more about what he was thinking when he drew Figure 17. Consider the following dialog that took place during the interview concerning this graph.

*Excerpt 49*

I: Am I suppose to take this graph in total or am I suppose to take just a piece of the graph?...

MNP PHILIP: I would have preferred to have more graphs so that I could separate all these different graphs that I've drawn on here, because, that's a graph, that's a graph, that's a graph, that's a graph [pointing to the different graphs]

I: Yeah.

MNP PHILIP: So I'd rather have, I'd rather have a bunch of graphs uh, so that I can express each of these.

I: So am I understanding you then to be saying? [Drew three graphs close together. Each graph contained a different Taylor polynomial.]

MNP PHILIP: Correct. That's how I was taught.... I got a good understanding when I was shown each graph and each uh, each step I guess.

I: So.

MNP PHILIP: It's like uh, it's like progressing [makes a sweeping moving with his left arm]... The changes from this, to this, to this [individually pointing to the three graphs by increasing order] are showing me a convergence...

I: ...Would you say that graph of Taylor series is consisting of all of that [circles the three graphs] or would you say it's consisting of just one piece of this?

MNP PHILIP: ...If you're calling it a series, then it's all [sweeps his right hand from left to right] of the different ones. Because the first term in the series would be the first graph, and then the second term in the series would be the second graph, the third term would be the third graph.

This participant indicated that he thought of Taylor series, not as the completion of some dynamic polynomial converging to a given generating function, but as the collection of all the Taylor polynomials. He referred to the graphs of all the Taylor polynomials when he said, "If you're calling it a series, then it's all of the different ones."

MNP PHILIP seemed to indicate that the polynomial graphs were created through a progression of adding more terms to a Taylor polynomial, "The changes from this, to this, to this [individually pointing to the three graphs by increasing order] are showing me convergence." This hinted toward a dynamic polynomial image of convergence, but the series was viewed as the collection of all polynomials that were built up through a dynamic process. Perhaps this type of visual image of Taylor series as the collection of all Taylor polynomials might be influenced by the figures found in textbooks (for example see some of the figures in Stewart, 2008, pp. 776-785). These figures are static images containing the generating function and multiple approximating Taylor

polynomials. More research is needed to fully understand the influence of such figures on student understanding.

*Termwise.* The next section on the third layer of analysis will better reveal how pervasive the termwise conception of convergence was within the novice participant group. Of all the emergent images, the termwise image was the most affirmed by novices within the questionnaire (see Table 24). But, indicators of this image only clearly appeared once during the high inference interview tasks. Immediately following his comments on what it meant for Taylor series to converge in Excerpt 46, NNP STEVE added the following:

*Excerpt 50*

NNP STEVE: So each figure gets smaller and smaller and smaller and is adding less and less and less until it's basically adding almost nothing.

I: Okay.

NNP STEVE: That's what I think convergence is, huh-huh.

In Excerpt 50, NNP STEVE referenced the individual terms of the Taylor series as getting smaller and smaller when he said that “each figure gets smaller and smaller and smaller.” He went on to conclude that “convergence is” indicated by terms of higher degree that are “adding almost nothing.” Therefore, STEVE indicated that Taylor series converge because the limit of the terms of the series go to zero.

The next section will reveal that the termwise image was affirmed within the questionnaire by the majority of novices. At first glance the interview data might seem to suggest otherwise, especially since only one novice used a termwise image. Furthermore, as the next subsection will show, the fourth layer of analysis indicated that an image like the dynamic partial sum, approximation, or pointwise might be heavily used by some of the novices interviewed (see Figures 18, 21, and 19, respectively). Based on my personal

experience, I was a little surprised that the termwise image did not show up as a primary image. As I have noted before, my interview sample size was small. Upon a closer investigation of the interviewed participants, none of the participants affirmed a termwise image on their questionnaire. Therefore, had I had a larger interview sample, it would be conceivable that it would have been possible that some novices might have demonstrated a termwise image as being primary, especially since the questionnaire revealed that many novices affirmed a termwise image. Thus, the termwise image is certainly worth exploring in more detail in future research.

*Primary Focus and Interactions between Images.* An analysis of the novice responses to the high inference interview tasks included a detailed focus analysis as described in Chapter 5. The focus analysis allowed me to verify a participant's primary focus and determine the interactions between indicators for different images. As with the experts, to illustrate individual participant tendencies I created focus diagrams, many of which can be seen in Appendix D. I also used the focus diagrams to define a participant's primary focus as a focus that was pronounced in all columns of the participant's focus diagram except one. Images that directly interacted were images that occurred during the same instance of focus, represented by images linked by vertical lines in the focus diagrams.

Like experts, some novice participants demonstrated tendencies similar to the experts, see Figure 18 and 19. Both of these novice participants responded with a remainder image at some point during their responses to Tasks 6 and 7. Also, both of these novices utilized a partial sum image when they responded to Task 11. Their responses to Tasks 6 and 7, and 11 were the same responses that typified the expert

participant group, especially the experienced expert group. A closer analysis of Figures 18 and 19 revealed that NNP JORDAN demonstrated a primary focus on the partial sum image and that NNP ANDY was one instance shy of demonstrating a primary focus on the pointwise image. Like the experts, both participants referenced multiple images during the same instance of focus. 63 percent (12 of 19) of JORDAN's instances of focus contained multiple images, and 35 percent (6 of 17) of ANDY's instances of focus contained multiple images. 75 percent (6 of 8) of JORDAN's instances of focus that indicated the approximation image were directly linked to the partial sum image, and 67 percent (4 of 6) of ANDY's approximation images were linked to the pointwise image. An analysis of all the novice focus diagrams revealed that the novice participant group linked images in around 37 percent (41 of 112) of all instances of focus.

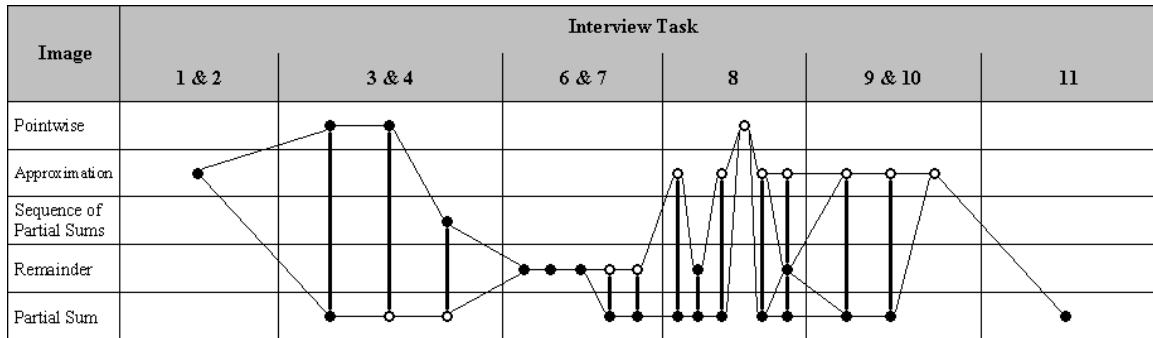


Figure 18. NNP JORDAN's Focus Analysis Diagram

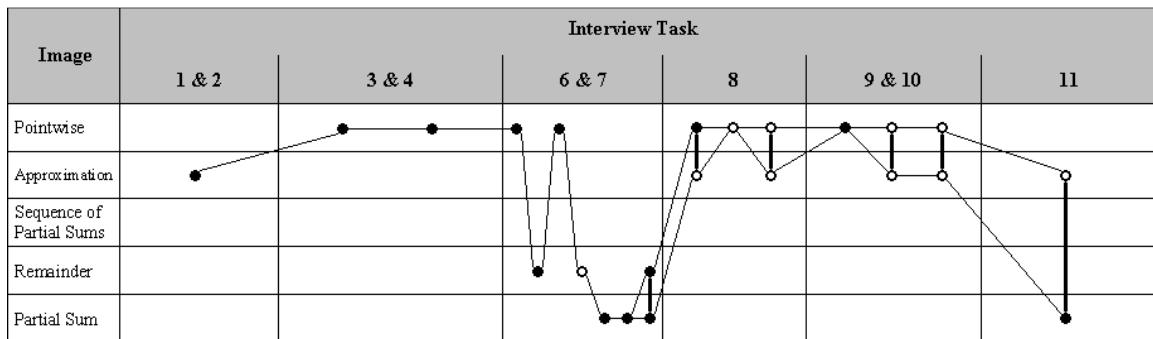


Figure 19. NNP ANDY's Focus Analysis Diagram

Other participants demonstrated tendencies that were slightly different from JORDAN's and ANDY's, see Figures 20 and 21. Both of these novice participants appeared to have long fixations onto a singular image. NNP BRIAN fixated on a pointwise image until prompted with an approximation image, and MNP RUSS fixated on an approximation image. NNP BRIAN nearly demonstrated a primary focus on the pointwise image, while MNP RUSS clearly demonstrated an approximation primary focus since the approximation image was indicated in all of the columns of his focus analysis diagram. In contrast to the primary image, noticeably missing from BRIAN's and RUSS's focus diagrams were the sequence partial sums and remainder images. Only 13 percent (2 of 15) of the instances of focus in BRIAN's responses to all the high inference type tasks contained multiple images. While RUSS linked 75 percent of any non-approximation image to an approximation image.

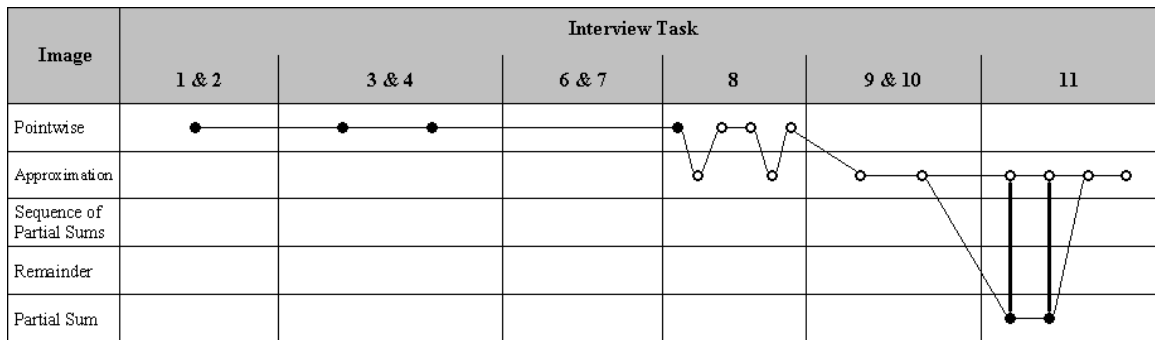


Figure 20. NNP BRIAN's Focus Analysis Diagram

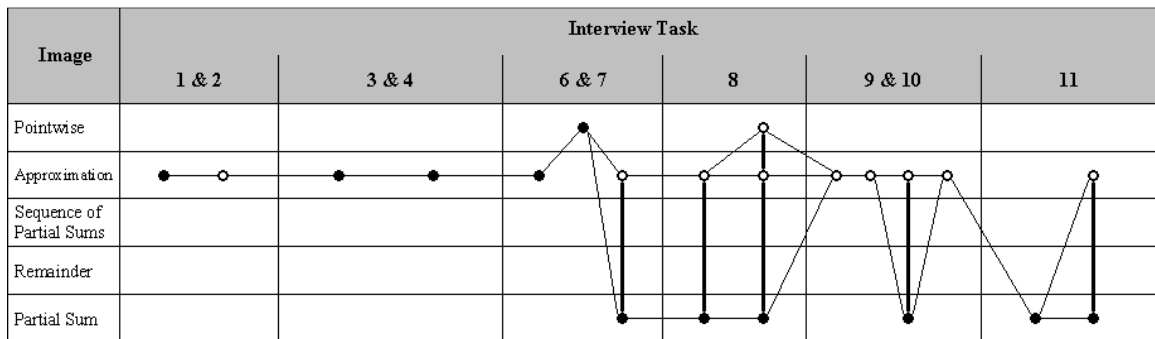


Figure 21. MNP RUSS' Focus Analysis Diagram

Table 22 revealed the number of novices who spontaneously indicated images in response to particular interview task or tasks. Like Table 21 in Chapter 5, Table 22 helped to disclose the tendencies of a participant group, the novice participant group in this case. This table revealed that the novices tended to rely upon the partial sum and pointwise images. If I had additionally tallied prompted indicators, the open circles in the focus diagrams, the approximation image would also be heavily relied upon by the novices. Like the experts, all novices utilized a partial sum image in response to Task 11, but unlike the experts, only 50 percent (4 of 8) of the novices interviewed employed a remainder image in response to Tasks 6 and 7. Noticeably missing from all but one novice response to the high inference type tasks was the sequence of partial sums image. Only 1 novice relied upon the sequence of partial sums image during the high inference type tasks even though three of the interviewed participants indicated a sequence of

partial sums image during the questionnaire. In addition, the remainder image was heavily relied upon by the expert group, but Table 22 seemed to suggest that this image was not heavily relied upon by the novice group. In fact, three novices made no reference to the remainder image during their responses to the high inference tasks.

Table 22  
*Tallies of Novices Who Demonstrated Spontaneous\* Images during Responses to the High Inference Interview Tasks*

Image	Interview Tasks					
	1 & 2	3 & 4	6 & 7	8	9 & 10	11
Pointwise	1	4	4	6	5	0
Approximation	4	5	2	NA	NA	NA
Sequence of Partial Sums	0	1	0	0	0	0
Remainder	0	1	3	4	1	0
Partial Sum	0	4	3	6	5	8
Termwise	0	1	0	0	0	0

\* As opposed to images that appeared because participants were prompted to discuss the image. Tasks 8 – 11 contained approximation language and therefore, any participant reference to an approximation image was prompted and therefore is not applicable (NA) to tally.

The focus analysis reinforced that no two novices think alike concerning the convergence of Taylor series. Some novices appeared very robust in their thinking, (see NNP JORDAN’s focus diagram in Figure 18), while others tended to fixate on one image (see NNP BRIAN’s focus diagram in Figure 20). Of the images considered in the focus analysis, the pointwise, approximation, and partial sum images were predominately used by the novices, whereas the sequence of partial sums image was only used by one novice on one occasion. Therefore, the focus diagrams have helped to illuminate the story of novice conceptions of the convergence of Taylor series. For more focus diagrams of



novice participants, see Appendix D. The next section will further reveal the novice conception of convergence of Taylor series by revealing the analysis of the novice questionnaire.

### *Layer 3: Questionnaire Analysis*

In total, 131 novices from OU and from RCC individually completed an approximately 60 minute questionnaire consisting of 33 tasks (for the complete questionnaire, see Appendix C). The tasks fell into four categories; identification and background information, short answer type questions, multiple choice type questions, and true / false type questions. As described in Chapter 3, the questionnaire tasks attempted to partially account for the effects of the interval of convergence (IOC), notational effect, and the effect of a generating function explicitly stated or unstated. For the questionnaire matrices used to aid analysis of these effects, see Tables 13 and 14 in Chapter 3. In most cases, these effects were accounted for using multiple choice type questions where novices were instructed to circle all responses that they could correctly conclude based on information given on an adjacent page.

This section concerning the third layer of analysis is divided into four subsections based on the analysis of the existing and emerging images, and the analysis of each of the three effects. The analysis of both the images and the effects mainly consisted of a combination of cross-tabs ("Cross tabulation," 2007) and cluster analysis ("Hierarchical cluster analysis," 2007; Johnson & Wichern, 2002). In proximity to some of the descriptive statistics, dendograms will be presented throughout this section to help graphically illustrate how different images and effects tended to merge in the minds of the novices as indicated by their questionnaire responses. I will begin this section with a

discussion of how the existing and emerging images appeared within the questionnaire. Then I will address the effects of the IOC, different notations, and the effect of an explicitly stated generating function.

*Existing and Emerging Images within the Questionnaire.* Tasks designed to determine and explain different conceptions of convergence in the context of Taylor series were given within the questionnaire. Many of these tasks were designed based on results from the analysis of the data from the experts. Therefore, these tasks were specifically designed with some of the previously discussed existing and emerging images in mind.

The questionnaire matrix indicating the questionnaire tasks designed to investigate existing images of convergence was given in Table 12 in Chapter 3. Table 22 indicates two sets of tasks where the sets were distinguished by their intervals of convergence (IOC); either finite or infinite. By presenting the existing image tasks in two different ways it gave insight into the possible effect that the IOC may have caused within the novice group. More on the effect will be discussed later in this section. Table 23 contains the common language between the two sets of tasks related to existing images as they appeared in the questionnaire. It turned out that in most cases the language used in the questionnaire was not only very consistent with the language used by experts, but it was also consistent with the language used by novices. This language can be found within many of the excerpts for both the experts and novices located throughout this chapter and the previous.

Table 23

*Common Language in Existing Images as It Appeared in the Questionnaire*

Existing Image	Task Numbers		Language
	Finite IOC	Infinite IOC	
Dynamic Reachable	16 d	19 k	The series gets closer and closer to the function until the function is reached.
Dynamic Unreachable	16 e	19 l	The series gets close to but never reaches the function.
Approximation	16 f	19 m	The series is an approximation to the function that can be made as accurate as you wish.
Exact	16 g	19 n	The series is identical to the function.

*Note.* Use the task numbers to locate the actual multiple choice task within the questionnaire in Appendix C.

Table 24 counted the number of novices, broken down into the two participant subgroups, who affirmed existing image tasks. Each participant is only counted once for each image if they responded to either the finite or infinite IOC task. This table seems to demonstrate that the approximation and the dynamic unreachable images were the most often relied upon by the novice participant group, whether NNPs or MNPs. 68 percent of the novices indicated an approximation image by their response to either the finite or infinite IOC tasks, and 59 percent of the novices indicated a dynamic unreachable image. The two lowest existing images indicated were the exact and the dynamic unreachable images at 34 percent and 32 percent, respectively. It should be noted that there is a considerable difference between the number of novice participants who indicated an approximation or dynamic unreachable image and those participants who indicated an exact or dynamic reachable image. Also differences appeared between the novice participant group, but two proportion z tests applied to each of the four images revealed that there is not any significant difference between the NNPs and MNPs. The p-values for each test, taking into account the implied direction, were 0.366, 0.252, 0.216, and 0.423 for the dynamic reachable, dynamic unreachable, approximation, and exact,

respectively. It should be noted that the sample size is small and in three of the cases, the number of affirmations or rejections were less than ten, therefore a plus four method was applied (D. Moore, S., 2007).

In addition, it should be noted that responses were not independent between different images. For example, the same participant could affirm both the dynamic reachable and the exact images, or the dynamic unreachable and approximation images, or any other combination of images. Since responses were not independent between images, some typical statistical tests, such as the  $\chi$ -square or ANOVA, were not applicable when comparing images. However, as the previous paragraph indicated, a two proportion z test could be applicable when comparing the two novice participant groups since the NNPs and MNPs were disjoint. Plus, when counts of affirmations or rejections were less than ten within either or both novice participant groups, a plus four method for the two proportion z-test was used (D. Moore, S., 2007).

Table 24  
*Novices Tallies Affirming an Existing Image to Either the Finite or Infinite IOC Questionnaire Tasks*

Existing Image	Novice Classification		Total
	NNP	MNP	
Dynamic Reachable	34 (33%)	8 (29%)	42 (32%)
Dynamic Unreachable	59 (57%)	18 (64%)	77 (59%)
Approximation	68 (66%)	21 (75%)	89 (68%)
Exact	36 (35%)	9 (32%)	45 (34%)

*Note.* Percent within each novice participant group are given in parenthesis within the “NNP” and “MNP” columns. Percents of all novices are given in parenthesis in the “Total” column. Recall that N=131. The sample sizes for the NNPs and MNPs were 103 and 28, respectively.

Unlike the existing image tasks found within the questionnaire, the emerging image tasks were not presented with reference to any interval of convergence. All

emergent image tasks, except for the termwise task, were related to the Maclaurin series for the exponential function  $e^x$  and began by stating that “You decide the convergence of the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  to  $e^x$  by...” Following this statement, each of the four emergent image tasks made a statement particular to that given image. See Table 25 for the common language used in each of the emergent image tasks as they appeared in the questionnaire. It should be noted that this language is consistent with language that was used by some of the novices and can be found within the excerpts.

Table 25  
*Common Language in Emergent Images as It Appeared in the Questionnaire*

Emergent Image	Task Number	You decide the convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ to $e^x$ by...
Pointwise	19 a	...plugging in numbers for $x$ and noticing that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$ converges to $e^x$ for each number $x$ .
Sequence of Partial Sums	19 b	...considering $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$ and noticing that this converges to $e^x$ .
Dynamic Partial Sum	19 c	...considering $1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$ and adding more and more terms and noticing that $1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$ converges to $e^x$ as $n$ goes to infinity.
Remainder	19 d	...considering the difference between $e^x$ and $1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$ and noticing that this difference goes to zero as $n$ goes to infinity.
Termwise*	28	A series $\sum_{n=1}^{\infty} a_n$ converges whenever $\lim_{n \rightarrow \infty} a_n = 0$ .

\* Termwise appeared as a different task and did not use the same language as the other four emergent images.

Table 26 counted the number of novices, broken down into NNPs and MNPs, who affirmed the emergent image tasks. This table seemed to demonstrate that the termwise image is the most often relied upon image. 59 percent of the novices, 58 percent of NNPs and 61 percent MNPs, indicated that termwise convergence to zero was a valid sufficient condition for determining series convergence. While the termwise image was the highest affirmed image by the NNPs, it should be noted that for the MNPs, the dynamic partial sum and the termwise images tied for the highest affirmed image. At 46 percent, the dynamic partial sum image was the next highest emergent image indicated by all novices in response to questionnaire tasks. When a directional two proportion z test was applied to the dynamic partial sum image, the difference between the percentages of NNPs, 42 percent, and the MNPs, 61 percent, was significant at the 0.05  $\alpha$ -level with a p-value of 0.037. Therefore this data suggested that there may be a difference between the number of NNPs and MNPs who affirm a partial sum image. It should be noted that the respective sample sizes, especially for the mature novice participants, are small. The two lowest emergent images indicated were the sequence of partial sums and the pointwise image at 34 percent and 37 percent, respectively. I found it interesting that the lowest affirmed image, the sequence of partial sums image, was the image that is most consistent with the formal definition of series convergence. The number of affirmations of the remainder image fell between the two highest and the two lowest. The remainder image, which was often relied upon by the experts when answering questions pertaining to proving Taylor series convergence, was affirmed by 42 percent of the novices. Since there appeared to be a noticeable difference between the number of NNPs and MNPs who affirmed the remainder image, a directional two

proportion z-test was applied, but the difference between the participant groups was insignificant (p-value = 0.1662). A two proportion z test was applied to each of the remaining three emergent images, but all were highly insignificant with the lowest p-value being 0.4072. Perhaps surprisingly, there was no significant difference between the NNPs and the MNPs in the number affirmations of the termwise image. Therefore, this data suggested that for this participant group, mathematical maturity appeared to have had little to no affect on the number of rejections of termwise convergence to zero as a sufficient condition for series convergence. In almost all emergent image categories, except pointwise, a higher percentage of MNPs affirmed each image. It appeared that the MNPs may have accepted a wider variety of various images, but as they accepted new images, these new images may not have necessarily replaced old images.

Table 26  
*Novice Affirmations of Emergent Images in Questionnaire Tasks*

Emergent Image	Novice Classification		Total
	NNP	MNP	
Pointwise	39 (38%)	10 (36%)	49 (37%)
Sequence of Partial Sums	35 (34%)	10 (36%)	45 (34%)
Dynamic Partial Sum	43 (42%)	17 (61%)	60 (46%)
Remainder	41 (40%)	14 (50%)	55 (42%)
Termwise	60 (58%)	17 (61%)	77 (59%)

*Note.* Percent within each novice participant group are given in parenthesis within the “NNP” and “MNP” columns. Percents of all novices are given in parenthesis in the “Total” column. Recall that N=131. The sample sizes for the NNPs and MNPs were 103 and 28, respectively.

*The Effect of the Interval of Convergence.* When answering a question about Taylor series there were many factors that may influence a response. One of these factors was the interval of convergence (IOC). All multiple choice questions had either series

with finite IOCs of the form  $(a, b)$  or an infinite IOC. When an infinite IOC was involved, any reference to an IOC was omitted. To account for the effect of the IOC, when possible, different versions of a multiple choice selection existed. Table 23 has already demonstrated how tasks concerning the existing images accounted for the finite and infinite IOCs. In particular each of the four multiple choice selections for the existing themes in the context of Taylor series, had two versions, one version with a finite and one with an infinite IOC. Table 27 gives the existing image response tallies to the questionnaire tasks that were specifically designed to reveal a participant's notion of limit in the context of Taylor series while taking into account the effect of the IOC.

Table 27  
*Novice Affirmations of Existing Images Accounting for Different Intervals of Convergence (IOCs)*

Interval of Convergence	Existing Images			
	Dynamic Reachable	Dynamic Unreachable	Approximation	Exact
Finite IOC	37 (28%)	58 (44%)	72 (55%)	30 (23%)
Infinite IOC	17 (13%)	59 (45%)	73 (56%)	28 (21%)

*Note.* N = 131. For each cell, the percentage of N are reported in parenthesis.

After dividing the existing image novice data into finite verses infinite IOCs, the approximation image remained as the most oft used existing image for both IOCs, 55 percent and 56 percent, respectively. As indicated by questionnaire responses, the dynamic unreachable image was the second most relied upon existing image, and the exact and dynamic reachable images were the least relied upon. It should be noted that based on Table 27, it appeared that the IOC caused little difficulty with three of the images, while it may have effected responses to the dynamic reachable image. Since the image and IOC categories are not independent, neither the  $\chi$ -square nor the two proportion z-test can be used to determine if there is a significant difference between



different IOCs for any image, including the dynamic reachable image. Furthermore, Table 28, created using cross-tabular descriptive statistics, indicates that Table 27 may be a little bit misleading when it comes to accounting for the effect of the IOC.

Table 28  
*Novice Tallies of Existing Image Interactions between Different Intervals of Convergence (IOCs)*

		Finite IOC				Infinite IOC			
	Convergent Taylor Series Images	Reachable	Unreachable	Approximation	Exact	Reachable	Unreachable	Approximation	Exact
Finite IOC	Reachable	<b>37</b>							
	Unreachable	7	<b>58</b>						
	Approximation	24	37	<b>72</b>					
	Exact	15	8	16	<b>30</b>				
Infinite IOC	Reachable	<b>12</b>	7	7	10	<b>17</b>			
	Unreachable	10	<b>40</b>	32	9	1	<b>59</b>		
	Approximation	21	39	<b>56</b>	17	8	35	<b>73</b>	
	Exact	12	10	14	<b>13</b>	11	7	12	<b>28</b>

*Note.* Bold diagonals indicate responses to similar existing images. The upper diagonals are left blank since the table is symmetric.

The main diagonal of Table 28 contains Table 27, and the bold off diagonal represents the number of novices that were consistent in their responses to each of the existing image tasks across the two different intervals of convergence. According to Table 28, novices were not consistent in their responses to the finite and infinite IOC tasks concerning the existing images. Only 12, 40, 56, and 13 novices were consistent in their responses to the dynamic reachable, dynamic unreachable, approximation, and exact image tasks, respectively. Only 32 percent of the novices that affirmed a dynamic reachable image in the context of a finite IOC also affirmed a dynamic reachable image in the context of an infinite IOC. On the other end, about 77 percent of the novices that affirmed an approximation image affirmed both the finite and infinite IOCs. All other

comparisons between responses to particular images across different IOCs range between 32 percent and 77 percent.

To help better understand the interaction between the different IOCs and the images, using SPSS I performed a hierarchical cluster analysis on the questionnaire data. In a cluster analysis, the cluster procedure attempted to cluster similar questionnaire tasks based on a measure of the tendencies that participants demonstrated when responding to particular tasks. If participants had a tendency to respond in the same way to a certain group of tasks, then those tasks were clustered together. Initially, the cluster analysis began by treating each task as a separate cluster. At the next step of the analysis, multiple clusters of tasks could have been formed to reveal how participants responded to certain groups of tasks. Once these clusters were formed, an analysis was performed on the new clusters to determine what clusters could have been combined to form larger clusters containing more tasks. At each step of the analysis, the criterion for the measure of the distance between each cluster was relaxed, allowing for clusters to join together. Therefore, tasks that were less related, took longer to cluster together. This analysis continued in a recursive manner until only one cluster remained, thus creating a hierarchy of clusters.

There were several different ways to measure the distance between clusters (i.e. measure the tendencies of particular tasks to cluster together). Since participants were limited to only affirming (1) or not affirming (0) particular image tasks, a binary measure was used. In addition, this binary measure could be set to consider two images as similar based on only affirmations to both (Jaccard measure), or it could be set to consider two

images as similar based on either affirmations to both or non-affirmations to both (simple matching).

Figure 22 contains dendrograms illustrative of the tendencies of particular questionnaire tasks to cluster together within the novice participant group based on novice responses. According to the SPSS Online Help and Tutorial, a dendrogram is “a visual representation of the steps in a heirarchical clustering solution that shows the clusters being combined and the values of the distance coefficients at each step” (“Dendrogram,” 2007). In SPSS, all measures of the distances between clusters as presented in dendrograms were rescaled between 0 and 25 to preserve the ratio between steps of the cluster analysis.

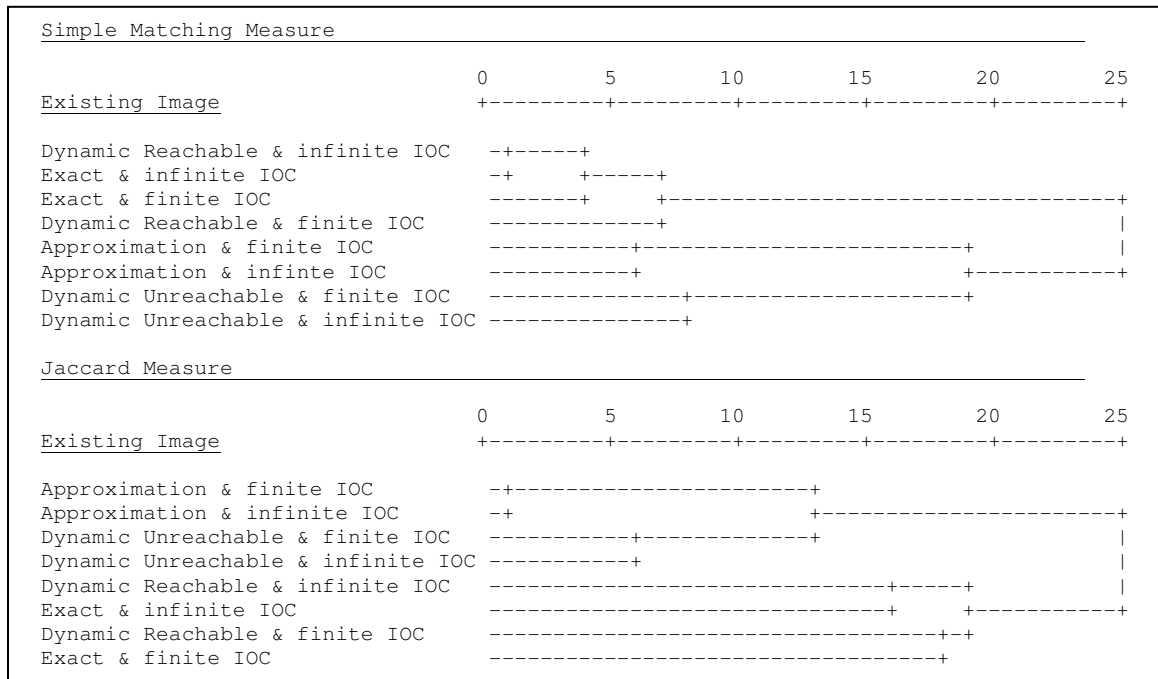


Figure 22. Dendrograms of Existing Images Based on Novice Responses to Corresponding Questionnaire Tasks

The dendrograms of Figure 22 illustrate the tendencies of certain existing images to combine together. In the simple matching dendrogram, the reachable and infinite IOC

and the exact and infinite IOC tasks combined into one cluster in the first measure of the cluster analysis. By the fourth measure, the exact and finite IOC image combined with the reachable and exact with infinite IOCs to form a new cluster. In the seventh measure, all the reachable and exact images combined no matter what IOC was considered.

Since the Jaccard measure and the simple matching measures were different, the dendrograms were different. The simple matching dendrogram not only illustrated similarities between affirmations, but non-affirmations. The simple matching dendrogram showed that the approximation tasks took until the 7<sup>th</sup> measure to cluster together. Since the Jaccard measure only illustrated similarities between affirmations, the dendrogram for the Jaccard measure indicated that the two approximation images were joined together in the first measure. This suggests that novice participants who affirmed one of the approximation images tended to affirm both approximation images, no matter if the IOC was infinite or finite. Following the approximation image, the unreachable images were the next to combine in the 6<sup>th</sup> measure of the Jaccard dendrogram. It took until the 19<sup>th</sup> measure for the reachable and exact images to combine. When considering Tables 27 and 28, the approximation and unreachable images had the largest percentage of participants constantly affirming both IOCs once they affirmed one IOC, but the reachable and exact have the lowest percentages. Therefore, the Jaccard dendrogram was consistent with the findings from Tables 27 and 28.

Both dendrograms illustrated that novice participants were inconsistent in their responses to the existing images based on the different IOCs. I am reluctant to conclude that the different IOCs are the root cause for such inconsistencies. There may be other lurking variables that were not specifically accounted for. For example, the infinite and

finite IOC tasks were posed in two different contexts. The two series with finite and infinite IOCs were generated by two different functions,  $\frac{1}{1-x}$  and  $e^x$ , respectively.

Other research has already shown that students rely on different images in different contexts (e.g., Oehrtman, 2002). Furthermore, during interviews it became clear that some novice participants struggled to interpret certain tasks. Therefore, for each novice, either (1) the IOC caused an effect, (2) they relied on different images in different contexts, or (3) they did not correctly interpret the tasks and hence, may have guessed.

During interviews, none of the novice participants appeared to pay particular attention to the IOC unless I instructed them to do so. Because many failed to take into account the IOC, the IOC appeared to not factor into many responses, in particular in responses to the existing image tasks. In addition, none of the novice interview participants appeared to struggle with correctly interpreting the tasks related to the existing images. When comparing the finite and infinite IOC existing image tasks, most novices became consistent in their responses during interviews even though they may not have been consistent during the questionnaire. When taking the questionnaire, the finite and infinite IOC existing image questions were separated by a few pages that would take several minutes to complete. Although I had a small sample of novices for the interviews, based on all the data, it appeared that it is more likely that novices were relying on different images at different times and that few novices failed to correctly interpret the tasks, even fewer, if any, had existing images that were greatly effected by the IOCs.

*Interactions with Existing and Emerging Images.* Both Table 28 and the dendrograms of Figure 22 showed that more is going on than just a possible IOC effect.

All images are interacting with each other. The dendrograms of Figure 22 showed two main clusters, reachable / exact and unreachable / approximation. Depending on which measure was used, the reachable / exact cluster was formed no earlier than the 8<sup>th</sup> measure and the unreachable / approximation cluster was formed no earlier than the 13<sup>th</sup> measure. Although it took a while for these two main clusters to form, it appeared that novices who demonstrated a dynamic reachable image may also tend to demonstrate an exact image, and novices that demonstrated a dynamic unreachable image may also tend to demonstrate an approximation image.

