UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

EQUIVARIANT PIECEWISE-LINEAR TOPOLOGY AND COMBINATORIAL APPLICATIONS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

Degree of

DOCTOR OF PHILOSOPHY

By

JAMES ROBERT DOVER
Norman, Oklahoma
2011

EQUIVARIANT PIECEWISE-LINEAR TOPOLOGY
AND COMBINATORIAL APPLICATIONS

A DISSERTATION APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

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DEDICATION

to

My parents

Larry F. Dover and Jacqueline Dover

For

Giving me a strong foundation in life, values, and mathematics

My wife

Kristen Dover

For

Her love, support, and patience

Acknowledgements

I want to thank my advisor, Professor Murad Özaydın, for how much he has taught me, for his guidance and encouragement, and for his patience and exceeding generosity with his time and energy. I will always be grateful for the opportunity to learn from and work with him.

I would also like to thank Professor Max Forester, whose courses started me on the road towards topology, and Professors Ralf Schmidt and Ravi Shankar, whose courses in algebraic geometry, representation theory, and differential geometry gave me valued alternative perspectives that aided me in this dissertation.

I want to express my gratitude to all of my committee members, the aforementioned and Professors Przebinda and Altan, for their valued time and support.

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Abstract

For G a finite group, we develop some theory of G–equivariant piecewise-linear topology and prove characterization theorems for G–equivariant regular neighborhoods. We use these results to prove a conjecture of Csorba that the Lovász complex $\mathrm{Hom}(C_5, K_n)$ of graph multimorphisms from the 5–cycle C_5 to the complete graph K_n is equivariantly homeomorphic to the Stiefel manifold, $V_{n-1,2}$, the space of (ordered) orthonormal 2–frames in $\mathbb{R}^{n-1}$, with respect to an action of the cyclic group of order 2.

Chapter 1

Introduction

1.1 Piecewise-Linear Topology

The primary objects in piecewise-linear (PL) topology are polyhedra. These are topological spaces that admit triangulations; that is, a polyhedron is homeomorphic to some simplicial complex. The morphisms between polyhedra are piecewise-linear functions, functions that, after some subdivisions, send simplices to simplices.

The PL-manifolds comprise an important subset of polyhedra. These are polyhedra whose triangulations satisfy the condition that the link of any vertex is a PL-sphere of the correct dimension. PL-manifolds provide a stepping stone between topological manifolds and smooth manifolds. Any smooth (differentiable) manifold admits a unique piecewise-linear structure [W40]; that is, there exists a triangulation of the manifold satisfying the above link condition, and any two triangulations share a common subdivision. On the other hand, there exist topological manifolds for which no triangulation satisfies the link condition [KS77] (so they are polyhedra but not PL-manifolds), and there exist topological manifolds which admit no triangulations [Fr82].

When G is a finite group, a smooth manifold M is called a G–manifold if the group elements act on M by differentiable maps (see [tD87], [Ka91]). We will develop a concept of a G–manifold in the category of PL-manifolds. Just requiring the group elements to act by PL-homeomorphisms is not sufficient to produce the desired analogous piecewise-linear results. For example, a smooth action of G on a differentiable manifold M restricts to a linearizable action of the stabilizer G_x on the link of a point $x \in M$. This means, for instance, that the G_x–fixed point set in the link cannot be a single point. However, this can happen in the case of a piecewise-linear action [O75].

1.1.1 Regular Neighborhoods

Let M be a smooth manifold with M_1 a submanifold. A neighborhood N of M_1 diffeomorphic to the normal bundle of M_1 in M is called a tubular neighborhood. Tubular neighborhoods always exist and are unique up to isotopy [La02].

The analog of a tubular neighborhood in the piecewise-linear category is a regular neighborhood. In a triangulated polyhedron (not necessarily a PL–manifold), a regular neighborhood of a subcomplex is constructed by taking a simplicial neighborhood in an appropriate subdivision. Like their smooth counterparts, regular neighborhoods always exist and are unique up to isotopy [RS82]. In the unique PL-structure on a smooth manifold, the regular neighborhood of a submanifold is a tubular neighborhood.

The main goal of our discussion of equivariant piecewise-linear topology is to develop the theory of G–regular neighborhoods. As we will show, G–regular neighborhoods always exist in a G–polyhedron, and they are unique up to G–

homeomorphism. We prove two important characterization theorems for G–regular neighborhoods in the interiors of G–manifolds: 3.20, the Simplicial G–Neighborhood Theorem, which identifies them with certain simplicial neighborhoods, and 3.30, the G–Collapsing Criterion, which recognizes them by equivariant collapses.

1.2 Lovász Complexes

A (simple) graph Γ is a collection of edges (two element subsets) on a set of vertices V_Γ, a purely combinatorial object. A morphism from a graph Γ to another graph Λ is a map from V_Γ to V_Λ that sends every edge of Γ to an edge of Λ. We consider $\mathrm{Hom}(\Gamma, \Lambda)$, the Lovász multimorphism complex, a bifunctor (contravariant in the first variable and covariant in the second) assigning a regular cell complex to the pair of graphs Γ and Λ [BK06]. A cell of this complex is a graph **multimorphism** ϕ, an assignment to each vertex v of Γ a nonempty set $\phi(v)$ of vertices of Λ such that choosing a single vertex from each $\phi(v)$ defines a graph morphism. Thus, the 0–cells of $\mathrm{Hom}(\Gamma, \Lambda)$ are themselves graph morphisms.

Symmetries on the graphs Γ and Λ induce symmetries on the cell complex $\mathrm{Hom}(\Gamma, \Lambda)$. A simple example is the complex $\mathrm{Hom}(K_2, K_n)$ where K_n is the complete graph on n vertices. This multimorphism complex is homeomorphic to an $(n-2)$–dimensional sphere. The edge K_2 has the involution of interchanging its two vertices, and this induces the antipodal action on the sphere $\mathrm{Hom}(K_2, K_n)$.

1.2.1 Applications

In his proof of the Kneser conjecture [Lo78], Lovász gave a lower bound for the chromatic number of a graph using what was essentially the edge complex functor $\mathrm{Hom}(K_2, -)$. Because this is a covariant functor with respect to graph morphisms, an n–coloring of a graph Λ induces a map from $\mathrm{Hom}(K_2, \Lambda)$ to $\mathrm{Hom}(K_2, K_n)$, which is equivariant with respect to the involutions coming from the reflection of K_2. Since the involution on $\mathrm{Hom}(K_2, \Lambda)$ does not fix any points, if $\mathrm{Hom}(K_2, \Lambda)$ is m–connected, there exists an equivariant map from S^{m+1} (with the antipodal action) to $\mathrm{Hom}(K_2, \Lambda)$. Hence, by utilizing the Borsuk-Ulam theorem, Lovász showed that if the edge complex of Λ is m–connected, the graph Λ is not $(m + 2)$–colorable.

Having seen that the connectivity m of the edge complex of a graph provides a lower bound of $m + 3$ for its chromatic number, Lovász conjectured that the connectivity of the odd cycle complex $\mathrm{Hom}(C_{2r+1}, \Lambda)$ would also give a lower bound; specifically, if $\mathrm{Hom}(C_{2r+1}, \Lambda)$ is m–connected, then Λ is not $(m + 3)$–colorable. This conjecture was first proven by Babson and Kozlov in [BK07], who showed that there is no equivariant map from S^{k+1} to $\mathrm{Hom}(C_{2r+1}, K_{k+3})$ with respect to the (fixed-point free) involution induced by any reflection on the odd cycle C_{2r+1}. Their proof then follows from the same logic as Lovász's.

In [S09-1] and [S09-2], C. Schultz reduced the Lovász conjecture to a computation involving the equivariant (with respect to the involution induced by a reflection on the cycle) cohomology of $\mathrm{Hom}(C_{2r+1}, K_n)$ (Another proof was then given by Kozlov in [Ko06-1]). Viewed in this light, the bound on the chromatic number of Λ given by the cycle complex does not end up being any better than that given by its edge complex [S09-2]. However, this calculation only makes use of the symmetry on the cycle complex coming from a single reflection. The cycle C_{2r+1} has, of course, an action of the dihedral group D_{2r+1}. Thus, studying the D_{2r+1}–equivariant cohomology of $\mathrm{Hom}(C_{2r+1}, K_n)$ provides hope for better lower bounds. The nonequivariant cohomology of the complexes $\mathrm{Hom}(C_m, K_n)$

has been computed [Ko08], but not the equivariant.

Multimorphism complexes have found use in other areas besides chromatic numbers or even graph theory. Interestingly, well before $\mathrm{Hom}(\Gamma, \Lambda)$ was defined, the underlying spaces in the family of complexes $\mathrm{Hom}(K_m, K_n)$ figured prominently in two unrelated applications of equivariant algebraic topology: in Alon's elegant Necklace Splitting Theorem (with m prime) [A87] and the proof of the prime power case [Ö87] of the Bárány-Shlossman-Szücs conjecture [BSS81].

Alon's theorem addressed the purely combinatorial question of equitably distributing the jewels from a necklace: Suppose m thieves steal an unclasped necklace made up of a sequence of k different types of jewels. There are ma_i identical jewels of type i for $1 \leq i \leq k$. The thieves want to split the necklace so that each type of jewel is distributed evenly amongst them. What is the least number of cuts required to accomplish this for all possible arrangements of the jewels? Simply grouping the jewels of the same type together produces a necklace that requires $k(m-1)$ cuts for an equitable splitting. Alon's theorem is that $k(m-1)$ cuts are sufficient for any necklace.

Alon constructs a topological space, which turns out to be the complex $\mathrm{Hom}(K_m, K_n)$, with $n = k(m-1)+1$, whose points correspond to necklace splittings using $k(m-1)$ cuts where each thief receives an equal length of necklace. The m thieves are represented by the vertices of the graph K_m, and the pieces of a split necklace are represented by the vertices in the target graph $K_{k(m-1)+1}$. For a multimorphism ϕ in this complex, the set $\phi(t)$ represents the collection of necklace-pieces given to thief t in a particular splitting. Coordinates in the cell indexed by ϕ describe the sizes of all the pieces. From this space $\mathrm{Hom}(K_m, K_n)$, we define a map to the space $\mathbb{R}^{m(k-1)}$ giving the distribution of the first $k-1$ types of jewels among the m thieves. This map is equivariant with respect to cyclic permutation of the thieves. Furthermore, if we assume that there is no equitable splitting, the image of the map falls in the complement of the diagonal $\mathbb{R}^{k-1}$ in $\mathbb{R}^{m(k-1)}$. Alon shows that no such equivariant map exists in the case that m is prime. Thus, when there is a prime number of thieves, there must be an equitable splitting with the correct number of cuts. The general case reduces to the prime case by an elementary argument.

The Bárány-Shlossman-Szücs conjecture states that, given any continuous map f from an $n = (k-1)(d+1)$–dimensional simplex to the Euclidean space $\mathbb{R}^d$, we can find k disjoint faces of the simplex whose images intersect. Assuming to the contrary that there is an f for which k such faces cannot be found, Özaydın in [Ö87] constructs a map from $\mathrm{Hom}(K_k, K_{n+1})$ to the sphere $S^{d(k-1)-1}$ which is equivariant with respect to an action of the permutation group Σ_k. He then shows that such a map cannot exist if k is a prime power. In this application, the complex $\mathrm{Hom}(K_k, K_{n+1})$ represents the space of k–tuples of disjoint faces of the simplex Δ^n.

1.2.2 Csorba Conjecture

For $n \geq 3$, the Lovász complex $\mathrm{Hom}(C_5, K_n)$ is the only manifold among the family of cycle complexes $\mathrm{Hom}(C_m, K_n)$ [CL06]. In fact, $\mathrm{Hom}(C_5, K_n)$ provides a combinatorial model for the geometrically defined Stiefel manifold $V_{n-1,2}$, the space of (ordered) orthonormal 2–frames in the Euclidean space $\mathbb{R}^{n-1}$ [J77]. Both spaces admit obvious involutions: $\mathrm{Hom}(C_5, K_n)$ inherits any reflection of the pentagon C_5 (any two of which are equivalent), while any reflection of the plane R^2 (any two of which are also equivalent via rotation) induces an involution on $V_{n-1,2}$.

In his thesis [C05], Csorba proved that $\mathrm{Hom}(C_5, K_n)$ and $V_{n-1,2}$ were equivariantly homeomorphic with respect to these involutions for small n and proved that they were homotopy equivalent for every n. He conjectured that they were equivariantly homeomorphic for all n. In [S08], Schultz proved the nonequivariant version of Csorba's conjecture as well as that $\mathrm{Hom}(C_5, K_n)$ is equivariantly homotopy equivalent to $V_{n-1,2}$ for all n.

Schultz's proof made use of the fact that $V_{n-1,2}$ is the boundary of a regular neighborhood $\{(x, y) \in S^{n-2} \times S^{n-2} \,|\, x \cdot y \geq 0\}$ of the diagonal in the manifold $S^{n-2} \times S^{n-2}$. By passing to a smaller cell complex using restrictions of multimorphisms, he was able to realize $\mathrm{Hom}(C_5, K_n)$ as the boundary of a regular neighborhood of the diagonal in some triangulation of $S^{n-2} \times S^{n-2}$. The result followed by the PL-homeomorphic equivalence of regular neighborhoods.

Our equivariant generalizations of regular neighborhood results in piecewise-linear topology provide us with the tools to prove the equivariant Csorba conjecture. In Schultz's model, the regular neighborhood in question is not an equivariant regular neighborhood. To rectify this, we use a different model for $\mathrm{Hom}(C_5, K_n)$ as well as a different (but equivalent) involution on $V_{n-1,2}$.

Chapter 2

Preliminaries

2.1 G–actions

Let G be a group and X a set. Let ψ be a map $G \times X \to X$. We denote $\psi(g, x) \in X$ as gx. We say that ψ is a (left) G**–action** on X if (i) $1x = x$ for all $x \in X$ and (ii) $(g_1 g_2)x = g_1(g_2 x)$ for all $g_1, g_2 \in G$ and $x \in X$. If X is a set with a G–action, we say X is a G–set. The **(**G**–)orbit** of a point $x \in X$ is the G–subset $Gx := \{gx \,|\, g \in G\}$. Similarly, we may refer to the G–orbit GU of a set $U \subseteq X$: $GU := \psi(G \times U) = \{gu \,|\, g \in G, u \in U\}$.

For an element x in a G–set X, its **stabilizer** is the subgroup $G_x := \{g \in G \,|\, gx = x\}$. The stabilizer of a subset $U \subseteq X$ is the subgroup of group elements that fix U setwise: $G_U := \{g \in G \,|\, gU = U\}$.

A G**–map** from a G–set X to a G–set Y is a map $f \colon X \to Y$ such that for any $g \in G$ and $x \in X$, $f(gx) = gf(x)$. We say that such a map f is **equivariant** with respect to G. To denote a G–map, we may write $f \colon X \to_G Y$.

For G a topological group, a G**–space** is a topological space X with a G–action such that the map $G \times X \to X$ given by $(g, x) \mapsto gx$ is continuous. In the case that G is a discrete group, this simply means that X is a G–set with each map $x \mapsto gx$ continuous. In fact, they must be homeomorphisms since each g is invertible with g^{-1} also continuous. A G–map between G–spaces is the same as for G–sets with the additional condition of continuity.

2.2 Simplicial and Cellular G–Complexes

An (abstract) **simplicial complex** K on a vertex set V is a collection of finite subsets of V such that if $\sigma \subseteq \tau \in K$, then $\sigma \in K$. A set $\sigma \in K$ is called a **simplex**; the non-empty simplices of K are also called the **faces** of K. The faces of K form a poset under inclusion, called the face poset $F(K)$. A simplicial **subcomplex** L of K (denoted $L \leq K$) is a subcollection of simplices of K which is itself a simplicial complex.

The **dimension** of a simplex σ in K, $\dim \sigma$ is defined to be one less than the cardinality of σ. The n–**skeleton** K^n of the simplicial complex K is the subcomplex made up of all simplices of K with dimension at most n. However, we will abuse the notation and use K^0 to also mean $\bigcup\{\sigma \,|\, \sigma \in K\}$; that is, it will mean the set of vertices of V that are actually contained in simplices of K.

We denote by Δ^n the simplicial complex consisting of a single n–simplex and all of its faces. The subcomplex $\partial\Delta^n$ consists of only its lower dimensional simplices. Further, Δ^{n-1} can be regarded as a subcomplex of $\partial\Delta^n$.

We now define some important types of subcomplexes. For $\sigma \in K$, the **link** of σ in K is the subcomplex $\mathrm{lnk}_K(\sigma) := \{\tau \in K \,|\, \tau \cap \sigma = \emptyset, \tau \cup \sigma \in K\}$, and the **star** of σ in K is $\mathrm{st}_K(\sigma) := \{\tau \in K \,|\, \tau \cup \sigma \in K\}$. For a subcomplex $L \leq K$, its **simplicial neighborhood** in K is $N_K(L) := \{\,\sigma \in K \,|\, \exists \tau \in K \text{ s.t. } \sigma \leq \tau \text{ and } \tau \cap L^0 \neq \emptyset\}$. Also, define $\dot{N}_K(L) := \{\,\sigma \in N_K(L) \,|\, \sigma \cap L^0 = \emptyset\}$, a combinatorial approximation of the boundary of $N_K(L)$.

A map $f\colon K^0 \to L^0$ is a **simplicial map** if for any $\sigma \in K$, $f(\sigma) \in L$. For such a simplicial map f, the image of f, $f(K)$ is a subcomplex of L.

A subcomplex L of K is called **full** if for any $\sigma \in K$ with $\sigma \subseteq L^0$, then $\sigma \in L$. A full subcomplex is completely determined by its 0–skeleton.

A frequent construction is the **join** of two simplicial complexes: If K and L are simplicial complexes, we define $K * L$ to be the simplicial complex consisting of all simplices $\sigma \cup \tau$ where $\sigma \in K$ and $\tau \in L$. The join of K with a single 0–simplex, Δ^0, is called a **cone** on K (often abbreviated as $K * v$). The join of K with $\partial\Delta_1$ (i.e., S^0) is the **suspension** of K.

The **geometric realization** of K is the topological space

$$|K| := \{ \sum_{v\in\sigma} t_v\delta_v \in \mathbb{R}^V \mid \sum_{v\in\sigma} t_v = 1,\, t_v > 0,\, \sigma \in K \setminus \{\emptyset\}\}$$

where δ_v is the standard basis vector of $\mathbb{R}^V$ corresponding to $v \in V$. In practice, we write v for δ_v in these affine combinations. The underlying topological space of a face σ of K is homeomorphic to a disk of dimension $\dim\sigma$ and is the subspace of $|K|$ given by

$$|\sigma| := \{ \sum_{v\in\sigma} t_v\delta_v \in \mathbb{R}^V \mid \sum_{v\in\sigma} t_v = 1,\, t_v \geq 0\}.$$

Note that we are using vertical bars to denote both the cardinality of a finite set and the geometric realization of a simplex or a simplicial complex; which one is meant should be clear from the context.

For an n–simplex $\sigma = \{v_0, \ldots, v_n\} \in K$, denote its barycenter in $|\sigma|$ by $\bar{\sigma} = \frac{1}{n+1}v_0 + \ldots + \frac{1}{n+1}v_n$.

A **regular cell structure** on a (compact Hausdorff) topological space X is a (finite) collection $\{c\}$ of subspaces (called cells or faces), each homeomorphic to a (closed) disk of dimension d for some d, such that (1) X is the disjoint union of the relative interiors of its cells (called the open cells and denoted by int c), and (2) the boundary of each cell is a union of (lower dimensional) cells. A topological space X together with a regular cell structure is a **regular cell complex**. For any simplicial complex K, the collection $\{|\sigma|\}_{\sigma \in K \setminus \{\emptyset\}}$ gives a regular cell structure on $|K|$. If X is a regular cell complex, a simplicial complex K is a **simplicial subdivision of** X if $|K|$ is homeomorphic to X with the image of each simplex $|\sigma|$ lying entirely within a cell c of X. We may also say a simplicial complex L subdivides another simplicial complex K if L is a simplicial subdivision of the regular cell complex $|K|$, in which case there is understood to be a specific homeomorphism between $|L|$ and $|K|$ in mind, so we regard $|L|$ as equal to $|K|$.

For any poset P, its **order complex** ΔP is the simplicial complex whose simplices are chains $a_0 < a_1 < \ldots < a_d$, with $a_i \in P$. The faces of a regular cell complex under inclusion form a poset whose order complex is its barycentric subdivision ([BLSWZ99] p200, [LW69] Ch 3 §1). Thus, the space X is determined up to homeomorphism by the face poset of its regular cell structure. For this reason, we will often identify a regular cell complex with its face poset. In the case of a simplicial complex K, K is itself a poset, $K = F(K) \cup \{\emptyset\}$, and while $\Delta F(K)$ is a simplicial complex isomorphic to the barycentric subdivision of K, ΔK is the cone on $\Delta F(K)$ with a vertex corresponding to the empty simplex.

Let G be a finite group. Suppose X is a G–space with a regular cell structure such that the G–action permutes the cells. Recall that the stabilizer of a cell c is the subgroup G_c consisting of group elements fixing c setwise. The G–space X is called a **regular cellular G–complex** if every closed cell c is G_c–homeomorphic to a cone on ∂c with the apex of the cone stabilized by G_c. As an example, we also define a **simplicial G–complex** as a simplicial complex K equipped with a permutation action of G on its vertex set V so that the induced action on the subsets of V sends simplices to simplices. Then G acts on the geometric realization $|K|$ by linearly extending the action on basis elements $g\delta_v \mapsto \delta_{gv}$, which makes $|K|$ into a regular cellular G–complex since the stabilizer of a simplex $|\sigma|$ always fixes its barycenter. If the stabilizer of each cell (or simplex) fixes the cell pointwise, the G–complex is called **admissible**. For example, if G acts on a poset P (preserving the partial order), then ΔP is an admissible simplicial G–complex since the only way to fix a chain setwise while preserving the order is to fix each element in the chain.

Just as in the nonequivariant case ([BLSWZ99] p200, [LW69] Ch 3 §1), the face poset of a regular cellular G–complex determines its G–homeomorphism type:

Lemma 2.1. *If X is a regular cellular G-complex with face poset F, then X is G–homeomorphic to $|\Delta F|$.*

Proof. The proof is by induction on the number of cells in X. There is nothing to prove if X consists of a single orbit of 0–cells. Now, choose a maximal cell c and define $Y := X \setminus \bigcup_{g\in G} \operatorname{int} gc$ with the induced G–cell structure. By the induction hypothesis, Y is G–homeomorphic to $|\Delta(F \setminus Gc)|$. Also, ∂c is G_c–homeomorphic to $|\Delta F_{<c}|$. Then we take the cone of ∂c, as in the definition of a regular cellular G–complex, with its apex being a point $x \in \operatorname{int} c$ fixed by G_c. Extending this coning equivariantly to $G\partial c$ gives the homeomorphism from $|\Delta F|$ to X. $\square$

2.3 The Equivariant Piecewise-Linear Category

Let K and L be simplicial complexes. A continuous function $f\colon |K| \to |L|$ is called **piecewise-linear** (PL) if there exist subdivisions K' and L' of K and L respectively such that, for any simplex σ of K', f maps $|\sigma|$ onto a simplex $|\tau|$ of $|L'|$, and f restricted to $|\sigma|$ is a linear function. In other words, f is induced by a simplicial map from K' to L'. Hence, if $|K| \subset |L|$ is a PL-inclusion, we may extend a subdivision K' of K to a subdivision L' of L.

We refer to a regular cell complex X with face poset F as a **polyhedron** when we identify X with the homeomorphic space $|\Delta F|$, thereby giving it a PL structure. Then, a **triangulation** K of a polyhedron X is a simplicial complex with $|K|$ PL-homeomorphic to X. Note here that a single abstract simplicial complex K might triangulate X in multiple ways through different homeomorphisms; when we say K is a triangulation of X, we are implicitly choosing one such homeomorphism. In the case X is a G–complex, we identify it with the G–homeomorphic $|\Delta F|$ as in Lemma 2.1 and call it a G**–polyhedron**. Clearly, G acts on X via PL-homeomorphisms. We further define a G**–triangulation** of a G–polyhedron X to be a simplicial G–complex K with $|K|$ G–PL-homeomorphic to X. We now want to show that any triangulation of a G–polyhedron can be subdivided into a G–triangulation. To do this, we will first show that any collection of triangulations of a polyhedron have a common subdivision.

Let $\mathcal{K} = \{K_1, K_2, \ldots, K_s\}$ be a collection of simplicial complexes, each one triangulating the same polyhedron X. We may assume, by subdividing if necessary as in the definition of piecewise-linearity, that for all i, any simplex in K_i embeds linearly into X. For $\phi \in K_1 \times \ldots \times K_s$, let $|\phi| := \bigcap_{1\leq i\leq s} |\phi_i|$. Define an equivalence relation on $K_1 \times \ldots \times K_s$ by $\phi \sim \psi \Leftrightarrow |\phi| = |\psi|$. We define a poset $C_{\mathcal{K}} := \{\phi \in K_1 \times \ldots \times K_s \,|\, |\phi| \neq \emptyset\}/\sim$ with the partial order $[\phi] \leq [\psi] \Leftrightarrow |\phi| \subseteq |\psi|$. We will identify an equivalence class $[\phi]$ with the geometric realization $|\phi|$ of its representatives.

Proposition 2.2. *Given a collection $\mathcal{K} = \{K_1, \ldots, K_s\}$ of simplicial complexes that triangulate a polyhedron X, then $C_{\mathcal{K}}$ is a regular cell structure on X, so that $|\Delta C_{\mathcal{K}}|$ is a triangulation of X and a common subdivision of $K_1, \ldots, K_s$.*

Proof. We show first that the open cells of $C_{\mathcal{K}}$ are disjoint. In this discussion, let $|\phi| = \bigcap_{1 \leq i \leq s} |\sigma_i|$ and $|\psi| = \bigcap_{1 \leq i \leq s} |\tau_i|$. Note that if $|\phi| \cap |\psi| \neq \emptyset$, we have that $|\phi| \cap |\psi| = \bigcap_{1 \leq i \leq s}(|\sigma_i| \cap |\tau_i|) = \bigcap_{1 \leq i \leq s} |\sigma_i \cap \tau_i|$. Thus, the intersection of any two closed cells is a closed cell. If $[\phi] < [\psi]$, then, without changing $[\psi]$ or $[\phi]$, we may replace each τ_i with its minimal face containing $|\psi|$ and each σ_i with $\sigma_i \cap \tau_i$. Then $|\psi| \cap \operatorname{int}(|\tau_i|)$ is always nonempty, and for some i we have $|\phi| \subseteq |\partial \tau_i|$. Therefore we have that $|\phi| \subseteq \partial |\psi|$, implying that the intersection of any two distinct closed cells must occur on the boundary of at least one of them, so any two distinct open cells are disjoint.

Each $|\psi|$ is a (nonempty) compact, convex polytope, yielding that $|\psi| \approx D^m$ for some $m \geq 0$. Furthermore, $\partial |\psi| = \bigcup_{[\phi] < [\psi]} |\phi|$: If $x \in \partial |\psi| \subseteq X$, there is a unique $\sigma_i \in K_i$ such that $x \in \operatorname{int}(|\sigma_i|)$, giving some ϕ with $x \in |\phi|$ and $[\phi] \leq [\psi]$. However $[\phi] < [\psi]$ because x is not in $\operatorname{int}(|\psi|)$, so there is an i with x not in $\operatorname{int}(|\tau_i|)$, i.e., $\sigma_i \neq \tau_i$. Conversely, if $x \in \bigcup_{[\phi] < [\psi]} |\phi|$, we have $x \in |\phi| \subseteq \partial |\psi|$ as above. This proves that $C_{\mathcal{K}}$ is a regular cell structure on X. Therefore, we have that $|\Delta C_{\mathcal{K}}| \approx X$. $\square$

Corollary 2.3. *If X is a G–polyhedron and K is a triangulation of X, then there is an admissible G–triangulation of X which is a subdivision of K.*

Proof. Let K be a triangulation of X via the PL-homeomorphism $f\colon |K| \to X$. We again assume, by subdividing, that each simplex of K embeds linearly into a simplex of $|\Delta F|$, where F is the face poset of a regular cellular G–structure on X. For each $g \in G$, we have a PL-homeomorphism $g\colon X \to X$ given by $x \mapsto gx$. Thus, K also triangulates X via the PL-homeomorphism $g \circ f\colon |K| \to X$. We call this triangulation gK for convenience, noting that G does not act on the simplicial complex K. Also note that since G acts affinely on each simplex of $|\Delta F|$, every simplex of gK embeds linearly in X as well.