When considering these interactions with the NNPs and also with the MNPs, the respective dendrograms for each novice group behaved very similar to those found in Figure 22, except for one. The Jaccard measure dendrogram for the MNPs failed to break into two main clusters each containing multiple tasks. One large cluster for the MNPs using the Jaccard measure contained all the dynamic unreachable and approximation tasks, but the dynamic reachable and exact tasks failed to cluster together. Instead, they clustered into the unreachable / approximation cluster. Upon a closer examination, out of a possible 56 affirmations by MNPs between the infinite and finite IOC tasks, the dynamic reachable / exact tasks were only jointly affirmed by three participants. Since the Jaccard measure counted two tasks as being similar based on only joint affirmations, it did not count the dynamic reachable and exact images as being similar due to a lack of joint affirmations. Furthermore, it should be noted that the MNP group had a small sample size, and thus, additional MNPs could change the clustering in that group's Jaccard dendrogram. For the simple matching and Jaccard measure dendrograms for both novice participant groups, see Appendix E.

Additionally, I considered the differences between those participants who were completely consistent and those who were never consistent in their responses to the existing image tasks across the two different IOCs. Participants who were completely consistent, always marked the same existing images for both the finite and infinite IOC. In contrast, those who were never consistent, never marked the same existing image twice. In total, I had 43 (33%) novice participants, 29 (40%) NNPs and 14 (50%) MNPs who were completely consistent. And I had 41 (31%) novice participants, 33 (32%) NNPs and 8 (29%) MNPs who were never consistent. It turned out that the proportion of completely consistent MNPs was significantly higher than the proportion of completely consistent NNPs at the 0.05  $\alpha$ -level ( $p=0.0145$ ). But the difference between the number of MNPs and NNPs that were never consistent was not significant ( $p=0.4025$ ). Therefore the data suggested that more MNPs tended to be completely consistent in their responses to the existing image tasks, while there was no difference between the MNPs and NNPs who were never consistent in their responses to existing image tasks.

Both the simple matching and the Jacard measure dendrograms for the completely consistent novice participants were similar to the dendrograms in Figure 22. Both dendrograms for the completely consistent novices illustrated two main clusters formed by the dynamic unreachable / approximation tasks and the dynamic reachable / exact tasks. In stark contrast, both dendrograms for the never consistent novice participants failed to form two main clusters in any apparently meaningful way. For example, for both measures, a dynamic unreachable task and reachable task joined together in the first measure for the completely inconsistent participant group. In addition, an approximation task joined early with a reachable task using both measures. This further illustrated that

some novices are either relying on vastly different images at different times, or they were not understanding the tasks. Based on interviews with novice participants I still conclude that it was less likely that the IOC had much of an effect. The dendrograms for the simple matching and Jaccard measures for the completely consistent and the never consistent novice participants can be found in Appendix E.

Table 29 further illustrates the types of interactions between existing images as indicated by responses to the corresponding questionnaire tasks. According to Table 29, out of any two image interaction, the approximation / dynamic unreachable images were the most oft affirmed interaction of existing images at 58 total affirmations (22%). The approximation only and dynamic unreachable only images were the next most oft affirmed interactions at 42 (16%) and 39 (15%), respectively. No other interaction came close to the top three in total number of affirmations, and there were small gains in the top three interactions between the finite and infinite IOCs, possibly indicating some kind of small effect.

Table 29 was also used to count the number of instances of particular existing image interactions. By adding up all the instances of the approximation image, it appeared in 145 of the 262 affirmations possible (55%). Compare this to dynamic unreachable image (45%), exact (22%), and the dynamic reachable image (21%). This reaffirms that the approximation image was the most often relied upon existing image by the novice participant group, followed closely by the unreachable image. Adding up all the instances of the dynamic unreachable / approximation image interaction, this dual interaction appeared 72 times (27%). Compare this to the dynamic reachable /



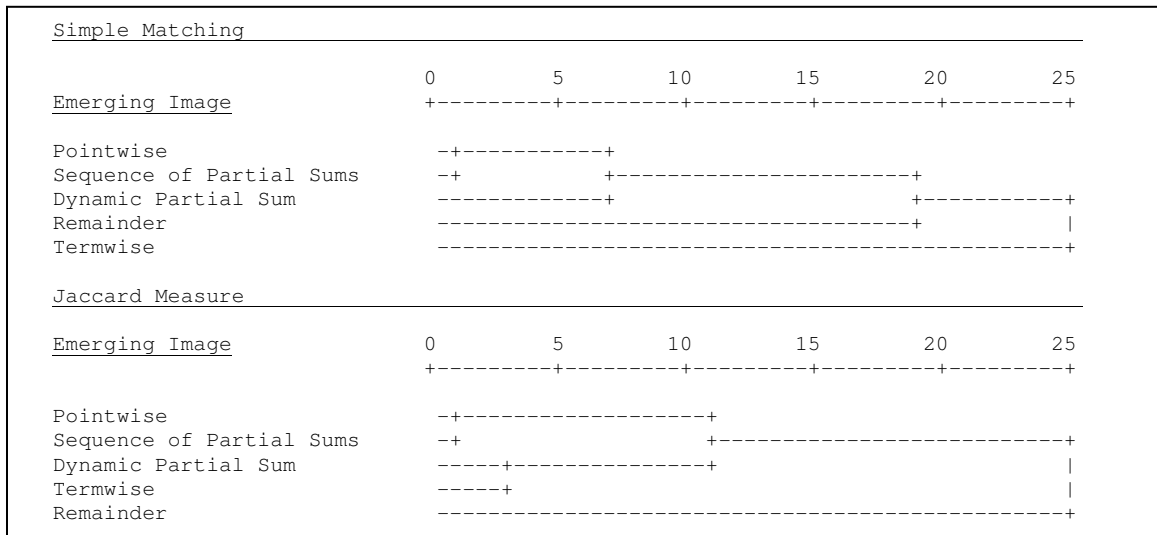
Table 29

*Existing Image Inaction Based on Questionnaire Response Tallies*

Existing Image Interaction	Interval of Convergence		Total Affirmations
	Finite IOC	Infinite IOC	
<b>1 Image</b>			
Reachable Only	5	4	9
Unreachable Only	17	22	39
Approximation Only	13	29	42
Exact Only	6	9	15
<b>2 Image Interactions</b>			
Reachable / Unreachable	2	0	2
Reachable / Approximation	12	2	14
Reachable / Exact	6	5	11
Unreachable / Approximation	28	30	58
Unreachable / Exact	2	2	4
Approximation / Exact	3	2	5
<b>3 Image Interactions</b>			
Reachable / Unreachable / Approximation	3	0	3
Reachable / Unreachable / Exact	0	0	0
Reachable / Approximation / Exact	7	5	12
Unreachable / Approximation / Exact	4	4	8
<b>4 Image Interaction</b>			
Reachable / Unreachable / Approximation / Exact	2	1	3
Total Affirmations	110	115	225
Total Affirmations Possible	131	131	262

approximation pair, 32 times (12%); approximation / exact, 28 times (11%); dynamic reachable / exact, 26 times (10%); dynamic unreachable / exact, 15 times (6%); and dynamic reachable / dynamic unreachable, 8 times (3%).

In addition to the existing image interactions, emerging images also interact. Figure 23 contains the simple matching and the Jaccard measure dendrograms for the novice participant group emerging image questionnaire tasks as indicated by novice responses. These dendrograms show that the pointwise and the sequence of partial sums emerging images tended to quickly cluster together. Whereas the remainder and the termwise images may not cluster with the other images depending on which measure was used. The dynamic partial sum image tended to join with the pointwise and sequence of partial sums images no later than the 12<sup>th</sup> measure.



*Figure 23.* Dendrograms of Emerging Images Based on Novice Responses to Corresponding Questionnaire Tasks

Interactions between existing and emergent images were considered. Simple matching and Jaccard measure dendrograms for all the novice participants, including NNPs, MNPs, completely consistent novice participants, and never consistent novice participants were created. Each of the ten dendrograms was different from the others and initially appeared to provide only little indication of how all the images were interacting. Figure 24 contains the simple matching and Jaccard measure dendrograms for the

interactions between the existing and emergent images for the entire novice participant group. This figure illustrates that for the novice participant group, the emergent images did not stand disjoint from the existing images but intermingled with the existing images.

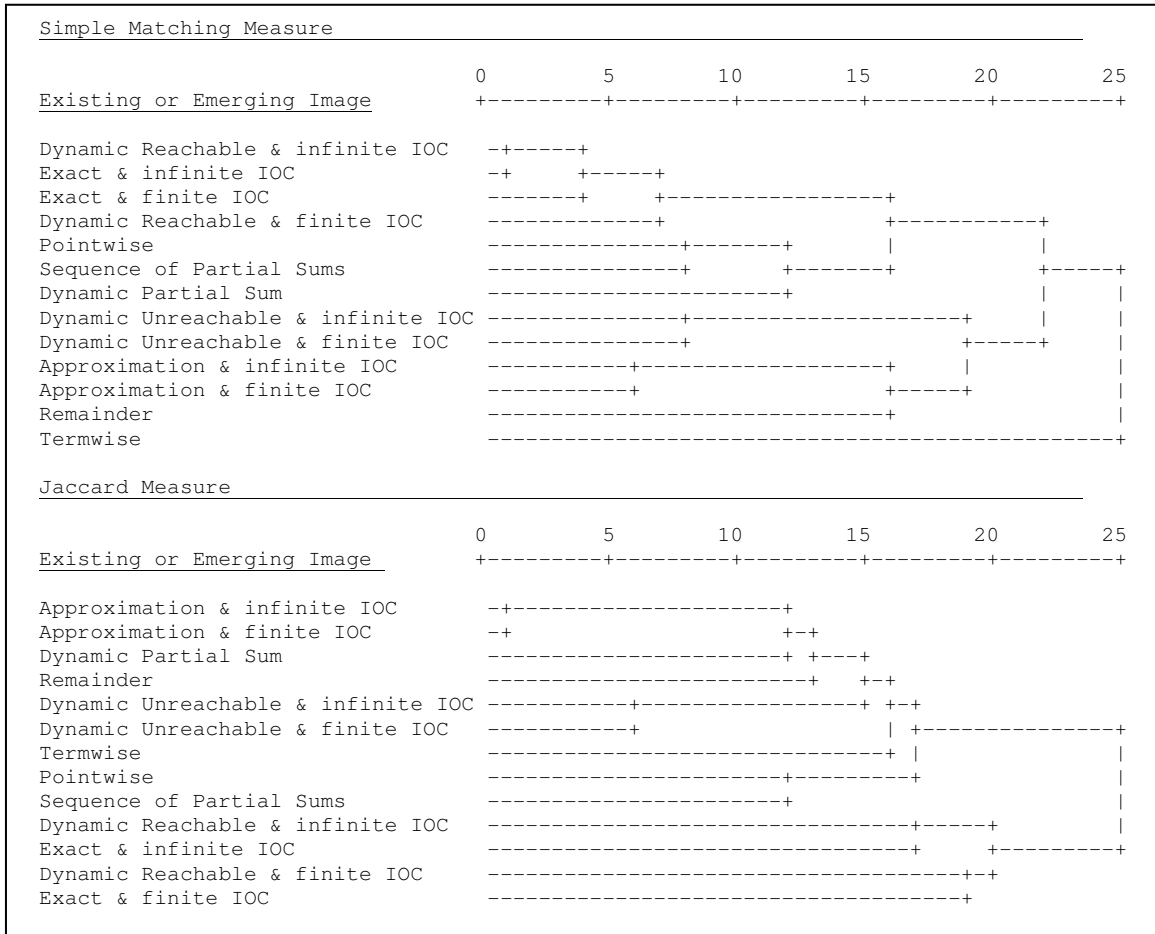


Figure 24. Dendrograms for the Interactions between Existing and Emergent Images for all Novice Participants.

A closer investigation of all ten dendrograms revealed that seven dendrograms illustrated the dynamic reachable / exact images and the dynamic unreachable / approximation images being contained within two separate large clusters. As Figure 24 illustrated, these two large clusters may take a while to form, but they formed nonetheless. Of the seven dendrograms that have these two large clusters, the existing images would be distributed amongst the two separate large clusters, depending mostly

on the measure being used. Table 30 tallied the number of the seven dendrograms based on the interactions of the emergent images with the two main clusters. The Jaccard measure reported that for those novice participants the emergent images interacted fairly evenly with the dynamic unreachable / approximation images but very little with the dynamic reachable / exact images. The simple matching measure showed that the pointwise and sequence of partial sums images interacted mainly with the dynamic unreachable image, while the other three emergent images interacted with both of the main clusters.

Table 30  
*Dendrogram Tallies\* of the Emergent and Existing Image Interactions Based on Different Measures Joining Two Large Clusters*

Two Large Existing Image Clusters	Measure	Emergent Image				
		Pointwise	Sequence of Partial Sums	Dynamic Partial Sum	Remainder	Termwise
Dynamic Reachable / Exact	Simple Matching	4	4	3	1	1
	Jaccard	0	0	0	0	0
Dynamic Unreachable / Approximation	Simple Matching	0	0	1	3	1
	Jaccard	3	3	3	3	3

\* Tallies are out of the seven novice dendrograms.

A study of the complete consistency group should provide the most useful information about how all these images interacted when participants comprehended the corresponding tasks because the complete consistency group has arguably demonstrated the best understanding of the existing images. It was probably less likely that participants from the completely consistent group guessed on or misunderstood the existing image tasks. Therefore, it was more likely that interactions depicted in their dendrograms represent reliance upon multiple images. No matter what measure was being used, the dynamic reachable / exact images and the dynamic unreachable / approximation images

clustered to form two large clusters. The emergent images were then distributed amongst those two clusters, and didn't cluster with any existing image until the 16<sup>th</sup> measure. This illustrated that while some emergent images may have tended to slowly merge with a particular cluster, they were fairly equally distributed amongst the existing images and do not stand disjoint from the existing images even when participants demonstrated a higher level of understanding of the existing images. The eight dendrograms corresponding to the NNPs, MNPs, never consistent, and the complete consistency groups all appear in Appendix E.

*The Notational, Stated Generating Function, and Equated Series Effects.* In addition to the IOC potentially influencing novice participant responses, other factors included the notational, stated generating function, and the equated series effect. These effects were described in detail in Chapter 3. Tables 13 and 14 in Chapter 3 listed the individual questionnaire tasks that were used to account for these effects, and Tables 31 and 32 give the corresponding tallies to the matching tasks from the tables in Chapter 3.

Table 31  
*3D Questionnaire Tallies Matrix for Notational Effect with Finite IOC*

	Equated Series Effect	Notational Effect	
		$n^{\text{th}}$ Partial Sum Notation	Sequence of Partial Sums Notation
Stated equated generating function (series is replaced by generating function)			
“Converges to” series		89 (68%)	69 (53%)
“Converges”		66 (50%)	40 (31%)
No stated equated generating function			
“Converges to” series		63 (48%)	50 (38%)
“Converges”		80 (61%)	57 (44%)

Table 32

*3D Questionnaire Tallies Matrix for Notational Effect with Infinite IOC*

	Equated Series Effect	Notational Effect	
		$n^{\text{th}}$ Partial Sum Notation	Sequence of Partial Sums Notation
Stated equated generating function (series is replaced by generating function)			
	“Converges to” series	82 (63%)	48 (37%)
	“Converges”	69 (53%)	49 (37%)
No stated equated generating function			
	“Converges to” series	58 (44%)	55 (42%)
	“Converges”	84 (65%)	61 (47%)

According to Tables 31 and 32, there appeared to be a notational, stated generating function, and equated series effects. Novices tended to prefer the  $n^{\text{th}}$  partial sum notation over the sequence of partial sums notation. Depending on if a generating function was explicitly stated or not, novices might prefer “converges to” over “converges” when considering the equated series effect. The data seemed to suggest that the novices tended to select “converges to” over simply “converges” when a generating function was stated, but in contrast, they tended to select simply “converges” over “converges to” when a specific generating function was not equated to the Taylor series. These tables only give a partial picture of the interaction. For example, they do not fully indicate which of these effects, notational, stated generating function, or equated series, could have been the most influential on participant responses to the corresponding tasks. To get a more complete picture, I considered dendrograms applied to the corresponding questionnaire tasks. The dendrograms in Figure 25 used the simple matching measure to illustrate how these factors interacted in novice participant responses.

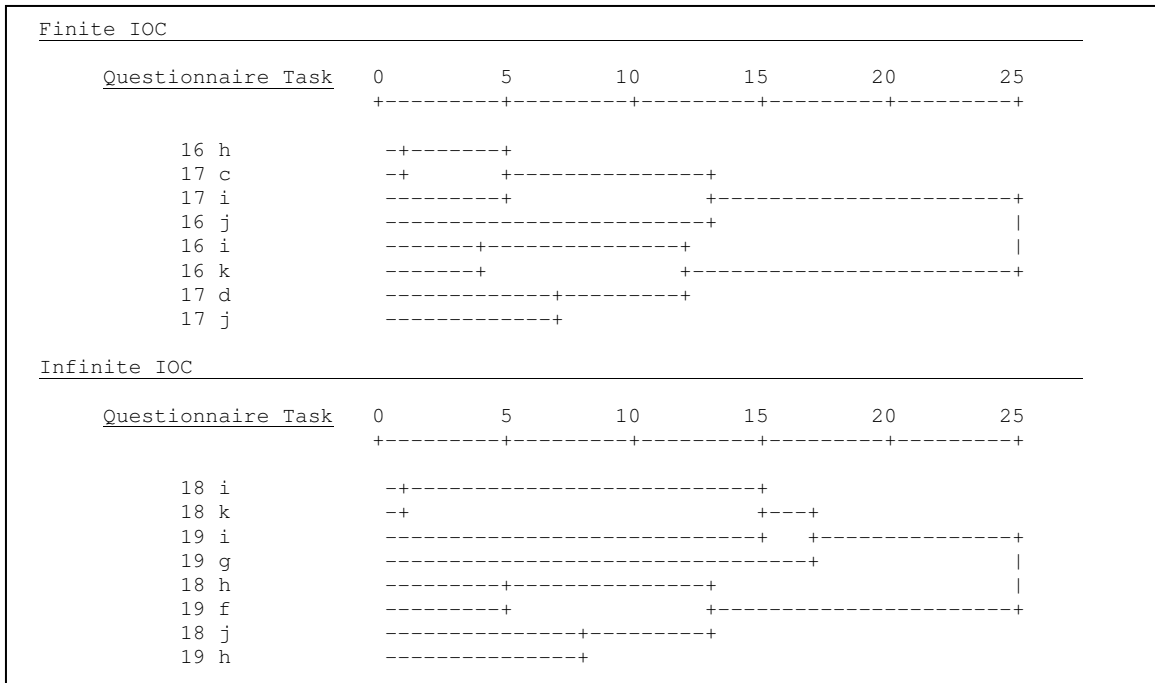


Figure 25. Dendrograms for the Notational, Generating Function, and Equated to Series Effect Based on the IOC Using Simple Matching Measure

According to Figure 25, there were two large clusters in each dendrogram, and when comparing these large clusters to Tables 13 and 14 in Chapter 3, it became clear that these large clusters were divided based on the two possible choices of the equated series effect. These large clusters indicated that there is an effect of “convergence to” verses simply “convergence,” and that this effect is more pronounced than the notational or the generating function effects. Following the equated series effect, the next most pronounced effect likely depended on if either “convergence to” or simply “convergence” had been selected by a given novice participant. If “convergence to” was selected, then both dendrograms suggested that the generating function effect was the second most influential effect. Whereas, if simple “convergence” was selected, then both dendrograms suggested that the notational effect was the second most influential. The Jaccard measure dendrograms yielded similar results.

At the surface level, some of these effects that I accounted for appeared related to some of the existing and emerging images. For example, the  $n^{\text{th}}$  partial sum notation might be perceived as corresponding to the dynamic partial sum image, while the sequence of partial sums notation might correspond to the sequence of partial sums image. In total, there was 49 novices who affirmed the  $n^{\text{th}}$  partial sum notation and the dynamic partial sum image, and 24 novices who affirmed the sequence notation and the sequence of partial sums image. In both cases, there were novices who affirmed one but not the other. To determine if any interaction was occurring, a cluster analysis was performed on the questionnaire tasks corresponding to these effects and the images. No noticeable interaction appeared when considering either the simple matching or Jaccard measures. In some cases, the image tasks failed to cluster at all with the effect tasks.

In an even finer analysis, I considered novice participants who were consistent in their responses to the  $n^{\text{th}}$  partial sum and / or sequence affirmations corresponding to the notational effect. To be considered as consistently responding to a particular notational effect, a participant who affirmed the  $n^{\text{th}}$  partial sum notation while responding to one problem number, must affirm the  $n^{\text{th}}$  partial sum notation in all subsequent problem numbers to be considered as consistent in their affirmations of the  $n^{\text{th}}$  partial sum notation. Therefore, in tasks numbered 16, 17, 18 and 19, a novice participant must affirm the  $n^{\text{th}}$  partial sum notation at least once in each task to be considered as consistent in their responses to the  $n^{\text{th}}$  partial sum notation, and similarly for the sequence notation. 78 (60%) novices were completely consistent in their affirmations of the  $n^{\text{th}}$  partial sum notation. Plus, the number of MNP affirmations of the  $n^{\text{th}}$  partial sum notation, 22 (79%), was significantly higher than the number of NNP affirmations, 56 (54%), at the  $\alpha$ -



level of 0.05 ( $p = 0.0139$ ). The number of novices that were completely consistent in their affirmations of the sequence notation was 45 (34%). In this case, there was no significant difference between the NNPs, 34 (33%), and the MNPs, 11 (39%) ( $p = 0.2676$ ). Perhaps a more substantial result is found in the 45 novice participants who were completely consistent in their responses to the sequence notation tasks. Of the 45, 44 were amongst the 78 that were completely consistent in their responses to  $n^{\text{th}}$  partial sum notation tasks. Therefore, it appeared that novices who consistently affirmed the sequence notation, were very likely to consistently affirm the  $n^{\text{th}}$  partial sum notation, but those who did not consistently affirm the sequence notation need not be consistent in their affirmations of the  $n^{\text{th}}$  partial sum notation. Furthermore, even when considering novice participants who were consistent in their notational response, the dynamic partial sum and sequence of partial sums images had little to no interaction with the notational effects. Therefore, it appeared that no particular image seemed to be influencing novice participant responses when accounting for the notational, generating function, and equated series effects. This does not mean that the existing or emergent images cannot influence any of these effects for the individual novice, but that for the novice group as a whole, the images appeared to have no influence on the effects.

### *Novice Potpourri*

In this section, I will discuss an assortment of interesting results that emerged from the novice data. Some of the topics addressed in this section did not fit into any of the previous themes addressed in the preceding sections. To be consistent with the “Expert Potpourri” section found in the previous chapter, many of the subsections of the “Novice Potpourri” section match subsections from the “Expert Potpourri” section. I will

begin this section by discussing those matching topics; visualization, approximation and function replacement, interval of convergence and re-centering, remainder as tail and Cauchy images. Since novices are different from experts, I did not limit myself to merely discussing what I had previously discussed with the experts. Therefore, additional interesting topics to those discussed in the expert chapter that emerged in the novice data are considered later in this section.

*Novices and Visualizing.* Like the experts, some novices demonstrated an ability to visualize Taylor series convergence, but unlike the experts, not all novices demonstrated this ability. Questionnaire Tasks 5 and 6, presented novice participants with the graph of sine and asked them to produce graphs of two different Taylor polynomials and a graph of the Taylor series, respectively (see Appendix C for the complete questions). Table 33 indicated that only 14 (11%) novices were able to correctly produce graphs for both the Taylor polynomials and the Taylor series given a graph of a generating function. Only 23 (18%) novices were able to correctly graph two Taylor polynomials, and only 20 (15%) novices were able to correctly graph a Taylor series given the graph of a generating function. Furthermore, only 77 (59%) produced any graph at all.

Implicit in Table 33 was that 48 (37%) novices attempted either the Taylor polynomial graphs or the Taylor series graph but were unable to produce a correct graph. These novices demonstrated various ways of visualizing convergence. One of the ways that novices visualized Taylor series convergence has already been shown by NNP

Table 33  
*Correct Novice Responses to Questionnaire Tasks 5 and 6*

Responses to Questionnaire Tasks 5 & 6	Novice classification		Total
	NNP	MNP	
Both 5 & 6 correct	8 (8%)	6 (21%)	14 (11%)
Only 5 correct	5 (5%)	4 (14%)	9 (7%)
Only 6 correct	6 (6%)	0 (0%)	6 (5%)
No response to only 5	5 (5%)	1 (4%)	6 (5%)
No response to only 6	14 (14%)	3 (11%)	17 (13%)
No response to both 5 & 6	40 (40%)	14 (50%)	54 (37%)

*Note.* Percent within each novice participant group is given in parenthesis within the “NNP” and “MNP” columns. Percents of all novices are given in parenthesis in the “Total” column. Recall that N=131. The sample sizes for the NNPs and MNPs were 103 and 28, respectively.

STEVE. In Figure 15, STEVE demonstrated how he viewed “Taylor polynomials” as growing in amplitude toward the generating function. Eight (6%) novices produced graphs similar to STEVE’s growing amplitude graph.

The growing amplitude “Taylor polynomials” were not the only Taylor polynomial graphs repeated by novices. Ten (8%) novices produced graphs similar to Figure 26. Although no interviewed novice produced a graph like that found in Figure 26, the graphs seemed to indicate Taylor polynomials viewed as sine curves with a progressively shrinking period. The shrinking period conception was not the most often observed incorrect visual conception. Twenty-one (16%) novices produced graphs similar to Figure 27. In this figure, it appeared that novices may have been conceptualizing convergence graphically as a horizontal shift, the more terms you add to the Taylor polynomial, the closer the graph moves horizontally toward the generating function.

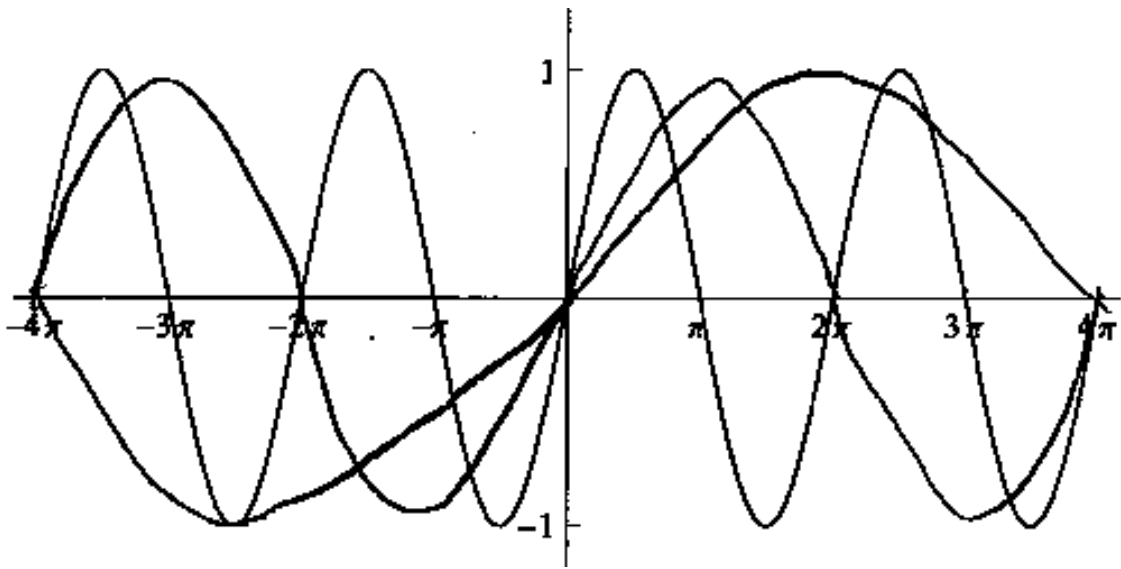


Figure 26. Possible Shrinking Period Graphical Image of Taylor Polynomial Convergence

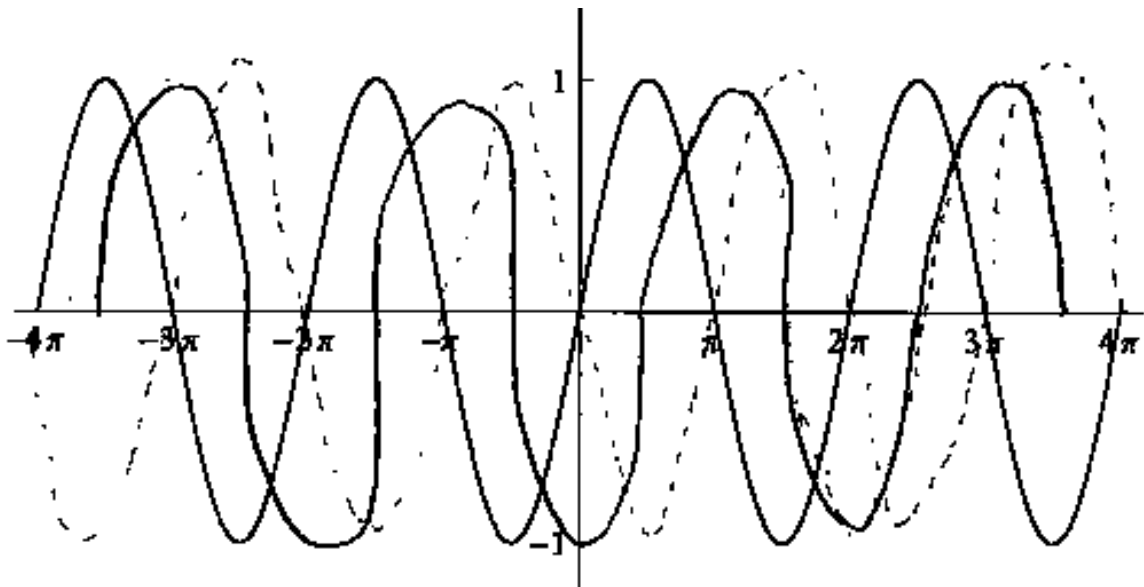
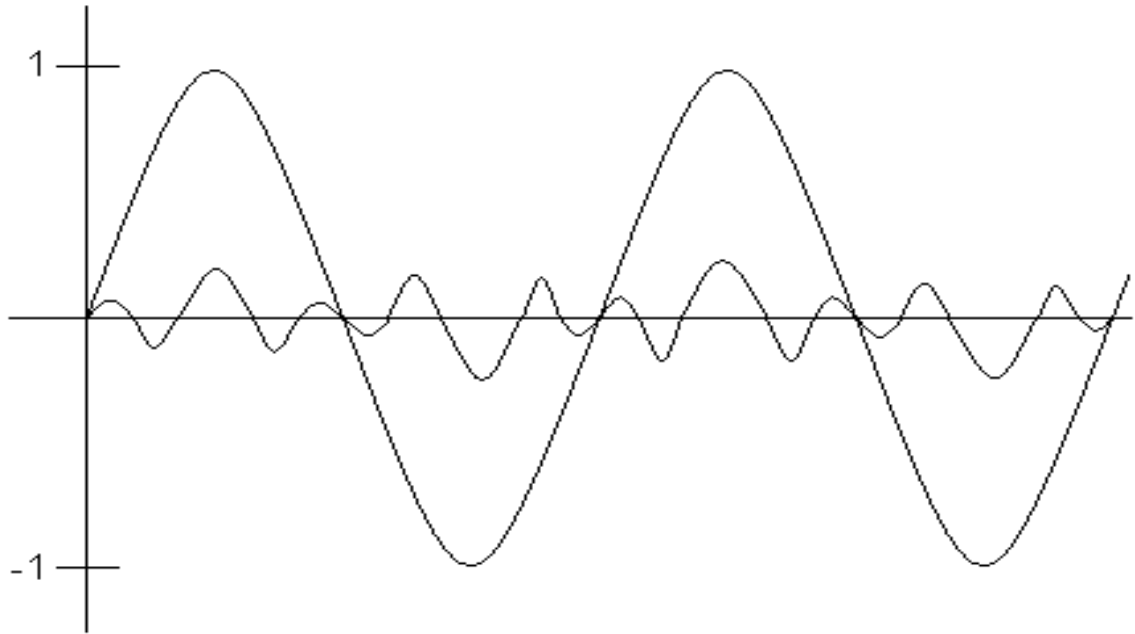


Figure 27. Possible Horizontal Shifting Graphical Image of Taylor Polynomial Convergence

Other interesting graphs that only a few participants produced, includes NNP STEVE's "squiggly line" found in Figure 15. STEVE may have originally conceived of Taylor polynomial convergence as a "squiggly line" that became less squiggly as one added more terms to the Taylor polynomial. When asked to sketch two Taylor polynomials, one NNP, instead of drawing graphs of two approximating polynomials, or even two approximating functions, simply drew two dots directly on the graph of sine. It is possible that he was viewing the sequence of Taylor polynomials as a collection of dots. This particular participant correctly responded to Questionnaire Task 6 and traced over the graph of sine when asked to graph the Taylor series for sine. Therefore, it seemed possible that he may have conceived of the limit of the sequence of Taylor polynomials as a limit that added more and more dots until one has the graph of the generating function. Similarly another NNP traced over a small portion of the generating function curve and correctly responded to Questionnaire Task 6. It seemed possible that this NNP may have conceived of the limit of the sequence of Taylor polynomials as a limit that added to the length of the curve that was traced over. Three participants produced graphs that had similar elements to that found in Figure 28 which appeared to be a formation of a possible resonance graph. Based on their background information, two may have studied resonance in either differential equations or physics. Based on the questionnaire data alone, it was unclear if they were really associating Taylor polynomials with the idea of resonance.



*Figure 28.* Reproduction of a Possible “Resonance” Graphical Image of Taylor Polynomial Convergence

As with NNP STEVE’s growing amplitude graph, some of these graphical conceptualizations of convergence may be restricted to the context of oscillating generating functions. If the graph of the generating function had not been oscillating, it may have produced different categories of visual responses from the novices. More research is needed to determine the amount of the influence that the oscillating generating function had on the visual conceptions of convergence demonstrated by novices. It certainly appeared to have had some effect.

Furthermore, out of these graphical images of Taylor series convergence mentioned in this section, only NNP STEVE’s increasing amplitude graphs (see Figure 15) were observed during an interview. The rest of the proposed graphical images of Taylor polynomial convergence were based on questionnaire data alone. Therefore, it is not completely clear exactly how each participant graphically viewed convergence, I can

only speculate based on the data available. Therefore, more research is needed in this area before one can be fully assured of the description of these types of graphical images of convergence.

To get a better insight into the graphical understanding of novice participants, I later gave them tasks in which I asked for an interpretation of what they were seeing given graphs related to Taylor series (see tasks 11 and 12 in the questionnaire in Appendix C). Unlike Questionnaire Task 5, in which novices were asked to produce Taylor polynomial graphs from only a graph of a generating function, tasks 11 and 12 already contained Taylor polynomial graphs as well as graphs of generating functions. According to Table 33, 60 novice participants left task 5 blank. Of the 60 novices who left task 5 blank, 35 at some level associated Taylor polynomial graphs to Taylor series during their responses to tasks 11 and 12. Furthermore, in their responses novices might have indicated an understanding of the graphical effect of adding more terms or of re-centering Taylor polynomials. In addition, they might have recognized the graph of the difference between the Taylor polynomials and the generating function as a graph of the error function. Novices may have only indicated one of these three observations (adding more terms, re-centering, or error) or they may have indicated all three while responding to tasks 11 and 12. Being able to make these observations along with the ability to produce Taylor polynomial graphs seemed to indicate different levels of graphical understanding of the concept of convergence of Taylor series. At the low end, you have those who are unable to graph Taylor polynomials and don't indicate any of the three observations. At the high end, you have those who are able to correctly graph Taylor polynomials and indicate all three observations.