Let $\mathcal{K} = \{gK \mid g \in G\}$. Using the notation from the proof of 2.2, each cell $[\phi]$ is given by a map $\phi\colon G \to K$. Then $|\phi| = \bigcap_{g\in G} g|\phi(g)| \subseteq X$. For $h \in G$, define $(h\phi)(g) := \phi(h^{-1}g)$. This induces an order-preserving G–action on $C_{\mathcal{K}}$ because, for any ϕ and any h in G, $|h\phi| = \bigcap_{g\in G} g|\phi(h^{-1}g)| = \bigcap_{g\in G} hg|\phi(g)| = h\bigcap_{g\in G} g|\phi(g)| = h|\phi|$. Lastly, $C_{\mathcal{K}}$ is a regular cellular G–complex because, since any cell $[\phi]$ of $C_{\mathcal{K}}$ is contained in a simplex of the admissible G–complex ΔF, any group element that stabilizes $[\phi]$ must fix it pointwise. The result follows by Lemma 2.1. $\square$

2.4 G–Collapses and Discrete Morse Theory

Let $Y \subset X$ be polyhedra with $X = Y \cup D^m$ with $m \geq 1$ and $Y \cap D^m = D^{m-1}$ where D^{m-1} is an $(m-1)$–disk and there is a PL-homeomorphism $D^m \to D^{m-1} \times I$ which maps D^{m-1} homeomorphically to $D^{m-1} \times \{0\}$. In this situation, we say there is an (m–dimensional) **elementary collapse** from X to Y. If there is a sequence $X = X_0, X_1, X_2, \ldots, X_k = Y$ of polyhedra with an elementary collapse from X_{i-1} to X_i for $1 \leq i \leq k$, we say X **collapses** to Y, or $X \searrow Y$. An elementary collapse (and therefore also a collapse) yields a deformation retraction from the larger polyhedron to the smaller. Observe that in a collapse the dimensions of the individual elementary collapses in the sequence may vary. A collapse is called m–dimensional if every elementary collapse in the sequence is of dimension $\leq m$.

Now let $Y \subset X$ be G–polyhedra with $X = Y \cup GD^m$ where D^m (and therefore also gD^m for each $g \in G$) is an m–disk as in the above definition of an elementary collapse. If we also have that (1) $gD^m \neq D^m$ implies that $gD^m \cap D^m \subset Y$ and (2) there exists a point $y \in D^{m-1}$ fixed by the stabilizer G_{D^m} such that D^{m-1} is G_{D^m}–homeomorphic to a cone with apex y on some G_{D^m}–complex and D^m is G_{D^m}–homeomorphic to $D^{m-1} \times I$, then we say there is an **elementary G–collapse** from X to Y. (Note then that any point $x \in \{y\} \times (0, 1]$ will have stabilizer $G_x = G_{D^m}$.) A sequence of these is called a G–collapse, denoted $X \searrow_G Y$.

A particular collapse is easier to describe when it arises from an underlying triangulation of the polyhedron X, so now we will focus on collapses within a simplicial complex K.

Let K be a finite simplicial complex. We use the notation $\sigma \lessdot \tau$ if $\sigma < \tau$ and dim $\sigma = \dim \tau - 1$. A simplex σ is called a **free face** of τ if τ is the only simplex such that $\sigma \lessdot \tau$. When σ is a free face of τ, we say there is an **elementary simplicial collapse** of K onto $K \setminus \{\sigma, \tau\}$. A sequence of such elementary collapses is a **simplicial collapse**. Simplicial collapses clearly induce collapses on geometric realizations.

Note that if K is an admissible G–complex, if σ is a free face of τ, for all $g \in G$, $g\sigma$ is a free face of $g\tau$. Thus K collapses simplicially to $K \setminus \bigcup_{g \in G}\{g\sigma, g\tau\}$. This is an **elementary simplicial G–collapse**. A sequence of such is a **simplicial G–collapse** and clearly induces a G–collapse on geometric realizations.

Our chosen method for describing simplicial collapses is Robin Forman's Discrete Morse Theory. More thorough discussions of the subject can be found in

[Fo98] and [Ko07].

By a **vector**, we mean a pair $(\sigma \lessdot \tau)$, where τ is thought of as the head of the vector and σ the tail. A **discrete vector field** on K is defined to be a collection of vectors $V = \{\, (\sigma_i \lessdot \tau_i) \,|\, i \in I\}$ such that each simplex $\rho \in K$ belongs to at most one element of V, either as a head or a tail of a vector.

Given a discrete vector field V on K, we have the notion of a **path**, which is a sequence of simplices in K of the form:

$$\sigma_0 \lessdot \tau_0 \gtrdot \sigma_1 \lessdot \tau_1 \gtrdot \ldots \gtrdot \sigma_{s-1} \lessdot \tau_{s-1} \gtrdot \sigma_s$$

where $\forall\, i :\; 0 \leq i < s$, $(\sigma_i \lessdot \tau_i) \in V$ and $\sigma_i \neq \sigma_{i+1}$. We say a path as above has length s. By a **cycle** we mean a path as above with $\sigma_s = \sigma_0$.

A **Morse matching** (or a **discrete gradient field**) is a discrete vector field V with no cycles. The simplices which are unpaired in V are called **critical**.

An equivalent concept to a Morse matching is a height function on K. A **height** (or **Morse**) **function** is a map $h\colon K \to \mathbb{R}$ satisfying $\forall\, \sigma \in K$,

$$|\{\rho \lessdot \sigma \,|\, h(\rho) \geq h(\sigma)\} \cup \{\tau \gtrdot \sigma \,|\, h(\tau) \leq h(\sigma)\}| \leq 1$$

Given a height function h, the corresponding Morse matching is the collection of pairs $(\sigma \lessdot \tau)$ for which $h(\sigma) \geq h(\tau)$. Conversely, it is not difficult to construct a height function inducing a given Morse matching ([Fo98], [Ko07]). This height function is clearly not unique. In fact, we may adjust it to be one-to-one and to take values in $\mathbb{N}$ (without changing the Morse matching).

When K is an admissible G-complex, a G–**matching** on K is a Morse matching V such that whenever $(\sigma \lessdot \tau) \in V$, so too is $(g\sigma, g\tau) \in V$ for any $g \in G$.

A G–matching may always be realized by a G–invariant height function. This height function may be adjusted to give a one-to-one function on the collection of G–orbits.

The following is the key lemma from discrete Morse theory we will use in this paper. A more general (though nonequivariant) version for cell complexes can be found in [Ko07], Theorem 11.13.

Lemma 2.4. *Let K be a finite admissible simplicial G–complex with a Morse G–matching whose critical simplices form a subcomplex L. Then K simplicially G–collapses to L.*

Proof. Let $h\colon K \to \mathbb{N}$ be a G–invariant height function corresponding to the given G–matching under which each orbit takes a unique value in $\mathbb{N}$. Define a new height function $\tilde{h}\colon K \to \mathbb{N}$ as follows: For $\sigma \in L$, set $\tilde{h}(\sigma) = \dim(\sigma)$. For $\sigma \notin L$, set $\tilde{h}(\sigma) = h(\sigma) + \dim(L)$. Under this new height function, all of the simplices in L remain critical, and the relative heights of all the simplices outside of L are unchanged, preserving their pairings. It also preserves the G–invariance. Therefore, $\tilde{h}$ corresponds to the same G–matching as h, and $\tilde{h}$ is one-to-one on orbits in $K \setminus L$.

Now, for $m \in \mathbb{N}$ define

$$K(m) := \{\sigma \in K \mid \exists\, \tau \geq \sigma \text{ such that } \tilde{h}(\tau) \leq m\}$$

Note that $K(\dim(L)) = L$. For $m \geq \dim(L)$, either $K(m+1) = K(m)$ or $K(m+1) = K(m) \cup G\{\rho, \tau\}$ where $\rho \lessdot \tau$ and $\tilde{h}(\tau) = m+1 < \tilde{h}(\rho)$. In the latter case, $K(m+1)$ G–collapses to $K(m)$ along the free faces $g\rho$ for $g \in G$. Since $K = K(\max\{\tilde{h}(\sigma) \,|\, \sigma \in K\})$, K G–collapses to L via a sequence of these collapsings. $\square$

Lemma 2.5. *Let $L \leq K$ be simplicial G–complexes with $K = L * v$. Then K G–collapses to v.*

Proof. Define a G–matching by pairing each simplex $\sigma \in L$ with $\sigma \cup \{v\}$. This pairs every simplex of K except the vertex v. That there are no cycles follows because all vectors lead to a simplex containing v, and no simplex containing v is the tail of a vector. $\square$

The following lemma serves to illustrate the use of discrete Morse theory to describe simplicial G–collapses. It is essentially an equivariant version of Theorem 3.1 in [Ko06-2]. We will make further use of it later.

Lemma 2.6. *Let G be any group and P be a finite poset, $h\colon P \to P$ an order-preserving poset map such that for any $x \in P$, $h(x) \geq x$ (or $h(x) \leq x$). Define Q to be the set of fixed points of h. Then ΔP collapses simplicially to ΔQ. In the case that h is a G–poset map, $P \searrow_G Q$.*

Proof. We prove it for the case that $h(x) \geq x$, the other case being almost identical. Since P is finite, we may choose N large enough so that for all x in P, we have $h^N(x) \in Q$. Now let $\sigma \in \Delta P$ be a chain $x_0 < x_1 < \ldots < x_m$. If $\exists\ i : 0 \leq i \leq m$ such that $x_i \notin Q$, let k be the largest such i. Then we may insert $h^N(x_k)$ into the chain immediately following x_k because $x_k < h^N(x_k) \leq h^N(x_{k+1}) = x_{k+1}$ if $k < m$. Associate to σ the chain obtained by inserting $h^N(x_k)$ or by deleting it in the case $h^N(x_k) = x_{k+1}$. Since it is an element of Q being inserted or deleted, the selection of k is not affected, and x_k uniquely determines the other chain in the pair. Therefore, this matching is well-defined. Also, this matching is equivariant if h is a G–map.

Suppose there is a cycle

$$\sigma_0 \lessdot \tau_0 \gtrdot \sigma_1 \lessdot \tau_1 \gtrdot \sigma_2 \lessdot \ldots \lessdot \tau_{s-1} \gtrdot \sigma_s = \sigma_0$$

We have for $1 \leq i \leq s$ that $\sigma_i = \tau_{i-1} \setminus \{y_i\}$ for some $y_i \in P$. Then there must be some pair $\sigma_j \lessdot \tau_j = \sigma \cup \{y_i\}$, so $y_i \in Q$ for all i. Thus every simplex in the cycle has all the same elements of $P \backslash Q$, so $\exists\, x \in P \backslash Q$ that is the greatest such element in every simplex. Hence $\tau_j = \sigma_j \cup \{h^N(x)\}$ for all j, and $y_i = h^N(x)$ for all i. This is a contradiction because the same element is being added and deleted in consecutive steps. Therefore, we have a Morse matching whose critical simplices are exactly the elements of ΔQ, a subcomplex of ΔP. Thus, ΔP G–collapses to ΔQ. $\square$

Chapter 3

G–Regular Neighborhoods

Now we follow the discussion of Rourke and Sanderson [RS82] to develop the theory of equivariant regular neighborhoods. In this chapter, all simplicial complexes are finite, all polyhedra are compact, and all inclusions of polyhedra are piecewise-linear.

Let $Y \subset X$ be polyhedra triangulated by L and K respectively, with $L \leq K$. The **derived subdivision of K near L** is the simplicial complex K' with vertex set $K^0 \cup \{v_\tau \mid \tau \in K \setminus L, \tau \cap L^0 \neq \emptyset\}$, and the simplices are of the form $\sigma \cup \{v_{\tau_1}, \ldots v_{\tau_m}\}$ where $\sigma \in L$ or $\sigma \in K$ with $\sigma \cap L^0 = \emptyset$ and $\sigma < \tau_1 < \ldots < \tau_m$. Geometrically, K' subdivides K by selecting, for each $\tau \in K \setminus L$ that intersects L^0, the new vertex v_τ in the interior of $|\tau|$ and then, in ascending order of dimension, replacing each $|\tau|$ with the cone (with apex v_τ) on its boundary (which has already been subdivided in the previous steps).

Suppose that, in addition, L is full in K and $|\dot{N}_K(L)|$ is the boundary of $|N_K(L)|$ in X. Let K' be a derived subdivision of K near L. Then $N = |N_{K'}(L)|$ is called a **regular neighborhood** of Y in X. If K and L are both admissible G–complexes and, when defining K', the set of new vertices $\{v_\tau\}$ is chosen to be

G–invariant, we say N is a G–**regular neighborhood** of Y in X.

Our first goal is to show that, as in [RS82] with non-equivariant regular neighborhoods, a G–regular neighborhood of Y in X is unique up to G–homeomorphism. It is clear that, whenever two G–regular neighborhoods arise from the same G–triangulations L and K, they are G–homeomorphic since the underlying simplicial G–complex of the two derived subdivisions is the same. We now show that any subdivisions of L and K can give rise to the same G–regular neighborhood.

To do this, we will make use of a specific map that will be helpful in many later contexts. Given a simplicial complex K and a subcomplex L, we define a map $f = f_{L,K} \colon |K| \to [0,1]$ as follows. First let $f(v) = 0$ if $v \in L^0$ and $f(v) = 1$ if $v \in K^0 \setminus L^0$. Now linearly extend f on simplices. Using this map, we may define an ϵ–**neighborhood of** $|L|$ **in** $|K|$ for any $\epsilon \in (0,1)$ as $f^{-1}[0,\epsilon]$.

Lemma 3.1. *Let $Y \subset X$ be G–polyhedra with G–triangulations $L \leq K$ with L full in K. Let $L_1 \leq K_1$ be G–subdivisions of L and K respectively. Then there are derived G-subdivisions K' and K_1' of K and K_1 near L and L_1 such that $|N_{K'}(L)| = |N_{K_1'}(L_1)|$.*

Proof. We follow the proof of the non-equivariant version, Lemma 3.7 in [RS82], and clarify some details with the combinatorial definition of derived subdivisions. Define $f = f_{L,K}$ as above. Choose ϵ small enough so that no vertex of $K_1 \setminus L_1$ is contained in $f^{-1}[0,\epsilon]$. Then choose derived G–subdivisions K' and K_1' of K and K_1 near L and L_1 respectively with all the new vertices v_τ lying in $f^{-1}(\epsilon)$. We can choose these vertices equivariantly because f is G–invariant (L being a G–complex) and K is admissible. Now we will show that $|N_{K'}(L)| = f^{-1}[0,\epsilon] = |N_{K_1'}(L_1)|$.

The map f takes values of 0 or ϵ on all of the vertices of $N_{K'}(L)$ and $N_{K_1'}(L_1)$, so both of these neighborhoods are contained in $f^{-1}[0,\epsilon]$. Now let x be a point in $f^{-1}[0,\epsilon] \subset |K'| = |K_1'|$; say x is in the interior of the simplex $\sigma \cup \{v_{\tau_1}, \ldots, v_{\tau_k}\}$ of K' (respectively K_1') as in the combinatorial definition of a derived subdivision. Then $x = s_0 v_0 + \ldots + s_k v_k + t_1 v_{\tau_1} + \ldots + t_m v_{\tau_m}$ where $\sigma = \{v_0, \ldots, v_k\}$ and $s_0 + \ldots + s_k + t_1 + \ldots + t_m = 1$. Suppose σ is not in L (resp. L_1). Then $f(v_i) = 1$ (resp. $f(v_i) > \epsilon$) for $i = 0, \ldots, k$. Meanwhile, $f(v_{\tau_j}) = \epsilon$ for $j = 1, \ldots, m$. We have then, in both cases, that $f(x) > (s_0 + \ldots + s_k)\epsilon + (t_1 + \ldots + t_m)\epsilon = \epsilon$. This is a contradiction, and σ must be in L (resp. L_1), yielding that x is in $|N_{K'}(L)|$ (resp. $|N_{K_1'}(L_1)|$). □

Now we can prove that any two G–regular neighborhoods are G–homeomorphic.

Theorem 3.2. *If N_1 and N_2 are G–regular neighborhoods of Y in X, then there exists a G–homeomorphism $h\colon X \to_G X$ that maps N_1 to N_2 and is the identity on Y.*

Proof. The proof mirrors that of the non-equivariant version, Theorem 3.8 in [RS82]. We are given two G–triangulations K_1 and K_2 of X with subcomplexes L_1 and L_2 G–triangulating Y. Also, for each i, there is a derived G–subdivision K_i' of K_i near L_i giving $N_i = |N_{K_i'}(L_i)|$. By 2.2, we can find K, a common subdivision of K_1 and K_2 with a subcomplex L triangulating Y, and by 2.3, we can assume K is a G–subdivision.

Applying 3.1, for each i, we can find derived G–subdivisions $\tilde{K}_i$ and K^i of K_i and K near L_i and L respectively such that $|N_{\tilde{K}_i}(L_i)| = |N_{K^i}(L)|$. Therefore we have that $N_1 = |N_{K_1'}(L_1)| \approx_G |N_{\tilde{K}_1}(L_1)| = |N_{K^1}(L)| \approx_G |N_{K^2}(L)| = |N_{\tilde{K}_2}(L_2)| \approx_G |N_{K_2'}(L_2)| = N_2$. Each G–homeomorphism in this sequence comes from a self-G–homeomorphism on X given by simply changing the placement of each derived vertex v_τ within the interior of a simplex $|\tau|$ touching Y while fixing the placement of every other vertex in the subdivision, including those vertices on Y itself. Thus, each step in the chain fixes Y pointwise, so the composition is the identity on Y. $\square$

3.1 G–Collars

Let I be the closed unit interval $[0,1]$. Then for a G-space X, give $X \times I$ the G–action $g(x,t) \mapsto (gx,t)$. If X is a regular cellular G–complex, so too is $X \times I$.

Proposition 3.3. *If X is a regular G-complex, then $X \times I$ has an admissible G–triangulation with no new vertices in $X \times (0,1)$.*

Proof. Let F be the face poset of X. Then the order complex $\Delta(F \times \{0,1\})$ of the product poset $F \times \{0,1\}$ (which is the face poset of $X \times I$) satisfies $|\Delta(F \times \{0,1\})| \approx_G |\Delta F| \times |\Delta\{0,1\}| \approx_G X \times I$. Hence $\Delta(F \times \{0,1\})$ gives the desired G–triangulation. $\square$

Lemma 3.4. *If $L \leq K$ are simplicial G–complexes with $K = L * v$, then for any $\epsilon \in (0,1)$, there exists a G–homeomorphism $h\colon |L| \times [0,\epsilon] \to f^{-1}_{L,K}[0,\epsilon]$, with $h(x,0) = x$.*

Proof. For $x \in |\sigma|$ with $\sigma \in L$ and $t \in [0,\epsilon]$, define $h(x,t) = tv + (1-t)x$. This lies in the simplex $\sigma \cup \{v\} \in K$. The map is G–equivariant since v is a fixed point. The inverse is given by $h^{-1}(u) = (\frac{u-f(u)v}{1-f(u)}, f(u))$. $\square$

Let $Y \subset X$ be G–polyhedra. A G**–collar** on Y in X is a G–embedding $C\colon Y \times I \to_G X$ such that $C(y,0) = y$ and $C(Y \times [0,1))$ is an open neighborhood of Y in X. Suppose that for every $a \in Y$ there are (closed, polyhedral) neighborhoods U and V of a in X and Y respectively, with $U \cap Y = V$, such that for any

$g \in G$, $gU \cap U \neq \emptyset$ implies that $ga = a$ and $gU = U$, and suppose further that $U \approx_{G_a} V \times I$, with $v \mapsto (v, 0)$ on V. Then we say that Y is **locally G–collarable** in X, and we have that $GU \approx_G GV \times I$. Local G–collarability is equivalent to G-collarability using an identical argument to the non-equivariant version, Theorem 2.25 in [RS82]:

Theorem 3.5. *If $Y \subset X$ is locally G-collarable, then there is a G–collar on Y in X.*

Proof. Construct a new G–polyhedron $Z := X \cup Y \times [-1, 0]$ by attaching a G–collar to Y outside of X, identifying $Y \subset X$ with $Y \times \{0\}$. We will construct a G–homeomorphism between X and Z which carries Y to $Y \times \{-1\}$. Then the preimage of $Y \times [-1, 0]$ will be a G–collar on Y in X.

For each $a \in Y$, let $GV_a \times I$ be a local G–collar at a. Using compactness, cover Y with the interiors of finitely many $GV_{a_1}, \ldots, GV_{a_k}$. Then for each i, we will define a G–homeomorphism $h_i \colon Z \to Z$ which maps the interior of $V_{a_i} \times \{0\}$ into the interior of $V_{a_i} \times [-1, 0]$ and is the identity outside of $GV_{a_i} \times [-1, 1]$. To do this, we consider a G–triangulation K_{a_i} of GV_{a_i}. Taking the product of this triangulation with an interval defines a regular G–cell structure on $GV_{a_i} \times [-1, 1]$. We now equivariantly subdivide this cell complex by its order complex: We must choose a new vertex v_c in the interior of each cell c. First, equivariantly select a point y_σ in the interior of each simplex $\sigma \in K_{a_i}$. For the cell $|\sigma| \times \{1\}$, choose $v_c = (y_\sigma, 1)$. Likewise, for the cell $|\sigma| \times \{-1\}$, choose $v_c = (y_\sigma, -1)$. Finally, for the cell $|\sigma| \times [-1, 1]$, choose $v_c = (y_\sigma, 0)$.

Define a different subdivision simply by moving v_c to $(y_\sigma, -\frac{1}{2})$ when c intersects the relative interior of $GV_{a_i} \times [0, 1]$ and making no change in the placement for v_c otherwise. Then the G–homeomorphism h_i is given by mapping the first

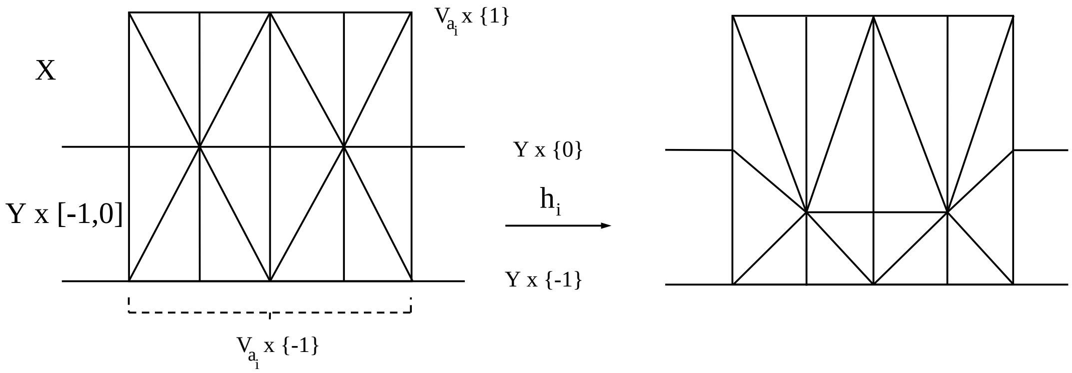

Figure 3.1: Construction of h_i

subdivision to the second, since they are both realizations of the same order complex. Note that h_i is the identity except on the relative interior of $GV_{a_i} \times [-1, 1]$ where $h_i(y, t) = (y, s)$ with $s < t$. Thus, h_i maps all of the relative interior of $GV_{a_i} \times \{0\}$ into $GV_{a_i} \times (-1, 0)$.

Now define h to be the composition $h_k \circ \ldots \circ h_1$. Since the interiors of the local G–collars cover all of Y, h maps all of $Y \times \{0\}$ into $Y \times (-1, 0)$. Let K G–triangulate Y, and thus also $h(Y \times \{0\})$. Then we consider $h(X) \cap [-1, 0]$. It has a regular cellular G–structure with a face poset isomorphic to that of $|K| \times [-1, 0]$. The cells in the former come in three types: simplices of K triangulating $Y \times \{0\}$ (which correspond to the same in the latter complex), simplices of K triangulating $h(Y \times \{0\})$ (which correspond to simplices of K triangulating $Y \times \{-1\}$), and the intersection of $h(X)$ with cells $|\sigma| \times [-1, 0]$ for $\sigma \in K$ (which correspond to the cells $|\sigma| \times [-1, 0]$). Therefore, $h(X) \cap [-1, 0]$ is G–homeomorphic to $Y \times [-1, 0]$, fixing $Y \times \{0\}$. We extend this homeomorphism by the identity to the rest of X to get the desired G–homeomorphism $\tilde{h}\colon X \to Z$ carrying Y to $Y \times \{-1\}$. □

Theorem 3.6. *If $Y \subset X$ is locally G–collarable, then a G–regular neighborhood of Y in X is a G–collar.*

Proof. By 3.5, we have that Y has a G–collar. Let L be an admissible G–triangulation of Y, and K be the G–triangulation of $Y \times I$ as in 3.3. Now choose a derived G–subdivision K' of K near L such that all of the new vertices lie in $Y \times \{\frac{1}{2}\}$. Then $|N_{K'}(L)| = Y \times [0, \frac{1}{2}] \approx_G Y \times I$. The result now follows from 3.2. $\square$

We will also make use of the notion of bicollarability. We say $Y \subset X$ is G–**bicollarable** in X if there exists a G–embedding of $Y \times [-1, 1] \to_G X$ with $(y, 0) \mapsto y$ for all $y \in Y$ and $Y \times (-1, 1)$ maps to an open neighborhood of Y in X.

Theorem 3.7. *If $N = |N_{K'}(L)|$ is a G–regular neighborhood of Y in X, then $|\dot{N}_{K'}(L)|$ is G–bicollarable in X.*

Proof. By 3.2, it suffices to consider the case $N = f^{-1}_{L,K}[0, \epsilon]$ for some $\epsilon \in (0, 1)$. That is, the derived vertices $\{v_\tau\}$ of K' were chosen in $f^{-1}(\epsilon)$. Let $0 < \epsilon_1 < \epsilon < \epsilon_2 < 1$. Equivariantly, choose alternate derived vertices $\{v^1_\tau\}$ and $\{v^2_\tau\}$ in $f^{-1}(\epsilon_1)$ and $f^{-1}(\epsilon_2)$ respectively, giving derived G–subdivisions K'_1 and K'_2 of K near L. Then there are the natural homeomorphisms $h_i \colon |\dot{N}_{K'}(L)| \to_G |\dot{N}_{K'_i}(L)|$ given by sending each v_τ to v^i_τ. We now define a G–bicollar $C \colon |\dot{N}_{K'}(L)| \times [-1, 1] \to_G \mathrm{cl}(|N_{K'_2}(L)| \setminus |N_{K'_1}(L)|)$ by setting

$$C(x,t) = \begin{cases} |t|h_1(x) + (1-|t|)x, & -1 \leq t \leq 0 \\ th_2(x) + (1-t)x, & 0 \leq t \leq 1 \end{cases}$$

$\square$

Note that this G–bicollarability can alternatively be expressed as $|\dot{N}_{K'}(L)|$ being G–collarable in both N and in $\mathrm{cl}(X \setminus N)$.

Lemma 3.8. *If N is a G–invariant neighborhood of Y in X with $N \cap \mathrm{cl}(X \setminus N)$ G–collarable in $\mathrm{cl}(X \setminus N)$, then any admissible G–triangulation L of N can be extended to an admissible G–triangulation K of X (i.e., L is a subcomplex of K).*

Proof. The theorem will follow after we show that two different G–triangulations of a polyhedron Z can be "reconciled" within a G–collar. That is, there exists a G–triangulation of $Z \times I$ inducing the two given triangulations on $Z \times \{0\}$ and $Z \times \{1\}$ respectively.

First, let J be an admissible G–subdivision of Δ^n. Using induction, we will show that there exists a G–triangulation of $|\Delta^n| \times I$ with $|\Delta^n| \times \{0\}$ triangulated by Δ^n and $|\Delta^n| \times \{1\}$ triangulated by J. If $n = 0$, the result is obvious. Otherwise, by the induction hypothesis, we know we can reconcile $\partial\Delta^n$ with ∂J, yielding a triangulation of $|\Delta^n| \times \{0\} \cup |\partial\Delta^n| \times I \cup |\Delta^n| \times \{1\}$. We complete the triangulation by coning with the point $(\overline{\Delta^n}, \frac{1}{2})$.

Now that we can reconcile subdivisions of individual simplices, we can reconcile a simplicial G–complex with a G–subdivision. Let J' be a G–subdivision of J. In increasing order of dimension, we may triangulate each $|J^m| \times I$ to reconcile the skeleta of J with their subdivisions.

Finally, given two G–triangulations J_1 and J_2 of a polyhedron Z, let J be a common G–subdivision. Then by the previous construction, we may G–triangulate $Z \times [0, \frac{1}{2}]$ so that J_1 triangulates $Z \times \{0\}$ and J triangulates $Z \times \{\frac{1}{2}\}$. Likewise, we may triangulate $Z \times [\frac{1}{2}, 1]$ to reconcile J with J_2. Put together, these yield the desired G–triangulation of $Z \times I$.

Now, to prove the result, let $C = Z \times I$ be a G–collar on $Z = N \cap \mathrm{cl}(X \setminus N)$ in $\mathrm{cl}(X \setminus N)$. Let L be an admissibile G–triangulation of N, inducing the triangulation J on $Z = Z \times \{0\}$. Let K_1 be a G–triangulation of $\mathrm{cl}(X \setminus C \cup N)$

with J_1 the induced triangulation of $Z \times \{1\}$. We know we can G–triangulate C with a simplicial complex K_2 to reconcile J with J_1. Now let $K = L \cup K_1 \cup K_2$. $\square$

3.2 G–Regular Neighborhoods in Manifolds

3.2.1 Manifolds and G–Manifolds

A polyhedron X is a **(PL)** n**–manifold** (with boundary) if every point $x \in X$ has a closed (polyhedral) neighborhood PL-homeomorphic to the n–disk $|\Delta^n|$. If, under this map, x lies in $|\partial\Delta^n|$, then we say $x \in \partial X$. Then ∂X is an $(n-1)$–manifold: We may choose a sufficiently small neighborhood of x in $|\partial\Delta^n|$ that contains only points of ∂X and is homeomorphic to Δ^{n-1}.

A simplicial complex K is a **combinatorial** n**–manifold** if for every simplex $\sigma \in K$, $|\operatorname{lnk}_K(\sigma)|$ is PL-homeomorphic to $|\partial\Delta^{n-\dim\sigma}|$ or to $|\Delta^{n-\dim\sigma-1}|$. The latter case means σ lies on the boundary of K, ∂K.