After a closer analysis of the data, it appeared that there was a separation caused by the ability to indicate a recognition of the graphical effect of adding more terms to Taylor polynomials. Of the 60 participants who left task 5 blank, 23 did not associate the graphs of tasks 11 and 12 to Taylor series in any way. Some even specifically indicated that they did not understand what the graphs of tasks 11 and 12 represented. Of the 71 novice participants that did not leave Questionnaire Task 5 blank, only nine did not associate the graphs of tasks 11 and 12 to Taylor series. According to an appropriate directional z test applied to these two proportions, novices who failed to even attempt to draw Taylor polynomial graphs were significantly more likely ( $\alpha$ -level of 0.01,  $p=4.246 \times 10^{-4}$ ) to be unable to recognize Taylor polynomial graphs. In addition, if novices were able to at least indicate an understanding of the graphical effect of adding more terms, then they appeared slightly more likely ( $\alpha$ -level of 0.1,  $p=0.09415$ ) to be able to graph Taylor polynomial graphs. Furthermore, the data suggested that MNPs are significantly more likely to indicate an understanding of the graphical effect of adding more terms than NNPs ( $\alpha$ -level 0.01,  $p=0.0029$ ).

*Approximation and Function Replacement.* Questionnaire Tasks 19 e and 20 1 (see Appendix C) provided some interesting insights into novice understanding of the use of Taylor series in the context of function replacement. In Questionnaire Task 20 1, novice participants were asked to affirm an approximation as true. In this approximation “equation”  $e^x$  had been replaced by the linear approximating Taylor polynomial  $1+x$ . This questionnaire task can be compared to the expert interview low inference true / false Task 31 (see Table 9 in Chapter 3). Both expert Task 31 and novice Questionnaire Task



20 l contained two term polynomial approximations to a given function and both contained a square root of a three term equation with the middle term including a “ $\pi$ ” coefficient. Unlike the task presented to the experts, the polynomial replacement was done correctly in the novice task. Furthermore, the novice task did not contain the speed of light coefficient “C.” By making these changes I avoided unnecessary lurking variables and could better form conclusions specific to novice understanding of Taylor polynomial approximation and function replacement based on questionnaire responses alone. In addition, with this goal in mind, I added task 19 e to the novice questionnaire. In 19 e novices were asked to affirm that the three term Taylor polynomial,  $1 + x + \frac{1}{2}x^2$ , truncated from the Maclaurin series for  $e^x$ , estimated  $e^x$ .

Only 37 (28%) novices, 26 (25%) NNPs and 11 (39%) MNPs, affirmed 20 l, indicating that they likely understood how Taylor polynomials may be used to replace certain functions to obtain approximations. A two proportion z-test applied to the two novice participant groups suggested that the MNPs were slightly more likely to affirm this task ( $\alpha$ -level of 0.1 and  $p=0.0716$ ). One potential cause for novices to fail to affirm this task might have been linked to a failure to recognize that a truncated Taylor series approximates the series’ generating function. If a novice recognized a truncated series’ approximation properties, then an affirmation of task 19 e should have been an indicator. 82 (63%), 59 (57%) NNPs and 23 (82%) MNPs, affirmed task 19 e. A two proportion z-test, using a plus four method (D. Moore, S., 2007), revealed that the MNPs were more likely to affirm task 19 e ( $\alpha$ -level of 0.05 and  $p=0.0114$ ). It should be noted that in both 19 e and 20 l, novices were given the Maclaurin series for  $e^x$  and were told that this

series equals  $e^x$ . Therefore, a novice's ability to recall the formula for the Maclaurin series for  $e^x$  should have been a nonissue.

It turned out that 34 of the 37 who affirmed the function replacement approximation task, 20 l, were from the 82 who affirmed the truncation task, 19 e. Therefore, the data suggested that in relation to Taylor series, there were some properties / applications that many novices needed to learn first prior to being able to understand other properties / applications. In particular, this data suggested that novices should understand simple Taylor polynomial approximation properties prior to using Taylor polynomials to replace functions contained within complicated formulas. Function replacement in complicated formulas is one of the main reasons that Taylor series are studied in other disciplines, but this data suggested that many novices were not correctly applying this property. Maybe even more surprising was that 48 (59%) of the 82 novices who affirmed the truncation task, 19 e, failed to affirm the function replacement approximation task, 20 l. One of the reasons that novices affirmed one but not the other may be due to the fact that the "equation" in 20 l is more complicated than the formula in 19 e. Literature has already revealed that students struggle more and more as problems become more abstracted (e.g., White & Mitchelmore, 1996). In summary, this data suggested that many novices struggled with a property that numerous experts would claim as relatively easy to understand and the main reason that Taylor series are studied in calculus.

*Interval of Convergence and Re-Centering.* For a detailed discussion of the interval of convergence's (IOC's) interaction with novice conceptions of convergence of Taylor series, see the previous section on Layer 3 analysis. In that section, I concluded

that the IOC most likely had little effect on novice existing images of Taylor series convergence. Instead various contexts of different tasks may have been the primary cause for novices to rely on different images at different times within the questionnaire.

Besides the IOC's interaction with concept images of Taylor series convergence, the IOC is also related to the domain of the Taylor series when the Taylor series is viewed as a function. Contained within the questionnaire were two tasks designed to address this issue and take into account the effect of a stated generating function.

Questionnaire Task 16 stated that " $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$  on the interval  $(-1,1)$ ."

Novices were then able to affirm that "the domain of  $1 + x + x^2 + x^3 + \dots$  was  $(-1,1)$ ."

Questionnaire Task 17 stated that the series  $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$  "converged on the interval  $(-2,2)$  and

diverged otherwise," and novices could then choose to affirm that "the domain of  $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

was  $(-2,2)$ ." Notably missing from Questionnaire Task 17 was any reference to a generating function.

In total, 58 (44%) novices affirmed that the IOC of the series was equal to the domain in either task. 51 of the 58 specifically affirmed that the IOC was equal to domain in Task 17. While only 38 of the 58 specifically affirmed that the IOC was equal to domain in Task 16. Therefore, at first glance the data seemed to suggest that if a novice specifically affirmed that the IOC was equal to domain in the context of a generating function, that novice more likely affirmed that the IOC was equal to domain when a generating function was not stated. But, I also had to take into account that in

Task 16, participants could also affirm that the “domain of  $\frac{1}{1-x}$  was  $(-1,1)$ .” It should be noted that the domain of this function was not  $(-1,1)$ , but 50 (38%) of the novices affirmed that it was. These 50 novices might have been associating  $\frac{1}{1-x}$  with the completion of the Taylor series, and if so,  $(-1,1)$  in this task would have been associated with the IOC. Thus, more research is needed to determine the effect of the generating function on novice tendencies to associate the IOC with domain. If I allowed for either question in Questionnaire Task 16 to indicate an association of the IOC with domain, then 83 (63%) novices, instead of 58, potentially affirmed that the IOC was equal to domain. This implied that at least 48 (37%) of the novices made no association of the IOC to domain. Therefore, a large percentage of the novice participant group were demonstrating difficulty in associating Taylor series to functions by not indicating an association of the IOCs to Taylor series domains. More on the relationship between functions and Taylor series will be discussed later. It should also be noted that I considered differences between the two novice participant groups, but none appeared significant.

Not all novice interview participants alluded to the re-centering properties of Taylor polynomial approximations during their responses to the high inference type tasks. As with the experts, portions of the interview protocol were specifically designed to allow a re-centering conception to have ample opportunity to emerge. See the corresponding section in Chapter 5 for a more detailed discussion of this protocol. When considering this protocol applied to high inference Task 8, only one novice, NNP JORDAN, alluded to a re-centering image, and this reference only occurred after a

prompt. Only on a couple more occasions did any novice demonstrate a re-centering image during responses to the high inference tasks. The data suggests that re-centering was not foremost on the minds of almost all of the novice interview participants.

Just because novices did not clearly demonstrate a notion of re-centering during interviews, it did not mean that novices did not have a conception of re-centering. 110 (84%) novices indicated some conception of re-centering during their responses to Questionnaire Task 20. 88 of the 110 affirmed that second degree Taylor polynomials centered at zero and one both estimate their generating function. 44 of the 110 affirmed that the polynomial centered at one should estimate the generating function “better” at  $x = 2$ . 98 of the 110 affirmed that the two corresponding Taylor series converged to the “same thing.” Graphically, 102 (78%) novices associated the two series as at least being similar, and 36 of the 102 affirmed that the graphs were “identical.” During the short answer type Tasks 11 and 12, 62 (47%) indicated a re-centering image when comparing two sets of graphs containing re-centered Taylor polynomials. Therefore, even though re-centering may not be foremost in the minds of many novices, some novices were able to demonstrate at least a simple graphical and algebraic understanding of re-centering.

*Remainder as Tail and Cauchy Images.* As with the experts, these images appeared to be secondary images that were infrequently, if at all, evoked by participants on an as needed basis during responses to high inference interview tasks. The remainder as tail image was characterized by a focus on the “tail” of the Taylor series. A Cauchy image of Taylor series convergence was an image of the Cauchy criterion of convergence applied to Taylor series. For a more detailed description of these two images, see the corresponding “Expert Potpourri” section in the previous chapter.

The remainder as tail image materialized when a participant viewed the limit of the “tail” of a series equaling zero as a sufficient condition for series convergence.

Consider NNP JORDAN’s comment’s below:

*Excerpt 51*

Task 7: What are the steps in proving that  $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$ ?

I: Could you elaborate on the remainder?

NNP JORDAN: Um. So if you cut it off at this point, uh [marks two vertical lines separating the first four terms of the Taylor series from the rest of the equation], if you cut it off and use just the terms available to you, the first four terms, this is a number and uh, anything after that [circles the ellipses to the right end of the Taylor series], which would include plus one over uh er, nine, yeah, that's right, [while writing “

$+\frac{1}{9!}x^9 - \frac{1}{11!}x^{11}$ ”] nine factorial, x to the ninth, uh, over eleven factorial,

minus, uh, x to the eleventh, these are all trailing terms [circles the terms that he had just written] that also have value if you add them together [holds both hands to the right as if holding something between them], and the remainder is what you didn't [waves right hand to the right] pick up.

So, say if we have a graph [begins graphing the graph found in Figure 29] of this as it goes to infinity and you only add up, um [mumbles "y, x, whatever" while drawing the y and x axis]. Um, if you only cut off this amount of term [makes the vertical line half way on x-axis above the axis and below his graph] and this goes to infinity [highlights the right portion of the graph under the curve], this has value and this is your remainder [writes “R” above the portion of the graph that he just highlighted].

Instead of the remainder being viewed as the difference between Taylor polynomials and a generating function, JORDAN viewed the remainder as those “trailing terms” that were not “picked up” by given Taylor polynomials. Furthermore, by graphing the picture found in Figure 29, NNP JORDAN demonstrated that the remainder as tail image could be influenced by an apparent application of the Integral Test (see Stewart, 2008, p. 735). In his graph, JORDAN drew a function resembling exponential decay. He then divided the function into two pieces by drawing a vertical line under the graph about half way down the x-axis. JORDAN appeared to have tacitly associated the

area under the left half of the graph with a given Taylor polynomial that had been “cut off” to some specific “amount of terms.” He then noted that area under the right half of the graph was associated with an “infinite” amount of terms corresponding to the remainder of the series.

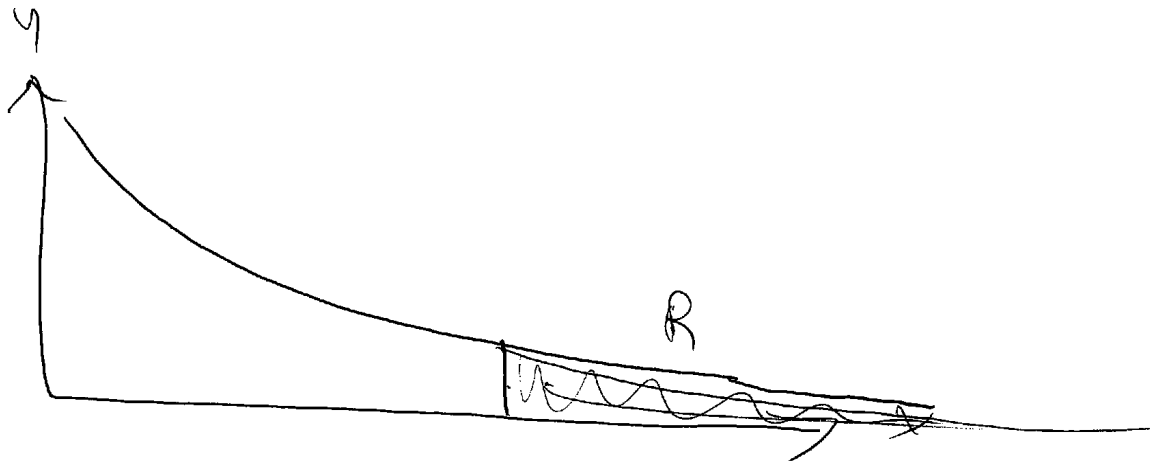


Figure 29. NNP JORDAN's Graph in Response to High Inference Task 6

NNP JORDAN was not the only novice to indicate a remainder as tail image of convergence of Taylor series. From the questionnaire data, 73 (56%) novices affirmed

that since  $\lim_{m \rightarrow \infty} \left( \frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \frac{x^{m+2}}{(m+2)!} + \dots \right) = 0$  for all numbers  $x$ ,  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ . In

essence, these novices were indicating that a remainder as tail image of convergence was a sufficient condition for Taylor series convergence to generating functions. To conclude this discussion of the remainder as tail image, it should be noted that differences between the two participant groups were considered but no noticeable differences emerged from the data.

It appeared that some novices demonstrated a partial Cauchy image during interviews. Novices that indicated a partial Cauchy image appeared to view the number of terms between two varying polynomials as being fixed. Novices failed to indicate a

full Cauchy image by demonstrating a conceptualization of being able to independently vary the degree of the Taylor polynomials indefinitely while making comparisons between the two polynomials. For example, in Excerpt 52, NNP ANDY referred to the “difference between those two sums.” The first cue to a Cauchy image used by ANDY was when he referred to partial sums “getting closer to each other” during his comments about the meaning of remainder. When asked about what he meant by “getting closer,” ANDY again potentially indicated a Cauchy image when he commented on the “difference between those two sums.” ANDY went on to elaborate using what appeared to be the harmonic series as an example. For ANDY, the difference between “two sums” of the harmonic series was a single term; “one”, “a half,” and “point three.” By making the difference between the partial sums fixed to a single term, ANDY was actually using a termwise conception of convergence as a particular instance of a Cauchy image. Clearly, ANDY lacked a full Cauchy image in this context since he never alluded to allowing the number of terms between the partial sums to vary.

*Excerpt 52*

Task 6: What is meant by the word “prove” if you were asked to, “Prove that sine is equal to its Taylor series”?

I: You brought up this word, remainder, what does remainder mean?

NNP ANDY: ...At least the partial sums should keep getting closer to each other...

I: So what do you mean by getting closer to each other?

NNP ANDY: Like, the difference between those two sums, they should, like keep decreasing. Like, so it gets closer to zero. Like you might have the partial sum between one of them to be one, and then maybe it'd be like a half, and then like point three, and just keep getting closer to zero until eventually the number is so small that it's almost zero.

As with the remainder as tail image, the questionnaire contained a task designed to elicit a Cauchy image of convergence. True / False Questionnaire Task 30 (see Appendix C) was identical to expert interview Task 29 which gave participants the



opportunity to affirm a type of Cauchy image relative to series. 58 (44%) novices indicated that the Cauchy image was true, but 23 of the 58 pointed out that they guessed. 48 (37%) novices did not respond with either a “true” or a “false.” Only 25 (19%) novices indicated that this criterion was false, and 13 of the 25 pointed out that they guessed. Based on these high numbers of guesses and omissions indicated, and based on my own personal experience, I believe that it was very likely that many more of the novices guessed than just those who indicated so. It is very likely that the majority of novice participants, especially the NNPs, did not fully comprehend this task since they have had little to no exposure with Cauchy sequences. A further analysis is needed to better understand the extent to which the novice participant group comprehended this task. To conclude this discussion of the Cauchy image, it should be noted that differences between the two novice participants groups were considered for the Cauchy image, but no significant differences were observed.

*Collapsing Taylor Series Image?* Chapter 2 has already discussed how other researchers have already observed students working on various problems modeling different physical situations who through a limiting process “collapse” away one of the dimensions inherent to the problem situation (Oehrtman, 2002; Thompson, 1994). Recall that Oehrtman (2002) defined a collapse metaphor for a changing quantity as a limiting process that was characterized by “imagining a physical referent for the changing quantity collapsing along one of its dimensions, yielding an object that was one dimension smaller” (p. 150). Furthermore, in relationship to Taylor series, Oehrtman (2002) observed students who used a collapse image in relation to a remainder image when they viewed the remainder as collapsed to zero when  $n$  went to infinity (p. 165).

An apparent type of collapse image of Taylor series appeared during the interviews with two novices. One in particular, NNP BRIAN used the collapse image of Taylor series often during his responses to the high inference interview tasks. For both novices, this image of Taylor series appeared to collapse the Taylor series into one “absolute” point. Therefore, the completion of the series was not a two dimensional object, a function, but a one dimensional object, a point. The IOC might have even been viewed as collapsing down into the center of the series, or some other point within the IOC.

The collapse image may be related to a pointwise image dominance over other images. NNP BRIAN, who relied more heavily on a collapse image, repeatedly talked about single potential input values sporadically throughout the first seven high inference interview tasks. Through the first six tasks, BRIAN never clearly alluded to any emergent image besides pointwise, and only used a dynamic unreachable image once. The approximation image only appeared after the prompt in Task 7, and the dynamic partial sum image only appeared in Task 10. No other images clearly appeared for this novice participant during the high inference tasks.

Indications for the collapse image for NNP BRIAN first appeared during his response to high inference Task 3. At first he related Taylor series to Riemann sums, and then he said that if he were to add up all the terms of cosine’s Maclaurin series, he would get a “definite point.” The completion of a convergent Riemann sum results in a “definite point,” i.e. a number, characteristically representative of the area under a curve above the  $x$ -axis. He initially appeared to allow the idea of Riemann sums to influence his conception of Taylor series, and falsely conclude that the resultant of a Taylor series

was a point and not a function. It is worth noting that he did refer to the Taylor series as an “infinite function,” but it appeared that for NNP BRIAN an “infinite function” may have been a “definite point.”

*Excerpt 53*

Task 3: What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$

when  $x$  is any real number?”

NNP BRIAN: I’m thinking it has something to do with like a Riemann sum. That’s just what comes to mind. Uh, if I add up, if I were to add up all of these it would give me a definite point after using what is pretty much indefinite, an infinite function.

Shortly following NNP BRIAN’s comments in Excerpt 53, BRIAN continued with the collapse image of Taylor series. In Excerpt 54, BRIAN again alluded to the Taylor series giving “one single point” that it “converges into.” This time the one single point was viewed as a point that encapsulated a “summary” of the series. When asked what he meant by convergence, he reiterated that the series was “going to come to a point.” He then began to give an example in which the series converged to one. He appeared to call the series a function as he did in the previous excerpt. He concluded that the series was “just gonna be that point.” BRIAN may be using function language, but every time, the Taylor series did not converge to a function, but to some point representative of the series.

*Excerpt 54*

Task 3: What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$

when  $x$  is any real number?”

NNP BRIAN: I’m thinking that because these are all fractions of I guess cosine curve or function, it’s gonna give me one single point. Um, kinda of a sum-, it’s gonna, it’s gonna give me a summary more or less is what I’m thinking... I’m thinking it’s gonna converge into something.

I: When you say, "converge into something," can you elaborate on what it means to converge?

NNP: Converge. It's going to come to a point. If it's, um, if it's converging to one, if some function is converging to one. Um. Might tell me that one, whether it's met at one or just close enough to one to see, you know, rounded up to one, it's just gonna be that point.

In the next high inference type task, NNP BRIAN persisted to allude to a collapsed image of Taylor series convergence into a single point. At first he exhibited a type of pointwise image when he alluded to being unable to plug one into the generating function because the function “didn’t exist” at one. He continued with the pointwise image by plugging in both endpoints of the IOC into the generating function. NNP BRIAN then incorrectly concluded that the series might be equal to any “infinite amount of numbers” from the IOC, including negative one. It appeared that at this spot, he might have been viewing the IOC as the set of possible numbers that the Taylor series could equal. After my prompt for him to discuss the “infinite amount of numbers” in more detail, he referred to his responses in the previous two excerpts. He then displayed a collapse image of series convergence again when he concluded that the “infinite amount of numbers” would cause the series to “mass” at “one finite number.”

*Excerpt 55*

Task 4: What is meant by the “(-1,1)” in, “ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$  when  $x$  is in the interval  $(-1,1)$ ?”

NNP BRIAN: I know it can't be, I'm thinking it can't be one for this particular function [pointing to  $1/(1-x)$ ] just because it doesn't exist.

I: Okay.

NNP BRIAN: Negative one would be two so I'm thinking it might be equal to a negative one and just up to one. So, just any infinite amount of numbers in between there.

I: Okay. What can we do with this infinite amount of numbers between there?

NNP BRIAN: Well, like in number 3, I'm thinking that in this infinite amount of numbers you're going to find some type of mass. I'm mean it's gonna be, if I were to add up all of them it would somehow equal one finite number [holds both hand up as if holding something between].

One of the factors that may have caused BRIAN to demonstrate a collapse image was revealed in his use of particular analogies. For example, in Excerpt 56, NNP BRIAN related Taylor series convergence to exam grades. He alluded to a fictional student that may flunk one exam, but ace all others. BRAIN stated that based on just the F and one A exam “you’re not going to know which one the student was.” He then notes that the other A exams would clarify the matter and reveal that the student was “more of an A student.”

*Excerpt 56*

NNP BRIAN: If it's to converge [referring to a Taylor series], I think the more numbers that you have, you know, the better feel you get for what type of convergent it's going to go to. It's like, it's like a grading scale, you know, you can have two tests in one semester and if you, you know, if both are, if one's an F and one's an A, you're not going to know which one the student was. If you got a couple more, you know, well maybe this student's more of an A student and that one F was just a fluke.

In the context of Taylor series, a difficulty potentially arose from this analogy. For a course in which multiple exams are given, there is only one grade at the end of the semester, multiple grades are not allowed. Therefore, this analogy did not take into account the variability of the “ $x$ ” contained within the domain of the Taylor series. The analogy may have served NNP BRIAN well with series convergence in general, but it appeared that it may have been leading him to erroneous conclusions in the context of Taylor series convergence.

*Influence of Functions on Taylor Series Convergence.* One of the influencing factors on the concept of convergence of Taylor series was the concept of limit of functions. On multiple occasions novices related the convergence of Taylor series to limits of functions. For example, in Excerpt 46, NNP STEVE related the idea of Taylor series convergence to  $e^x$ . When explaining why series converge, it was as if STEVE

was explaining why a function approaches its horizontal asymptote as  $x$  goes to infinity. Adding “more pieces” to the series, instead of causing him to approach a function, caused STEVE to approach a “number.” In so doing, STEVE also displayed a collapse image of Taylor series convergence. Therefore, this collapse image appeared to have resulted from a limit of function misapplication to Taylor series convergence.

Prior to Excerpt 57, NNP ANDY had brought up the word “limit” in the context of Taylor series convergence. In Excerpt 57, when asked what “limit” meant, ANDY responded with a function type definition for limit instead of relating limit to series. Just like with NNP STEVE, ANDY related series convergence to that of a function approaching a horizontal asymptote as  $x$  went to infinity.

*Excerpt 57*

Task 3: What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$ ” when  $x$  is any real number?”

I: Okay. So you use this word “limit.” So what does, could you elaborate on what “limit” means?

NNP ANDY: Limit, um. If you take a function and if you just keep getting closer to like infinity [moves right hand to the right], or negative infinity, then it keeps moving closer to a number. It might not actually reach that number but it will just keep getting closer.

It is also worth noting that ANDY relied very heavily on a pointwise image throughout the high inference tasks, he was only one image away from being officially considered as having a primary singular focus on pointwise. In fact he used a pointwise image just prior to making the comments found in Excerpt 57. Since the notion of plugging something in is inextricably associated with the concept of function, participants that relied heavily on the pointwise image may have done so because of their reliance upon their understanding of functions. In this case the concept of limit of function would

be linked to the pointwise image of convergence, and this could explain the source of ANDY's comments in Excerpt 57.

It is worth noting that novice participants in this study, especially NNPs, should have had more prolonged exposure to convergence in the context of limits of functions, then in the context of limits of series. Also, as the previous section demonstrated, pointwise was one of the top three images utilized by the novice participant group. Whereas, as Chapter 5 showed, the pointwise image was not amongst the top images utilized by the expert participant group. Therefore, the data seemed to suggest that as participants have more exposure to series convergence, they may have relied less upon pointwise images. In so doing they may have been relying less on their notion of limits of functions, and more on other concepts of convergence of Taylor series. Now that these images of convergence of Taylor series have been defined, more research can be done on the influence of the concept of function on student conceptions of convergence of Taylor series.

*Lack of Need for Proof.* Some novices indicated that they did not feel the need to prove that Taylor series converge to their generating functions. Consider Excerpts 58 and 59 in which NNP STEVE responded to tasks concerning Taylor series convergence proofs:

*Excerpt 58*

Task 6: What is meant by the word “prove” if you were asked to, “Prove that sine is equal to its Taylor series.”

NNP STEVE: Well, I just think prove is uh, what a professor would do on the board. I'm gonna prove that it's, sine is equal to this series and so he'll do half an hour worth of chalk board work, and then prove that what he said is true. And we can go on with the lesson.

I: Okay

NNP STEVE: And uh, it just kinda goes and goes and goes until he gets what he wants.

*Excerpt 59*

Task 7: What are the steps in proving that

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots?$$

NNP STEVE: And uh, I don't really know the steps. That's one of those things when a teacher starts proving something that I already accept as truth, I just kinda glaze over and think about what's going on this weekend.

In Excerpt 58, NNP STEVE seemed to indicate that he saw proofs as disjoint from the lesson. Instead proofs were seen as “what a professor would do on the board” for “half an hour” while the lesson was stopped. After the professor was done “getting what he wanted,” the lesson could “go on.” In this excerpt, STEVE seemed to ascribe the professor as the recipient of the benefit of the proof and not the student. In Excerpt 59, I learned why the lesson stopped for STEVE during Taylor series convergence proofs, STEVE had “glazed over.” Certainly, STEVE did not receive the benefit of seeing a Taylor series convergence proof while being “glazed over.” What was interesting was why STEVE had “glazed over.” STEVE had already accepted that Taylor series converge to their generating functions. By making this assumption, STEVE was committing an apparently epistemological error. As Chapter 4 showed, this “mistake” had been committed throughout the early history of Taylor series.

It should be noted that some of STEVE’s comments appeared to be linked to proofs in general and not specific to Taylor series. Therefore, experts should not only expect many students to fail to understand Taylor series convergence proofs, but they should expect some students to see no need for these proofs, and for proofs in general. As STEVE has indicated, students that do not see a need may “glaze over” during the “half hour” of class used to prove that sine is indeed equal to its Macluarin series. In the case of Taylor series convergence proofs, this lack of need for proof may result from



limited exposure to functions that do not equal their respective Taylor series. Perhaps a larger emphasis on non-analytic functions would foster a sense of need for convergence proofs? Unfortunately as STEVE appeared to have indicated, this lack of need for proof most likely runs much deeper than just Taylor series. Clearly, more research is needed to see how to foster a need for proof in the context of proving Taylor series convergence.

### *Conclusion*

This chapter has analyzed and described the different ways in which novices, NNPs and MNPs, conceptualized the convergence of Taylor series. At the beginning of the chapter I discussed the two main categories of conceptual images found within the novice data: existing and emergent images (see Tables 16 and 18 for a description of these images). To aid this discussion, I included results from the first, second, and fourth layers of analysis. Following this, I elaborated on the third layer of analysis which gave a detailed look into some of the interactions between images and possible influencing factors. Images contained within the existing and emergent image categories were not the only images to emerge from within the novice data. Additional images that appeared less influential were also addressed in a potpourri section that contained other interesting results.

The language used by novices to express each of the concept images was similar to, if not identical to the language used by experts. A difference between experts and novices was found in the meaning behind some of their utterances. Using actual participant graphs, I showed an example where a novice viewed approximating Taylor polynomials as increasing in amplitude toward the generating function, sine (see Figures 14 and 15). Furthermore, other graphs from other participants indicated approximation

images of Taylor “polynomials” for sine potentially containing images of sine curves shrinking in period or shifting horizontally (Figures 26 and 27, respectively). These particular visual images of Taylor polynomial convergence appeared intimately linked to the generating function sine. In one case, a novice’s sequence of partial sums image related to Taylor series contained a visual image of the graphs of all the Taylor polynomials on one axis (Figure 17). All of these graphical examples, together with corresponding transcript excerpts, illustrated how novices used the same utterances as experts but the meaning behind their words may have been drastically different.

The dynamic unreachable and the approximation images were the most oft used existing images by the novice participant group as indicated by both the questionnaire analysis (Layer 3) and the focus analysis (Layer 4) (see Tables 24 and 22). Novices tended to embed the notion of unreachable in the context of time, not in the context of approximation as the experts did. Some novices expressed an awareness of two different ways of thinking about the attainability of limits, but these excerpts were short and they did not go into detail about the different ways of thinking.

The third layer of analysis revealed that the termwise image was the most affirmed emergent image by the novice participant group (see Table 26). This layer of analysis went on to reveal that the dynamic partial sum and remainder were the next two most affirmed images. The fourth layer of analysis showed that the pointwise, dynamic partial sum, and approximation images were heavily relied upon by the interview participants (see Table 22). Missing from the most used images from the fourth layer were the termwise and remainder images. A closer analysis of the novice data showed that none of those who were interviewed displayed a termwise image in the

questionnaire. Therefore, the termwise image may have been excluded as a heavily relied upon image from the fourth layer results due to sampling issues. Since the remainder image was often affirmed in the questionnaire but was not heavily used by interview participants, this may indicate that novices were aware of the remainder image but infrequently employed it. Both layers of analysis showed that the sequence of partial sums image was the least relied upon.

Dendrograms used in the third layer of analysis helped reveal how images linked together. In some cases the dynamic reachable and exact existing images tended to link together, as well as the approximation and dynamic unreachable (see Figure 22). Not only were the dynamic unreachable and the approximation images the most often affirmed existing images, they were the most often combined existing images within the questionnaire data. For the emergent images, the pointwise and sequence of partial sums quickly joined together but the remainder and termwise might not join with other images depending on the measure used (see Figure 23). Layer 3 analysis revealed that most of all of the existing and emergent images interacted together within the novice participant group. Layer 4 analysis revealed that some novices may tend to link some images together more so than other novices. Furthermore, even though the third layer of analysis revealed that novices may affirm multiple images at the same time, the focus diagrams showed that the novice group linked multiple images in only 37 percent of all their instances of focus. This suggested that novices may be aware of multiple ways of conceptualizing Taylor series convergence, in fact they may be able to discuss multiple images at one time (e.g., see Figure 18), but they may also tend to discuss Taylor series with a singular focus on one image at a time (e.g., see Figure 20).

The questionnaire attempted to account for the notational, stated generating function, and equated series effects (see Chapter 3 for a description of each of these effects). Based on Layer 3 analysis the equated series effect was the most pronounced and the other two effects depended on participant responses to the equated series tasks. These effects may have influenced responses to Taylor series questions on a surface level, but the analysis revealed that no particular effect seemed to influence conceptual images of Taylor series convergence. As indicated by the questionnaire, the effects did not interact much, if at all, with concept imagery of Taylor series convergence even though some of these effects seemed ontologically related to images.

When considering potential differences between the two novice participant groups, I was surprised to learn that there was no significant difference between the two novice participant groups in the proportion of affirmations of the termwise image. In fact, the percentage of MNPs affirming the termwise image was slightly higher than the percentage of NNPs. For most of the images, the percentage of MNPs affirming an image was slightly higher than the percentage of NNPs who affirmed the same image. Of all the images, only the dynamic partial sum was significantly higher. In addition, those images that had a lower MNP percentage, were only slightly lower. The data appeared to suggest that the more mature novices may have had a wider variety of images at their disposal, but as new images were incorporated into their schemata, old images may not have been replaced. Graphically, MNPs were significantly more likely than NNPs to indicate an understanding of the graphical effect of adding more terms to a Taylor polynomial. Thus, the MNPs demonstrated that they had additional visual images at their disposal. Algebraically, the data suggested that MNPs were significantly more likely

than NNPs to demonstrate an understanding that Taylor polynomials approximate generating functions and that they can be used to simplify more complicated equations. Hence, the MNPs demonstrated that they had additional algebraic images at their disposal.

Additional images used by the novices included the remainder as tail, Cauchy, and the collapsing Taylor series images. The remainder as tail and Cauchy images were both observed in the expert group. 56 percent of the novices affirmed that “tail” convergence to zero was a sufficient condition to show that a series converged to its generating function. 44 percent of the novices affirmed a Cauchy image in the questionnaire, but the data suggested that many novices may not have fully comprehended the corresponding Cauchy task. The collapsing Taylor series image was characterized by converging Taylor polynomials that collapsed down to a single “definite point.” Further analysis revealed that in some cases, the collapse image may be related to a pointwise image dominance over other images.

In the potpourri section I demonstrated how a pointwise image may have been related to a participant’s concept of function. Alcock and Simpson (2004) observed one student participant who moved between different notations of Taylor series in what they believed to be an effort by the student to move to a more familiar object (p. 11). Similarly, while learning about Taylor series convergence, novices may have been relying heavily on familiar things, such as the concept of functions. For example, the data suggested that the concept of limit applied to functions appeared to influence some comments about Taylor series convergence. Following the potential associations of the

pointwise image to the collapsing image, the concept of function might have been influencing the use of the collapsing Taylor series image.

As already discussed in this section, novices need not have a visual understanding of Taylor series convergence. Also in the potpourri section, I showed that very few novices were able to graph Taylor polynomials and Taylor series given the graph of a particular generating function. Those novices who failed to attempt to draw Taylor polynomial graphs were significantly more likely than those who attempted to be unable to recognize Taylor polynomial graphs. In relation to function replacement, the data showed that few novices understood that Taylor polynomials could be used to simplify complicated equations. The analysis went on to suggest that novices may need to first understand simple Taylor polynomial approximation properties, such as the relationship of Taylor polynomials to generating functions, prior to using Taylor polynomials to replace functions in complicated equations. In relation to proving Taylor series convergence, some novices did not see a need for a proof. One even said that he “glazed over” during class when proofs were presented because he had already accepted that all Taylor series converge to their generating functions.

Now that the results from both the expert and novice participant groups have been presented and discussed in Chapters 5 and 6, this allows me to conclude this study in the next chapter by commenting on some of the commonalities and differences between the expert and novice participant groups. In addition, in the final chapter, using what I’ve learned, I will also discuss some potential strategies for instruction. Finally, I will conclude this study by revealing several directions for future research.

## CHAPTER 7

### Bringing It All Together A Discussion of Commonalities and Differences between Experts and Novices

This was a study of expert and novice conceptions of the convergence of Taylor series. Novices were defined as undergraduate students from The University of Oklahoma (OU) and from a regional community college (RCC), and experts were defined as graduate students or faculty from the Department of Mathematics at OU. In this chapter I will analyze and describe some of the commonalities and differences between the expert and novice participants. I will begin this chapter with a brief review of the study purposes and procedures. After the review, I will discuss the commonalities and differences between the expert and novice participant groups by responding to the research questions from Chapter 1. Following the research questions, I will elaborate on how this study informed pedagogy. Then I will present several directions for future research, and I will subsequently conclude this chapter and this study with some final remarks.

#### *Review of the Study Purposes and Procedures*

Taylor series is a topic that is briefly covered in most university calculus sequences. This may be because that in many cases Taylor series constitutes four or fewer sections of a traditional calculus textbook. With this limited exposure, what do calculus students really understand about the convergence of Taylor series? Do they think of Taylor series convergence as a sequence of converging polynomials? Do they

think of convergence as a remainder going to zero? Do they think the Taylor series for sine really “equals” sine, or is it merely a good estimation for sine? Furthermore, how might experts respond to these questions? It was questions like these that provided the impetus for this research into expert and novice conceptualizations of the convergence of Taylor series.

As Chapter 4 revealed, Taylor series greatly influenced the development of calculus. Once the experts started using Taylor series, it took around 150 years to suitably address the issue of convergence of Taylor series as it is taught today. Even prior to Cauchy’s definition of convergence of series, experts like James Gregory, Isaac Newton, Gottfried Leibniz, Brook Taylor, Colin Maclaurin, Leonard Euler, and Joseph Louis Lagrange used Taylor series for its approximation properties. As many of today’s expert participants reported, Taylor series are studied in today’s calculus for several reasons, most notably for their approximation abilities.

Functions, limits, derivatives, intervals of convergence, series remainders, and specific tests for convergence, such as the ratio and root tests, are typically discussed during lectures on Taylor series. Furthermore, a detailed lesson on Taylor series, and series in general, may involve complicated formal definitions and theorems, such as  $\epsilon - N$  definitions, and the Lagrange remainder theorem. It is clear that Taylor series incorporates many topics from calculus, and some of these topics may be more complex than others. Together, they can make the learning of the concept of convergence of Taylor series a very difficult endeavor for anyone to undertake.

Of these topics, the limit concept is one that is inextricably related to the concept of convergence of Taylor series. While the effects of the limit concept on student



understanding have been well documented over the last few decades (e.g., Bergthold, 1999; Cornu, 1991; Cottrill et al., 1996; Davis & Vinner, 1986; Monaghan, 1991; Oehrtman, 2002; Roh, 2008; Williams, 1991), the notion of convergence of Taylor series has remained relatively unexplored. As presented in Chapter 2, there have been only a few studies involving Taylor series, most of which focused on some broader topic that included Taylor series, such as limit (Oehrtman, 2002), approximation techniques (Kidron, 2004), visualizers and non-visualizers (Alcock & Simpson, 2004, 2005), or animation (Kidron & Zehavi, 2002). None of these studies specifically sought to classify how different people think about Taylor series convergence. Therefore, the problem addressed in this study was to analyze and describe the different ways in which people, both novices and experts, conceptualize convergence in the context of Taylor series. The focus of this study was on identifying and categorizing particular types of conceptualizations and on how these conceptions may differ depending on the extent of a participant's mathematical background. A secondary focus was to gain insights into what might be influencing particular types of knowledge.

Since only a few studies have made reference to student understanding of Taylor series (e.g., Alcock & Simpson, 2004, 2005; Kidron, 2004; Kidron & Zehavi, 2002; Oehrtman, 2002), it was necessary to create an appropriate qualitative study using exploratory methods. Together, Tall and Vinner's work on concept images (see Tall & Vinner, 1981) and Williams' work on limit models (see Williams, 1991) both helped to provide an appropriate framework for this study.

In Tall and Vinner's work of 1981, entitled *Concept Image and Concept Definition in Mathematics with Particular Reference to Limit and Continuity*, they

introduced the notions of concept images and concept definitions to the mathematics education community. They defined the concept image as “all the mental pictures and associated properties and processes” associated with a mathematical concept (Tall & Vinner, 1981, p. 152). Different individuals can have different conceptual images associated with the same mathematical concept, and images can change within the individual as he or she learns and matures. Furthermore, these images need not make sense within an individual’s entire conceptual schemata. The individual may remain unaware of conflicting concept images until contradicting images are evoked simultaneously. In the same work, Tall and Vinner went on to categorize student thinking about limit based on dynamic and static imagery as indicated by students’ written and oral responses to particular questions involving limits. Over the years, the notion of concept images has proven useful for identifying and categorizing particular types of conceptualizations relative to various topics within mathematics (e.g., Cornu, 1991; Mason, 2008; R. C. Moore, 1994; Vinner, 1983, 1991; Weber et al., 2008).

In 1991, Williams wrote an article entitled *Models of Limit Held by College Calculus Students*. In this paper, he reported a study in which he used questionnaires to categorize different models of limit used by calculus students. He called these models of limit dynamic-theoretical, acting as boundary, formal, unreachable, acting as approximation, and dynamic-practical (see Table 2 in Chapter 2 or a description of each model). Each of these models of limit corresponds to a particular conceptual image relating to limit. For example, the unreachable model is associated with a concept image that considers the limit as a number or point that a function gets close to but never reaches. Since Williams’ work of 1991, his limit model questionnaire has been used by

other researchers to help illuminate student understanding of limit in different contexts (e.g., Fisher, 2008; Lauten et al., 1994; Oehrtman, 2002). In this study I adapted some questions from Williams (1991), and I used those questions to help identify different conceptual images that participants were using concerning the convergence of Taylor series. I called these images dynamic, dynamic unreachable, dynamic reachable, and approximation (see Table 16 in Chapter 3 for a description of each of these images in the context of Taylor series). Together with an additional image called exact, I referred to these five images as “existing images” since four had already been noted by Williams and were a part of his “preexisting” framework in the context of the limit of functions.

In total, 16 experts and 131 novices participated in this study into the different conceptions of convergence of Taylor series. By classifying participants into experts and novices, this study was able to distinguish between different levels of exposure that participants had with regard to Taylor series, and series in general. To help further differentiate between participant levels of exposure to Taylor series additional subgroups were created. Expert and novice participant groups were each divided into two subgroups, experienced expert participants (EEP) and capable expert participants (CEP), and new novice participants (NNP) and mature novice participants (MNP), respectively. NNPs were students who had just seen Taylor series once, while MNPs were students who had been exposed to series results within at least two university classes. CEPs were mathematics graduate students or faculty who were capable of teaching a course containing or doing research with series, but had not done so. EEPs were mathematics graduate students or faculty who had recently taught a course containing series or frequently used series results in their own research. Therefore, using these subcategories,

NNPs had the least amount of exposure to Taylor series, while EEPs had the most amount of exposure.

Data were collected from these participant groups using three phases of data collection. Phase 1 consisted of focus group and individual interview data collected from experts, while Phase 2 consisted of questionnaire data collected from novices, and Phase 3 consisted of individual interview data collected from novices. Going into Phase 1, it was believed that experts should have a more robust mental imagery of Taylor series, have more numerous images, have an ability to efficiently and effectively switch between different images, and be more cognizant of how they and others think about Taylor series (Carlson & Bloom, 2005; Hiebert & Carpenter, 1992; Lester, 1994). In most cases, the data suggested that this was true for the participants in the expert group, especially for the EEPs. Since this was true it allowed for the creation of an improved questionnaire to help better illuminate novice understanding during Phases 2 and 3. In Phase 2, effects, such as the interval of convergence and various sequential limit notations, were accounted for within the questionnaire to help determine their influence on novices' conceptions of Taylor series. In Phase 3, many tasks presented to experts during interviews were repeated during interviews with novices to allow for better comparisons between experts and novices. During each phase of data collection, many tasks were arranged from high inference to low inference type questions to enable more reliable inferences about a given participant's spontaneous primary focus on images and probe deeper into their understanding of such images. For a more detailed description of the high and low inference type questions, as well as a more thorough description of each of the phases of data collection, see Chapter 3.

Data for this study consisted of background data, interview transcripts, interview handouts, and questionnaires. The background data was used to distinguish between the different groups of participants. The analysis of the transcripts and questionnaires was done in four layers. In the first layer, the focus group / interview data were coded for the existing themes that were adapted from Williams (1991) (see Table 16 in Chapter 3). The second layer of analysis consisted of an open coding scheme (Strauss & Corbin, 1990) applied to the focus group / interview data. The open coding of Layer 2 allowed for the pointwise, sequence of partial sums, dynamic partial sum, remainder, and termwise images to emerge from the data (see Table 18 in Chapter 5 for a description of each emergent image). Many instances of these images permeated the transcripts, but without the open coding, these additional images would not have emerged. Since these images were not dependent on Williams' preexisting framework and since they emerged from the data, I referred to these images as "emergent images." In addition to the images that I referred to as the "emergent images," other images emerged from the second layer of analysis. The remainder as tail, Cauchy, and collapsing Taylor series images emerged from some of the participants but they were not nearly as pervasive as the images categorized as "emergent images." The third layer of analysis consisted of an analysis of the questionnaire data for both the existing and emerging themes. In addition, descriptive statistics, such as cross tabs ("Cross tabulation," 2007) and cluster analysis ("Hierarchical cluster analysis," 2007; Johnson & Wichern, 2002), were done in this layer to help better illuminate certain tendencies of the novice participant group. The third layer did not apply to the experts since the experts did not participate in questionnaires. The fourth layer of analysis was a focus analysis (Sfard, 2000, 2001) describing the interactions

between the different images working together within each participant's conception as they progressed through the interview tasks. Using Layer 4 of the analysis, I created focus diagrams for each participant to help tell the "story" of how they responded to particular Taylor series tasks. This layer was very useful in helping to identify some of the similarities and differences between experts and novices which will be discussed in more detail in the next section. For a more elaborate discussion of the data and the four layers of analysis, refer to Chapter 3.

*Commonalities and Differences between Experts and Novices:*

*Discussing the Research Questions*

In this study I reported qualitative research methods incorporating the three phases of data collection and the four layers of analysis to analyze and describe the different conceptualizations of the convergence of Taylor series within the expert and novice participant groups. At the end of Chapters 5 and 6, I discussed the findings of each participant group in each chapter's "Conclusions" section. In this section, I bring the results from the two chapters together to compare and contrast the participant groups. As an aid, I will use the research questions as presented in Chapter 1 as a guide.

*Images of the Convergence of Taylor series.* Both experts and novices demonstrated a variety of ways in which they conceptualized the convergence of Taylor series. In Chapter 5, I discussed the two categories of conceptual images found within the expert data: existing images and emergent images. See Tables 16 and 18 for a description of the existing and emergent images. The first and second layers of analysis showed that both the expert and the novice participant groups utilized all of the existing and emergent images to various degrees. In this subsection I will briefly describe the

different concept images related to Taylor series convergence and then list some main conclusions about expert and novice conceptions.

The existing images were categorized as dynamic, dynamic unreachable, dynamic reachable, approximation, and exact. All dynamic images were typically expressed by words such as “gets close to,” “tends to,” “approaches,” “goes to,” and “moves.” In contrast, exact images were devoid of dynamic utterances. Dynamic reachable and unreachable images were characterized by utterances indicating the attainment or lack of attainment of an end result to the limiting process. The approximation image was revealed by participants who indicated that they were able to make Taylor polynomials as accurate as desired. For a more detailed description of all five existing image categories, see Chapter 5.

The emerging images were categorized as pointwise, sequence of partial sums, dynamic partial sum, remainder, and termwise. A participant’s focus on a point-by-point convergence was indicative of the pointwise image of Taylor series convergence. In some cases, participants used the pointwise image as a way of communicating divergence. The dynamic partial sum image was an instance of the preexisting dynamic image mentioned above, but the dynamic partial sum image was limited to a focus on adding more and more terms to a Taylor polynomial. In contrast the sequence of partial sums image focused on the limit of the entire sequence as opposed to the limit of a single polynomial on which terms were progressively added. The remainder image was characterized by a focus on the limit of the remainders equaling zero, and the termwise image was identified by a focus on the limit of terms of the series equaling zero. The sequence of partial sums is arguably the image that is the most in line with Cauchy’s

definition of series convergence, whereas the termwise image is inconsistent with Cauchy' definition because termwise convergence to zero is a necessary but not sufficient condition for series convergence. For a more detailed description of all of the emergent images, see Chapter 5.

Besides the existing and emergent images, additional images emerged from the data but were rarely used by the experts and novices. These additional images were the remainder as tail, Cauchy, and collapsing Taylor series images. The remainder as tail image was related to the remainder image, but in this image, the remainder was viewed as the "tail" of the series and not as the difference between the Taylor polynomials and the generating function. Although the remainder and the remainder as tail images may be mathematically the same, they represented two different conceptualizations, and thus, I separated them into two different image categories. The Cauchy image was also related to the remainder image, but in an instance of the Cauchy image the difference was not between the Taylor polynomials and the generating function, but between two Taylor polynomials. An individual with a full Cauchy image would see convergence to zero for the difference between two Taylor polynomials as the indicator for Taylor series convergence. A collapsing limit image has been observed by Oehrtman (2002), and in the context of Taylor series, it was characterized by one who viewed the Taylor polynomials as collapsing to one "absolute" point instead of converging to the generating function. In essence, participants who used this image collapsed a 2-D object, a function, down to a 1-D object, a point. For a more detailed description of these additional emergent images, see the "Potpourri" sections of Chapters 5 and 6.



Some of the main conclusions from the expert and novice data relative to the different concept images of Taylor series convergence are summarized in the following numbered list. For a more comprehensive description of each number, refer to Chapters 5 and 6.

1. Both experts and novices conceptualized the convergence of Taylor series in a variety of ways. No one individual was limited to just one way of conceptualizing convergence, even though a few novices came close.
2. The approximation image was the most often used existing image by the expert participant group. The novice participant group also frequently used the approximation image and an additional image: the dynamic unreachable existing image. Several experts commented about how the approximation image was one of the main reasons, if not the main reason, that Taylor series are studied in calculus. Novices might also make this claim, but they did so less frequently. The dynamism indicated by experts and novices was usually in the context of adding more terms to a Taylor polynomial that approximated a generating function. Dynamic images tended to also appear in the expert participant group in the context of a remainder function that went to zero, but this tendency was not as pronounced in the novice participant group. The dynamic unreachable image may have appeared more within the novice participant group because of a potential view of infinity, and a lack of familiarity with actual infinity. As Alcock and Simpson (2004) observed, students may try to relate their understanding of Taylor series to more familiar objects (p. 11). Therefore, since potential infinity has been related to finite

realities, time, it was the more familiar object (Monaghan, 2001; Sierpiska, 1987; Tall, 2001). The data clearly showed novices who linked a dynamic unreachable image to time, but in contrast, when experts used a dynamic unreachable image, they did so mostly in the context of approximation where reaching the completion of the series was viewed as unnecessary. In addition, some experts compared and contrasted two types of “limit knowledge” that appeared to be related to potential and actual infinity. This awareness of two different ways of thinking about infinity seemed to separate some experts from most novices who did not indicate such an awareness.

3. The dynamic partial sum and remainder images were the most often used emergent images by the expert participant group. The pointwise and dynamic partial sum images were the most often used emergent images by the interviewed novices, but the termwise image was the image most affirmed in the questionnaire, followed by the dynamic partial sum and remainder images. Since these images did not match between the two data collection methods used for the novices, it suggested that novices might be aware of certain images, but infrequently actually employ them. The pointwise image may have more frequently appeared in the novice participant group during interviews because novices were attempting to relate their understanding of Taylor series to more familiar objects, in particular, functions (Alcock & Simpson, 2004, p. 11).
4. The sequence of partial sums image was the least relied upon emergent image for both the expert and the novice participant groups. Since the sequence of

partial sums image is the most in line with the formal definition of series convergence, this result demonstrated that even the more mathematically mature may continue to rely upon informal reasoning unless otherwise prompted. The termwise image did not fully appear within the expert group, certainly never as a primary image. In addition the remainder image was also infrequently used by the novices. It seemed more natural for the novices to fixate on a dynamic partial sum image and not pay attention to the differences between the partial sums and the generating function.

5. Because of the history behind the use of the termwise image, this image demonstrated itself as an epistemological obstacle (see Cornu, 1991) in the context of Taylor series convergence. Therefore, it was only natural that the novice participant group demonstrated a termwise image more than the expert participant group. In addition, the historicity of the debate between the dynamic unreachable and dynamic reachable has also demonstrated itself as an epistemological obstacle that is still very much alive.
6. Both experts and novices tended to use the same language when discussing Taylor series convergence, but the experts tended to be a little more precise with their language usage. The meaning behind utterances from novices might vary from novice to novice and may even change within a particular novice. This was indication that the novices were still building their formal theory.
7. Even though the novice participant group as a whole, demonstrated nearly the same variety of images as the expert participant group, some individual

novices tended to use fewer images. The data further suggested that individual MNPs were more likely to have a wider variety of images at their disposal than individual NNPs. This result seemed to imply that new images encountered during the learning process need not replace old images.

8. It appeared that some novices might fixate with a singular focus on one image, while experts were more prone to link different conceptions together more frequently. The fourth layer of analysis revealed that experts used multiple images in 51 percent (89 of 175) of all instance of focus, while novices used multiple images in only 37 percent (41 of 112) of all instances of focus. Therefore, the data suggested that some of the more mathematically mature participants may not only have had a wider variety of images at their disposal, but they tended to use multiple images at the same time. The third layer of analysis revealed that when novices linked multiple images together they may tend to link the dynamic unreachable and approximation images as well as the dynamic reachable and exact, and the pointwise and sequence of partial sums. Approximation and unreachable seemed to coincide because when participants considered approximating Taylor series, they would not be concerned with actually adding more terms to a series completion was reached, but they would only be concerned with adding enough terms to yield a satisfactory estimation. Likewise, if a participant viewed the Taylor series as reaching the generating function, then the generating function and the series were viewed as exactly the same. Unfortunately, the interviews did not provide a good explanation as to why pointwise and sequence of partial sums

emergent images were frequently liked together. Therefore, more research is needed in this area to better explain such interactions.

9. Another factor that seemed to separate the expert and novice participant groups was in how they used the images. Most experts seemed to be able to move with ease between the different ways of thinking about Taylor series convergence as demonstrated by their use of different images when responding to different tasks. When discussing how to prove convergence, the remainder image was used by all the EEPs but the novices were infrequent in their utilization of this image for this task. Both the experts and the novices used a dynamic partial sum image in the context of getting a better approximation.
10. Some experts, especially EEPs, tended to have what appeared to be a well developed and readily accessible example space (Watson & Mason, 2005). Mason (2008) referred to an example space as “the class of examples which come to mind in association with a concept or technique” (p. 257). In particular, one of the more noticeable instances of expert ability to use their example spaces came in the form of a counterexample. Several experts disproved that termwise convergence to zero was a sufficient condition for series convergence by recalling the harmonic series. In contrast, fewer novices were able to allude to the harmonic series as a counterexample to the sufficiency of termwise convergence implying series convergence.
11. In addition to making more comments about how the approximation properties of Taylor series was one of the main reasons for studying Taylor

series, experts also demonstrated that they better understood the roll of such approximations. 37 percent of novices participating in the questionnaire failed to affirm that a three term Taylor polynomial approximated a given generating function. In a more complicated formula in which a two term Taylor polynomial replaced its generating function, a staggering 72 percent of novices failed to indicate that the formula with the Taylor polynomial approximated the formula with the generating function. Instruction may help in this area because in both cases, NNPs were significantly more likely than MNPs to fail to affirm these two tasks. The difference between the groups in their responses to these two tasks was likely due to the slightly more complicated formula of the second task (White & Mitchelmore, 1996). Furthermore, the data suggested that novices may need to understand the simple truncation properties to Taylor series prior to engaging in the more complicated formulas involving function replacement.

12. Chopping motion, downward motion, and growing interval gestures (Figures 8, 9, and 13) were commonly used by many experts and novices to denote adding more terms to a Taylor polynomial, convergence to zero, and intervals of accuracy, respectively. These gestures are representative of a type of embodiment of the concept of convergence of Taylor series.

*Graphical Images of the Convergence of Taylor Series.* Experts demonstrated little to no difficulties in interpreting graphs related to Taylor series. They were able to correctly answer questions concerning errors, and interpret the effects of adding more terms to and re-centering Taylor polynomials. In stark contrast, the vast majority of

novices were unable to correctly produce graphs related to Taylor series. Instead of Taylor polynomial graphs, some novices produced graphs of functions with growing amplitudes, shrinking periods, horizontal shifts, and even a few similar to resonance. These graphs demonstrated how novices may have been attempting to grasp at previous graphical knowledge from which to build their conception of Taylor polynomial graphs.

Some additional novices were able to at least recognize Taylor polynomial graphs and comment on the graphical effects of adding more terms and re-centering. Some novices even noted the error between Taylor polynomials and the corresponding generating function. In so doing, the novice participant group indicated different levels of graphical understanding. At the low end were those who were unable to produce or recognize Taylor polynomial graphs. At the high end were those who correctly graphed Taylor polynomials and indicated a graphical understanding of adding more terms, re-centering, and error. A closer analysis of the data indicated that novices who did not attempt to draw Taylor polynomials graphs were significantly more likely to not recognize already drawn Taylor polynomial graphs.

A graphical understanding of Taylor series may be the single most notable effect separating novices from experts. It also may be the conception that is easily influenced through instruction which gives more exposure to the graphical attributes of Taylor series (e.g., Cory, 2005, 2008; Kidron & Zehavi, 2002; Roh, 2008). The progression in understanding of the graphical conceptions of Taylor series at different levels of mathematical maturity was more transparent than for other conceptions. Novices struggled with formulating and indentifying Taylor polynomials graphs, but MNPs were significantly more likely to be able to indicate an understanding of the graphical effect of

adding more terms than NNPs, and experienced experts did not struggle at all with graphical interpretations. I will discuss more about Taylor polynomial graphs and student understanding in the next two sections.

*Making the Connection to Functions.* Both experts and novices related Taylor series to functions. At first I thought that experts were more prone to relate Taylor series to functions. For example, experts would mention differentiating and integrating Taylor series like a function, and some referred to Taylor series as an infinite polynomial.

Experts would perform function operations on Taylor series. For example, experts

identified the series  $\sum_{n=0}^{\infty} \frac{5^n}{n!}$  as equal to  $e^5$ , and in so doing, they had to recognize that

$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$  and treat  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  as a function by plugging 5 into the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ . In doing

operations similar to these, experts appeared to be relating the interval of convergence (IOC) to domain, but only one expert specifically said so during the interviews.

After a closer analysis of the data, I found that novices may have been relating Taylor series to functions just as much as experts. Therefore, the difference was not in how much each group related Taylor series to functions, but in how they did so. Novices made some of the same comments as experts. For example, some novices related Taylor series to differentiation, integration, and to an infinite polynomial, and some equated

$\sum_{n=0}^{\infty} \frac{5^n}{n!}$  to  $e^5$  but they appeared to do so less frequently. Like the experts, many novices

performed operations on Taylor series that one could interpret as relating the IOC to domain, but 37 percent of the novices made no specific association of the IOC to domain when directly asked in the questionnaire.



More frequently than the experts, novices tended to rely upon a pointwise image of convergence. As discussed in the previous section, the pointwise image may be an indication of reliance upon function conceptions because a pointwise conception of functions has already been characterized by other researchers (e.g., Dubinsky & Harel, 1992; Ferrini-Mundy & Graham, 1994; Monk, 1992; Oehrtman et al., 2008). In addition, the collapse image that appeared in the novice participant group may be related to a function conception because it displayed indicators similar to the pointwise conception. In addition, the growing amplitude, shrinking period, and horizontal shift graphical images as mentioned in the previous subsection appeared to be related to a graphical understanding of functions in general and trigonometric functions in particular. I believe that more research is needed in this area to better unpack how experts and novices are relating their understanding of Taylor series to their function conceptions, but this study has shown that experts and novices both related Taylor series to functions and it has indicated some ways in which they were doing so.

*Interval of Convergence (IOC) and Re-Centering.* Both the experts and novices demonstrated an understanding of the IOC. For the novices, the IOC, whether finite or infinite in measure, appeared to affect their responses to tasks concerning Taylor series, but this effect seemed small and perhaps not as prominent as some of the other effects considered in the next subsection. In the context of the images, it appeared more likely that novices were relying on different images when responding to different image tasks with various IOCs and that only a few novices were directly affected by the IOC. The novices may have been directing their focus more toward the series and its generating

function, and thus, overlooking how the IOCs may change depending on the generating function.

Both experts and novices failed to consistently rely on re-centering as a tool for obtaining better approximations. Only two of the nine interviewed experts initially brought up re-centering as a viable option for estimating sine by using its Taylor series (see Task 8 in Table 8 in Chapter 3). After some prompts designed to give a re-centering image ample opportunity to emerge, four experts still did not clearly mention re-centering. Furthermore, only one novice alluded to re-centering in response to the same task, and he only did so after prompts. I am not saying that re-centering did not appear elsewhere from experts and novices, for example all interviewed experts were able to identify the effect of graphically re-centering Taylor polynomials, and many novices indicated an understanding of re-centering within their questionnaires. But the data suggested that for both participant groups the notion of re-centering the Taylor polynomial to obtain better approximations was not a primary attribute in many of the participants' approximation concept images. This may be due to the influence of Maclaurin series and its use in exercises and applications in current curriculum (Hass et al., 2007; Stewart, 2008).

*Various Other Effects.* The convergence of Taylor series can be expressed in many different ways. In Chapter 3, I describe how I attempted to account for these effects in the novice questionnaire. I called the effects that I accounted for the IOC, notational, generating function, and equated series effects. The IOC effect has already been discussed in the previous subsection. The other effects were not accounted for in the expert participant group during the focus group or interviews since I believed that

these effects would be nonexistent, or at least very minimal, in expert conceptions because of expert experience with these factors throughout their mathematical careers.

Within the novice participant group, the third layer of analysis revealed that the equated series effect was the most prominent of the non-IOC effects, but none of these effects greatly influenced novice existing or emergent concept images. The effects did not interact much, if at all, with concept image questionnaire tasks even though some of these effects seemed ontologically similar to some of the images. These results did not mean that novices didn't struggle with interpreting the various ways of writing Taylor series or that different ways of writing Taylor series couldn't cause them to respond differently to particular tasks. But it seemed to imply that these different ways of writing Taylor series, did not much effect their conceptions of Taylor series convergence. Therefore, I cannot claim that if one student affirmed a particular way of writing Taylor series convergence, then that student would be more likely to indicate a particular concept image of Taylor series and vice versa.

### *Informing Pedagogy*

This study informed pedagogy by adding to the body of knowledge concerning student understanding of Taylor series in answering the research questions addressed above. This study was not intended to purport one teaching method over another, nor can it do so with any certainty since no instructional treatment was actually done and no student was actually followed during their learning process. But students were observed at different stages of their mathematical careers, and experts were studied. Differences were noticed between these groups, and therefore I can make some pedagogical

observations based on these differences, but more research is needed before one teaching method can be conclusively chosen over another.

*Graphing.* When novices displayed a graphical understanding of Taylor series, they might also have demonstrated themselves as having an abundance of conceptual images linked together in a way resembling that of an expert. The most notable occurrence of such a display took place during the interview with NNP JORDAN. Consider his response to the question, “What if you wanted to estimate sine at 103 radians?” asked during high inference interview Task 8.

*Excerpt 60*

NNP JORDAN: I remember this. [Begins drawing] Uh. If we say we have a curve [drawn Step 1 of Figure 30], uh, and we add up one term, say this is at zero [drawn Step 2 of Figure 30], and this is our linear approximation, well then uh, it's incredibly accurate in a small range [drawn Step 3 of Figure 30] but as we extrapolate this forward there's a huge gap [drawn Step 4 of Figure 30]. So it would be better if out here, we're at 103 [draw Step 5 of Figure 30], um, we take this estimation closer to here, where a [drawn Step 6 of Figure 30] linear approximation, or quadratic [Step 7 of Figure 30], or cubic [Step 8 of Figure 30] or whatever has um the remainder is um, less. So yeah, if we tried to do that approximation of sine 103 out here [points to the linear approximation centered at zero] and this was for example our estimation, we would have so much more error between here and here [Figure 31] than we would if we centered that approximation here [highlights the polynomial approximations centered at 103]. Yeah.

Just because a participant was categorized as a novice, didn't mean that the participant looked like a novice when comparing their conceptions about the convergence to that of experts. When comparing JORDAN's focus analysis diagram (See Figure 18 in Chapter 6) to the experts' focus diagrams (see Chapter 5 and Appendix D), JORDAN looked much like an experienced expert. Like an expert, JORDAN utilized multiple images; in fact, he used all of them. Like an expert, JORDAN incorporated multiple images in most of his instances of focus. Like an expert, JORDAN, referenced a

remainder image during his response to Tasks 6 and 7, and he alluded to a dynamic partial sum image during his response to Task 11. Why might JORDAN look so much like an expert? For one thing, he had a well developed graphical image of Taylor series that was linked to and in line with his conceptual understanding of Taylor series. The development of his graphical understanding may have been pivotal in developing those links with other concept images and building his conceptual understanding of Taylor series.

So how does a student's graphical understanding of Taylor series develop? As noted in Chapter 2, Kidron and Zehavi (2002) have already demonstrated that, when used appropriately, a computer algebra system (CAS) can be used to effectively spur students to engage in self motivated discovery of properties of Taylor series prior to the development of a formal theory. Kidron and Zehavi (2002) and Kidron (2004) both observed students engaging in a process of asking questions, using the CAS to investigate their questions, and asking more questions based off of the CAS' results. Although this type of student engagement was an encouraging result for using a CAS as an instructional tool, Kidron and Zehavi (2002) and Kidron (2004) also reported some negative results caused by the CAS. Kidron had her students program the CAS to do animations of convergence, but in some cases the dynamic graphs did not help the students who had not analyzed the actual mathematics (Kidron, 2004), and may even lead students to reverse the roles of epsilon and  $N$  (Kidron & Zehavi, 2002).

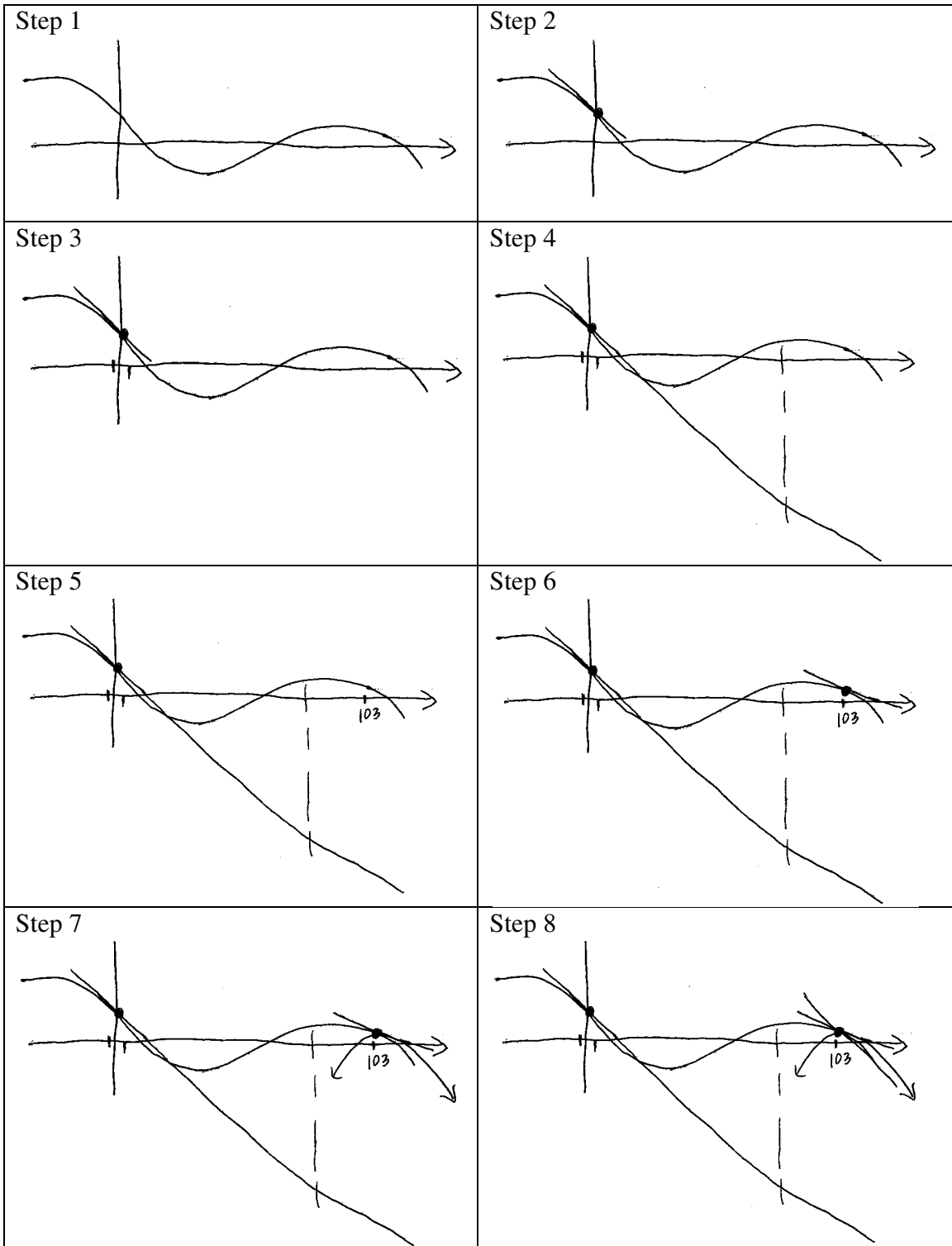


Figure 30. The Reproduced\* Progression of NNP JORDAN's Graph in Response to High Inference Task 8.

\*All images in this figure are reproduced based on the individual's finished graph. The order of the steps were determined based on the transcript and the audio and video evidence.

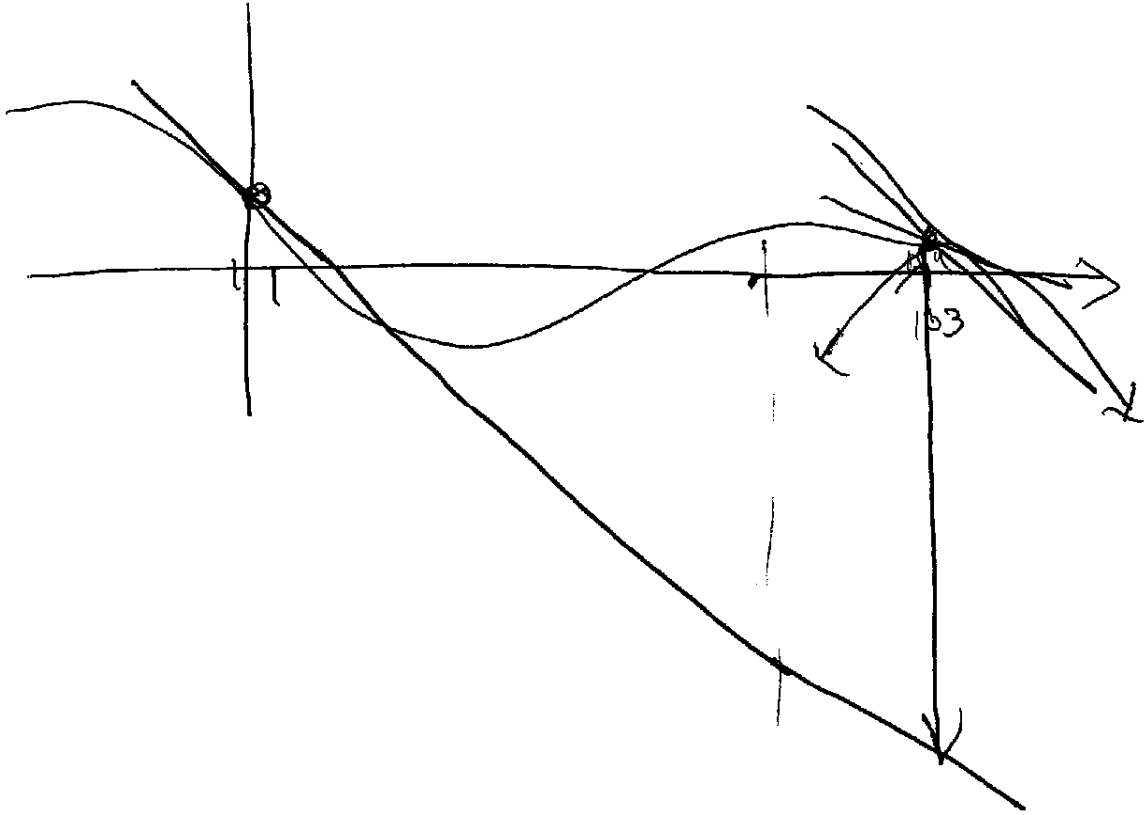


Figure 31. Copy of NNP JORDAN's Actual Graph in Response to High Inference Task 8

Just because Kidron and Zehavi (2002) and Kidron (2004) noted some difficulties related to using a CAS, this does not mean that one should shy away from using a CAS to produce animations of Taylor series in an effort to develop students' graphical understanding. In relation to the limit of function and sequences, Cory (2008) found that animations helped students develop their formal understanding and retain their understanding long after the instructional treatments had ceased. Cory did not have her participants do the actual programming of the animations, but they could manipulate the dynamic graphs in a way that was consistent with the formal mathematical theory. Also in relation to sequences, Roh (2008) deviated from the use of the CAS, and used "ε-strips" traced on transparencies to demonstrate how students could build their formal

understanding using graphs (pp. 220 - 222). Therefore, it seems possible that in some cases the programming required from Kidron's students may have been too excessive, and thus her students may have become more concerned with the syntax of the program and less concerned with the mathematical structure. Animations or graphs need to mirror the formal mathematical structure to help nurture correct mathematical conceptions. I can speculate that it seems highly likely that future research on Taylor series will reveal that when graphs, whether dynamic or static, are used correctly they will be a great aid in developing student understanding of this complex topic.