To define equivariant manifolds, we consider G–polyhedra and G–complexes that are manifolds and that have particularly well-behaved G–actions.

Consider an orthogonal representation $\rho\colon G \to O_n(\mathbb{R})$. We denote by $S(\rho)$ and $D(\rho)$ the unit sphere and disk respectively in the corresponding representation space. Further, denote by $S_+(\rho)$ the hemisphere with final coordinate nonnegative and similarly for $D_+(\rho)$. Each of these has a unique piecewise-linear structure coming from its smooth structure [I00].

We now inductively define a **combinatorial** G**–sphere**. S^0 with a G–action is a combinatorial 0–dimensional G–sphere. An admissible simplicial G–complex K with $|K|$ G–homeomorphic to $S(\rho)$ for some $\rho\colon G \to O_{n+1}(\mathbb{R})$ is an n–dimensional combinatorial G–sphere if for every $v \in K^0$, $\operatorname{lnk}_K(v)$ is an $(n-1)$–dimensional combinatorial G_v–sphere, itself G_v–homeomorphic to $S(\mathbb{R}v^\perp)$, where $\mathbb{R}v^\perp$ is the orthogonal complement in $\rho|_{G_v}$ of the trivial representation $\mathbb{R}v$.

Similarly, we may define a **combinatorial G–hemisphere** by substituting $S_+(\rho)$ and allowing links of vertices to be n–dimensional G–spheres or G–hemispheres in the above definition. Finally, a **combinatorial G–disk** is simply the cone on a G–sphere or G–hemisphere with a G–fixed point.

A simplicial G–complex K is an n–dimensional **combinatorial G–manifold** if for every $v \in K^0$, $\operatorname{lnk}_K(v)$ is an $(n-1)$–dimensional combinatorial G_v–sphere or hemisphere. When its link is a hemisphere, a vertex lies on the boundary of K.

A G–polyhedron X is an n–dimensional **(PL) G–manifold** (with boundary) if every point $x \in X$ has a closed neighborhood U_x which is G_x–homeomorphic to the geometric realization of a combinatorial n–dimensional G_x–disk with x corresponding to the point 0. If x lies on the boundary of U_x, then $x \in \partial X$ and U_x must have been the cone on a hemisphere. Taking V_x to be the cone on the equator gives a closed neighborhood of x in ∂X G_x–homeomorphic to an $(n-1)$–G–sphere so that ∂X is an $(n-1)$–dimensional G–manifold.

We make the following observation.

Lemma 3.9. *Let K be an n–dimensional combinatorial G–sphere or hemisphere. If there exists a G–fixed vertex $y \in K^0$, then the cone $|x * K|$ (with x a G–fixed point) is G–homeomorphic to $|\operatorname{st}_K(y)| \times I$.*

Proof. Let $\rho\colon G \to O_n(\mathbb{R})$ be the representation for which $|K|$ is a G–sphere or hemisphere. Then we have that $\mathbb{R}y$ is a trivial subrepresentation of ρ. Also, $|\operatorname{lnk}_K(y)|$ is G–homeomorphic to either $S(\mathbb{R}y^\perp)$ or $S_+(\mathbb{R}y^\perp)$. Call this sphere or hemisphere S. Thus, we have that $|K|$ is G–homeomorphic to either a cone on S

with y (which is G–homeomorphic to the union of $S \times I$ with the cone of $S \times \{0\}$ with y) or a suspension of S by the points y and $-y$ in $\mathbb{R}y$ (G–homeomorphic to the union of $S \times I$ with cones on both $S \times \{0\}$ and $S \times \{1\}$). In either case, coning with a G–fixed point x yields a G–disk G–homeomorphic to $(S * y) \times I$, as required. $\square$

We will prove shortly that combinatorial G–manifolds are exactly the G–triangulations of PL G–manifolds. To do this, we will require the following lemma.

Lemma 3.10. *Let v be a vertex of a simplicial G–complex K. If K' is a derived G–subdivision of K near v, then $|\operatorname{lnk}_K(v)|$ is G_v-homeomorphic to $|\operatorname{lnk}_{K'}(v)|$.*

Proof. Assume that the derived vertices are chosen in $f_{v,K}^{-1}(\epsilon)$ for some $\epsilon \in (0,1)$. Then a point in $|\operatorname{lnk}_{K'}(v)| = f^{-1}(\epsilon)$ is of the form $\epsilon u + (1-\epsilon)v$ where $u \in |\operatorname{lnk}_K(v)|$. Mapping this point to u gives the desired homeomorphism. $\square$

Lemma 3.11. *In an n–dimensional combinatorial G–manifold K, $lnk_K(\sigma)$ is a combinatorial G_σ–sphere or hemisphere of dimension $n - \dim\sigma - 1$ for any nonempty $\sigma \in K$.*

Proof. We use induction on $\dim\sigma$. If σ is a vertex, the result is true by definition. If $\dim\sigma > 0$, let v be a vertex of σ and $\tau = \sigma \setminus \{v\}$. Then $\operatorname{lnk}_K(\sigma) = \operatorname{lnk}_{\operatorname{lnk}_K(\tau)}(v)$. Let $H = G_\tau$. By hypothesis, $\operatorname{lnk}_K(\tau)$ is an $(n - \dim\tau - 1)$–dimensional H–sphere or hemisphere. Therefore, the link $\operatorname{lnk}_{\operatorname{lnk}_K(\tau)}(v)$ is an $(n - \dim\tau - 2) = (n - \dim\sigma - 1)$–dimensional H_v–sphere or hemisphere. Since K is admissible, $H_v = G_\sigma$. $\square$

In light of 3.11, it is now convenient to characterize ∂K: It is the collection of simplices of K whose links are hemispheres. ∂K is a subcomplex of K: Consider

$\sigma \in \partial K$. Let $\sigma = \tau \cup \{v\}$. Suppose $\operatorname{lnk}_K(\tau)$ is a sphere. Then $\operatorname{lnk}_K(\sigma) = \operatorname{lnk}_{\operatorname{lnk}_K(\tau)}(v)$ must be a sphere, contradicting $\sigma \in \partial K$. Repeating this argument shows that the link of any face of σ must be a hemisphere, so each face of σ lies in ∂K.

Proposition 3.12. *If X is a G–polyhedron triangulated by an admissible G–complex K, then X is an n–dimensional G–manifold if and only if K is an n–dimensional combinatorial G–manifold. When both are manifolds, ∂K G–triangulates ∂X.*

Proof. Suppose first that K is a combinatorial G–manifold. Then any $x \in X$ lies in the interior of some simplex $\sigma \in K$. Then $\operatorname{lnk}_K(\sigma)$ is an $(n - \dim\sigma - 1)$–G_σ–sphere by 3.11. Since K is admissible, $G_\sigma = G_x$. Define $U_x = |\operatorname{lnk}_K(\sigma) * \partial\sigma * x|$. This is an n–G_x–disk since $\partial\sigma$ is fixed pointwise by G_x. Therefore, X is an n–G–manifold. Note that x lies in the boundary of U_x if and only if $\operatorname{lnk}_K(\sigma)$ was a G–hemisphere. Thus $\partial X = |\partial K|$.

Now suppose that X is an n–manifold with triangulation K. Let v be a vertex of K and U_v be a closed neighborhood of v in X which is a G_v–disk.

Assume first that v is not in ∂X. Consider a triangulation of X with U_v triangulated as a subcomplex. We may alter this triangulation by replacing the subcomplex triangulating U_v with the cone on its boundary by v. Call this new triangulation K_1. Let K' and K_1' be derived subdivisions of K and K_1 respectively near v. Then the stars of v in these two subdivisions are both G_v–regular neighborhoods of v in X and therefore G_v–homeomorphic by 3.2, and thus so are the links. We also have by 3.10, that the G_v–homeomorphism type of the link of v is invariant under derived subdivision. Therefore, we have that $|\operatorname{lnk}_K(v)| \approx_{G_v} |\operatorname{lnk}_{K_1}(v)|$, an $(n-1)$–G_v–sphere.

If v is in ∂X and thus ∂U_v, then we obtain K_1 by triangulating U_v with a cone on a subdivision of a G_v–hemisphere with v. Then following the same logic as before, we conclude that $|\operatorname{lnk}_K(v)| \approx_{G_v} |\operatorname{lnk}_{K_1}(v)|$, an $(n-1)$–G_v–hemisphere. Thus, we have that K is a combinatorial G–manifold. $\square$

Proposition 3.13. *If M is a G–manifold, then ∂M is G–collarable in M.*

Proof. By 3.5, it suffices to show that ∂M is locally G–collarable.

Let x be a point in ∂M. Let K be a G–triangulation of M with x as a vertex and L the subcomplex triangulating ∂M. By subdividing, we can ensure that $\operatorname{st}_K(gx) \cap \operatorname{st}_K(x) \neq \emptyset$ happens only when $gx = x$.

We have that $\operatorname{lnk}_K(x)$ is an $(n-1)$–G_x–hemisphere $S_+(\rho)$ and $\operatorname{lnk}_L(x)$ is an $(n-2)$–G_x–sphere $S(\rho_1)$, for a subrepresentation ρ_1 of ρ. Then $\rho_1^\perp$ must be the trivial representation since the upper hemisphere is fixed. Therefore, $|\operatorname{st}_K(x)| \approx_{G_x} |\operatorname{st}_L(x) * w|$ for some G_x–fixed point w. The latter cone is G_x–homeomorphic to $|\operatorname{st}_L(x)| \times I$. Extending this construction to the G–orbit of x yields a local G–collar. $\square$

The next proposition provides us with a simple way to find n–dimensional submanifolds in a G–manifold which are themselves G–manifolds. We will require the following lemma.

Lemma 3.14. *A quadrant of a G–sphere in which the two nonnegative coordinates give trivial subrepresentations is a G–hemisphere.*

Proof. Consider a G–sphere S^1 with the trivial G–action. We will define a piecewise-linear function $S^1 \to S^1$ sending the first quadrant homeomorphically to the upper hemisphere. Triangulate S^1 in two ways: Let K be an octagon on the vertex set $\mathbb{Z}_8$ with 1–simplices $\{i, i+1\}$ for $0 \leq i \leq 7$; within this triangulation, the edges $\{0, 1\}$ and $\{1, 2\}$ form the first quadrant. Let L be the square with vertex set $\mathbb{Z}_4$ with 1–simplices $\{i, i+1\}$ for $0 \leq i \leq 3$; the upper hemisphere consists of the edges $\{0, 1\}$ and $\{1, 2\}$. Then we map K onto L via the two-fold covering sending i to $i \bmod 4$.

Now, if ρ is a trivial, 2–dimensional subrepresentation of some representation of G, $S(\rho \oplus \rho^{\perp})$ is G–homeomorphic to $S(\rho) * S(\rho^{\perp})$. Mapping the first quadrant of $S(\rho)$ to the upper hemisphere as above maps the first quadrant of $S(\rho) * S(\rho^{\perp})$ to its upper hemisphere. $\square$

Proposition 3.15. *Let M be an n–dimensional G–manifold and M_1 be an n–dimensional G–invariant submanifold with* $\mathrm{cl}(\partial M_1 \cap \mathit{int}\ M)$ *G–bicollarable in M. Then M_1 is a G–manifold.*

Proof. Let K be a G–triangulation of M with subcomplexes (K_1, L) triangulating $(M_1, \partial M_1)$. We need to show that the link of any vertex $v \in K_1^0$ is a G_v–sphere or hemisphere. We consider the case $v \in \mathrm{cl}(\partial M_1 \cap \mathrm{int}\ M)$. The link of any other vertex of K_1 is the same in both K and K_1.

Since $\mathrm{cl}(\partial M_1 \cap \mathrm{int}\ M)$ is G–bicollarable, we may consider a closed G_v–invariant neighborhood $|\,\mathrm{st}_L(v)| \times [-1, 1] = U_v$ with $(x, 0) = x$ for all $x \in |\,\mathrm{st}_L(v)|$ and $|\,\mathrm{st}_{L_1}(v)| \times [0, 1] \subset M_1$. Triangulate U_v in the following way: First triangulate $|\,\mathrm{lnk}_L(v)| \times [-1, 1]$. Add to it by coning $|\,\mathrm{lnk}_L(v)| \times \{-1\}$ and $|\,\mathrm{lnk}_L(v)| \times \{1\}$ with $(v, -1)$ and $(v, 1)$ respectively. Let J be this triangulation of $|\,\mathrm{lnk}_L(v)| \times [-1, 1] \cup |\,\mathrm{st}_L(v)| \times \{-1, 1\}$. Then $J * v$ triangulates $|\,\mathrm{st}_L(v)| \times [-1, 1]$.

Now subdivide K so that it contains a subdivision of $J * v$ as a subcomplex. Choose a derived G_v–subdivision K' near v. From the construction in the proof of 3.1, we may assume $|\,\mathrm{st}'_K(v)|$ is an ϵ–neighborhood of v in $|J * v|$. Thus, by

3.10, $|\operatorname{lnk}_L(v)| \times [-1,1] \cup |\operatorname{st}_L(v)| \times \{-1,1\}$ is G_v–homeomorphic to $|\operatorname{lnk}'_K(v)|$, which we know is an $(n-1)$–G_v–sphere or hemisphere since M is a G–manifold.

Consider the point $w = (v,1)$. Its stabilizer in G_v is all of G_v. Hence, $\operatorname{lnk}_J(w)$ is an $(n-2)$–G_v–sphere or hemisphere. But $|\operatorname{lnk}_J(w)|$ is G_v–homeomorphic to $|\operatorname{lnk}_L(v)|$, giving us that $|J|$ is the suspension of $|\operatorname{lnk}_L(v)|$ by w and $(v,-1)$, and thus a G_w–sphere or hemisphere. In conclusion, $|\operatorname{lnk}_L(v)| \times [0,1] \cup |\operatorname{st}_L(v)| \times \{1\}$, which is G_v–homeomorphic to $|\operatorname{lnk}_{K_1}(v)|$ is a G_v–hemisphere. (Note that when $\operatorname{lnk}_L(v)$ is a G_v–hemisphere, $|\operatorname{lnk}_L(v)| \times [0,1] \cup |\operatorname{st}_L(v)| \times \{1\}$ is a quadrant of a G_v–sphere where the two nonnegative coordinates give trivial subrepresentations. By 3.14, this is a G_v–hemisphere.) $\square$

3.2.2 Simplicial G–Neighborhood Theorem

G–regular neighborhoods are particularly well-behaved within G–manifolds. We begin with some simple cases.

Lemma 3.16. *A G–regular neighborhood of a point x in an n–dimensional G–manifold M is an n–G_x–disk.*

Proof. In any derived subdivision K' near x, the $N_{K'}(x)$ is $\operatorname{st}_{K'}(x)$ which is a G–disk of dimension n. $\square$

We will use the following two non-equivariant facts.

Lemma 3.17. *Let σ be a proper face of Δ^n. Then a regular neighborhood of $|\sigma|$ in $|\Delta^n|$ is an n–disk.*

Proof. Let $f = f_{\sigma,\Delta^n}$. Then let K be a derived subdivision of Δ^n near σ along $f^{-1}(\epsilon)$, so that $f^{-1}[0,\epsilon]$ is a regular neighborhood of $|\sigma|$ as in the proof of 3.1. It remains to show that $f^{-1}[0,\epsilon]$ is convex and n–dimensional. Let x and y

lie in $f^{-1}[0,\epsilon]$. Then since f is linear on the simplex Δ^n, $f(tx+(1-t)y) = tf(x)+(1-t)f(y) \le \epsilon$, proving that the ϵ neighborhood is convex. It is n–dimensional because it is a closed neighborhood in the n–manifold $|\Delta^n|$, so any point in its interior has an n–disk neighborhood. □

Corollary 3.18. *Let σ be a face of $\partial\Delta^n$. Then a regular neighborhood of $|\sigma|$ in $|\partial\Delta^n|$ is an $(n-1)$–disk.*

Proof. Let K be the derived subdivision of Δ^n from 3.17. Then $\mathrm{lnk}_K(v_{\Delta^n})$ is a derived subdivision of $\partial\Delta^n$ near σ, and in this subdivision $\mathrm{lnk}_{N_K(\sigma)}(v_{\Delta^n})$ is the simplicial neighborhood of σ. Since $N_K(\sigma)$ is an n–manifold with v_{Δ^n} on its boundary (since it lies in $f^{-1}(\epsilon)$), $\mathrm{lnk}_{N_K(\sigma)}(v_{\Delta^n})$ is an $(n-1)$–disk. □

We will assume for the remainder of the chapter that $Y \subset M$ are polyhedra with M an n–manifold. Whenever M is a G–manifold, we will assume Y is G–invariant.

The first important property of regular neighborhoods within a G–manifold is that they are themselves G–manifolds of the same dimension.

Proposition 3.19. *A G–regular neighborhood N of Y in an n–G–manifold M is an n–G–manifold with boundary. If $Y \subset int\ M$, then $\partial N = |\dot{N}_{K'}(L)|$ where L and K' are as in the definition of regular neighborhood.*

Proof. By 3.7 and 3.15, it suffices to show that a G–regular neighborhood in M is an n–manifold.

Let $K_N = N_{K'}(L)$ be the induced triangulation of N in K', the derived subdivision of K near L. For $v \in L^0$, $\mathrm{lnk}_{K_N}(v) = \mathrm{lnk}_{K'}(v)$ which is an $(n-1)$–disk or sphere since K' is a combinatorial n–manifold.

For $v \in \dot{N}_{K'}(L)$, $v = v_\tau$ for some $\tau \in K \setminus L$ with $\tau \cap L^0 \neq \emptyset$. Then $\mathrm{lnk}_{K_N}(v_\tau)$ consists of simplices of the form $\rho \cup \{v_{\tau_1}, \ldots, v_{\tau_m}\}$ where $\rho \in L$, each $\tau_i \in K \setminus L$ with $\tau_i \cap L^0 \neq \emptyset$, and τ can be inserted somewhere into the chain $\rho < \tau_1 < \ldots < \tau_m$. Such a simplex is a join of two parts: $\rho \cup \{v_{\tau_1}, \ldots, v_{\tau_k}\}$ with $\tau_k < \tau$ and $\{v_{\tau_{k+1}}, \ldots, v_{\tau_m}\}$ with $\tau < \tau_{k+1}$. The first is a typical simplex of the simplicial neighborhood of $\sigma = \tau \cap L^0$ in the derived subdivision of $\partial\tau$ near σ. This simplicial neighborhood is a $(\dim\tau - 1)$–disk by 3.18. The second part $\{v_{\tau_{k+1}}, \ldots, v_{\tau_m}\}$ is in one-to-one correspondence to the simplex $\tau_{k+1} \setminus \tau < \ldots < \tau_m \setminus \tau$ in the barycentric subdivision of $\mathrm{lnk}_K(\tau)$, which is an $(n - \dim\tau - 1)$–sphere by 3.11 (It cannot be a disk because, from the definition of a regular neighborhood, τ must lie on the boundary of $|N_K(L)|$ in M so it does not lie on ∂M). Therefore, we have that $\mathrm{lnk}_{K_N}(v_\tau)$ is the join of a $(\dim\tau - 1)$–disk and an $(n - \dim\tau - 1)$–sphere and hence an $(n-1)$–disk.

Therefore, K_N is a combinatorial n–manifold. We now prove the last part of the proposition by noting that if Y lies in the interior of M, the link of any simplex of L in K_N will be a sphere. Also, the link of any simplex $\{v_{\tau_0}, \ldots, v_{\tau_m}\}$ of $\dot{N}_{K_N}(L)$ will include will be a join of complexes, one of which is the regular neighborhood of $\tau_0 \cap L^0$ in $\partial\tau_0$, a disk. Thus, the link must also be a disk. This proves that $\partial N = |\dot{N}_{K'}(L)|$. $\square$

We now come to the first characterization theorem for G–regular neighborhoods in the interior of a manifold. With suitable conditions, any G–invariant simplicial neighborhood turns out to be a G–regular neighborhood, not just in a derived G–subdivision.

Theorem 3.20 (Simplicial G–Neighborhood Theorem)**.** *Suppose N is a G–invariant neighborhood of a G–polyhedron Y in the interior of n–dimensional*

G–manifold M. Then N is a G–regular neighborhood of Y if and only if

(i) N is an n–manifold with boundary ∂N G–bicollarable in M,

(ii) there are admissible G–triangulations $(K,\ L)$ of $(M,\ Y)$ with L full in K, such that $N = |N_K(L)|$ and $\partial N = |\dot{N}_K(L)$

Proof. If N is a G–regular neighborhood of Y, the two conditions follow immediately from the definition, 3.19 and 3.7.

For the other direction, following the proof of the nonequivariant version (Theorem 3.11 in [RS82]), we construct a series of G–collars. Since ∂N is G–bicollarable in M, we may find a G–collar in $\mathrm{cl}(M \backslash N)$, denoting it $C_1 = \partial N \times [0,1]$ with $\partial N = \partial N \times \{1\}$.

Now choose a derived G–subdivision K' of K near L. Then $|N_{K'}(\dot{N}_K(L)|$ is a G–regular neighborhood of ∂N in N, so by 3.6 it is a G–collar $C_2 = \partial N \times [1,2]$. Finally, $N' = |N_{K'}(L)|$ is a G–regular neighborhood of Y in M and therefore, by 3.7, we can find a G–collar on its boundary $|\dot{N}_{K'}(L)| = |\dot{N}_{K'}(\dot{N}_K(L))| = \partial N \times \{2\}$. Call this G–collar $C_3 = \partial N \times [2,3]$. Let $C = \partial N \times [0,3]$ be the union of the three G–collars.

Define a G–homeomorphism $h\colon C \to C$ by

$$h(x,t) = \begin{cases} (x, \frac{t}{2}), & 0 \leq t \leq 2 \\ (x, 2t-3), & 2 \leq t \leq 3 \end{cases}$$

This is the identity on $\partial N \times \{0,3\}$, so it may be extended to all of M. Then we see that $h(N') = N$. Therefore, N is a G–regular neighborhood of Y. □

The following three corollaries are nonequivariant results directly from [RS82]. There is no additional need for equivariant versions.

Corollary 3.21. *If D is an n–disk in the interior of an n–manifold M, then D is a (nonequivariant) regular neighborhood of any point in its interior.*

Proof. Simply replacing the triangulation of D by coning ∂D with the interior point realizes D as a simplicial neighborhood of that point. $\square$

Corollary 3.22. *If D is an n-disk in an n-sphere S, then* $\mathrm{cl}(S \setminus D)$ *is an n-disk.*

Proof. D must intersect a maximal face $|\Delta^{n-1}|$ of the n–sphere $S = |\Delta^n|$. Choose a point x in $D \cap |\Delta^{n-1}|$. Both disks are regular neighborhoods of x, so there is a homeomorphism of S taking D to $|\Delta^{n-1}|$ by 3.2. This sends $\mathrm{cl}(S \setminus D)$ to $\mathrm{cl}(|\Delta^n| \setminus |\Delta^{n-1}|)$, which is clearly an n–disk. $\square$

Corollary 3.23. *If $N \subset$ int M are n–manifolds, then* $\mathrm{cl}(M \setminus N)$ *is an n–manifold with boundary the disjoint union of ∂M and ∂N.*

Proof. Let $L \leq K$ triangulate N and M respectively. Let K_1 be the subcomplex of K triangulating $\mathrm{cl}(M \setminus N)$. For $v \in K^0 \setminus L^0$, $\mathrm{lnk}_{K_1}(v) = \mathrm{lnk}_K(v)$, so it is an $(n-1)$–disk or sphere. For $v \in \partial L^0$, $|\mathrm{lnk}_{K_1}(v)| = \mathrm{cl}(|\mathrm{lnk}_K(v)| \setminus |\mathrm{lnk}_L(v)|$. This is an $(n-1)$–disk by 3.22. We further note that for a simplex σ in K_1, $\mathrm{lnk}_{K_1}(\sigma)$ is a disk only when σ is in ∂L or ∂K, by the same reasoning as for vertices. The two boundaries are disjoint since N lies in the interior of M. $\square$

Corollary 3.24. *If $N_1 \subset$ int N_2 are two G–regular reighborhoods of Y in the interior of a G–manifold M, then* $\mathrm{cl}(N_2 \setminus N_1)$ *is a G–collar on ∂N_2.*

Proof. Following the nonequivariant proof (Corollary 3.18 in [RS82]), let K_1 and K_2 be admissible G–triangulations of M yielding N_1 and N_2 respectively as the simplicial neighborhoods of Y as in the Simplicial G–Neighborhood Theorem 3.20. Choose a derived G–subdivision K of K_2 near Y. Let N be the resulting

G–regular neighborhood of Y. Then, as we saw in the proof of 3.20, $\mathrm{cl}(N_2 \setminus N)$ is a G–collar on ∂N_2. Then both N and N_1 are G–regular neighborhoods of Y in N_2, so there is a G–homeomorphism of N_2 sending N_1 to N, and hence $\mathrm{cl}(N_2 \setminus N_1)$ to $\mathrm{cl}(N_2 \setminus N)$. □

3.2.3 G–Collapsing Criterion

Let $M_1 \subset M$ be n–G–manifolds with an elementary G–collapse from $M = M_1 \cup GD^n$ to M_1 such that (1) $D^n \cap M_1 = D^{n-1}$ lies in a G–collarable subpolyhedron $W \subseteq \partial M_1$, (2) under the G_{D^n}–triangulation $K = y * L$ of D^{n-1}, if $gD^n \neq D^n$, then $gD^n \cap D^n \subset |L| \times \{0\}$, and (3) $y \in \partial D^{n-1}$ implies that, in the G–collar on W, $|y * \partial L| \times I \subset \partial M$. Then this collapse is called an **elementary G–shelling**, and we call a sequence of elementary G–shellings a **G–shelling**.

While collapsing only shows homotopy equivalence in general, shellings show homeomorphic equivalence.

Lemma 3.25. *If M G–shells to M_1, then there is a G–homeomorphism $h\colon M \to M_1$ which is the identity outside a given neighborhood of $M \setminus M_1$.*

Proof. As in the corresponding proof of Lemma 3.25 in [RS82], we need only to consider the case of an elementary G–shelling. Let $M = M_1 \cup GD^n$ give the elementary G–shelling. Denote G_{D^n} by H. Let $K = y * L$ be the H–triangulation of D^{n-1} from the definition of elementary G–collapse.

Choose a G–collar on GD^{n-1} in M_1 within the given neighborhood of $M \setminus M_1$. We may consider the disk $D^{n-1} \times [-1, 1]$ with $D^n = D^{n-1} \times [-1, 0]$, $E^n = D^{n-1} \times [0, 1] \subset M_1$, and $D^{n-1} = D^{n-1} \times \{0\}$. Then if $D^n \neq gD^n$, we have that $D^{n-1} \times [-1, 1]$ may only intersect $gD^{n-1} \times [-1, 1]$ in $|L| \times [0, 1]$. We will define an H–homeomorphism from $D^{n-1} \times [-1, 1]$ to E^n which is the identity

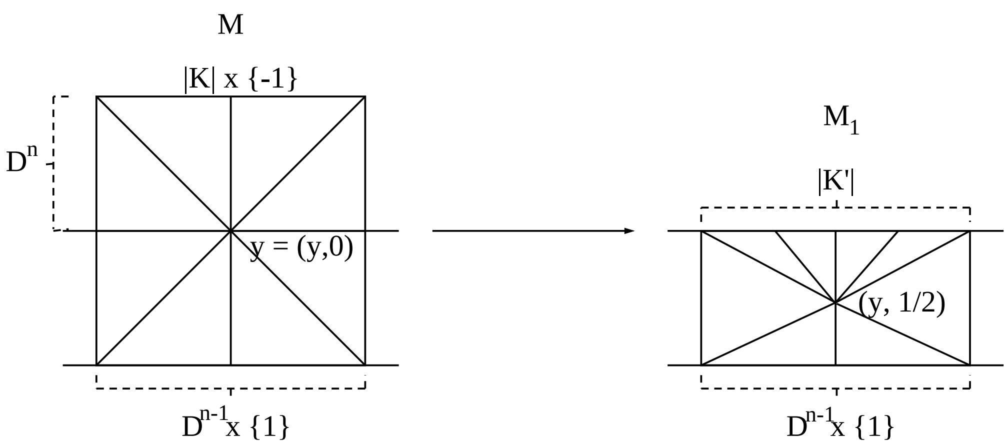

Figure 3.2: Shelling homeomorphism

on $|L| \times [0,1] \cup |K| \times \{1\}$; such a homeomorphism can then be extended, first equivariantly to all of $G(D^{n-1} \times [-1,1])$ and then by the identity to the rest of M. This last extension is possible because either $|L| \times I = \partial D^{n-1} \times I$ or $\mathrm{cl}(\partial D^{n-1} \setminus |L|) \times I = |y * \partial L| \times I \subset \partial M$.

Let K' be a derived H–subdivision of K near y. We have that $|K| \times \{-1\} \cup |L| \times [-1,0]$ is H–homeomorphic to $|K'| = |N_{K'}(y)| \cup |N_{K'}(L)|$ because they are both H–homeomorphic to $|K|$ with an H–collar attached outside to $|L|$. Therefore, we have an H–homeomorphism from $D^{n-1} \times \{-1,1\} \cup |L| \times [-1,1]$ to $D^{n-1} \times \{0,1\} \cup |L| \times [0,1]$. Coning the two polyhedra with $(y,0)$ and $(y, \frac{1}{2})$ gives the desired H–homeomorphism from $E^n \cup D^n$ to E^n. $\square$

Coupled with 3.25, the next theorem will show that when X and Y differ only by G–collapses, their G–regular neighborhoods are G– homeomorphic.