*Example Space.* According to Watson and Mason (2005), one of the important pedagogical principles of learning mathematics “consists of exploring, rearranging, and extending example spaces and the relationships between and within them” (p. 6). They went on to say that learners gain “fluency and facility in associated techniques and discourse” by “developing familiarity” with the example spaces associated with given concepts (p. 6). From this data, it seems that most experts, more so the EEPs, had a more developed “fluency and facility” than most novices in the context of Taylor series convergence, and series in general. Mason (2008), referencing Schoenfeld (1998), went on to warn that “mindless repetition does not always result in proficiency, nor even recall of having carried out that practice: how often do learners deny recall of a topic or technique that the teacher knows they have studied previously?” (p. 264). The harmonic series, which was so readily available to many experts as a prototypical counterexample, may not have been perceived by some students as being important, and therefore, having mindlessly used the harmonic series previously, they were unable to recall it later when needed. Mindless repetition due to rote memorization does not build a firm conceptual



understanding with an abundance of internal connections to other concepts that will likely withstand the test of time (Hiebert & Carpenter, 1992; Mason, 2008; Vinner, 1997).

If Watson and Mason (2005) were correct about the importance of the relationship between example spaces and learning, then instructors must take into account the type of examples that they present to their students as well as the degree to which their students are understanding the importance of such examples. Examples should be carefully chosen so that they aid the development of good conceptions of convergence of Taylor series and help students abandon their use of inaccurate conceptions. Watson and Mason (2005) warn that a few “good” examples need not be enough for the novice.

Novices are likely to need several teacher-provided examples so that their attention is drawn explicitly to important dimensions of possible variation, rather than distracted by unimportant ones... If learners are offered several examples, they can start making sense of them by doing what comes naturally, that is, seeing what is the same and what is different about them. (Watson & Mason, 2005, p. 106)

Watson and Mason (2005) also suggested that the instructor let the learner construct their own examples and justify their constructions. These examples need not be prototypical and some may even need to get students to break out of familiar patterns so that they can adapt their concept images (Watson & Mason, 2005). Therefore, the teacher needs to be prepared for a variety of responses and maybe even adapt their own conceptions. For a more detailed description of suggestions and other ideas for building example spaces, I suggest consulting Watson and Mason (2005).

*Oehrtman's Approximation Framework.* Another pedagogical suggestion is based on creating a mathematical framework that is both consistent with the formal mathematical theory and more accessible to students. Instead of the standard  $\delta - \epsilon$  or  $\epsilon - N$ , definitions coming early in an instructional sequence, Oehrtman (2008) suggested that

these formal definitions be held back until after “multiple rounds of instruction that reinforce the conceptual structure of limits in different settings” using an approximation framework (p. 72). Included in these “different settings” were topics such as differentiation, integration, and Taylor series. Oehrtman (2008) elaborated on an approximation schema that he reported in Oehrtman (2002):

The main components of students’ spontaneous use of approximation ideas to reason about limits consisted of an unknown quantity and approximations that are believed to be close in value to the unknown quantity. For each approximation, there is an associated error,

$$\text{error} = | \text{unknown quantity} - \text{approximation} |.$$

Consequently, a bound on the error allows one to use an approximation to restrict the range of possibilities for the actual value as in the inequality

$$\text{approximation} - \text{bound} < \text{unknown quantity} < \text{approximation} + \text{bound}.$$

An approximation is contextually judged to be accurate if the error is small, and a good approximation method allows one to improve the accuracy of the approximation so that the error is as small as desired. An approximation method is precise if there is not a significant difference among the approximations after a certain point of improving accuracy. (Oehrtman, 2008, p. 73)

After presenting the logical framework behind the approximation schema, Oehrtman (2008) went on to explain that the structure of the approximation framework was consistent with the mathematical structure found in the formal  $\delta - \epsilon$  and  $\epsilon - N$  definitions. Using the approximation framework, Oehrtman (2008) came up with an instructional method incorporating contextual, graphical, algebraic, and numerical examples designed to “systematize students’ spontaneous understandings” (p. 74). This “systematization” occurred through an ongoing process of reflection and abstraction that progressively provided “fewer step-by-step instructions” to allow students to “begin to remember or develop appropriate strategies to solve increasingly more sophisticated

problems” (Oehrtman, 2008, p. 74). Oehrtman (2008) pointed out that his instructional approach is in line with Piaget’s theory of abstraction (e.g., Piaget, 1970) and I will also point out that his approach is in line with Watson and Mason (2005) since it reinforces the concept of convergence of Taylor series using multiple types of examples that can develop a network of different conceptions strongly connected together. For a more detailed description of this approximation framework and some example tasks, see Oehrtman (2008).

### *Directions for Future Research*

If teachers desire for novices to look more like the experts, how do they best achieve this goal? Why does someone comprehend Taylor series in one way while someone else comprehends Taylor series differently? How do people utilize their comprehensions of Taylor series when solving nonstandard problems? While this study provided insights into its own research questions, additional questions like the ones above were not answered. It is questions like these that can provide a great impetus for future research. In this section, I will discuss some possible directions for future mathematics education research concerning Taylor series.

*The Effect of Visual Images.* It is very clear from the data that many students do not have a good visual image, if they have any visual image at all, of the convergence of Taylor series. Alcock and Simpson (2004, 2005) found that both visualizers and non-visualizers could have a sound understanding of formal theory, but they did not comment of the effect of being able to visualize, visualizer or non-visualizer. It seems likely that good non-visualizers may actually have a well developed graphical understanding, they just don’t tend to rely upon it. My data seemed to suggest that being able to correctly

visualize Taylor polynomial approximations to generating functions may aid in comprehension. In particular, this study revealed one novice participant, NNP JORDAN, who demonstrated a very good graphical understanding of the convergence of Taylor series, and his graphical understanding seemed to aid his conceptual understanding. Other research has suggested that visual images when used appropriately can be helpful in the learning of calculus and analysis (e.g., Cory, 2005, 2008; Kidron, 2004; Kidron & Zehavi, 2002; Navarro & Carreras, 2006; Pinto & Tall, 2002; Roh, 2008).

So how did NNP JORDAN and others who may be like him come to be in possession of such a multifaceted visual image? Was the visual image the main aid to the development of JORDAN's understanding or was there other variables influencing his development even more? How might dynamic computer generated images affect student understanding of the concept of Taylor series? Kidron and Zehavi (2002) and Kidron (2004) have already demonstrated how a CAS may be used as an instructional tool prior to the development of a formal theory. What about in conjunction with the formal theory? What if students do not program the CAS to produce the animations but instead manipulate animations that have been designed to better reflect the formal theory? What is the effect of graphical representations, dynamic and static, on the understanding of visualizers and non-visualizers?

*Influence of the Concepts of Function, Limit, and Infinity.* As this study has demonstrated, novices attempted to relate their understanding of Taylor series to previous conceptions in which they had more familiarity. As calculus students enter into an initial act of learning Taylor series, their conceptions of function, limit, and infinity will naturally influence their formation of a conception of Taylor series. How might different

conceptions of the notion of function affect their learning and understanding of the concept of Taylor series? How might different conceptions about the notion of limit affect their learning and understanding? In what ways can Oehrtman's approximation framework for limit be used to build various conceptions of the convergence of Taylor series? What about the influence of actual and potential views of infinity on the conceptions of Taylor series?

*The Influence of Covariational Reasoning.* In Chapter 1, I alluded to students having to keep track of multiple changing variables when learning about the convergence of Taylor series. The type of reasoning involved to keep track of “two varying quantities while attending to the ways in which they change in relation to each other” has been defined as covariational reasoning by Carlson et al. (2002, p. 354). In the context of Taylor series, there are multiple variables that can change: the degree of an approximating polynomial, the center of the series, the interval of convergence, the error, and the interval of accuracy. In addition every Taylor polynomial is a function with covarying quantities  $x$  and  $y$ . So how does learning the concept of convergence of Taylor series increase one's ability to reason covariationally? How does a student's ability to coordinate two varying quantities affect his ability to learn the concept of convergence applied to Taylor series? Consider MNP RUSS' response to high inference interview Tasks 9 and 10 in Excerpt 61:

*Excerpt 61*

Tasks 9 and 10: What is meant by the “approximation” symbol and what is meant by the “near” in “ $\sin(x) \approx x - \frac{x^3}{3!}$  = a Taylor polynomial for sine when  $x$  is near 0?”

MNP RUSS: Well, I would say the uh, approximation sign would have maybe more, would be a lot of polynomial, and then near, which would be maybe just a couple.

In Excerpt 61, MNP RUSS struggled with distinguishing between the approximation symbol and the word “near.” In this case, the approximation symbol was representative of a small difference between two  $y$  values associated with the generating function and the cubic Taylor polynomial. The word “near” was associated with  $x$  values in an unspecified neighborhood around zero. Instead of making these observations, RUSS associated the approximation symbol with a Taylor polynomial of large degree as indicated by his reference to “a lot of polynomial.” Then, tacitly implied by his reference to “just a couple,” he appeared to maintain his focus on the degree of the polynomial when he concluded that “near” was referring to a polynomial with only a few terms. He failed to associate the approximation sign and the word “near” with two different quantities. Therefore, he indicated potential difficulties with covariational reasoning in the context of Taylor series.

It should be noted that MNP RUSS’ focus analysis diagram (Figure 21 in Chapter 6) did not resemble the focus analysis diagrams for the experienced experts. RUSS fixated on particular images, did not link multiple images together in a large number of instances of focus, and did not utilize a remainder image during his responses to Tasks 6 and 7. Did he have difficulty with covariational reasoning? Based only on Excerpt 61, it is inconclusive if MNP RUSS struggled with covariational reasoning, but if so, was that one of the factors causing him to fail to look like an expert?

*Multivariational Reasoning?* At the end of their article, Oehrtman, Carlson, and Thompson (2008) alluded to extending the idea of covariational reasoning to higher dimensions. Because of its abundance of variables, I propose that comprehending Taylor series convergence requires more than just simple covariational reasoning, but

multivariational reasoning. To borrow from Carlson et al. (2002), I define multivariational reasoning to be the cognitive activities involved in coordinating multiple varying quantities while attending to the ways in which they change in relation to each other.

It is interesting to note that after reviewing some of the excerpts, participants seemed to reason about this multivariational convergence of Taylor series in covariational ways. That is, even though multiple variables were involved, participants might have coordinated only two variables at a time. For example, in Excerpt (1) of Table 19 in Chapter 5, EEP GRIFFIN said that “the remainder is something that as the polynomial [waves right hand left to right] increases in size, the remainder [two chops followed by waving right hand toward the right] decreases and goes to zero.” In this excerpt it appeared that GRIFFIN was coordinating the degree of the polynomial and the remainder, two variables. Therefore GRIFFIN was applying covariational reasoning in a multivariable situation. Or was he? In this case, the polynomial and the remainder are functions. Perhaps he was simultaneously coordinating their  $x$  and  $y$  values even though he didn’t verbalize that he was doing so? In the end, a framework for multivariational reasoning needs to be developed, and along with multivariable functions, Taylor series may prove to be a useful tool for developing such a framework.

#### *Final Remarks*

This has been the study of expert conceptions of the convergence of Taylor series, yesterday, today, and tomorrow. Using the notion of concept images and Williams’ models of limit applied to Taylor series, together with the participant groups, the three phases of data collection, and the four layers of analysis, I was able to successfully

analyze and describe different ways in which experts and novices conceptualized the convergence of Taylor series. Collectively, this framework and method proved to be useful in illuminating the various ways in which people conceptualize the convergence of Taylor series depending on the extent of their mathematical backgrounds. Compared to novices, many experts had a more robust mental imagery of Taylor series with more numerous images that they could efficiently and effectively switch between, and many proved to be cognizant of how they and others thought about Taylor series. This did not mean that an individual novice could not look like an expert or that an expert was beyond mistake. Compared to experts, some novices appeared to be almost desperately attempting to relate their relatively new found topic of Taylor series to other more familiar objects such as function and potential infinity.

With the knowledge gained from this study I challenge instructors to use the descriptions of the concept images, including the depiction of the visual elements, the implied connections to function conceptions, the effects of the interval of convergence and re-centering, and the account of the various other effects, to meet students where they are to maximize their learning opportunity. Taylor series taught well should not only add to students' understanding of Taylor series, but also lead students into confronting previous malformed notions of limit and complement their correct conceptions of limit. I am not saying that all of calculus should be reworked to contain more Taylor series, but that instructors should choose wisely in how they spend their allotted time with Taylor series. Instructors of higher level mathematics classes need to understand what conceptions of Taylor series that their students are bringing into the classroom. This study has shown that in many cases they may not be bringing very much and the



conceptions that they do bring may not match the formal theory. In the time that teachers are given for the instruction of Taylor series, I believe that they need to emphasize a conceptual understanding of Taylor series and not just train behavior through repetition of formulas. In the previous section I discussed some ideas for doing so. With that said, I leave the mathematical community with one excerpt found on a questionnaire from a new novice participant. This excerpt should challenge all mathematics instructors to not just train behavior, but teach the concepts.

*Excerpt 62*

NNP: Teaching me examples get me through the test, but I don't feel I've been taught the concepts behind the examples.

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## APPENDIX A

### Copies of Institutional Review Board (IRB) Forms

All copies of the IRB documents have been scaled to fit within the margins of this dissertation.

IRB Approval Form for this Study using Expert Participants .....	308
IRB Approval Form for this Study using Novice Participants .....	309
IRB Informed Consent Form for Expert Participants in a Focus Group .....	310
IRB Informed Consent Form for Expert Participants in Interviews Only .....	313
IRB Informed Consent Form for Novice Participants .....	316



*The University of Oklahoma*

OFFICE FOR HUMAN RESEARCH PARTICIPANT PROTECTION

IRB Number: 12084  
Approval Date: April 18, 2008

April 18, 2008

Jason Martin  
Mathematics  
601 Elm Avenue, PHSC 827  
Norman, OK 73019

**RE: Experts' Understanding of the Convergence of Taylor Series & Their Perceptions of Student Understanding**

Dear Mr. Martin:

On behalf of the Institutional Review Board (IRB), I have reviewed and granted expedited approval of the above-referenced research study. This study meets the criteria for expedited approval category 6, 7. It is my judgment as Chairperson of the IRB that the rights and welfare of individuals who may be asked to participate in this study will be respected; that the proposed research, including the process of obtaining informed consent, will be conducted in a manner consistent with the requirements of 45 CFR 46 as amended; and that the research involves no more than minimal risk to participants.

This letter documents approval to conduct the research as described:

Consent form - Subject Dated: April 17, 2008 Revised - Interviews  
Consent form - Subject Dated: April 17, 2008 Revised - Focus groups  
Other Dated: April 17, 2008 Revised - Verbal script (Experts)  
Other Dated: April 17, 2008 Letter of support - Dr. Murphy, course 5950  
Protocol Dated: April 17, 2008 Revised  
IRB Application Dated: April 17, 2008 Revised  
Survey Instrument Dated: April 02, 2008 Interview Handout for those not in a Focus Group  
Survey Instrument Dated: April 02, 2008 Interview Handout - for those in a Focus Group  
Survey Instrument Dated: April 02, 2008 Focus Group Handout  
Survey Instrument Dated: April 02, 2008 Quests for Non-Focus Group Participants  
Survey Instrument Dated: April 02, 2008 Interview Quests for people in a Focus Group  
Survey Instrument Dated: April 02, 2008 Focus Group Questions  
Other Dated: April 02, 2008 Verbal Script - Seminars

As principal investigator of this protocol, it is your responsibility to make sure that this study is conducted as approved. Any modifications to the protocol or consent form, initiated by you or by the sponsor, will require prior approval, which you may request by completing a protocol modification form. All study records, including copies of signed consent forms, must be retained for three (3) years after termination of the study.

The approval granted expires on April 17, 2009. Should you wish to maintain this protocol in an active status beyond that date, you will need to provide the IRB with an IRB Application for Continuing Review (Progress Report) summarizing study results to date. The IRB will request an IRB Application for Continuing Review from you approximately two months before the anniversary date of your current approval.

If you have questions about these procedures, or need any additional assistance from the IRB, please call the IRB office at (405) 325-8110 or send an email to [irb@ou.edu](mailto:irb@ou.edu).

Cordially,

E. Laurette Taylor, Ph.D.  
Chair, Institutional Review Board

Ltr\_Prot\_Fappv\_Exp 660 Parrington Oval, Suite 316, Norman, Oklahoma 73019-3085 PHONE: (405) 325-8110 FAX: (405) 325-2373



*The University of Oklahoma*

OFFICE FOR HUMAN RESEARCH PARTICIPANT PROTECTION

IRB Number: 12084

Amendment Approval Date: June 25, 2008

June 26, 2008

Jason Martin  
Mathematics  
601 Elm Avenue, PHSC 827  
Norman, OK 73019

**RE: IRB No. 12084: Your Understanding of the Convergence of Taylor Series**

Dear Mr. Martin:

On behalf of the Institutional Review Board (IRB), I have reviewed your protocol modification form. It is my judgement that this modification allows for the rights and welfare of the research subjects to be respected. Further, it has been determined that the study will continue to be conducted in a manner consistent with the requirements of 45 CFR 46 as amended; and that the potential benefits to subjects and others warrant the risks subjects may choose to incur.

This letter documents approval to conduct the research as described in:

- Other Dated: June 10, 2008 Verbal Recruitment Script for Undergraduates
- Survey Instrument Dated: June 10, 2008 Interview Questions
- Survey Instrument Dated: June 10, 2008 Questionnaire
- Other Dated: June 16, 2008 Approval Form - OCCC IRB
- Amend Form Dated: June 20, 2008 Revised
- Protocol Dated: June 20, 2008 Revised
- Consent form - Subject Dated: June 20, 2008 Compensation - Revised
- Consent form - Subject Dated: June 20, 2008 No Compensation - Revised

**Amendment Summary:**

- 1) Addition of a Research Site - Oklahoma City Community College.
- 2) Change in Procedure - Addition of undergraduate student participants at OU and Oklahoma City Community College. Participants will complete a written questionnaire and a follow-up face-to-face interview and possibly two task-based interviews.
- 3) Increase in Subject Enrollment - From a previously approved maximum enrollment of 30 to a new maximum enrollment of 500.
- 4) Addition of 2 Informed Consent Forms for undergraduate students. One for students who will receive course credit as compensation and one form for students who will not receive class credit.
- 5) Change in Study Title: From "Experts' Understanding of the Convergence of Taylor Series & Their Perceptions of Student Understanding" to "Your Understanding of the Convergence of Taylor Series".
- 6) Revised protocol to reflect changes in procedure.

This letter covers only the approval of the above referenced modification. All other conditions, including the original expiration date, from the approval granted April 18, 2008 are still effective.

If consent form revisions are a part of this modification, you will be provided with a new stamped copy of your consent form. Please use this stamped copy for all future consent documentation. Please discontinue use of all outdated versions of this consent form.

If you have any questions about these procedures or need additional assistance, please do not hesitate to call the IRB office at (405) 325-8110 or send an email to [irb@ou.edu](mailto:irb@ou.edu).

Cordially,

Aimee Franklin, Ph. D  
Vice Chair, Institutional Review Board

660 Parrington Oval, Suite 316, Norman, Oklahoma 73019-3085 PHONE: (405) 325-8110 FAX: (405) 325-2373



**University of Oklahoma  
Institutional Review Board  
Informed Consent to Participate in a Research Study**

**Project Title:** Experts' Understanding of the Convergence of Taylor Series & Their Perceptions of Student Understanding  
**Principal Investigator:** Jason Howard Martin  
**Department:** Department of Mathematics

You are being asked to volunteer for this research study. This study is being conducted at University of Oklahoma. You were selected as a possible participant because you are considered an expert in Taylor series.

Please read this form and ask any questions that you may have before agreeing to take part in this study.

**Purpose of the Research Study**

The purpose of this study is to understand how experts think about the convergence of Taylor series and how they think their students think about this topic.

**Number of Participants**

About 30 people will take part in this study.

**Procedures**

If you agree to be in this study, you will be asked to do the following:

- Participate in no more than three task-based individual interviews. Interview sessions will last no longer than 60 minutes each.
- Allow the researcher to use your oral and written responses to tasks presented during the interview. Portions of these documents may be used confidentially as illustrations of expert thinking in the final report of this investigation.

**Length of Participation**

No more than 5 weeks.

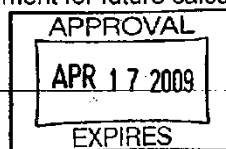
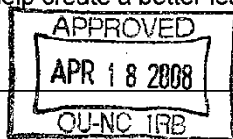
**This study has the following risks:**

There are no identifiable risks.

**Benefits of being in the study are**

This study will help expose you to mental images about Taylor series that you might not be aware of. This should instantly help aid you in better understanding how your students may be thinking about Taylor series.

Also, this study can add to epistemological understanding and aid in pedagogical issues. It has potential to help create a better learning environment for future calculus students.



**Benefits of being in the study are**

This study will help expose you to mental images about Taylor series that you might not be aware of. This should instantly help aid you in better understanding how your students may be thinking about Taylor series.

Also, this study can add to epistemological understanding and aid in pedagogical issues. It has potential to help create a better learning environment for future calculus students.

**Confidentiality**

In published reports, there will be no information included that will make it possible to identify you without your permission. Research records will be stored securely and only approved researchers will have access to the records.

There are organizations that may inspect and/or copy your research records for quality assurance and data analysis. These organizations include the OU Institutional Review Board.

**Compensation**

You will not be reimbursed for you time and participation in this study.

**Voluntary Nature of the Study**

Participation in this study is voluntary. If you withdraw or decline participation, you will not be penalized or lose benefits or services unrelated to the study. If you decide to participate, you may decline to answer any question and may choose to withdraw at any time.

**Waivers of Elements of Confidentiality**

Your name will not be linked with your responses unless you specifically agree to be identified. Please select one of the following options

I consent to being quoted directly.

I do not consent to being quoted directly.

**Audio Recording of Study Activities**

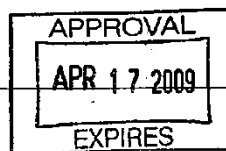
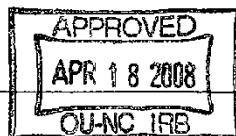
To assist with accurate recording of participant responses, interviews may be recorded on an audio recording device. You have the right to refuse to allow such recording without penalty. Please select one of the following options.

I consent to audio recording.  Yes  No.

**Video Recording of Study Activities**

To assist with accurate recording of your responses, interviews may be recorded on a video recording device. You have the right to refuse to allow such recording. Please select one of the following options:

I consent to video recording.  Yes  No.



701-A-1

**Contacts and Questions**

If you have concerns or complaints about the research, the researcher, Jason Martin, conducting this study can be contacted at 405-325-6711 or [jmartin@ou.edu](mailto:jmartin@ou.edu). The research advisor, Dr. Teri J. Murphy, can be contacted at 405-325-4071 or [tjmurphy@ou.edu](mailto:tjmurphy@ou.edu).

Contact the researcher(s) if you have questions or if you have experienced a research-related injury.

If you have any questions about your rights as a research participant, concerns, or complaints about the research and wish to talk to someone other than individuals on the research team or if you cannot reach the research team, you may contact the University of Oklahoma – Norman Campus Institutional Review Board (OU-NC IRB) at 405-325-8110 or [irb@ou.edu](mailto:irb@ou.edu).

***You will be given a copy of this information to keep for your records. If you are not given a copy of this consent form, please request one.***

**Statement of Consent**

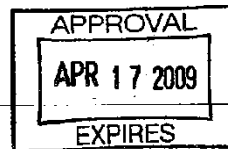
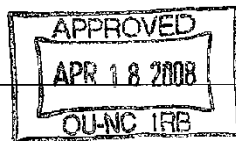
I have read the above information. I have asked questions and have received satisfactory answers. I consent to participate in the study.

---

Signature

Date

Revised 07/23/2007



Page 3 of 3

**University of Oklahoma  
Institutional Review Board  
Informed Consent to Participate in a Research Study**

**Project Title:** Experts' Understanding of the Convergence of Taylor Series & Their Perceptions of Student Understanding  
**Principal Investigator:** Jason Howard Martin  
**Department:** Department of Mathematics

You are being asked to volunteer for this research study. This study is being conducted at University of Oklahoma. You were selected as a possible participant because you are considered an expert in Taylor series.

Please read this form and ask any questions that you may have before agreeing to take part in this study.

**Purpose of the Research Study**

The purpose of this study is to understand how experts think about the convergence of Taylor series and how they think their students think about this topic.

**Number of Participants**

About 30 people will take part in this study.

**Procedures**

If you agree to be in this study, you will be asked to do the following:

- Participate in no more than three task-based individual interviews. Interview sessions will last no longer than 60 minutes each.
- Allow the researcher to use your oral and written responses to tasks presented during the interview. Portions of these documents may be used confidentially as illustrations of expert thinking in the final report of this investigation.

**Length of Participation**

No more than 5 weeks.

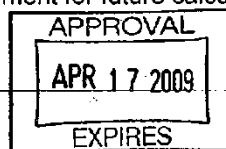
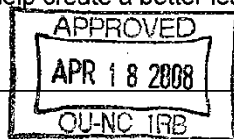
**This study has the following risks:**

There are no identifiable risks.

**Benefits of being in the study are**

This study will help expose you to mental images about Taylor series that you might not be aware of. This should instantly help aid you in better understanding how your students may be thinking about Taylor series.

Also, this study can add to epistemological understanding and aid in pedagogical issues. It has potential to help create a better learning environment for future calculus students.



**Confidentiality**

In published reports, there will be no information included that will make it possible to identify you without your permission. Research records will be stored securely and only approved researchers will have access to the records.

There are organizations that may inspect and/or copy your research records for quality assurance and data analysis. These organizations include the OU Institutional Review Board.

**Compensation**

You will not be reimbursed for your time and participation in this study.

**Voluntary Nature of the Study**

Participation in this study is voluntary. If you withdraw or decline participation, you will not be penalized or lose benefits or services unrelated to the study. If you decide to participate, you may decline to answer any question and may choose to withdraw at any time.

**Waivers of Elements of Confidentiality**

Your name will not be linked with your responses unless you specifically agree to be identified. Please select one of the following options

I consent to being quoted directly.

I do not consent to being quoted directly.

**Audio Recording of Study Activities**

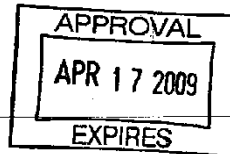
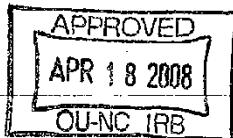
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I consent to audio recording.  Yes  No.

**Video Recording of Study Activities**

To assist with accurate recording of your responses, interviews may be recorded on a video recording device. You have the right to refuse to allow such recording. Please select one of the following options:

I consent to video recording.  Yes  No.



**Contacts and Questions**

If you have concerns or complaints about the research, the researcher, Jason Martin, conducting this study can be contacted at 405-325-6711 or [jmartin@ou.edu](mailto:jmartin@ou.edu). The research advisor, Dr. Teri J. Murphy, can be contacted at 405-325-4071 or [tjmurphy@ou.edu](mailto:tjmurphy@ou.edu).

Contact the researcher(s) if you have questions or if you have experienced a research-related injury.

If you have any questions about your rights as a research participant, concerns, or complaints about the research and wish to talk to someone other than individuals on the research team or if you cannot reach the research team, you may contact the University of Oklahoma – Norman Campus Institutional Review Board (OU-NC IRB) at 405-325-8110 or [irb@ou.edu](mailto:irb@ou.edu).

***You will be given a copy of this information to keep for your records. If you are not given a copy of this consent form, please request one.***

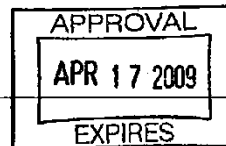
**Statement of Consent**

I have read the above information. I have asked questions and have received satisfactory answers. I consent to participate in the study.

---

Signature

Date



**University of Oklahoma  
Institutional Review Board  
Informed Consent to Participate in a Research Study**

**Project Title:** Your Understanding of the Convergence of Taylor Series  
**Principal Investigator:** Jason Howard Martin  
**Department:** Department of Mathematics

You are being asked to volunteer for this research study. This study is being conducted at University of Oklahoma and at Oklahoma City Community College. You were selected as a possible participant because of your previous exposure to Taylor series.

Please read this form and ask any questions that you may have before agreeing to take part in this study.

**Purpose of the Research Study**

The purpose of this study is to understand how people think about the convergence of Taylor series.

**Number of Participants**

About 500 people will take part in this study.

**Procedures**

If you agree to be in this study, you will be asked to do the following:

- Complete one 75 minute questionnaire about Taylor series. The primary investigator will contact you to set up a time to take the questionnaire.
- In addition, after completing the questionnaire you may choose to participate in no more than two one-hour interviews. Not everyone who agrees to be interviewed will be chosen for any interview. If you do this you will need to allow the researcher to use your oral and written responses to tasks presented during the interview. Portions of these documents may be used confidentially as illustrations of student thinking in any report of this investigation.

**Length of Participation**

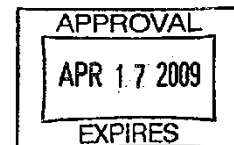
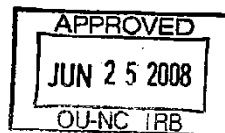
One 75 minute questionnaire and, for those who choose to be considered for interviews, up to two one-hour interviews that can occur over a six week period.

**This study has the following risks:**

There are no identifiable risks.

**Benefits of being in the study are**

There are no identifiable benefits.



**Confidentiality**

In published reports, there will be no information included that will make it possible to identify you without your permission. Research records will be stored securely and only approved researchers will have access to the records.

There are organizations that may inspect and/or copy your research records for quality assurance and data analysis. These organizations include the OU Institutional Review Board.

**Compensation**

Completion of the questionnaire will count as a small bonus assignment, one homework, or one quiz grade as stated by your instructor. For those who do not wish to participate or who are unable to participate, your instructor will provide you with a comparable assignment for the exact same grade incentive.

**Voluntary Nature of the Study**

Participation in this study is voluntary. If you withdraw or decline participation, you will not be penalized or lose benefits or services unrelated to the study. If you decide to participate, you may decline to answer any question and may choose to withdraw at any time.

**Audio Recording of Study Activities: For Participants Who Agree To Interviews**

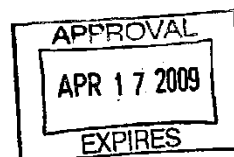
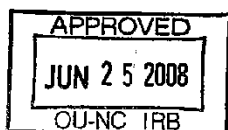
To assist with accurate recording of participant responses, interviews may be recorded on an audio recording device. You have the right to refuse to allow such recording without penalty. Please select one of the following options.

I consent to audio recording.  Yes  No.

**Video Recording of Study Activities: For Participants Who Agree To Interviews**

To assist with accurate recording of your responses, interviews may be recorded on a video recording device. You have the right to refuse to allow such recording. Please select one of the following options:

I consent to video recording.  Yes  No.





701-A-1

**Contacts and Questions**

If you have concerns or complaints about the research, the researcher, Jason Martin, conducting this study can be contacted at 405-325-6711 or jmartin@ou.edu. The research advisor, Dr. Teri J. Murphy, can be contacted at 405-325-4071 or tjmurphy@ou.edu.

Contact the researcher(s) if you have questions or if you have experienced a research-related injury.

If you have any questions about your rights as a research participant, concerns, or complaints about the research and wish to talk to someone other than individuals on the research team or if you cannot reach the research team, you may contact the University of Oklahoma – Norman Campus Institutional Review Board (OU-NC IRB) at 405-325-8110 or irb@ou.edu.

*You will be given a copy of this information to keep for your records. If you are not given a copy of this consent form, please request one.*

**Statement of Consent**

I have read the above information. I have asked questions and have received satisfactory answers. I consent to participate in the study.

---

Name (Print)

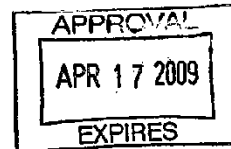
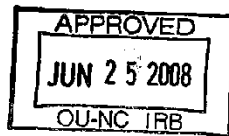
---

Signature

Date

---

e-mail Address (Necessary for contact to set up time to take questionnaire)



## APPENDIX B

### Data Collection Instruments for the Expert Participant Group

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## Expert Focus Group Protocol

Brackets indicate the layout of the handout and notes for me. In some cases font size has been reduced to allow the protocol to fit the margins of this dissertation.

Furthermore, to enhance the readability of the main body of this dissertation, a one-to-one correspondence between expert and novice interview tasks was created by changing some focus group task numbers to match the task numbers as presented in the main body of this dissertation (e.g., Table 8).

### **INTRODUCTION**

Thank you for your time and willingness to participate. As you know, I am interested in how people understand Taylor series. Particularly, I am trying to understand how people understand the convergence of Taylor series. Feel free to volunteer any detail you wish when answering a question. You also have the option of declining to answer – passing – on any of the questions.

You will be given a handout with tasks to complete during this interview.

Do you have any questions before we start? .

[“\*” indicate questions that I really like.]

[Handout begins]

### **DIRECTIONS:**

We will go over this handout one page at a time. At different times you may be asked to write on the handout to answer questions. You are also encouraged to think out loud.

[Say - Think, similar to Jeopardy.

1. I'll pose a question.
2. You answer it quietly and individually on paper
3. Reveal answers.
4. Discuss answers within the group]

**Please write down your first and last name.**

First and Last Name?

[NP = New page of the expert focus group handout]

**SHORT ANSWER:**

**Please reflect upon each question for about a minute. Write down any responses you may have. Then we'll discuss your responses.**

2) \*What are Taylor series?

[Expect a wide range of images.]

3) \*Why are Taylor series studied in calculus?

[NP]

4) \*What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$  when  $x$  is any real number?”

[Expect approximation and proximity images to come out if present. May even get a formal definition of convergence.]

5) \*What is meant by the “(-1,1)” in, “ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$  when  $x$  is in the interval (-1,1)?”

[Expect either images of domain to come out or more approximation images.]

[NP]

6) \*Find the exact sum of the series  $\sum_{n=0}^{\infty} \frac{5^n}{n!} = 1 + \frac{5}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \dots$ .

a) Follow-Up Question: What about this problem causes difficulty?

[It may be necessary to tell them that it is the series for  $e^x$ .]

[NP]

7) \*What is meant by the word “prove” if you were asked to, “Prove that sine is equal to its Taylor series.”

[Wanting to see remainder come up as well as other approximation images.]

8) \*What are the steps in proving that  $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$ ?

[Wanting to see remainder come up as well as other approximation images.]

[NP]

9) How can we estimate sine by using its Taylor series?

a) Follow up question: What if we wanted to estimate sine at 103? How might we use a Taylor series to do that? If confused about the measure, follow up with 103 radians.

[Expect approximation images. Looking to see if center of series images are also present.]

[NP]

10) \*What is meant by the “approximation” symbol in “ $\sin(x) \approx x - \frac{x^3}{3!}$  = a Taylor polynomial for sine when  $x$  is near 0?”

11) \*What is meant by the “near” in “ $\sin(x) \approx x - \frac{x^3}{3!}$  = a Taylor polynomial for sine when  $x$  is near 0?”

[NP]

12) \*How can we get a better approximation for sine than using  $x - \frac{x^3}{3!}$  ?

[Expected Answer: Add more terms]

[NP]

13) What is the difference between the following questions.

- a) \*What is the maximum error in using a given  $n^{\text{th}}$  degree Taylor polynomial to approximate  $f(x)$ ? Verses
- b) \*How large does  $n$  need to be to guarantee that the  $n^{\text{th}}$  degree Taylor polynomial is within 0.1 of  $f(x)$  ?

[NP]

Now we focus our attention from our personal understanding to student understanding.

14) What problems would you include in a lesson plan for **convergence** of Taylor series? Why would you include these problems?

15) A student asks you, “Why does sine equal its Taylor series?” How would you answer the student’s question? Why?

16) If there was one thing that students should take from Taylor series, what would that be?

17) What do you perceive to be the biggest problems that students have with Taylor series?

a) Follow-up: Why do they have these problems.

18) What mistakes do you commonly see students make when working with Taylor series? Why do you think they make these mistakes?

a) Follow-up: How would you characterize these mistakes? Basic math? Algebra? Geometry? Trigonometry? Calculus 1? Calc 2? Etc.

19) What other comments would you like to provide that you believe would inform the topic of convergence of Taylor series and student understanding.

[NP]

**TRUE OR FALSE:** [For individual interviews or at the end of focus group time if time allows]

**Circle True or False or DN for Don't know for each problem below.**

- 20)  $\overline{9} = 1$ . (Note  $\overline{9}$  means .999999 with 9's continuing to repeat). True / False / DN
- 21) Every bounded increasing sequence is convergent. True / False / DN
- 22) Every bounded decreasing sequence is convergent. True / False / DN
- 23) A convergent series of continuous functions is continuous. True / False / DN
- 24) \*Every function can be represented as a Taylor series. That is, every function equals its Taylor series. True / False / DN

- 25) \*The function producing a given Taylor series is unique. That is, if  $f(x)$  produces Taylor series  $T(x)$ , then no other function  $g(x)$  can produce the same Taylor series  $T(x)$ . True / False / DN
- 26) \*The Taylor series for a given function is unique. That is,  $f(x)$  cannot have two different Taylor series representations. True / False / DN

- 27) If  $f(x) = T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$  on the interval  $[0,1]$ , then on  $(0,1)$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx} \left( \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \right) \text{ and } \int_0^1 f(x) dx = \int_0^1 \left( \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \right) dx.$$

True / False / DN

- 28) \*A series  $\sum_{n=1}^{\infty} a_n$  converges if  $\lim_{n \rightarrow \infty} a_n = 0$ . True / False / DN

- 29) \*If  $a_m + a_{m+1} + a_{m+2} + \dots \rightarrow 0$  as  $m \rightarrow \infty$ , then  $\sum_{n=1}^{\infty} a_n$  converges. True / False / DN

- 30) \*If  $a_m + a_{m+1} + a_{m+2} + \dots + a_{k-1} + a_k \rightarrow 0$  as  $m, k \rightarrow \infty$ , then  $\sum_{n=1}^{\infty} a_n$  converges.

True / False / DN

- 31) \*Since  $\lim_{m \rightarrow \infty} \frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \frac{x^{m+2}}{(m+2)!} + \dots = 0$  for all real numbers  $x$ ,  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  for all real

numbers  $x$ .

True / False / DN

- 32) \*When  $C$  is some constant number  $\sqrt{C^2 + \pi C + \sin \phi} \approx \sqrt{C^2 + \pi C + 1 - \frac{1}{6} \phi^3}$  when  $\phi$  is near

zero. (Note:  $\approx$  means approximately)

True / False / DN

- a) \*Follow up question: Why did you answer (Insert True or False or DN) on problem 27?

[NP]

**BACKGROUND INFORMATION:**

[Included in either the individual interview or focus group as needed]

**Please write down your background information.**

33) What is your title? (GTA, assoc. professor, asst. professor...)

a) If graduate student, how many analysis courses have you taken?

34) How long have you taught college courses that contain series?

35) Do you have any lesson plans from when you taught this topic in the past?

[NP]

**Closing**

Now that we are done, do you have any questions you'd like to ask me about this research project? [If you want to contact me later, here is my contact information (supply participant with a card).] Also, I may need to contact you later for additional questions or clarification. Can I also have your follow-up contact information?

Participant Email: \_\_\_\_\_

Participant Address: \_\_\_\_\_  
\_\_\_\_\_  
\_\_\_\_\_  
\_\_\_\_\_

[End of Handout]

## Expert Focus Group Handout

In an effort to save space, I have condensed the interview handout by eliminating space between tasks. Horizontal lines have been placed in this appendix to indicate page breaks in the original expert focus group handout. Therefore, tasks that occurred on the same page of the original handout are NOT separated by a horizontal line. Furthermore, tasks that occurred on the same page were originally spaced out on the page to give the participant ample room to write. In some cases font size has been reduced to allow the focus group handout to fit the margins of this dissertation. In addition, to enhance the readability of the main body of this dissertation, a one-to-one correspondence between expert and novice interview tasks was created by changing some expert focus group task numbers to match the task numbers as presented in the main body of this dissertation (e.g., Table 8).

---

### **DIRECTIONS:**

We will go over this handout one page at a time. At different times you may be asked to write on the handout to answer questions. You are also encouraged to think out loud.

**Please write down your first and last name.**

First and Last Name?

---

### **SHORT ANSWER:**

**Please reflect upon each question for about a minute. Write down any responses you may have. Then we'll discuss your responses.**

- 1) What are Taylor series?
  - 2) Why are Taylor series studied in calculus?
- 
- 3) What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$  when  $x$  is any real number?”
  - 4) What is meant by the “(-1,1)” in, “ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$  when  $x$  is in the interval (-1,1)?”
-



---

5) Find the exact sum of the series  $\sum_{n=0}^{\infty} \frac{5^n}{n!} = 1 + \frac{5}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \dots$ .

---

6) What is meant by the word “prove” if you were asked to, “Prove that sine is equal to its Taylor series.”

7) What are the steps in proving that  $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$ ?

---

8) How can we estimate sine by using its Taylor series?

---

9) What is meant by the “approximation” symbol in “ $\sin(x) \approx x - \frac{x^3}{3!}$  = a Taylor polynomial for sine when  $x$  is near 0?”

10) What is meant by the “near” in “ $\sin(x) \approx x - \frac{x^3}{3!}$  = a Taylor polynomial for sine when  $x$  is near 0?”

---

11) How can we get a better approximation for sine than using  $x - \frac{x^3}{3!}$ ?

---

12) What is the difference between the following questions?

a) What is the maximum error in using a given  $n^{\text{th}}$  degree Taylor polynomial to approximate  $f(x)$ ?

b) How large does  $n$  need to be to guarantee that the  $n^{\text{th}}$  degree Taylor polynomial is within 0.1 of  $f(x)$ ? Verses

---

Now we focus our attention from our personal understanding to student understanding.

13) What problems would you include in a lesson plan for **convergence** of Taylor series? Why would you include these problems?

14) A student asks you, “Why does sine equal its Taylor series?” How would you answer the student’s question? Why?

15) If there was one thing that students should take from Taylor series, what would that be?

16) What do you perceive to be the biggest problems that students have with Taylor series?

17) What mistakes do you commonly see students make when working with Taylor series? Why do you think they make these mistakes?

18) What other comments would you like to provide that you believe would inform the topic of convergence of Taylor series and student understanding.

---

---

**TRUE OR FALSE:****Circle True or False or DN for Don't Know for each problem below.**

- 19)  $\overline{.9} = 1$ . (Note  $\overline{.9}$  means .999999 with 9's continuing to repeat). True / False / DN  
20) Every bounded increasing sequence is convergent. True / False / DN  
21) Every bounded decreasing sequence is convergent. True / False / DN  
22) A convergent series of continuous functions is continuous. True / False / DN  
23) Every function can be represented as a Taylor series. That is, every function equals its Taylor series. True / False / DN

24) The function producing a given Taylor series is unique. That is, if  $f(x)$  produces Taylor series  $T(x)$ , then no other function  $g(x)$  can produce the same Taylor series  $T(x)$ . True / False / DN

25) The Taylor series for a given function is unique. That is,  $f(x)$  cannot have two different Taylor series representations. True / False / DN

26) If  $f(x) = T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$  on the interval  $[0,1]$ , then on  $(0,1)$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx} \left( \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \right) \text{ and } \int_0^1 f(x) dx = \int_0^1 \left( \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \right) dx.$$

True / False / DN

27) A series  $\sum_{n=1}^{\infty} a_n$  converges if  $\lim_{n \rightarrow \infty} a_n = 0$ . True / False / DN

28) If  $a_m + a_{m+1} + a_{m+2} + \dots \rightarrow 0$  as  $m \rightarrow \infty$ , then  $\sum_{n=1}^{\infty} a_n$  converges. True / False / DN

29) If  $a_m + a_{m+1} + a_{m+2} + \dots + a_{k-1} + a_k \rightarrow 0$  as  $m, k \rightarrow \infty$ , then  $\sum_{n=1}^{\infty} a_n$  converges.

True / False / DN

---

30) Since  $\lim_{m \rightarrow \infty} \frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \frac{x^{m+2}}{(m+2)!} + \dots = 0$  for all real numbers  $x$ ,  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  for all real

numbers  $x$ .

True / False / DN

31) When  $C$  is some constant number  $\sqrt{C^2 + \pi C + \sin \phi} \approx \sqrt{C^2 + \pi C + 1 - \frac{1}{6} \phi^3}$  when  $\phi$  is near

zero. (Note:  $\approx$  means approximately)True / False / DN

---

---

**BACKGROUND INFORMATION:**

**Please write down your background information.**

32) What is your title? (GTA, assoc. professor, asst. professor...)

a) If graduate student, how many analysis courses have you taken?

33) How long have you taught college courses that contain series?

34) Do you have any lesson plans from when you taught this topic in the past?

**Closing**

Now that we are done, do you have any questions you'd like to ask me about this research project? I may need to contact you later for additional questions or clarification.

Can I also have your follow-up contact information?

Participant Email: \_\_\_\_\_

Participant Address: \_\_\_\_\_

\_\_\_\_\_

\_\_\_\_\_

\_\_\_\_\_

---

## Expert Interview Protocol

Brackets indicate the layout of the handout and notes for me. In some cases font size has been reduced to allow the protocol to fit the margins of this dissertation.

Furthermore, to enhance the readability of the main body of this dissertation, a one-to-one correspondence between expert and novice interview tasks was created by changing some expert interview task numbers to match the task numbers as presented in the main body of this dissertation (e.g., Table 8). When a computer image was used, some screen shots were added to aid one in understanding what participants were seeing.

### **INTRODUCTION**

Thank you for your time and willingness to participate. As you know, I am interested in how people understand Taylor series. Particularly, I am trying to understand how people understand the convergence of Taylor series. Feel free to volunteer any detail you wish when answering a question. You also have the option of declining to answer – passing – on any of the questions. You will be given a handout with tasks to complete during this interview. Do you have any questions before we start?

[“\*” Indicate questions that I really like!]

[Handout begins]

### **DIRECTIONS:**

We will go over this handout one page at a time. At different times you may be asked to write on the handout to answer questions. You are also encouraged to think out loud.

**Please write down your first and last name.**

First and Last Name?

[NP = New Page of Handout. I’ve condensed the handout for the IRB approval process.]

### **SHORT ANSWER:**

**Please reflect upon each question for about a minute. Write down any responses you may have. Then we’ll discuss your responses.**

1) \*What are Taylor series?

[Expect a wide range of images.]

2) \*Why are Taylor series studied in calculus?

[NP]

- 3) \*What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$  when  $x$  is any real number?”

[Expect approximation and proximity images to come out if present. May even get a formal definition of convergence.]

- a) Follow-Up Question: If they bring up convergence, have them elaborate on what it means to converge.

- 4) \*What is meant by the “(-1,1)” in, “ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$  when  $x$  is in the interval (-1,1)?”

[Expect either images of domain to come out or more approximation images.]

- a) Follow-Up Question: What happens outside the interval (-1,1)?

[NP]

- 5) \*Find the exact sum of the series  $\sum_{n=0}^{\infty} \frac{5^n}{n!} = 1 + \frac{5}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \dots$ .

- a) Follow-Up Question: What about this problem causes difficulty?

[It may be necessary to tell them that it is the series for  $e^x$ .]

[NP]

- 6) \*What is meant by the word “prove” if you were asked to, “Prove that sine is equal to its Taylor series.”

[Wanting to see remainder come up as well as other approximation images.]

- 7) \*What are the steps in proving that  $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$ ?

[Wanting to see remainder come up as well as other approximation images.]

[NP]

- 8) How can we estimate sine by using its Taylor series?

- a) Follow up question: What if we wanted to estimate sine at 103? How might we use a Taylor series to do that? If confused about the measure, follow up with 103 radians.

[Expect approximation images. Looking to see if center of series images are also present.]

[NP]

9) \*What is meant by the “approximation” symbol in “ $\sin(x) \approx x - \frac{x^3}{3!}$  = a Taylor polynomial for sine when  $x$  is near 0?”

10) \*What is meant by the “near” in “ $\sin(x) \approx x - \frac{x^3}{3!}$  = a Taylor polynomial for sine when  $x$  is near 0?”

a) Follow-Up Question: What happens when we are NOT near zero?

[NP]

11) \*How can we get a better approximation for sine than using  $x - \frac{x^3}{3!}$ ?

[Expected Answer: Add more terms]

a) Follow-Up Question: Can you think of anything else you might want to consider when answering this question?

[NP]

12) What is the difference between the following questions.

a) \*What is the maximum error in using a given  $n^{\text{th}}$  degree Taylor polynomial to approximate  $f(x)$ ? Verses

b) \*How large does  $n$  need to be to guarantee that the  $n^{\text{th}}$  degree Taylor polynomial is within 0.1 of  $f(x)$ ?

i) Follow-Up Question: Which of these questions is more difficult to answer?

ii) Follow-Up Question: Which of these questions is more important?

[NP]

Now we focus our attention from our personal understanding to student understanding.

- 13) What problems would you include in a lesson plan for **convergence** of Taylor series?  
Why would you include these problems?
- 14) A student asks you, “Why does sine equal its Taylor series?” How would you answer the student’s question? Why?
- 15) If there was one thing that students should take from Taylor series, what would that be?
- 16) What do you perceive to be the biggest problems that students have with Taylor series?
  - a) Follow-up: Why do they have these problems.
- 17) What mistakes do you commonly see students make when working with Taylor series? Why do you think they make these mistakes?
  - a) Follow-Up Question: How would you characterize these mistakes? Basic math? Algebra? Geometry? Trigonometry? Calculus 1? Calc 2? Etc.
  - b) Follow-Up Question: Do you think they are understanding the problems that they are getting correct?
- 18) What other comments would you like to provide that you believe would inform the topic of convergence of Taylor series and student understanding.

[NP]

**TRUE OR FALSE:**

**Circle True or False or DN for Don't Know for each problem below.**

- 19)  $\overline{.9} = 1$ . (Note  $\overline{.9}$  means .999999 with 9's continuing to repeat). True / False / DN  
20) Every bounded increasing sequence is convergent. True / False / DN  
21) Every bounded decreasing sequence is convergent. True / False / DN  
22) A convergent series of continuous functions is continuous. True / False / DN  
23) \*Every function can be represented as a Taylor series. That is, every function equals its Taylor series. True / False / DN

- 24) \*The function producing a given Taylor series is unique. That is, if  $f(x)$  produces Taylor series  $T(x)$ , then no other function  $g(x)$  can produce the same Taylor series  $T(x)$ . True / False / DN  
25) \*The Taylor series for a given function is unique. That is,  $f(x)$  cannot have two different Taylor series representations. True / False / DN

- 26) If  $f(x) = T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$  on the interval  $[0,1]$ , then on  $(0,1)$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx} \left( \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \right) \text{ and } \int_0^1 f(x) dx = \int_0^1 \left( \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \right) dx .$$

True / False / DN

- 27) \*A series  $\sum_{n=1}^{\infty} a_n$  converges if  $\lim_{n \rightarrow \infty} a_n = 0$ . True / False / DN

- 28) \*If  $a_m + a_{m+1} + a_{m+2} + \dots \rightarrow 0$  as  $m \rightarrow \infty$ , then  $\sum_{n=1}^{\infty} a_n$  converges. True / False / DN

- 29) \*If  $a_m + a_{m+1} + a_{m+2} + \dots + a_{k-1} + a_k \rightarrow 0$  as  $m, k \rightarrow \infty$ , then  $\sum_{n=1}^{\infty} a_n$  converges. True / False / DN

- 30) \*Since  $\lim_{m \rightarrow \infty} \left( \frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \frac{x^{m+2}}{(m+2)!} + \dots \right) = 0$  for all real numbers  $x$ ,  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  for all real numbers  $x$ . True / False / DN

- 31) \*When  $C$  is some constant number  $\sqrt{C^2 + \pi C + \sin \phi} \approx \sqrt{C^2 + \pi C + 1 - \frac{1}{6} \phi^3}$  when  $\phi$  is near zero. (Note:  $\approx$  means approximately) True / False / DN

a) \*Follow up question: Why did you answer (Insert True or False or DN) on problem 27?

b) Note: This answer is false because  $\sin \phi$  is NOT approximated by  $1 - \frac{1}{6} \phi^3$  !!! The question should be changed to a true question when asking students for answers.



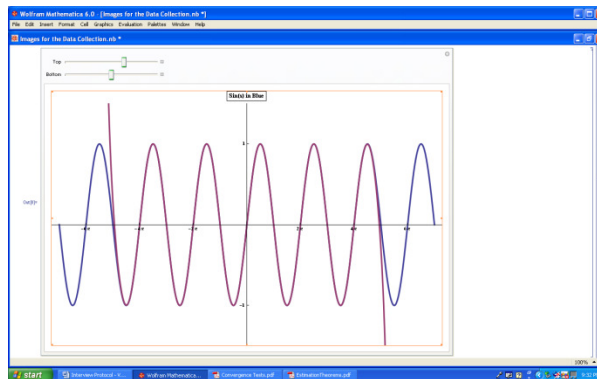
[NP] [Screen shots of images students will be manipulating is included in this copy for IRB approval – Screen shots will not be included in the task handout]

**SHORT ANSWER:**

**Please reflect upon each question for about a minute. Write down any responses you may have. Then we'll discuss your responses.**

32) \*Now I am going to show you an image.

- a) What are you seeing when I move the Top slider?
- b) What are you seeing when I move the Bottom slider?

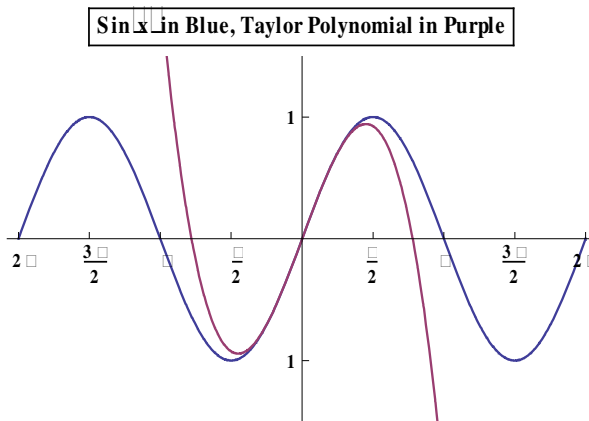


\*The participant may move the slider themselves if they want to.

[NP]

33) \*For part a) and part b), consider the graph below?

- a) Estimate the error in using the Taylor polynomial to approximate  $\sin\left(\frac{\pi}{4}\right)$ .
  - i) If confused about “error” follow up with, “error is the difference between using the Taylor polynomial and using actual value for  $\sin\left(\frac{\pi}{4}\right)$ .”
- b) For what values of  $x$  is the Taylor polynomial within 0.1 for  $\sin x$ .



[NP]

Attempt this problem without referring to any calculus book or notes.

34) \*What is the maximum error in using  $1 + \frac{x}{1!}$  to approximate  $e^x$  on (0,2)? Justify your steps?

a) Follow-Up Question: What about this task is giving you difficulty?"

If they eventually say the interval (0,2). Change the question to (-2,2).

They can also just give the general steps.

[NP]

Attempt these problems without referring to any calculus book or notes.

35) \*How large does  $n$  need to be to guarantee that the  $n^{\text{th}}$  degree Taylor polynomial

$1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$  is within 0.001 of  $e^x$  when  $x$  is in the interval (0,2)? Justify your steps?

a) Follow-Up Question: What about this task is giving you difficulty?

36) \*Is there a way to decrease the  $n$  you got on problem 36?

[Another question designed to get at their images of center of series. This may be one question too many]

a) Follow-Up Question: What should you take into account when attempting to decrease the  $n$ ?

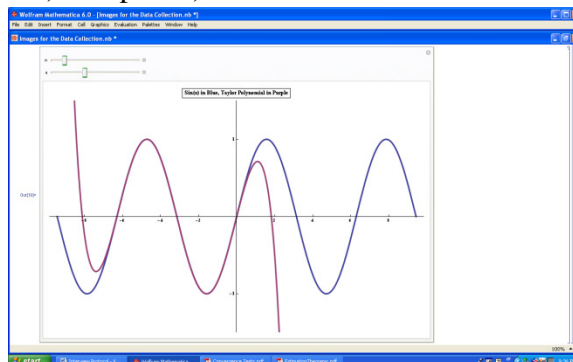
b) Follow-Up Question: How does problem 36 change if I ask you to prove that  $e^x$

**equals**  $1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$ .

[NP]

37) \*Use the image with sliders to answer part a) and part b). Please move the sliders to help get your answer.

a) How large does  $n$  have to be to get the Taylor polynomial function within 0.1 of  $\sin x$  when  $x$  is in the interval (-4,4)? What is the  $n$  doing to the Taylor polynomial formula?



b) Take the  $n$  value that you got from part a. What happens to the values of  $x$  that guarantee that the Taylor polynomial function is within 0.1 of  $\sin x$  when you move the "a" slider? What is the happening to the Taylor polynomial formula?

[NP]

38) What are Taylor series and why are they studied in Calculus?

- a) Follow Up Question: What do you think students take away from having studied Taylor series in Calculus III?
- b) Follow Up Question: You've seen some dynamic graphs during this interview process today, do you ever use anything similar in your presentation of Taylor series to your class? What do you do with your graphs?
  - i) Follow Up Question: Do you ever produce images that show them how the Taylor series moves when you change the center of the series?

[NP]

39) What other comments would you like to provide that you believe would inform the topic of convergence of Taylor series and student understanding.

[NP]

**BACKGROUND INFORMATION:**

**Please write down your background information.**

40) What is your title? (GTA, assoc. professor, asst. professor...)

- a) If graduate student, how many analysis courses have you taken?

41) How long have you taught college courses that contain series?

[NP]

**Closing**

Now that we are done, do you have any questions you'd like to ask me about this research project? [If you want to contact me later, here is my contact information (supply participant with a card).] Also, I may need to contact you later for additional questions or clarification. Can I also have your follow-up contact information?

Participant Email: \_\_\_\_\_

Participant Address: \_\_\_\_\_  
\_\_\_\_\_  
\_\_\_\_\_  
\_\_\_\_\_

[End of Handout]

## Expert Interview Handout

In an effort to save space, I have condensed the interview handout by eliminating space between tasks. Horizontal lines have been placed in this appendix to indicate page breaks in the original expert interview handout. Therefore, tasks that occurred on the same page of the original handout are NOT separated by a horizontal line. Furthermore, tasks that occurred on the same page were originally spaced out on the page to give the participant ample room to write. In some cases, font size has been reduced to allow the handout to fit the margins of this dissertation. In addition, to enhance the readability of the main body of this dissertation, a one-to-one correspondence between expert and novice interview tasks was created by changing some expert interview task numbers to match the task numbers as presented in the main body of this dissertation (e.g., Table 8).

---

### **DIRECTIONS:**

We will go over this handout one page at a time. At different times you may be asked to write on the handout to answer questions. You are also encouraged to think out loud.

**Please write down your first and last name.**

First and Last Name?

---

### **SHORT ANSWER:**

**Please reflect upon each question for about a minute. Write down any responses you may have. Then we'll discuss your responses.**

1) What are Taylor series?

2) Why are Taylor series studied in calculus?

---

3) What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$  when  $x$  is any real number?”

4) What is meant by the “(-1,1)” in, “ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$  when  $x$  is in the interval (-1,1)?”

---

5) Find the exact sum of the series  $\sum_{n=0}^{\infty} \frac{5^n}{n!} = 1 + \frac{5}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \dots$ .

---

---

6) What is meant by the word “prove” if you were asked to, “Prove that sine is equal to its Taylor series.”

7) What are the steps in proving that  $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$ ?

---

8) How can we estimate sine by using its Taylor series?

---

9) What is meant by the “approximation” symbol in “ $\sin(x) \approx x - \frac{x^3}{3!}$  = a Taylor polynomial for sine when  $x$  is near 0?”

10) What is meant by the “near” in “ $\sin(x) \approx x - \frac{x^3}{3!}$  = a Taylor polynomial for sine when  $x$  is near 0?”

---

11) How can we get a better approximation for sine than using  $x - \frac{x^3}{3!}$ ?

---

12) What is the difference between the following questions?

a) What is the maximum error in using a given  $n^{\text{th}}$  degree Taylor polynomial to approximate  $f(x)$ ?

verses

b) How large does  $n$  need to be to guarantee that the  $n^{\text{th}}$  degree Taylor polynomial is within 0.1 of  $f(x)$ ?

---

Now we focus our attention from our personal understanding to student understanding.

13) What problems would you include in a lesson plan for **convergence** of Taylor series? Why would you include these problems?

14) A student asks you, “Why does sine equal its Taylor series?” How would you answer the student’s question? Why?

15) If there was one thing that students should take from Taylor series, what would that be?

16) What do you perceive to be the biggest problems that students have with Taylor series?

17) What mistakes do you commonly see students make when working with Taylor series? Why do you think they make these mistakes?

18) What other comments would you like to provide that you believe would inform the topic of convergence of Taylor series and student understanding.

---

---

**TRUE OR FALSE:****Circle True or False or DN for Don't Know for each problem below.**

- 19)  $\overline{.9} = 1$ . (Note  $\overline{.9}$  means .999999 with 9's continuing to repeat). True / False / DN  
20) Every bounded increasing sequence is convergent. True / False / DN  
21) Every bounded decreasing sequence is convergent. True / False / DN  
22) A convergent series of continuous functions is continuous. True / False / DN  
23) Every function can be represented as a Taylor series. That is, every function equals its Taylor series. True / False / DN

- 24) The function producing a given Taylor series is unique. That is, if  $f(x)$  produces Taylor series  $T(x)$ , then no other function  $g(x)$  can produce the same Taylor series  $T(x)$ . True / False / DN  
25) The Taylor series for a given function is unique. That is,  $f(x)$  cannot have two different Taylor series representations. True / False / DN

- 26) If  $f(x) = T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$  on the interval  $[0,1]$ , then on  $(0,1)$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx} \left( \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \right) \text{ and } \int_0^1 f(x) dx = \int_0^1 \left( \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \right) dx .$$

True / False / DN

- 27) A series  $\sum_{n=1}^{\infty} a_n$  converges if  $\lim_{n \rightarrow \infty} a_n = 0$ . True / False / DN

- 28) If  $a_m + a_{m+1} + a_{m+2} + \dots \rightarrow 0$  as  $m \rightarrow \infty$ , then  $\sum_{n=1}^{\infty} a_n$  converges. True / False / DN

- 29) If  $a_m + a_{m+1} + a_{m+2} + \dots + a_{k-1} + a_k \rightarrow 0$  as  $m, k \rightarrow \infty$ , then  $\sum_{n=1}^{\infty} a_n$  converges. True / False / DN

- 30) Since  $\lim_{m \rightarrow \infty} \left( \frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \frac{x^{m+2}}{(m+2)!} + \dots \right) = 0$  for all real numbers  $x$ ,  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  for all real numbers  $x$ . True / False / DN

- 31) When  $C$  is some constant number  $\sqrt{C^2 + \pi C + \sin \phi} \approx \sqrt{C^2 + \pi C + 1 - \frac{1}{6} \phi^3}$  when  $\phi$  is near zero. (Note:  $\approx$  means approximately) True / False / DN

---

**SHORT ANSWER:**

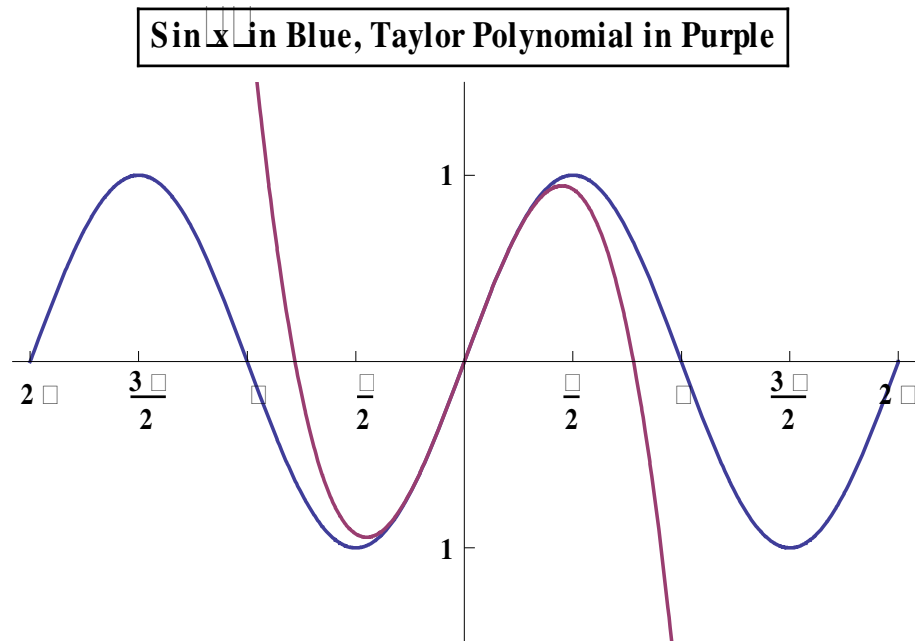
**Please reflect upon each question for about a minute. Write down any responses you may have. Then we'll discuss your responses.**

- 32) Now I am going to show you an image.  
a) What are you seeing when I move the Top slider?  
b) What are you seeing when I move the Bottom slider?
-

---

33) For part a) and part b), consider the graph below?

- a) Estimate the error in using the Taylor polynomial to approximate  $\sin\left(\frac{\pi}{4}\right)$ .
- b) For what values of  $x$  is the Taylor polynomial within 0.1 for  $\sin x$ .



---

Attempt this problem without referring to any calculus book or notes.

- 34) What is the maximum error in using  $1 + \frac{x}{1!}$  to approximate  $e^x$  on  $(0,2)$ ? Justify your steps?
-

Attempt these problems without referring to any calculus book or notes.

35) How large does  $n$  need to be to guarantee that the  $n^{\text{th}}$  degree Taylor polynomial

$1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$  is within 0.001 of  $e^x$  when  $x$  is in the interval  $(0,2)$ ? Justify your steps?

36) Is there a way to decrease the  $n$  you got on problem 35?

---



---

37) Use the image with sliders to answer part a) and part b). Please move the sliders to help get your answer.

- a) How large does  $n$  have to be to get the Taylor polynomial function within 0.1 of  $\sin x$  when  $x$  is in the interval  $(-4,4)$ ? What is the  $n$  doing to the Taylor polynomial formula?
- b) Take the  $n$  value that you got from part a. What happens to the values of  $x$  that guarantee that the Taylor polynomial function is within 0.1 of  $\sin x$  when you move the “a” slider? What is the happening to the Taylor polynomial formula?

---

38) What are Taylor series and why are they studied in Calculus?

---

39) What other comments would you like to provide that you believe would inform the topic of convergence of Taylor series and student understanding

---

**BACKGROUND INFORMATION:**

**Please write down your background information.**

40) What is your title? (GTA, assoc. professor, asst. professor...)

- a) If graduate student, how many analysis courses have you taken?

41) How long have you taught college courses that contain series?

---

**Closing**

Now that we are done, do you have any questions you’d like to ask me about this research project? Also, I may need to contact you later for additional questions or clarification. Can I also have your follow-up contact information?