Theorem 3.26. *Suppose $Y \subseteq X$ are G–polyhedra in a G–manifold M. If $X \searrow_G Y$, then a G–regular neighborhood of X G–shells to a G–regular neighborhood of Y in M.*

Proof. We follow the proof of the nonequivariant version, Theorem 3.26 in [RS82], checking that the conditions of G–shelling are satisfied. The proof uses induction on the dimension of the collapse from X to Y.

Suppose that the theorem holds when the G–collapse is $(m-1)$–dimensional. We now consider the case where there is an m–dimensional elementary G–collapse from X to Y. Let $X = Y \cup GD^m$, with $Y \cap D^m = D^{m-1} \times \{0\}$ where $D^m \approx_{G_{D^m}} D^{m-1} \times I$. For simplicity, we will from now on denote the subgroup G_{D^m} by H.

Let K be an admissible G–triangulation of M with full subcomplexes $L_2 \leq L_1$ triangulating Y and X respectively. Denote by J the subcomplex triangulating $Z = D^{m-1} \times \{1\} \subset D^m$, and by GJ, the resulting G–triangulation of its G–orbit, GZ. Finally, let y be the apex in the G_{D^m}–cone structure of D^{m-1}. Note then that $\{y\} \times I$ is fixed pointwise by H, and any point (y,t) with $t > 0$ has stabilizer exactly H. Let $x = (y, \frac{1}{2})$.

We may assume that there are no vertices of K^0 in $D^{m-1} \times (0,1)$: To see this, consider the projection $p\colon D^{m-1} \times [0,1] \to [0,1]$. We may subdivide ($H$–equivariantly) to make the map simplicial, so that I is partitioned into subintervals $[0 = \epsilon_0, \epsilon_1], [\epsilon_1, \epsilon_2], \ldots, [\epsilon_{k-1}, 1 = \epsilon_k]$. Then there is an elementary H–collapse of $p^{-1}[0, \epsilon_i]$ to $p^{-1}[0, \epsilon_{i-1}]$ for $1 \leq i \leq k$. Each one of these individual collapses and their orbits satisfies our assumption.

Now we choose a derived G–subdivision K' of K near $L_2 \cup GJ$. Choose the derived vertices for simplices in $L_1 \setminus (L_2 \cup GJ)$ in $Gp^{-1}(\frac{1}{2})$ ensuring that x is one of them, and denote by L' the new triangulation of X. Then $N_{K'}(L')$ gives a G–regular neighborhood of X, which is the union of $N_{K'}(L_2)$ and $N_{K'}(GJ)$, G–regular neighborhoods of Y and GZ respectively. By 2.5, there is an $(m-1)$–dimensional H–collapse from $|J|$ to $(y,1)$, so the induction hypothesis, 3.16, and 3.25 together imply that $|N_{K'}(J)|$ is an n–dimensional H–disk. Let $E^n =$

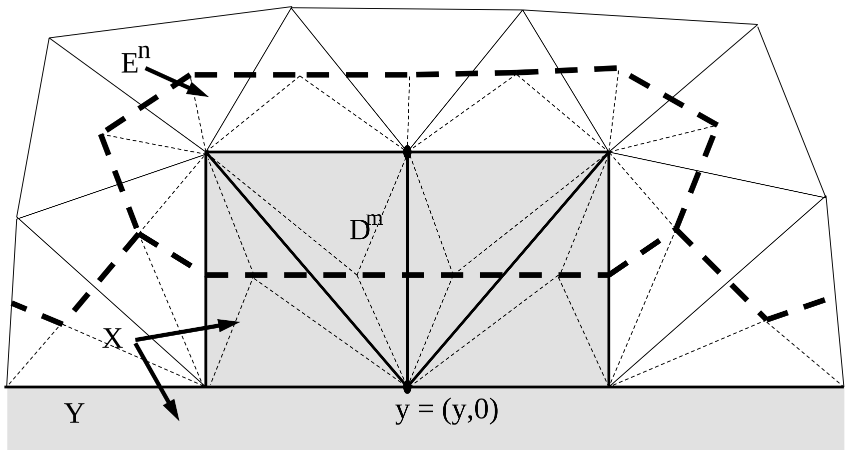

Figure 3.3: Regular neighborhood shelling

$|N_{K'}(J)|$. By 3.9, E^n is H–homeomorphic to $|\operatorname{st}_{\dot{N}_{K'}(J)}(x)| \times I$.

We will show that if $N_{K'}(gJ) \neq N_{K'}(J)$, the two subcomplexes must be disjoint. For such a g, suppose there exists a vertex $v = v_\tau \in N_{K'}(gJ) \cap N_{K'}(J)$. (Note that it must be a derived vertex since gJ and J are themselves disjoint.) Then $\tau \in K$ contains vertices u and w of gJ and J respectively. Thus, $\rho = \{u, w\} \in L_1$ since L_1 is a full subcomplex of K, but $|\rho|$ is not contained in Y and it is not contained in GD^m since a simplex in D^m may only contain vertices from L_2 and J, not gJ. This contradicts $X = Y \cup GD^m$, so $N_{K'}(gJ) \cap N_{K'}(J)$ must be empty. Since we have shown that $gE^n \neq E^n$ implies $gE^n \cap E^n = \emptyset$, it must be true that $G_{E^n} = H$.

We next prove that $|N_{K'}(J)| \cap |N_{K'}(L_2)|$ is an $(n-1)$–disk E^{n-1} which is H–homeomorphic to $|\operatorname{st}_{\dot{N}_{K'}(J)}(x)|$, giving that E^n is H–homeomorphic to $E^{n-1} \times I$ as required. To see this, we show that E^{n-1} is an H–regular neighborhood of $D^{m-1} \times \{\frac{1}{2}\}$ in the $(n-1)$–H–manifold $|\dot{N}_{K'}(J)|$, so that we may again invoke the induction hypothesis for $(m-1)$–dimensional collapses and 3.25 (since $D^{m-1} \times \{\frac{1}{2}\}$ H–collapses to x and an H–regular neighborhood of x is the desired star of x).

Let P be the subcomplex of K' triangulating $D^{m-1} \times \{\frac{1}{2}\}$ and let $Q = \dot{N}_{K'}(J)$ for brevity. The claim then is that $N_Q(P) = N_{K'}(L_2) \cap N_{K'}(J)$.

Let $\sigma \in N_{K'}(L_2) \cap N_{K'}(J)$, we easily see that σ cannot intersect L_2^0 or J^0 and must consist only of derived vertices of the form v_ρ. Then there must exist $u \in L_2^0$ and $w \in J^0$ such that $\sigma \cup \{u\}$ and $\sigma \cup \{w\}$ are both simplices of K'. This implies that there exists $v_\rho \in \sigma$ for some $\rho \in K$ containing both u and w. But then $\{u, w\} \in L_1$ due to the fullness of L_1. Thus, $v_{\{u,w\}}$ is in P and can be added to σ, so $\sigma \in N_Q(P)$. Hence, we have $N_{K'}(L_2) \cap N_{K'}(J) \subseteq N_Q(P)$.

For the other inclusion, if σ is in $N_Q(P)$, it means that there is a $v_\tau \in P^0$ such that $\sigma \cup \{v_\tau\}$ is in Q for some τ which contains vertices from both L_2 and J. We note again that σ consists only of derived vertices since it is in $Q = \dot{N}_{K'}(J)$, so let ρ be the minimal face such that $v_\rho \in \sigma \cup \{\tau\}$. Then $\rho \leq \tau$, so we have that $\rho \in L_1$. Since ρ was subdivided, it must contain some vertex $u \in L_2^0$. Therefore, u may be added to σ to get a simplex of K' intersecting L_2^0, i.e., $\sigma \in N_{K'}(L_2)$, and it is already in $Q \subset N_{K'}(J)$. This proves that $N_{K'}(L_2) \cap N_{K'}(J) = N_Q(P)$. This finishes the proof that E^{n-1} is an H–regular neighborhood of $D^{m-1} \times \{\frac{1}{2}\}$ in $|Q|$ and therefore H–homeomorphic to the $(n-1)$–disk $|\operatorname{st}_Q(x)|$ as explained.

Observe that $GE^n \cap |N_{K'}(L_2)| \subset |\dot{N}_{K'}(L_2)|$, which is G–collarable in $|N_{K'}(L_2)|$ by 3.7.

There is one remaining condition to check for this to be a G–shelling. Write $E^{n-1} = |\operatorname{st}_Q(x)|$. Then we must verify that $x \in \partial E^{n-1}$ implies that within the G–collar on $|\dot{N}_{K'}(L_2)|$ in $|N_{K'}(L_2)|$, $|x * \partial \operatorname{lnk}_Q(x)| \times I \subset \partial |N_{K'}(L')|$. It suffices for us to show that every simplex of $x * \partial \operatorname{lnk}_Q(x)$ lies on ∂M because the G–collar is given by moving derived vertices around within simplices of K. Thus, if a simplex σ consisting only of derived vertices lies on ∂M, then $\sigma \times I$ lies on ∂M, and hence also on $\partial |N_{K'}(L')|$.

Since E^{n-1} is an H–regular neighborhood of $x \in |Q|$, if $x \in \partial E^{n-1}$, then $x \in \partial|Q| = |Q| \cap \partial M$. Thus, $x \in \partial M$, forcing it to also belong to $\partial|N_{K'}(L')|$. Likewise, any simplex $\sigma \in x * \partial \operatorname{lnk}_Q(x)$ containing x lies in ∂E^{n-1} but not in $\dot{N}_Q(x)$, forcing σ to be in ∂M and therefore $\partial|N_{K'}(L')|$. Hence, we have proven the final condition that this constitutes a G–shelling of a G–regular neighborhood of X to a G–regular neighborhood of Y. □

We proved the following corollary during the previous proof's induction.

Corollary 3.27. *If $Y \subseteq M$ G–collapses to a point, any G–regular neighborhood of Y in M is an n–G–disk.*

Corollary 3.28. *If an n–G–manifold M G–collapses to a point, it is an n–G–disk.*

Proof. Let x be a point in M to which it G–collapses. M is a G–regular neighborhood of itself, so M G–shells to (and is therefore G–homeomorphic to) an n–disk. □

Corollary 3.29. *If X is in the interior of M and $X \searrow_G Y$, then a G–regular neighborhood of X in M is a G–regular neighborhood of Y in M.*

Proof. Let N_1 be a G–regular neighborhood of X in M. By 3.26, N_1 G–shells to N_2, a G–regular neighborhood of Y in M, so by 3.25, there is a G–homeomorphism of M mapping N_2 to N_1. The homeomorphism carries any G–triangulation of N_2 to a G–triangulation of N_1. Then, since any G–triangulation of N_1 can be extended to a G–triangulation of M by 3.8, we can apply 3.20 to get that N_1 is a G–regular neighborhood of Y in M. □

We can now prove our final goal for this chapter, a characterization theorem for G–regular neighborhoods based on G–collapses.

Theorem 3.30 (Collapsing Criterion for G–Regular Neighborhoods)**.** *Let M be an n–dimensional G–manifold, and let $N \subset int\ M$ be a G–invariant neighborhood of a G–polyhedron Y. Then N is G–regular if and only if*

(i) N is an n–manifold with boundary ∂N G–bicollarable in M,

(ii) $N \searrow_G Y$

Proof. The proof follows exactly the non-equivariant version (Corollary 3.30 in [RS82]). Let K be an admissible G–triangulation of M with $N = |N_K(L)|$ a G–regular neighborhood of $Y = |L|$. We already know N is an n–manifold by 3.19. Choose a derived G–subdivision of K near L with the new vertices in $f^{-1}_{L,K}(\epsilon)$ for some $\epsilon \in (0, 1)$, so that $N_1 = f^{-1}[0, \epsilon]$ is another G–regular neighborhood of Y. N_1 has a regular cellular G–structure whose cells are obtained by intersecting the interior simplices of $N_K(L)$ with $f^{-1}(0)$, $f^{-1}[\epsilon]$, and $f^{-1}[0, \epsilon]$. We may collapse, along with its orbit, each cell $|\sigma| \cap f^{-1}[0, \epsilon]$, $\sigma \in N_K(L) \setminus \dot{N}_K(L)$, and its face $|\sigma| \cap f^{-1}(\epsilon)$, in order of decreasing dimension. That this is a G–collapse follows from the admissibility of K. By 3.2, N G–collapses to Y as well.

For the other implication, suppose we have N satisfying conditions (i) and (ii). Let $C = \partial N \times [-1, 1]$ be a G–bicollar with $\partial N = \partial N \times \{0\}$. Then let $N_1 = N \cup \partial N \times [0, \frac{1}{2}]$, which constitutes a G–regular neighborhood of N in M because we can triangulate it to be a simplicial neighborhood. Therefore, by 3.29, since $N \searrow_G Y$, N_1 is also a G–regular neighborhood of Y. But we can define a G–homeomorphism on C fixing $\partial N \times \{-1, 1\}$ and carrying $\partial N \times \{\frac{1}{2}\}$ to $\partial N \times \{0\}$. We can extend this by the identity to all of M, mapping N_1 to N, showing that the latter is also a G–regular neighborhood of Y in M. $\square$

Chapter 4

Lovász Complexes

4.1 Graphs and Graph Multimorphisms

A **graph** Γ is a pair (V_Γ, E_Γ), where V_Γ is a set (called the **vertex set** of Γ) and E_Γ (the **edge set**) is a collection of cardinality 2 multisets of elements of V_Γ. If $\{v, w\} \in E_\Gamma$, we say the vertices v and w in V_Γ are **adjacent** in Γ. An edge $\{v, v\} \in E_\Gamma$ is called a **loop**. We call a graph with no loops a **simple graph.** For simplicity, we will consider only finite graphs.

Let $n \geq 1$. Denote by K_n the **complete graph** on n vertices. That is, $V_{K_n} = \{1, \ldots, n\}$ and E_{K_n} consists of all edges of the form $\{i, j\}$ with $i \neq j$. Denote by C_n the n–**cycle**. $V_{C_n} = \{1, \ldots, n\}$, E_{C_n} consists of all edges of the form $\{i, i+1\}$ as well as $\{1, n\}$. With these definitions, C_1 is a single vertex with a loop, and $C_2 = K_2$.

If Γ and Λ are graphs, a **graph morphism** from Γ to Λ is a function $f\colon V_\Gamma \to V_\Lambda$ such that $\{v, w\} \in E_\Gamma$ implies $\{f(v), f(w)\} \in E_\Lambda$. A morphism from Γ to K_n is called an n–**coloring** of Γ, and if such a morphism exists, Γ is called n–**colorable**. If Γ has any loops, it is not n–colorable for any n; there are no morphisms from a graph with loops to any simple graph.