Participant Email: \_\_\_\_\_

Participant Address: \_\_\_\_\_

\_\_\_\_\_

\_\_\_\_\_

\_\_\_\_\_

## APPENDIX C

### Data Collection Instruments for the Novice Participant Group

Questionnaire .....	344
Novice Interview Protocol .....	369
Novice Interview Handout.....	374



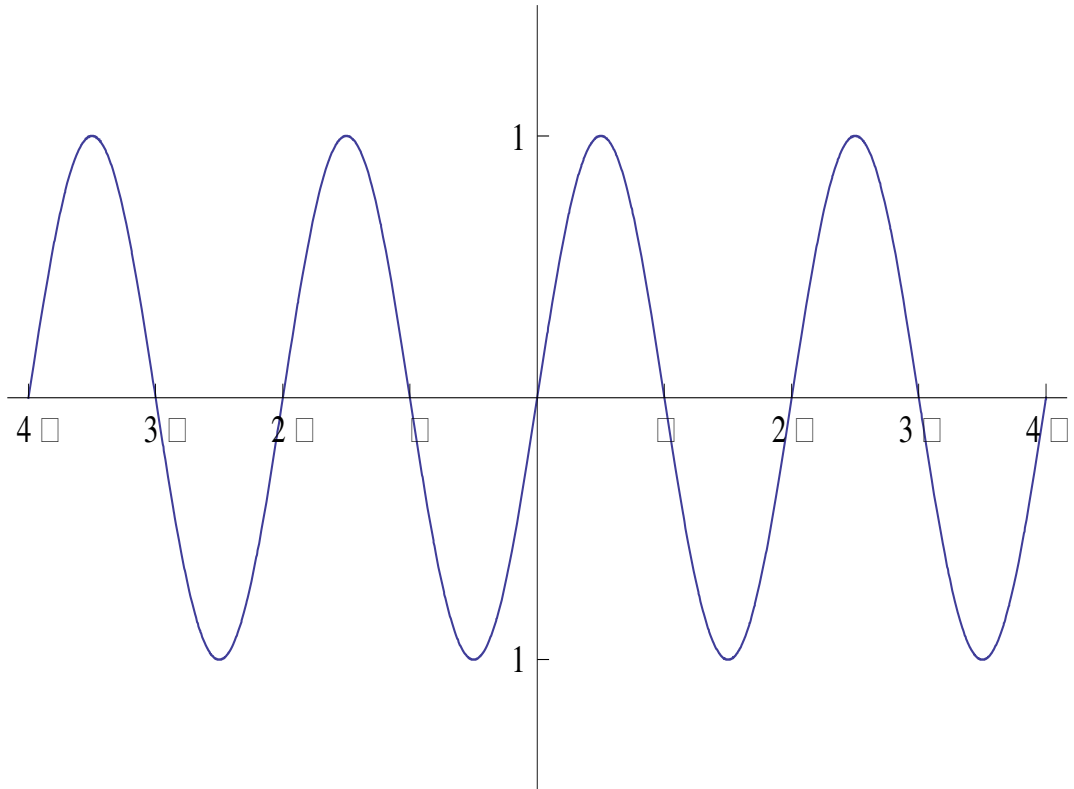
**SECTION 3: SHORT ANSWER**

---

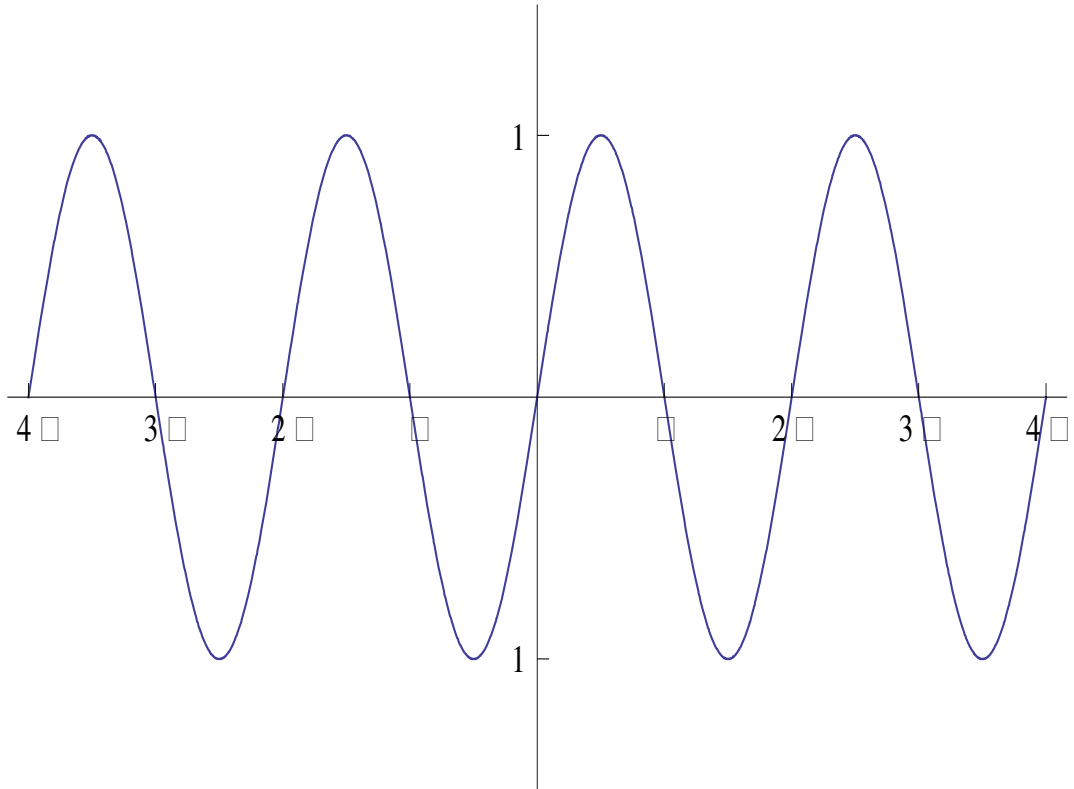
Legibly write down your answers to the following questions. Please show all your work. Feel free to write down anything that comes to your mind while attempting each task.

4) Find the exact sum of the series  $\sum_{n=0}^{\infty} \frac{5^n}{n!} = 1 + \frac{5}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \dots$ .

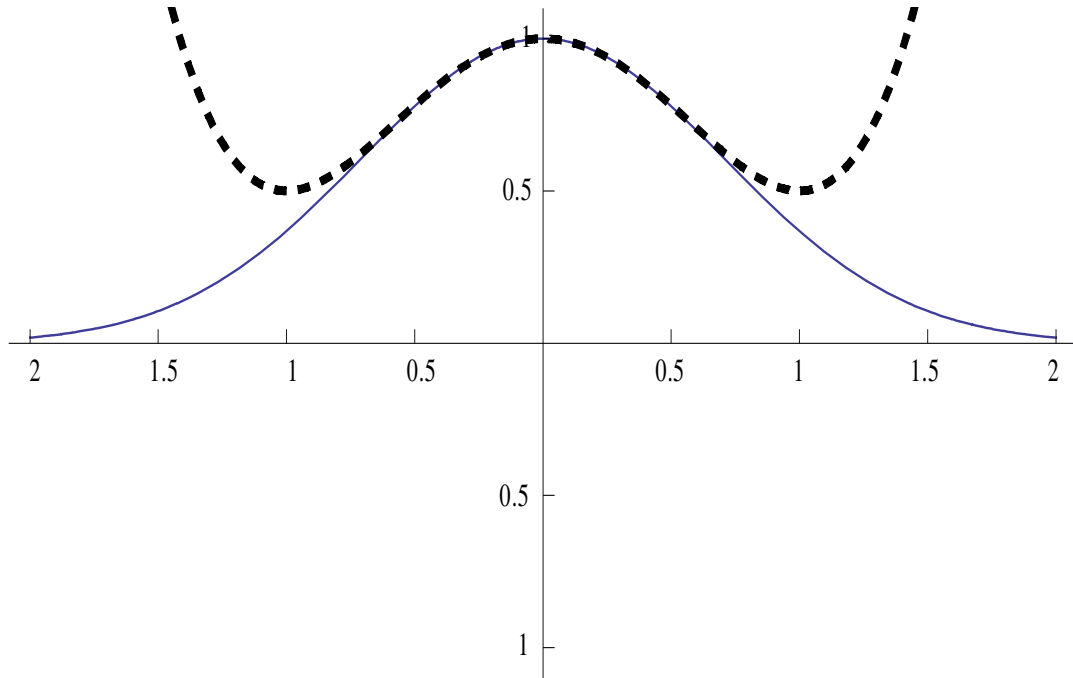
5) Using the graph of  $\sin x$  below, on the same axes sketch two different Taylor polynomials for sine.



6) Using the graph of  $\sin x$  below, on the same axes sketch the Taylor series for sine.



- 7) Use the graph to approximate the error in using the dashed curve to estimate the value of the solid curve at  $x = 1$ .



Error is approximately \_\_\_\_\_

8) What is the error in using  $y = \frac{1}{2} - \frac{1}{4}x$  to estimate  $y = \frac{1}{x+2}$  at  $x = 1$ ?

9) It can be shown that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \text{ for all numbers } x \text{ and that}$$

$$e^x = \sum_{n=0}^{\infty} e \frac{(x-1)^n}{n!} = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \cdots \text{ for all numbers } x$$

In a few sentences legibly answer each question below.

a) Describe the difference, if any, that exists between  $e^x$  and  $e + e(x-1) + \frac{e}{2}(x-1)^2$ . State that no difference exists if you believe that there is no difference.

b) Describe the difference, if any, that exists between  $1 + x + \frac{1}{2}x^2$  and  $e + e(x-1) + \frac{e}{2}(x-1)^2$ . State that no difference exists if you believe that there is no difference.

c) Describe the difference, if any, that exists between  $e^x$  and the infinite series  $\sum_{n=0}^{\infty} e \frac{(x-1)^n}{n!}$ . State that no difference exists if you believe that there is no difference.

d) Describe the difference, if any, that exists between the infinite series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  and the infinite series  $\sum_{n=0}^{\infty} e \frac{(x-1)^n}{n!}$ . State that no difference exists if you believe that there is no difference.



10) When attempting to answer a problem about  $\sum_{n=0}^{\infty} \frac{x^n}{2^n} = 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \dots$ , a student correctly writes the following:

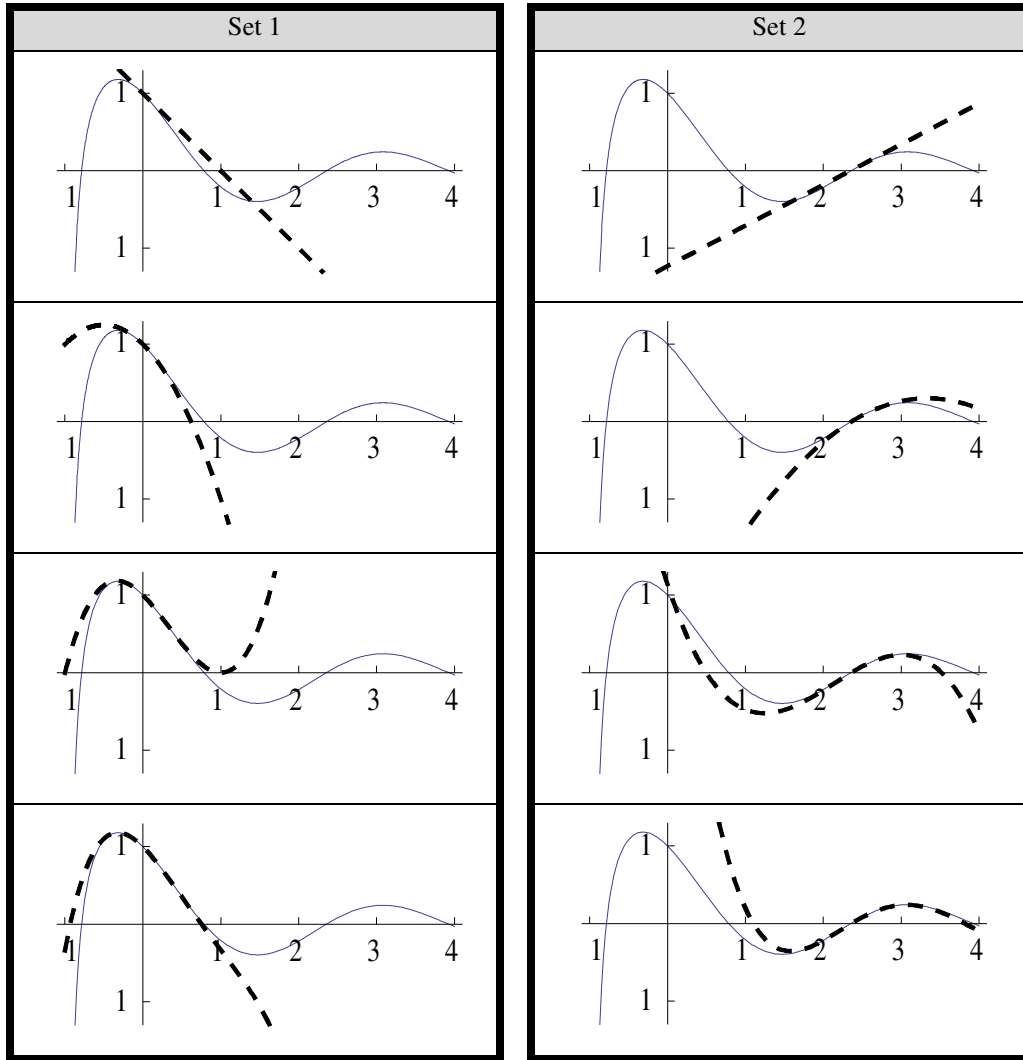
*The Root Test states that if  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = a \text{ number} < 1$  then the series  $\sum_{n=0}^{\infty} a_n$  converges, if  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$  then the series diverges, and if  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$  the Root Test is inconclusive.*

The student then correctly notes the following:

*Note that  $\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{x^n}{2^n} \right|} = \frac{|x|}{2}$  and note that  $\frac{|x|}{2} < 1$  implies that  $|x| < 2$ .*

In a few sentences legibly describe what you can or cannot conclude based off of what the student has written.

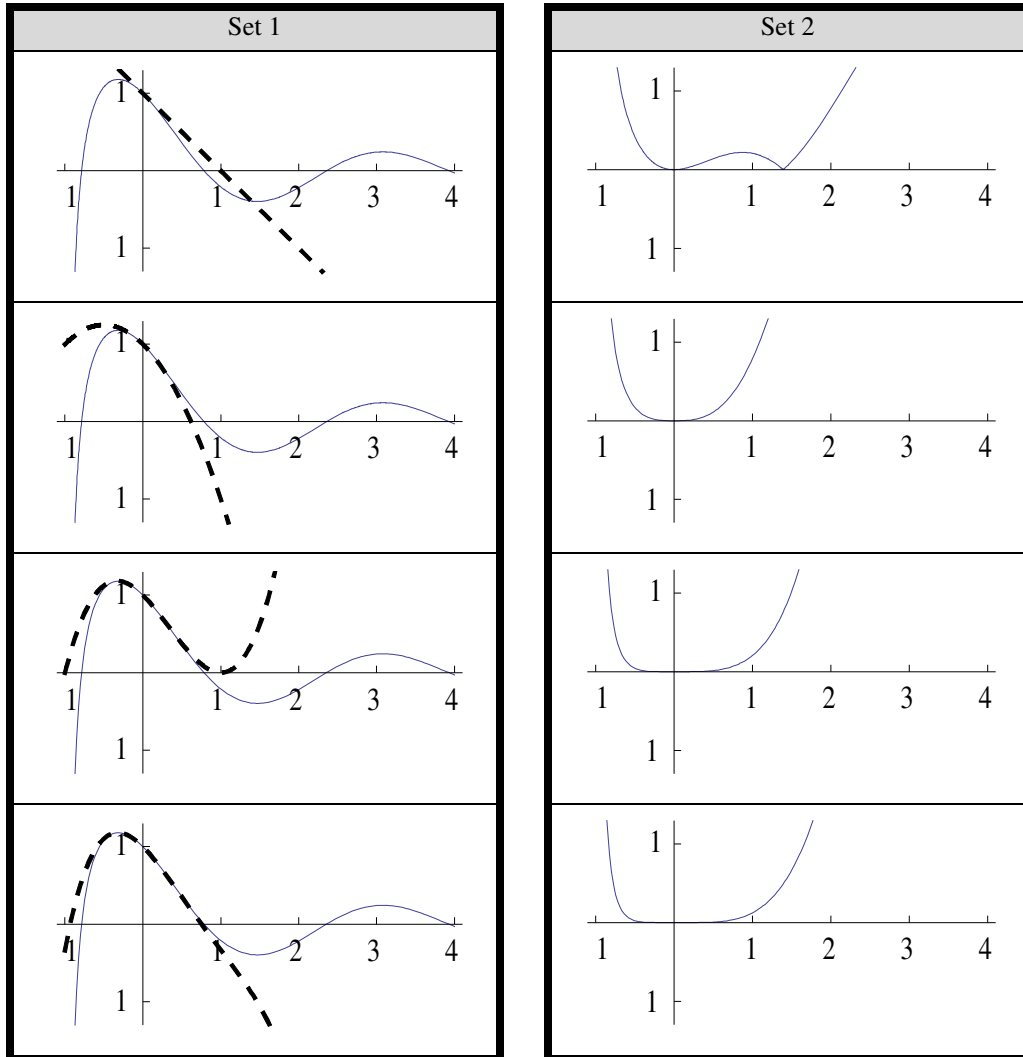
11) Consider the following two sets of graphs:



Write a few sentences to answer the following questions:

- What is the difference between Set 1 and Set 2?
- What are Set 1 and Set 2 illustrating?
- How are Set 1 and Set 2 related to Taylor Series if at all?

12) Consider the following two sets of graphs:



Write a few sentences to answer the following questions:

- What is the difference between Set 1 and Set 2?
- What are Set 1 and Set 2 illustrating?
- How are Set 1 and Set 2 related to Taylor Series if at all?

13) A student correctly writes the following:

*Taylor's Inequality states that if there exists a number  $M$  such that  $|f^{(n+1)}(x)| \leq M$  for all  $x$  in an interval, then the remainder function  $R_n(x)$  of the Taylor series for  $f(x)$  at  $x = a$  satisfies the inequality  $R_n(x) \leq \frac{M}{(n+1)!} |x-a|^{n+1}$  for all  $x$  in the interval.*

The student also correctly notes the following:

*The Taylor series for  $e^x$  is given by  $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$  and*

$$\lim_{m \rightarrow \infty} \left( \frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \frac{x^{m+2}}{(m+2)!} + \dots \right) = 0 \text{ for all numbers } x.$$

The student now relates Taylor's Inequality to the function  $e^x$  and correctly writes:

*Since  $e^x \leq 8$  for all  $x$  in the interval  $(-2,2)$ , the remainder function  $R_n(x)$  of the Taylor series for  $e^x$  at  $x = 0$  satisfies the inequality  $R_n(x) \leq \frac{8}{(n+1)!} |x|^{n+1}$  for all  $x$  in  $(-2,2)$ .*

The student then correctly notes the following:

$$\text{The } \lim_{n \rightarrow \infty} \frac{8}{(n+1)!} |x|^{n+1} = 0 \text{ for all numbers } x.$$

Based off of everything this student has written what can you conclude about...

a)  $M$  \_\_\_\_\_

\_\_\_\_\_

b)  $e^x$  \_\_\_\_\_

\_\_\_\_\_

c)  $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$  \_\_\_\_\_

\_\_\_\_\_

d)  $R_n(x)$  \_\_\_\_\_

\_\_\_\_\_

14) Write a few sentences to answer the following question.

What is meant by the “=” in, “ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$  on the interval  $(-1,1)$ ?”

15) A student correctly says the following:

*The series  $1 + x + x^2 + x^3 + \dots$  converges to  $\frac{1}{1-x}$  on the interval  $(-1,1)$ .*

Write a few sentences to answer the following questions:

a) What does the student mean when they say “converge?”

b) What happens to  $\frac{1}{1-x}$  and  $1 + x + x^2 + x^3 + \dots$  inside the interval  $(-1,1)$ ?

c) What happens to  $\frac{1}{1-x}$  and  $1 + x + x^2 + x^3 + \dots$  outside the interval  $(-1,1)$ ?

**SECTION 4: MULTIPLE CHOICE**

---

Please completely read each question and **ALL** responses before circling your answers. Then circle **ALL** responses that apply to each question. Feel free to provide any other responses that may have come to your mind. The real number system applies to all problems.

16) A student correctly writes the following:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \text{ on the interval } (-1,1)."$$

The student also correctly writes the following:

We can define  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  by  $T_0(x) = 1$ ,  $T_1(x) = 1 + x$ ,

$T_2(x) = 1 + x + x^2$ , ...,  $T_n(x) = 1 + x + x^2 + \dots + x^n$ , etc.

Please read **ALL** responses on the next page before circling.

Circle **ALL** responses that you can correctly conclude based off of what the student has written.

- a) The domain of  $\frac{1}{1-x}$  is  $(-1,1)$ .
- b) The domain of  $1 + x + x^2 + x^3 + \dots$  is  $(-1,1)$ .
- c)  $1 + x + x^2$  estimates  $\frac{1}{1-x}$  on the interval  $(-1,1)$ .
- d) The series  $1 + x + x^2 + x^3 + \dots$  gets closer and closer to  $\frac{1}{1-x}$  on the interval  $(-1,1)$  until  $\frac{1}{1-x}$  is reached.
- e) The series  $1 + x + x^2 + x^3 + \dots$  gets close to but never reaches  $\frac{1}{1-x}$  on the interval  $(-1,1)$ .
- f) The series  $1 + x + x^2 + x^3 + \dots$  is an approximation to  $\frac{1}{1-x}$  and can be made as accurate as you wish on the interval  $(-1,1)$ .
- g) The series  $1 + x + x^2 + x^3 + \dots$  is identical to  $\frac{1}{1-x}$  on the interval  $(-1,1)$ .
- h)  $1 + x + x^2 + x^3 + \dots + x^n$  converges on the interval  $(-1,1)$  as  $n$  goes to infinity.
- i)  $1 + x + x^2 + x^3 + \dots + x^n$  converges to  $\frac{1}{1-x}$  on the interval  $(-1,1)$  as  $n$  goes to infinity.
- j)  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  converges on the interval  $(-1,1)$  as  $n$  goes to infinity.
- k)  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  converges to  $\frac{1}{1-x}$  on the interval  $(-1,1)$  as  $n$  goes to infinity.
- l)  $1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots$  converges.
- m) I don't know what to circle.
- n) I cannot conclude anything.
- o) Something else came to my mind. \_\_\_\_\_



17) When attempting to answer a problem about  $\sum_{n=0}^{\infty} \frac{x^n}{2^n} = 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \dots$ , a student

correctly writes the following:

*This series converges on the interval  $(-2,2)$  and diverges otherwise.*

The student also correctly writes the following:

*We can define  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  by*

$$T_0(x) = 1,$$

$$T_1(x) = 1 + \frac{1}{2}x,$$

$$T_2(x) = 1 + \frac{1}{2}x + \frac{1}{4}x^2,$$

$\vdots$

$$T_n(x) = 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \dots + \frac{1}{2^n}x^n,$$

*etc.*

Please read ALL responses on the next page before circling.

Circle ALL responses that you can correctly conclude based off of what the student has written.

- a) The domain of  $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$  is  $(-2,2)$ .
- b)  $1 + \frac{1}{2}x + \frac{1}{4}x^2$  estimates  $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$  on the interval  $(-2,2)$ .
- c)  $1 + \frac{1}{2}x + \frac{1}{4}x^2 + \cdots + \frac{1}{2^n}x^n$  converges on the interval  $(-2,2)$  as  $n$  goes to infinity.
- d)  $1 + \frac{1}{2}x + \frac{1}{4}x^2 + \cdots + \frac{1}{2^n}x^n$  converges to  $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$  on the interval  $(-2,2)$  as  $n$  goes to infinity.
- e) There is no graph for the infinite series  $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$  on the interval  $(-2,2)$ .
- f) The series  $\sum_{n=0}^{\infty} \frac{(3)^n}{2^n}$  converges.
- g)  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  is a sequence of numbers.
- h)  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  is a sequence of functions.
- i)  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  converges on the interval  $(-2,2)$  as  $n$  goes to infinity.
- j)  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  converges to  $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$  on the interval  $(-2,2)$  as  $n$  goes to infinity.
- k)  $\lim_{n \rightarrow \infty} T_n(x)$  exists on the interval  $(-2,2)$ .
- l)  $\lim_{n \rightarrow \infty} T_n(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n}$  on the interval  $(-2,2)$ .
- m)  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  helps to determine if the series  $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$  converges.
- n) The student can define a function  $f$  with domain  $(-2,2)$  by defining  $f$  to be given by the formula
- $$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n}.$$
- o) The series  $\sum_{n=0}^{\infty} \frac{(x-7)^n}{2^n}$  converges on the interval  $(5,9)$ .
- p) I don't know what to circle.
- q) I cannot conclude anything.
- r) Something else came to my mind. \_\_\_\_\_

18) When attempting to answer a problem about  $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$ , a

student correctly writes the following:

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \text{ converges for all numbers } x.$$

By replacing  $x$  with  $x+4$  we get the series  $\sum_{n=0}^{\infty} \frac{(x+4)^{2n+1}}{(2n+1)!}$ .

The student also correctly writes the following:

We can define  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  by

$$T_0(x) = x,$$

$$T_1(x) = x + \frac{x^3}{3!},$$

$$T_2(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!},$$

$\vdots$

$$T_n(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!},$$

*etc.*

Please read ALL responses on the next page before circling.

Circle ALL responses that you can correctly conclude based off of what the student has written.

- a) The series  $\sum_{n=0}^{\infty} \frac{(x+4)^{2n+1}}{(2n+1)!}$  converges for all numbers  $x$ .
- b) The series  $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$  and  $\sum_{n=0}^{\infty} \frac{(x+4)^{2n+1}}{(2n+1)!}$  are different since they have different formulas.
- c) There is no difference between the series  $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$  and  $\sum_{n=0}^{\infty} \frac{(x+4)^{2n+1}}{(2n+1)!}$  since we only replaced  $x$  by  $x+4$ .
- d) The graphs of these two series are similar.
- e) The graphs of these two series are identical.
- f) You cannot graph either of these series.
- g)  $x + \frac{x^3}{3!} + \frac{x^5}{5!}$  estimates  $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ .
- h)  $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!}$  converges as  $n$  goes to infinity.
- i)  $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!}$  converges to  $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$  as  $n$  goes to infinity.
- j)  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  converges as  $n$  goes to infinity.
- k)  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  converges to  $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$  as  $n$  goes to infinity.
- l) I don't know what to circle.
- m) I cannot conclude anything.
- n) Something else came to my mind. \_\_\_\_\_

19) A student correctly writes the following:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \text{ for all numbers } x.$$

The student also correctly writes the following:

We can define  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  by  $T_0(x) = 1$ ,

$$T_1(x) = 1 + x,$$

$$T_2(x) = 1 + x + \frac{1}{2}x^2,$$

$\vdots$

$$T_n(x) = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n,$$

*etc.*

Please read ALL responses on the next page before circling.

Circle ALL responses that you can correctly conclude based off of what the student has written.

- a) You decide the convergence of the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  to  $e^x$  by plugging in numbers for  $x$  and noticing that
- $$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \text{ converges to } e^x \text{ for each number } x.$$
- b) You decide the convergence of the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  to  $e^x$  by considering
- $$\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$$
- and noticing that this converges to  $e^x$ .
- c) You decide the convergence of the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  to  $e^x$  by considering  $1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$  and adding more and more terms and noticing that  $1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$  converges to  $e^x$  as  $n$  goes to infinity.
- d) You decide the convergence of the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  to  $e^x$  by considering the difference between  $e^x$  and
- $$1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$$
- and noticing that this difference goes to zero as  $n$  goes to infinity.
- e)  $1 + x + \frac{1}{2}x^2$  estimates  $e^x$ .
- f)  $1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$  converges as  $n$  goes to infinity.
- g)  $1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$  converges to  $e^x$  as  $n$  goes to infinity.
- h)  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  converges as  $n$  goes to infinity.
- i)  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  converges to  $e^x$  as  $n$  goes to infinity.
- j)  $\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$ .
- k) The series  $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$  gets closer and closer to  $e^x$  until  $e^x$  is reached.
- l) The series  $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$  gets close to  $e^x$  but never reaches  $e^x$ .
- m) The series  $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$  is an approximation to  $e^x$  that can be made as accurate as you wish.
- n) The series  $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$  is identical to  $e^x$ .
- o) I don't know what to circle.
- p) I cannot conclude anything.
- q) Something else came to my mind. \_\_\_\_\_

20) A student correctly writes the following:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \text{ for all numbers } x, \text{ and}$$

$$e^x = \sum_{n=0}^{\infty} e \frac{(x-1)^n}{n!} = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \cdots \text{ for all numbers } x.$$

The student also correctly writes the following:

We can define  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  and

$\{S_0(x), S_1(x), S_2(x), \dots, S_n(x), \dots\}$  by  $T_0(x) = 1$ ,

$$T_1(x) = 1 + x,$$

$$T_2(x) = 1 + x + \frac{1}{2}x^2,$$

$\vdots$

$$T_n(x) = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n,$$

etc.

and

$$S_0(x) = e,$$

$$S_1(x) = e + e(x-1),$$

$$S_2(x) = e + e(x-1) + \frac{e}{2}(x-1)^2,$$

$\vdots$

$$S_n(x) = e + e(x-1) + \frac{e}{2}(x-1)^2 + \cdots + e \frac{(x-1)^n}{n!},$$

etc.

Please read ALL responses on the next page before circling.

Circle ALL responses that you can correctly conclude based off of what the student has written.

- a) The series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  and  $\sum_{n=0}^{\infty} e \frac{(x-1)^n}{n!}$  are different since they have different formulas.
- b) There is no difference between the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  and  $\sum_{n=0}^{\infty} e \frac{(x-1)^n}{n!}$  since they both equal  $e^x$ .
- c) The graphs of these two series are similar.
- d) The graphs of these two series are identical.
- e) You cannot graph either of these series.
- f)  $e + e(x-1) + \frac{e}{2}(x-1)^2$  and  $1 + x + \frac{1}{2}x^2$  both estimate  $e^x$ .
- g)  $e + e(x-1) + \frac{e}{2}(x-1)^2$  estimates  $e^x$  at  $x = 2$  better than  $1 + x + \frac{1}{2}x^2$ .
- h)  $1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$  and  $e + e(x-1) + \frac{e}{2}(x-1)^2 + \dots + e \frac{(x-1)^n}{n!}$  converge to the same thing as  $n$  goes to infinity.
- i) Both  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  and  $\{S_0(x), S_1(x), S_2(x), \dots, S_n(x), \dots\}$  converge to the same thing as  $n$  goes to infinity.
- j)  $\lim_{n \rightarrow \infty} T_n(x) = \lim_{n \rightarrow \infty} S_n(x)$
- k)  $e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$  for all numbers  $x$ .
- l)  $\sqrt{5.3^2 + 5.3\pi + e^x}$  can be approximated by  $\sqrt{5.3^2 + 5.3\pi + 1 + x}$ .
- m) I don't know what to circle.
- n) I cannot conclude anything.
- o) Something else came to my mind. \_\_\_\_\_



21) A student correctly writes the following:

*Taylor's Inequality states that if there exists a number  $M$  such that  $|f^{(n+1)}(x)| \leq M$  for all  $x$  in an interval, then the remainder function  $R_n(x)$  of the Taylor series for  $f(x)$  at  $x = a$  satisfies the inequality  $R_n(x) \leq \frac{M}{(n+1)!} |x - a|^{n+1}$  for all  $x$  in the interval.*

The student also correctly notes the following:

*The Taylor series for  $e^x$  is given by  $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$  and*

$$\lim_{m \rightarrow \infty} \left( \frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \frac{x^{m+2}}{(m+2)!} + \dots \right) = 0 \text{ for all numbers } x.$$

The student now relates Taylor's Inequality it to the function  $e^x$  and correctly writes:

*Since  $e^x \leq 8$  for all  $x$  in the interval  $(-2, 2)$ , the remainder function  $R_n(x)$  of the Taylor series for  $e^x$  at  $x = 0$  satisfies the inequality  $R_n(x) \leq \frac{8}{(n+1)!} |x|^{n+1}$  for all  $x$  in  $(-2, 2)$ .*

The student then correctly notes the following:

$$\text{The } \lim_{n \rightarrow \infty} \frac{8}{(n+1)!} |x|^{n+1} = 0 \text{ for all numbers } x.$$

Please read ALL responses before circling.

Circle ALL responses that you can correctly conclude based off of what the student has written.

- a) Since  $\lim_{m \rightarrow \infty} \left( \frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \frac{x^{m+2}}{(m+2)!} + \dots \right) = 0$  for all numbers  $x$ ,  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  for all numbers  $x$ .
- b) Since  $\lim_{m \rightarrow \infty} \left( \frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \frac{x^{m+2}}{(m+2)!} + \dots \right) = 0$  for all numbers  $x$ ,  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  when  $x$  is in the interval  $(-2, 2)$ .
- c) Since  $\lim_{m \rightarrow \infty} \left( \frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \frac{x^{m+2}}{(m+2)!} + \dots \right) = 0$  for all numbers  $x$ ,  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  converges for all numbers  $x$ .
- d) Since  $\lim_{m \rightarrow \infty} \left( \frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \frac{x^{m+2}}{(m+2)!} + \dots \right) = 0$  for all numbers  $x$ ,  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  converges when  $x$  is in the interval  $(-2, 2)$ .
- e) Since  $\lim_{n \rightarrow \infty} \frac{8}{(n+1)!} |x|^{n+1} = 0$  for all numbers  $x$ ,  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  for all numbers  $x$ .
- f) Since  $\lim_{n \rightarrow \infty} \frac{8}{(n+1)!} |x|^{n+1} = 0$  for all numbers  $x$ ,  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  when  $x$  is in the interval  $(-2, 2)$ .
- g) The  $\lim_{n \rightarrow \infty} \frac{8}{(n+1)!} |x|^{n+1}$  equaling 0 has nothing to do with  $e^x$  equaling the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ .
- h) The series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  equals  $e^x$  for all numbers  $x$ .
- i) The series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  approximates  $e^x$  for all numbers  $x$ .
- j) The series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  equals  $e^x$  when  $x$  is in the interval  $(-2, 2)$ .
- k) The series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  approximates  $e^x$  when  $x$  is in the interval  $(-2, 2)$ .
- l)  $R_n(x)$  goes to 0 for any number  $x$  as  $n$  goes to infinity.
- m)  $R_n(x)$  goes to 0 when  $x$  is in the interval  $(-2, 2)$  as  $n$  goes to infinity.
- n) Define  $T_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots + \frac{1}{n!}x^n$  for any number  $x$ . Then  $|e^x - T_n(x)|$  goes to 0 for any number  $x$  as  $n$  goes to infinity.
- o)  $|e^x - T_n(x)|$  goes to 0 when  $x$  is in the interval  $(-2, 2)$  as  $n$  goes to infinity.
- p) I don't know what to circle.
- q) I cannot conclude anything.

r) Something else came to my mind. \_\_\_\_\_

**SECTION 5: TRUE / FALSE**

Circle T for True or F for False. If you circle either T or F but you are not confident please also circle G for Guess. If unable to make an educated guess please only circle DK for Don't Know.

22)  $\overline{.9} = 1$ . (Note  $\overline{.9}$  means .999999 with 9's continuing to repeat)..... T / F / G / DK

23) Every bounded increasing sequence is convergent. .... T / F / G / DK

24) A convergent series of continuous functions is continuous. .... T / F / G / DK

25) Every function equals its Taylor series. .... T / F / G / DK

26) If  $f(x)$  produces Taylor series  $T(x)$ , then no other function  $g(x)$  can produce the same Taylor series  $T(x)$ . .... T / F / G / DK

27)  $f(x)$  cannot have two different Taylor series representations..... T / F / G / DK

28) A series  $\sum_{n=1}^{\infty} a_n$  converges whenever  $\lim_{n \rightarrow \infty} a_n = 0$  ..... T / F / G / DK

29) If  $a_m + a_{m+1} + a_{m+2} + \dots \rightarrow 0$  as  $m \rightarrow \infty$ , then  $\sum_{n=1}^{\infty} a_n$  converges..... T / F / G / DK

30) If  $a_m + a_{m+1} + a_{m+2} + \dots + a_{k-1} + a_k \rightarrow 0$  as  $m, k \rightarrow \infty$ , then  $\sum_{n=1}^{\infty} a_n$  converges.. T / F / G / DK

31) Since  $\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \dots$ ,

$\int \left( \frac{1}{\sqrt{1+x}} \right) dx = \int \left( 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \dots \right) dx$ ..... T / F / G / DK

32)  $25c^2 \left( \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} + \frac{5}{16} \frac{v^6}{c^6} + \dots \right)$  can be approximated by  $25c^2 \left( \frac{1}{2} \frac{v^2}{c^2} \right)$ ..... T / F / G / DK

33) I have seen graphs of Taylor polynomials that are similar to the graphs that I saw at the beginning of this questionnaire. .... T / F / G / DK

**SECTION 6: CONCLUSION**

Now that you are done, please turn in the questionnaire to the questionnaire administrator. If you have any questions that you would like to ask about this research project, please ask the questionnaire administrator. Thank you very much for your willingness to participate in this research project.

## Novice Interview Protocol

Brackets indicate the layout of the handout and notes for me. In some cases font size has been reduced to allow the protocol to fit the margins of this dissertation.

Furthermore, to enhance the readability of the main body of this dissertation, a one-to-one correspondence between expert and novice interview tasks was created by changing some novice interview task numbers to match the task numbers as presented in the main body of this dissertation (e.g., Table 15).

### **INTRODUCTION**

Thank you for your time and willingness to participate. As you know, I am interested in how people understand Taylor series. Particularly, I am trying to understand how people understand the convergence of Taylor series. Feel free to volunteer any detail you wish when answering a question. You also have the option of declining to answer – passing – on any of the questions. You will be given a handout with tasks to complete during this interview. Do you have any questions before we start?

[“\*” Indicate questions that I really like!]

[Handout begins]

### **DIRECTIONS:**

We will go over this handout one page at a time. At different times you may be asked to write on the handout to answer questions. You are also encouraged to think out loud.

1) \*What are Taylor series?

[Expect a wide range of images.]

2) \*Why are Taylor series studied in calculus?

[NP = New Page in interview handout]

- 3) \*What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$  when  $x$  is any real number?”

[Expect approximation and proximity images to come out if present. May even get a formal definition of convergence.]

- a) Follow-Up Question: If they bring up convergence, have them elaborate on what it means to converge.

- 4) \*What is meant by the “(-1,1)” in, “ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$  when  $x$  is in the interval (-1,1)?”

[Expect either images of domain to come out or more approximation images.]

- a) Follow-Up Question: What happens outside the interval (-1,1)?

[NP]

- 6) \*What is meant by the word “prove” if you were asked to, “Prove that sine is equal to its Taylor series.”

[Wanting to see remainder come up as well as other approximation images.]

- 7) \*What are the steps in proving that  $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$ ?

[Wanting to see remainder come up as well as other approximation images.]

[NP]

- 8) How can we estimate sine by using its Taylor series?

- a) Follow up question: What if we wanted to estimate sine at 103? How might we use a Taylor series to do that? If confused about the measure, follow up with 103 radians.

[Expect approximation images. Looking to see if center of series images are also present.]

[NP]

- 9) \*What is meant by the “approximation” symbol in “ $\sin(x) \approx x - \frac{x^3}{3!}$  = a Taylor polynomial for sine when  $x$  is near 0?”

- 10) \*What is meant by the “near” in “ $\sin(x) \approx x - \frac{x^3}{3!}$  = a Taylor polynomial for sine when  $x$  is near 0?”

- a) Follow-Up Question: What happens when we are NOT near zero?

[NP]

11) \*How can we get a better approximation for sine than using  $x - \frac{x^3}{3!}$ ?

[Expected Answer: Add more terms]

a) Follow-Up Question: Can you think of anything else you might want to consider when answering this question?

[NP]

12) What is the difference between the following questions.

a) \*What is the maximum error in using a given  $n^{\text{th}}$  degree Taylor polynomial to approximate  $f(x)$ ? Verses

b) \*How large does  $n$  need to be to guarantee that the  $n^{\text{th}}$  degree Taylor polynomial is within 0.1 of  $f(x)$ ?

i) Follow-Up Question: Which of these questions is more difficult to answer?

ii) Follow-Up Question: Which of these questions is more important?

[NP]

*Taylor's Inequality states that if there exists a number  $M$  such that*

*$|f^{(n+1)}(x)| \leq M$  for all  $x$  in an interval, then the remainder function  $R_n(x)$  of the Taylor series for  $f(x)$  at  $x = a$  satisfies the inequality*

$$R_n(x) \leq \frac{M}{(n+1)!} |x-a|^{n+1} \text{ for all } x \text{ in the interval.}$$

13) How is Taylor's Inequality useful?

a) Follow-Up Question: In what ways do you use Taylor's Inequality?

**Potential Follow-Up Questions to be Excluded if Time Does Not Permit!**

- b) \*What is the maximum error in using  $1 + \frac{x}{1!}$  to approximate  $e^x$  on (0,2)? Justify your steps?
- i) Follow-Up Question: What about this task is giving you difficulty?"  
If they eventually say the interval (0,2). Change the question to (-2,2).  
They can also just give the general steps.
- c) \*How large does  $n$  need to be to guarantee that the  $n^{\text{th}}$  degree Taylor polynomial  $1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$  is within 0.001 of  $e^x$  when  $x$  is in the interval (0,2)? Justify your steps?
- i) Follow-Up Question: What about this task is giving you difficulty?
- d) \*Is there a way to decrease the  $n$  you got in part c?  
[Another question designed to get at their images of center of series. This may be one question too many]
- i) Follow-Up Question: What should you take into account when attempting to decrease the  $n$ ?
- ii) Follow-Up Question: How does problem 36 change if I ask you to prove that  $e^x$  equals  $1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$ .

[NP]

14) Problem 19: A student correctly writes the following:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \text{ for all numbers } x.$$

The student also correctly writes the following:

$$\text{We can define } \{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\} \text{ by } T_0(x) = 1,$$

$$T_1(x) = 1 + x,$$

$$T_2(x) = 1 + x + \frac{1}{2}x^2,$$

⋮

$$T_n(x) = 1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n,$$

etc.

Please read ALL responses on the next page before circling.

Circle ALL responses that you can correctly conclude based off of what the student has written.

a) You decide the convergence of the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  to  $e^x$  by plugging in numbers for  $x$  and noticing

$$\text{that } \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \text{ converges to } e^x \text{ for each number } x.$$

b) You decide the convergence of the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  to  $e^x$  by considering

$$\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\} \text{ and noticing that this converges to } e^x.$$

c) You decide the convergence of the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  to  $e^x$  by considering  $1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$  and

adding more and more terms and noticing that  $1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$  converges to  $e^x$  as  $n$  goes to infinity.

d) You decide the convergence of the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  to  $e^x$  by considering the difference between  $e^x$

$$\text{and } 1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n \text{ and noticing that this difference goes to zero as } n \text{ goes to infinity.}$$

e) You decide the convergence of the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  to  $e^x$  by considering the terms  $a_n = \frac{x^n}{n!}$  of the

series and noticing that the limit of the terms go to zero,  $\lim_{n \rightarrow \infty} a_n = 0$ .

Follow-Up Question: If you could choose just one of these a – e, which would you choose as the one which you use most to determine that a series converges to a function?

[NP]

15) More from the Questionnaire.



## Novice Interview Handout

In an effort to save space, I have condensed the interview handout by eliminating space between tasks. Horizontal lines have been placed in this appendix to indicate page breaks in the original novice interview handout. Therefore, tasks that occurred on the same page of the original handout are NOT separated by a horizontal line. Furthermore, tasks that occurred on the same page were originally spaced out on the page to give the participant ample room to write. In addition, to enhance the readability of the main body of this dissertation, a one-to-one correspondence between expert and novice interview tasks was created by changing some novice interview task numbers to match the task numbers as presented in the main body of this dissertation (e.g., Table 15).

---

### **DIRECTIONS:**

We will go over this handout one page at a time. You may be asked to write on the handout to answer questions. You are also encouraged to think out loud.

- 1) What are Taylor series?
  - 2) Why are Taylor series studied in calculus?
- 
- 3) What is meant by the “=” in “ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$  when  $x$  is any real number?”
  - 4) What is meant by the “ $(-1,1)$ ” in, “ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$  when  $x$  is in the interval  $(-1,1)$ ?”
- 
- 6) What is meant by the word “prove” if you were asked to, “Prove that sine is equal to its Taylor series.”
  - 7) What are the steps in proving that  $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$ ?
- 
- 8) How can we estimate sine by using its Taylor series?
-

---

9) What is meant by the “approximation” symbol in “ $\sin(x) \approx x - \frac{x^3}{3!}$  = a Taylor polynomial for sine when  $x$  is near 0?”

10) What is meant by the “near” in “ $\sin(x) \approx x - \frac{x^3}{3!}$  = a Taylor polynomial for sine when  $x$  is near 0?”

---

11) How can we get a better approximation for sine than using  $x - \frac{x^3}{3!}$ ?

---

12) What is the difference between the following questions.

- a) What is the maximum error in using a given  $n^{\text{th}}$  degree Taylor polynomial to approximate  $f(x)$ ? Verses
  - b) How large does  $n$  need to be to guarantee that the  $n^{\text{th}}$  degree Taylor polynomial is within 0.1 of  $f(x)$ ?
- 

*Taylor’s Inequality states that if there exists a number  $M$  such that*

*$|f^{(n+1)}(x)| \leq M$  for all  $x$  in an interval, then the remainder function  $R_n(x)$  of the Taylor series for  $f(x)$  at  $x = a$  satisfies the inequality*

$$R_n(x) \leq \frac{M}{(n+1)!} |x-a|^{n+1} \text{ for all } x \text{ in the interval.}$$

13) How is Taylor’s Inequality useful?

---

More on next page

---

14) Problem 19: A student correctly writes the following:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \text{ for all numbers } x.$$

The student also correctly writes the following:

We can define  $\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  by  $T_0(x) = 1$ ,

$$T_1(x) = 1 + x,$$

$$T_2(x) = 1 + x + \frac{1}{2}x^2,$$

$\vdots$

$$T_n(x) = 1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n,$$

*etc.*

Please read ALL responses on the next page before circling.

Circle ALL responses that you can correctly conclude based off of what the student has written.

- a) You decide the convergence of the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  to  $e^x$  by plugging in numbers for  $x$  and noticing

$$\text{that } \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \text{ converges to } e^x \text{ for each number } x.$$

- b) You decide the convergence of the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  to  $e^x$  by considering

$\{T_0(x), T_1(x), T_2(x), \dots, T_n(x), \dots\}$  and noticing that this converges to  $e^x$ .

- c) You decide the convergence of the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  to  $e^x$  by considering  $1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$  and

adding more and more terms and noticing that  $1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$  converges to  $e^x$  as  $n$  goes to infinity.

- d) You decide the convergence of the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  to  $e^x$  by considering the difference between  $e^x$

and  $1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$  and noticing that this difference goes to zero as  $n$  goes to infinity.

- e) You decide the convergence of the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  to  $e^x$  by considering the terms  $a_n = \frac{x^n}{n!}$  of the

series and noticing that the limit of the terms go to zero,  $\lim_{n \rightarrow \infty} a_n = 0$ .

Follow-Up Question: If you could choose just one of these a – e, which would you choose as the one which you use most to determine that a series converges to a function?

---

15) More from the Questionnaire.

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## APPENDIX D

### Additional Focus Diagrams

Additional Novice Focus Diagrams.....	378
Additional Expert Focus Diagrams.....	379

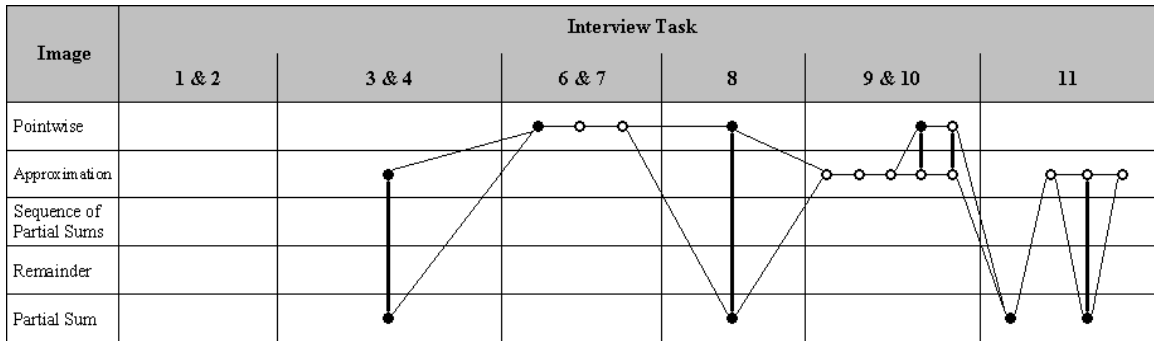


Figure 32. NNP DAVE's Focus Analysis Diagram

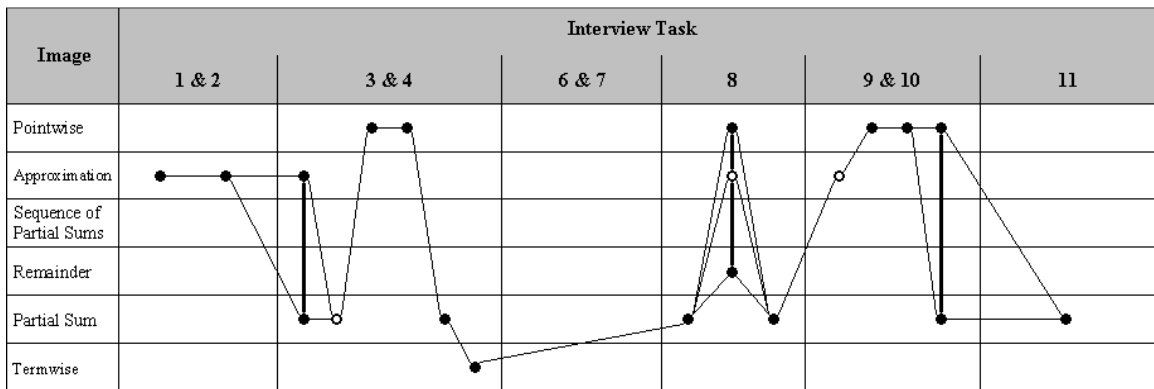


Figure 33. NNP STEVE's Focus Analysis Diagram

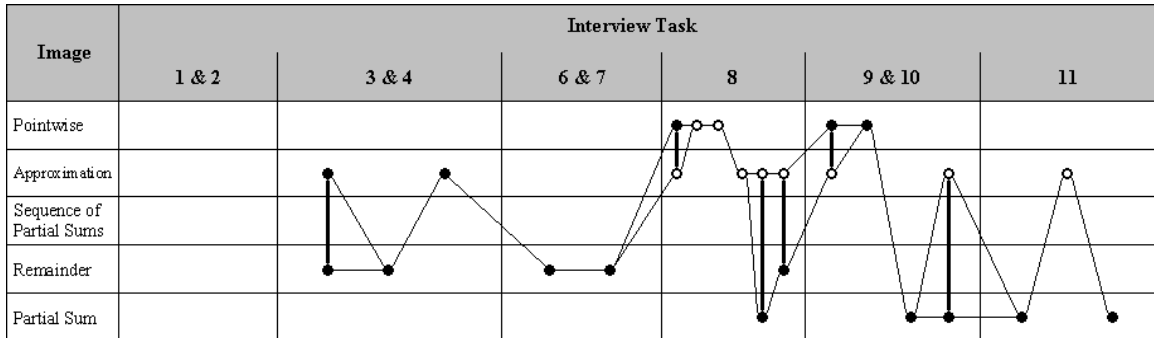


Figure 34. MNP PHILIP's Focus Analysis Diagram

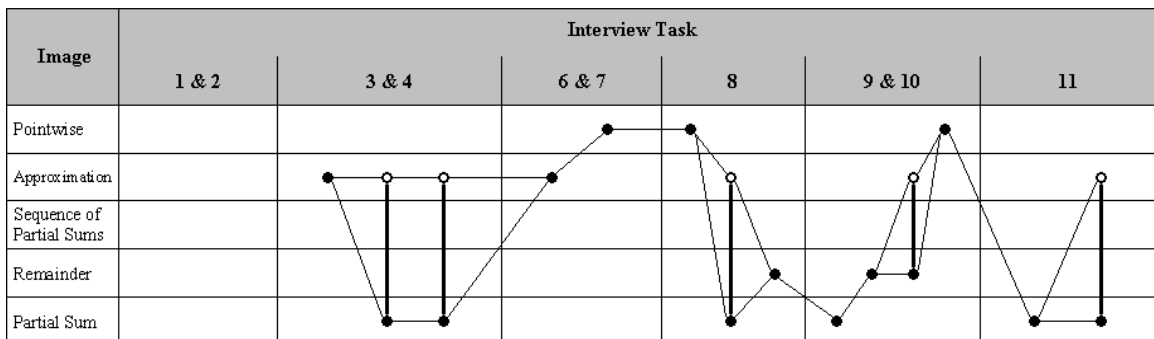


Figure 35. MNP TOMAS' Focus Analysis Diagram

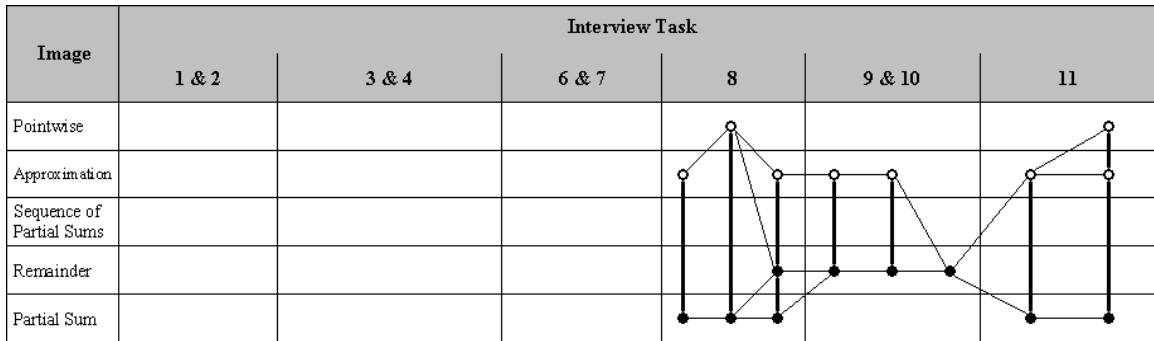


Figure 36. CEP KELLEN's Focus Analysis Diagram

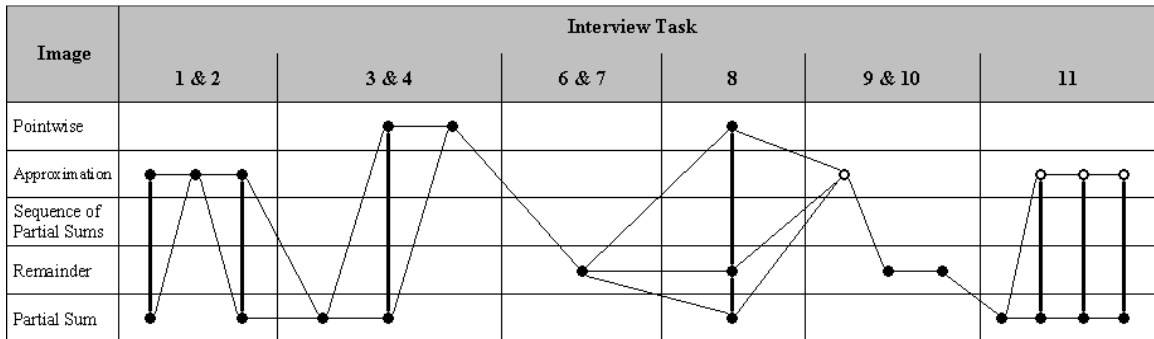


Figure 37. EEP CLARK's Focus Analysis Diagram

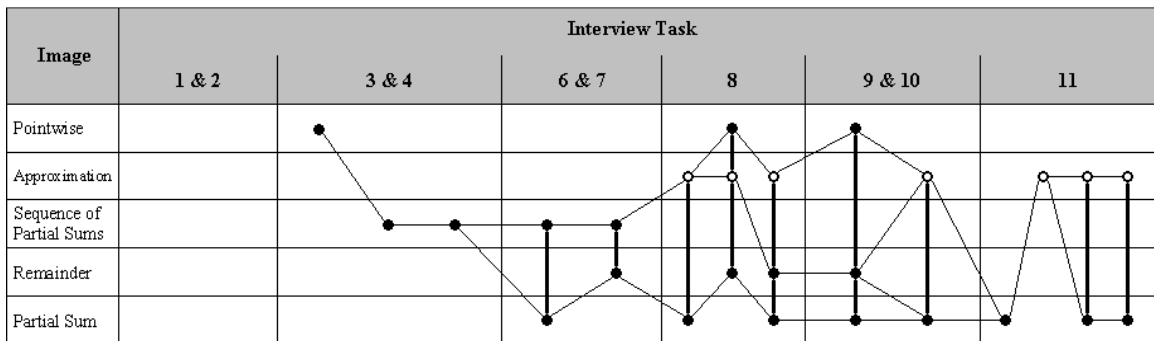


Figure 38. EEP DEAN's Focus Analysis Diagram

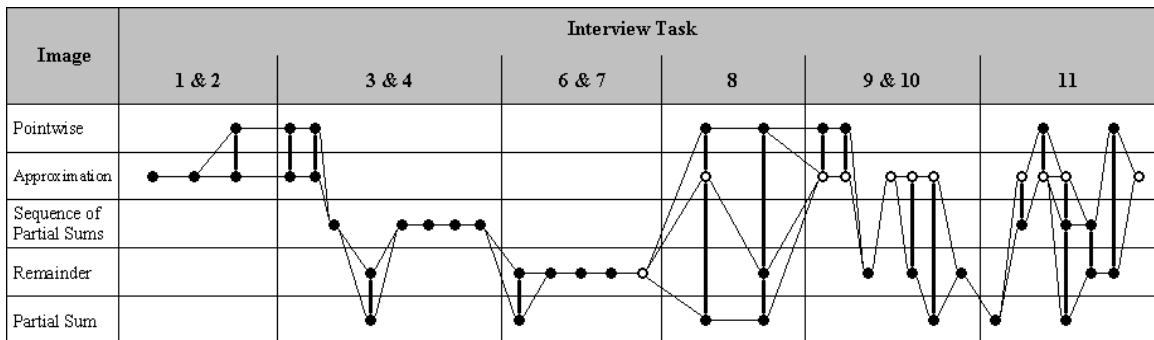


Figure 39. EEP DYLAN's Focus Analysis Diagram

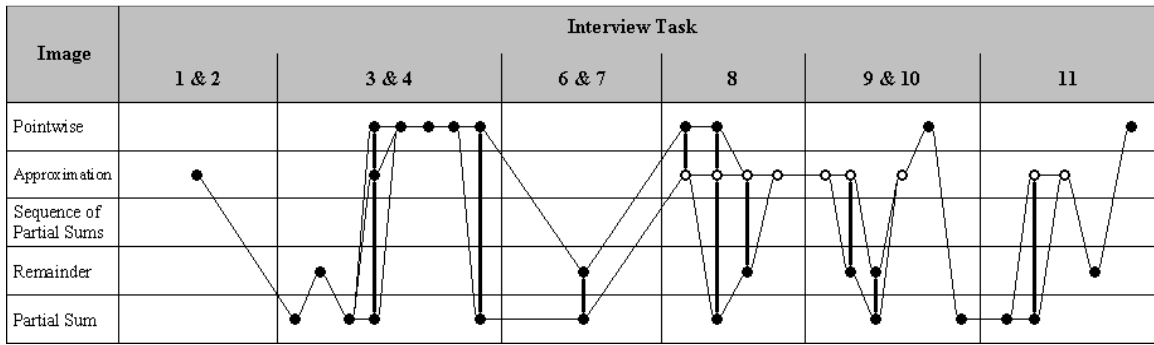


Figure 40. EEP JAMES' Focus Analysis Diagram

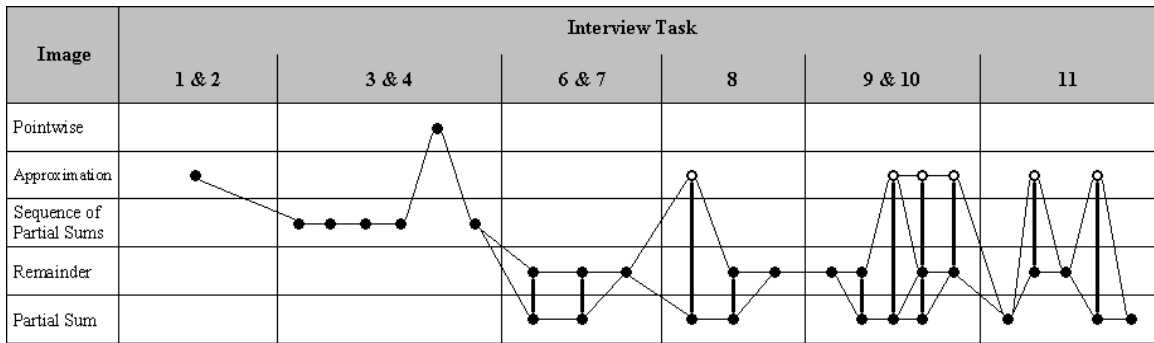


Figure 41. EEP LOGAN's Focus Analysis Diagram

APPENDIX E

Additional Novice Dendrograms

Dendrograms for the Interactions between Existing and Emerging Images for  
the NNPs .....382

Dendrograms for the Interactions between Existing and Emerging Images for  
the MNPs .....383

Dendrograms for the Interactions between Existing and Emerging Images for  
the Never Consistent Novice Participants.....384

Dendrograms for the Interactions between Existing and Emerging Images for  
the Completely Consistent Novice Participants.....385





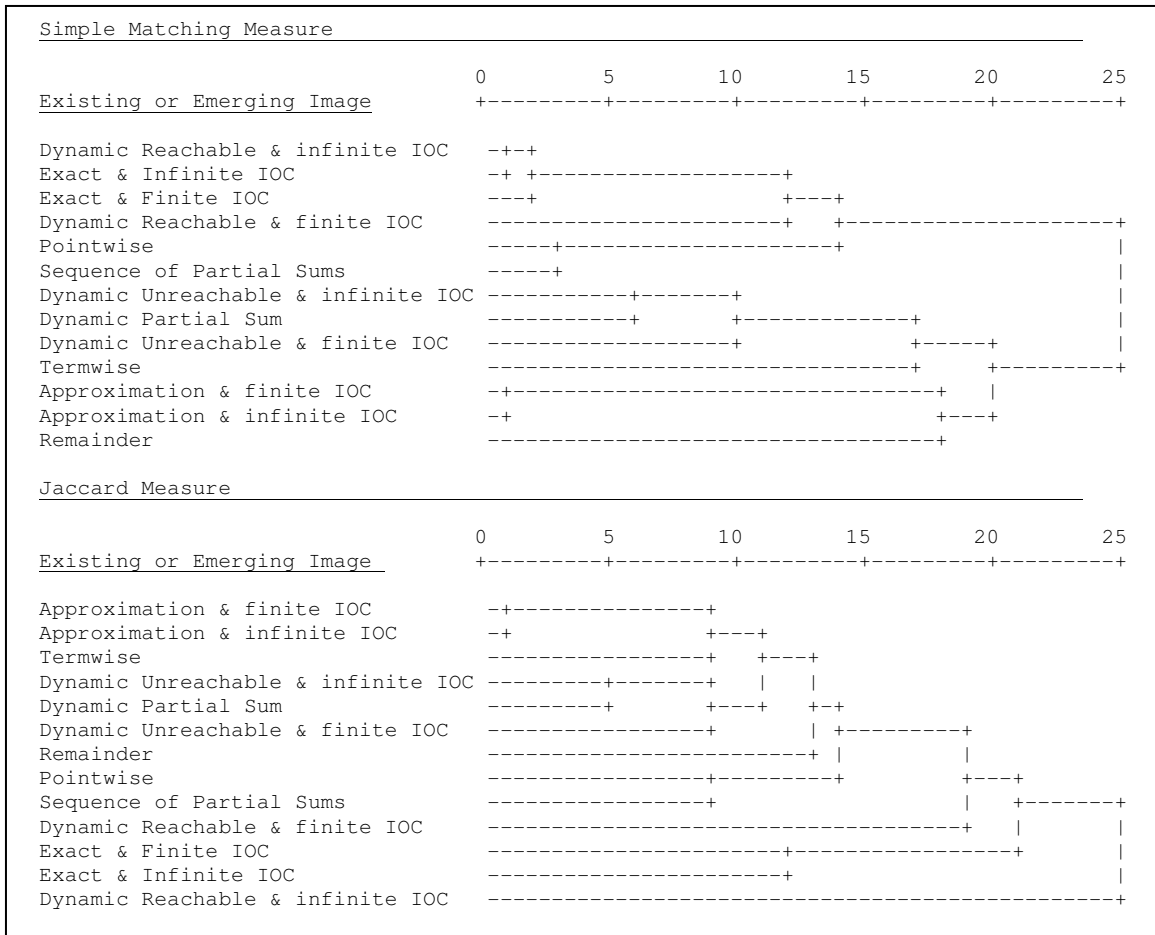
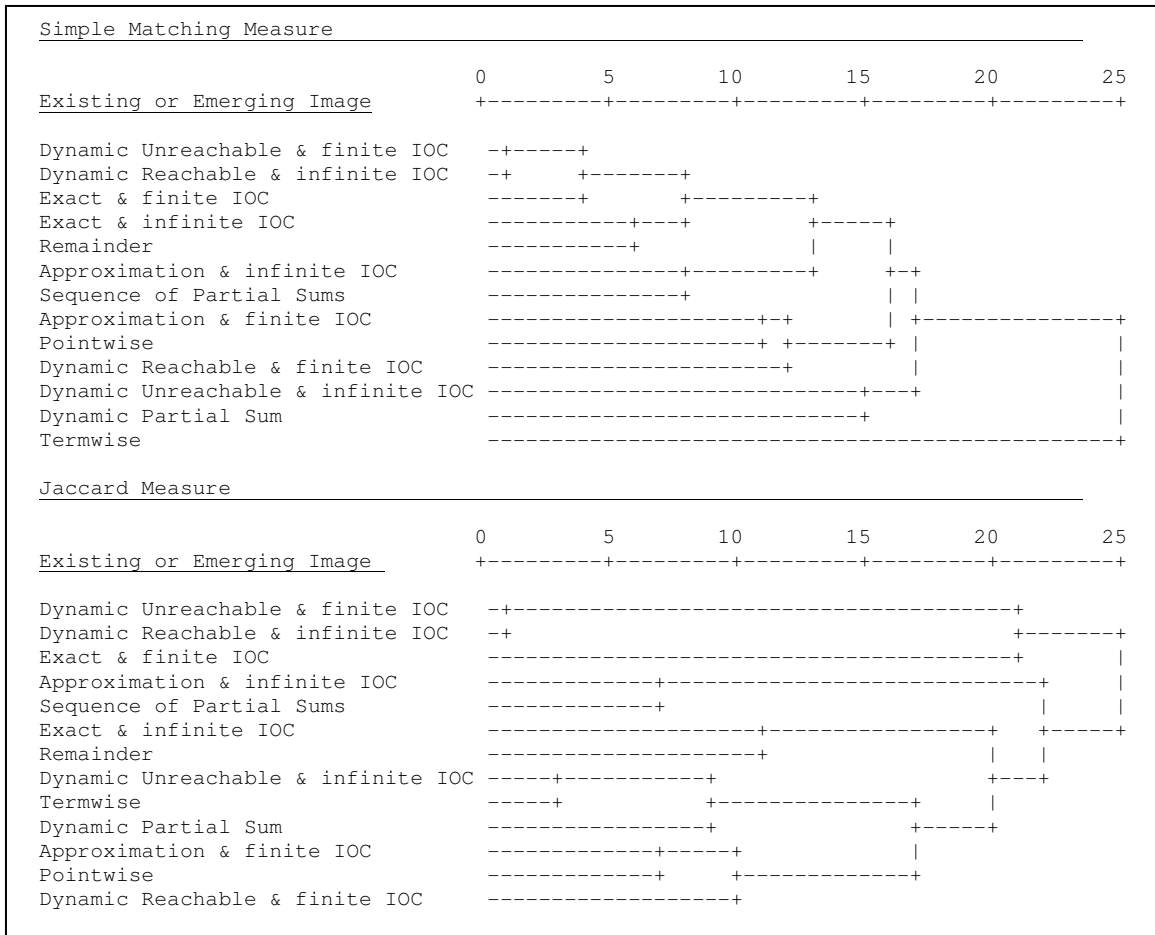


Figure 43. Dendrograms for the Interactions between Existing and Emergent Images for the MNPs



*Figure 44.* Dendrograms for the Interactions between Existing and Emerging Images for the Never Consistent Novice Participants

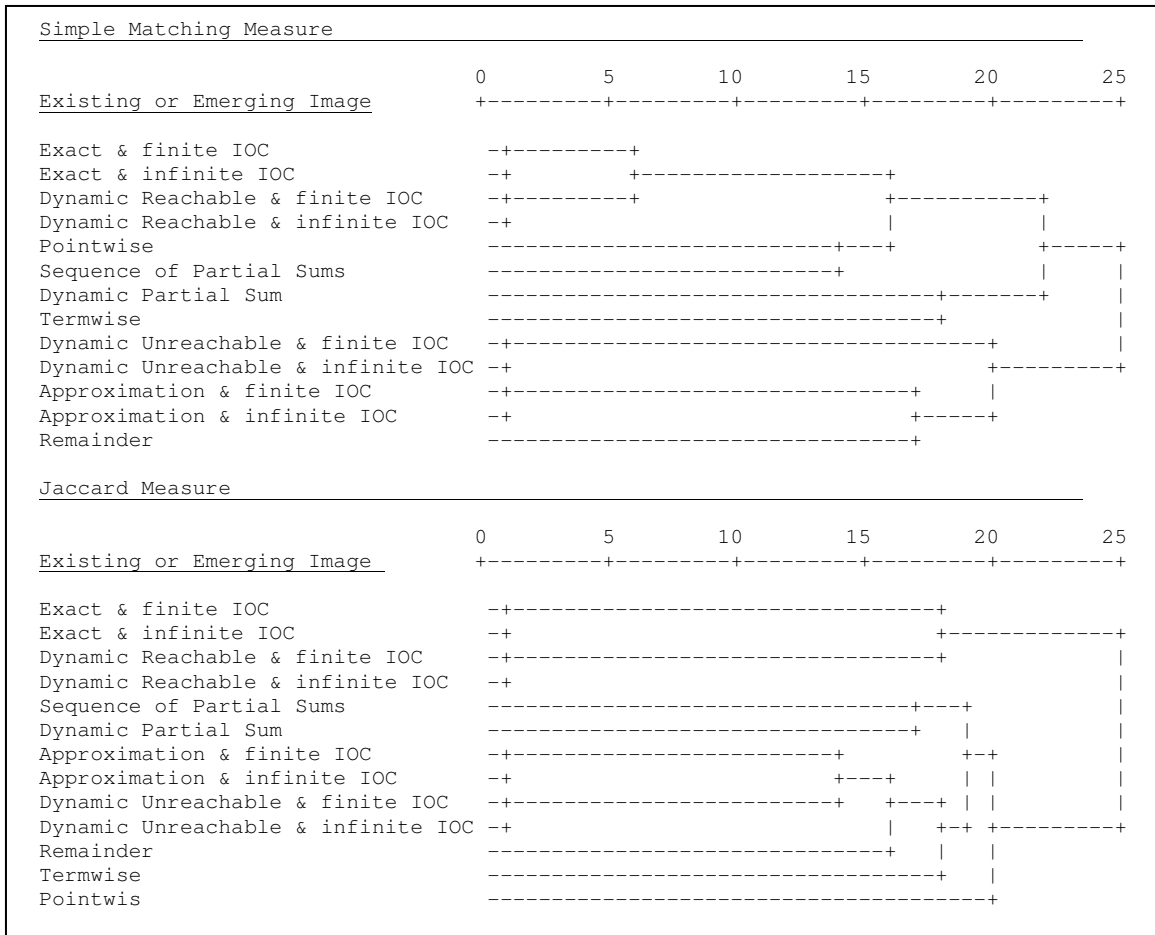


Figure 45. Dendrograms for the Interactions between Existing and Emerging Images for the Completely Consistent Novice Participants