A morphism which is bijective on vertices and on edges is a graph isomorphism. A group action on a graph Γ is an action of a group G on the set V_Γ with each group element inducing a graph isomorphism of Γ onto itself. We may define the **quotient graph** Γ/G to be the graph with vertex set $V_{\Gamma/G}$ the set of G–orbits in V_Γ with an edge $\{Gv, Gw\} \in E_{\Gamma/G}$ if and only if $\{g_1v, g_2w\} \in E_\Gamma$ for some $g_1, g_2 \in G$. Note that if a vertex in Γ is adjacent to any other vertex in its orbit, Γ/G will have a loop.

A **graph multimorphism** from Γ to Λ is a relation $\phi \subseteq V_\Gamma \times V_\Lambda$ such that (1) $\phi(v) := \{w \in V_\Lambda \mid (v, w) \in \phi\}$ is non-empty for all $v \in V_\Gamma$, and (2) any function $f \subseteq \phi$ is a graph morphism. In other words, ϕ is a multimorphism when there is a function $f \subseteq \phi$ and any function $f \subseteq \phi$ is a graph morphism. In particular, a graph morphism is also a multimorphism.

Two graph multimorphisms $\phi\colon \Gamma \to \Lambda$ and $\psi\colon \Lambda \to \Theta$ can be composed to obtain a multimorphism from Γ to Θ defined as $\psi \circ \phi := \{(u, w) \in V_\Gamma \times V_\Theta \mid \exists\, v \in \phi(u) \text{ with } w \in \psi(v)\}$. We check that this is a multimorphism. For each $u \in V_\Gamma$, we may choose $v \in \phi(u)$, and we may choose $w \in \psi(v)$. Thus, $\psi \circ \phi$ contains a function. Second, let $\{u_1, u_2\} \in E_\Gamma$. Since ϕ is a multimorphism, for any $v_1 \in \phi(u_1)$ and $v_2 \in \phi(u_2)$, $\{v_1, v_2\} \in E_\Lambda$, and further, since ψ is a multimorphism, for any $w_1 \in \psi(v_1)$ and $w_2 \in \psi(v_2)$, $\{w_1, w_2\} \in E_\Theta$. Therefore, $\psi \circ \phi$ is a graph multimorphism.

4.2 Lovász Complexes

Multimorphisms from a graph Γ to another graph Λ form a poset under inclusion, denoted $\mathrm{Hom}(\Gamma, \Lambda)$. It is the face poset of a regular cell complex, the **Lovász**

multimorphism complex. We will use $\mathrm{Hom}(\Gamma, \Lambda)$ interchangeably to mean both the face poset and the complex.

Consider ϕ, a multimorphism from Γ to Λ. For each vertex $v \in V_\Gamma$, $\phi(v)$ is a set of vertices in V_Λ. If we give the vertices of Λ any total ordering, we have that $\Delta\phi(v)$ is the full simplex on the set $\phi(v)$. Geometrically, the multimorphism ϕ can itself be regarded as a product of simplices:

$$\prod_{v \in V_\Gamma} |\Delta\phi(v)|$$

Since each simplex is convex, the product is homeomorphic to a disk of dimension $\sum_{v \in V_\Gamma}(|\phi(v)| - 1)$. Its boundary is comprised of the cells indexed by multimorphisms $\psi \subset \phi$.

$\mathrm{Hom}(-,-)$ is a bifunctor from the category of graphs and graph multimorphisms to the category of regular cell complexes and cellular maps. It is contravariant in the first variable: A multimorphism $\alpha\colon \Gamma' \to \Gamma$ induces a cellular map $\mathrm{Hom}(\alpha, \Lambda)\colon \mathrm{Hom}(\Gamma, \Lambda) \to \mathrm{Hom}(\Gamma', \Lambda)$ given by sending ϕ to $\phi \circ \alpha$. It is covariant in the second variable: $\beta\colon \Lambda \to \Lambda'$ induces a cellular map $\mathrm{Hom}(\Gamma, \beta)\colon \mathrm{Hom}(\Gamma, \Lambda) \to \mathrm{Hom}(\Gamma, \Lambda')$ by mapping ϕ to $\beta \circ \phi$.

If G and H are groups acting on Γ and Λ respectively, then $G \times H$ acts on $V_\Gamma \times V_\Lambda$, which induces an action on subsets of $V_\Gamma \times V_\Lambda$, restricting to an action on $\mathrm{Hom}(\Gamma, \Lambda)$ since the multimorphism conditions are preserved by any isomorphism of Γ or Λ. Explicitly, the action on $\mathrm{Hom}(\Gamma, \Lambda)$ is given by $((g, h)\phi)(v) = h\phi(g^{-1}v)$. We will only make use of the equivariance in the first variable. In particular, a graph Γ equipped with a G–action defines a functor $\mathrm{Hom}(\Gamma, -)$ from the category of graphs and multimorphisms to regular cellular G–complexes and cellular G–maps.

Theorem 4.1. *Let G be a group acting on a graph Γ. Let Λ be a simple graph. Then the induced G–action on $\operatorname{Hom}(\Gamma, \Lambda)$ is fixed point free if and only if there are no morphisms from the quotient graph Γ/G to Λ. The induced action is free if and only if, for every nontrivial $H \leq G$, there are no morphisms from Γ/H to Λ.*

Proof. Suppose there is a morphism $\phi\colon \Gamma/G \to \Lambda$. Then there exists a morphism $\tilde{\phi}\colon \Gamma \to \Lambda$ which is constant on G–orbits. Thus $\tilde{\phi}$ is a fixed point in $\operatorname{Hom}(\Gamma, \Lambda)$.

In the other direction, if ϕ is a fixed point, it must be constant on each orbit in V_Γ. This induces a multimorphism $\phi_G\colon \Gamma/G \to \Lambda$. A morphism may be obtained from ϕ_G by choosing any of its 0–cell.

For the second part of the theorem, if the G–action is free, then the action of any subgroup H of G is fixed point free, so there are no morphisms from Γ/H to Λ. If the G–action is not free, then there is a non-trivial stabilizer G_ϕ. The G_ϕ–action is not fixed point, so there is a morphism from Γ/G_ϕ to Λ. □

Corollary 4.2. *Let G be a group acting on a graph Γ. The induced action on $\operatorname{Hom}(\Gamma, K_n)$ is fixed point free if and only if the quotient graph Γ/G is not n–colorable. The induced action is free if and only if, for every nontrivial $H \leq G$, Γ/H is not n–colorable.*

From now on, we will assume all graphs are simple unless they are quotient graphs. Thus, the presence of loops in Γ/G will immediately tell us that G acts on $\operatorname{Hom}(\Gamma, \Lambda)$ without fixed points.

We consider as an example the **edge complex functor**, $\operatorname{Hom}(K_2, -)$. Here, let G be the group $\{\pm 1\}$ of automorphisms of the edge K_2. We denote the vertices of K_2 by $+$ and $-$, then the nontrivial group element -1 interchanges $+$ and $-$, and so acts as an involution on $\operatorname{Hom}(K_2, \Lambda)$ by switching the subsets $\phi(+)$ and

$\phi(-)$ (of V_Λ). This action is free by 4.1 (again, assuming that Λ is simple) because K_2/G is a loop.

An important case is the edge complex $\mathrm{Hom}(K_2, K_n)$. As a poset, it consists of pairs (A, B) of nonempty disjoint subsets of $\{1, \ldots, n\}$ (Here, $A = \phi(+)$ and $B = \phi(-)$), ordered by component-wise inclusion. It can easily be shown that $\mathrm{Hom}(K_2, K_n)$ is an $(n-2)$–sphere.

Lemma 4.3. $\mathrm{Hom}(K_2, K_n)$ *is* $\{\pm 1\}$*–homeomorphic to* S^{n-2} *with the antipodal action.*

Proof. We denote the vertices of K_n by $v_1, v_2, \ldots, v_n$. Geometrically, a point in a cell (A, B) of $\mathrm{Hom}(K_2, K_n)$ looks like $(a_1 v_1 + \ldots + a_n v_n, b_1 v_1 + \ldots + b_n v_n)$, where $a_i = 0$ if $v_i \notin A$, $b_i = 0$ if $v_i \notin B$, and $a_1 + \ldots + a_n = 1 = b_1 + \ldots + b_n$.

We define a map $h\colon\ \mathrm{Hom}(K_2, K_n) \to \mathbb{R}^n$ by sending $(a_1 v_1 + \ldots + a_n v_n, b_1 v_1 + \ldots + b_n v_n)$ to $(a_1 - b_1, \ldots, a_n - b_n)$. This map is injective since A and B are disjoint. In the image of this map, the sum of the coordinates is always zero, so the image lies in the hyperplane orthogonal to the diagonal vector $(1, \ldots, 1)$. Also note that the sum of the absolute values of the coordinates is always two, so the image lies in the $(n-2)$–sphere of radius two with respect to the L^1 norm (inducing the taxicab metric) within this hyperplane. An inverse map can be defined from this $(n-2)$–sphere by assigning the coordinates with positive values and negative values to $\phi(+)$ and $\phi(-)$ respectively. Thus, we see both that the map is a homeomorphism and that the antipodal action on S^{n-2} corresponds exactly to switching $\phi(+)$ and $\phi(-)$ as claimed. $\square$

A multimorphism $\phi \in \mathrm{Hom}(\Gamma, \Lambda)$ is also determined by specifying $\phi^{-1}(w) := \{v \in V_\Gamma \mid (v, w) \in \phi\}$ for each $w \in V_\Lambda$. For any ϕ, each $\phi^{-1}(w)$ must be an **independent** subset of vertices (i.e., no two elements are adjacent in Γ). The

independence complex $\mathrm{ind}(\Gamma)$ of the graph Γ is the simplicial complex with vertex set V_Γ and simplices the independent subsets.

In the case when $\Lambda = K_n$, the only condition on each $\phi^{-1}(w)$ is independence, so $\mathrm{Hom}(\Gamma, K_n)$ consists of relations $\phi \subset V_\Gamma \times V_{K_n}$ such that: (1) $\phi(v)$ is nonempty for all $v \in V_\Gamma$, and (2) $\phi^{-1}(j) \in \mathrm{ind}(\Gamma)$ for $j = 1, \ldots, n$. The link of ϕ in $\mathrm{Hom}(\Gamma, K_n)$ is naturally identified with the join over all $1 \leq j \leq n$ of the links of the simplices $\phi^{-1}(j)$ in $\mathrm{ind}(\Gamma)$. Hence, when $\mathrm{ind}(\Gamma)$ is a combinatorial (G–)sphere, $\mathrm{Hom}(\Gamma, K_n)$ is a closed (G–)manifold. The converse is also true [C05].

As an example, consider the complexes $\mathrm{Hom}(C_m, K_n)$. The independence complexes of cycles are as follows: $\mathrm{ind}(C_3)$ consists of 3 disjoint points, $\mathrm{ind}(C_4)$ is a pair of disjoint edges, $\mathrm{ind}(C_5)$ is a pentagon (a PL-sphere), and for $m > 5$, $\mathrm{ind}(C_m)$ has maximal simplices of different dimensions and thus cannot be a sphere. Therefore, among these cycle complexes, $\mathrm{Hom}(C_5, K_n)$ is the only manifold.

4.3 Restricted Lovász Complexes

Let Γ and Λ be finite simple graphs. Let I be an independent set of vertices of Γ. Consider the graph morphism $\Gamma \backslash I \to \Gamma$ given by inclusion. This induces a cellular map $\mathrm{Hom}(\Gamma, \Lambda) \to \mathrm{Hom}(\Gamma \setminus I, \Lambda)$ defined by restricting each multimorphism to $V_\Gamma \setminus I$. Define $\mathrm{Hom}_I(\Gamma, \Lambda)$ to be the image of this map. Thus, $\mathrm{Hom}_I(\Gamma, \Lambda)$ is the subcomplex of $\mathrm{Hom}(\Gamma \setminus I, \Lambda)$ whose cells are the multimorphisms $\phi \colon \Gamma \setminus I \to \Lambda$ that can be extended to all of Γ. Another way of viewing this complex is by taking the multimorphisms from Γ to Λ and identifying two of them if they differ only on the set I.

If G is a group acting on Γ, $\mathrm{Hom}_I(\Gamma, \Lambda)$ inherits the G–action only when I is (setwise) G–invariant.

The projection from $\mathrm{Hom}(\Gamma, \Lambda)$ to $\mathrm{Hom}_I(\Gamma, \Lambda)$ is a homotopy equivalence since the fibers are contractible [C05]. In fact, for $\Lambda = K_n$, it was proven by Schultz [S08] that the two complexes are homeomorphic (but not via the aforementioned projection) whenever $\mathrm{ind}(\Gamma)$ is a PL-sphere. We do not prove the full equivariant version of that result, but we will need the following version of one of Schultz's lemmas in [S08].

Lemma 4.4. *Let Γ be a graph with a G-action, $n \geq 1$, and I a G–invariant, independent subset of the vertex set V_Γ. For all $v \in V_\Gamma$, define*

$$A_v := \{J \in \mathrm{ind}(\Gamma) \,|\, v \in J\}$$

$$B_v := \{J \setminus I \,|\, J \in A_v\}$$

If there is a G–homeomorphism $h\colon\ |\Delta\,\mathrm{ind}(\Gamma)| \to |\Delta\,\mathrm{ind}(\Gamma \setminus I)|$ such that $h(|\Delta A_v|) = |\Delta B_v|$ for all $v \in V_\Gamma$, then $\mathrm{Hom}(\Gamma, K_n)$ is G–homeomorphic to $\mathrm{Hom}_I(\Gamma, K_n)$.

Proof. Following exactly [S08], we consider the equivariant poset embedding

$$f : \mathrm{Hom}(\Gamma, K_n) \to \prod_{i=1}^{n} \mathrm{ind}(\Gamma) = \mathrm{ind}(\Gamma)^{\{1,\ldots,n\}}$$

given by $\phi \mapsto (\phi^{-1}(i))_i$. Then $\mathrm{Hom}(\Gamma, K_n) \approx_G |\Delta \mathrm{im}\ f|$. The poset $\mathrm{ind}(\Gamma)^{\{1,\ldots,n\}}$ can naturally be identified with those relations $\phi \subseteq V_\Gamma \times \{1, \ldots, n\}$ that are multimorphisms from the induced subgraph on the vertices with $\phi(v)$ nonempty to the complete graph K_n. The additional condition that no $\phi(v)$ can be empty yields the following description:

$$\text{im } f = \bigcap_{v \in V_\Gamma} \bigcup_{j=1}^{n} \prod_{i=1}^{n} \begin{cases} A_v, & i = j \\ \text{ind}(\Gamma), & i \neq j \end{cases}$$

All the A_v satisfy the condition that if $x \in A_v$ and $x \leq y$, then $y \in A_v$. Therefore, taking the order complex commutes with unions, and we obtain that

$$\text{Hom}(\Gamma, K_n) \approx_G \bigcap_{v \in V_\Gamma} \bigcup_{j=1}^{n} \prod_{i=1}^{n} \begin{cases} |\Delta A_v|, & i = j \\ |\Delta \, \text{ind}(\Gamma)|, & i \neq j \end{cases}$$

We use a similar argument for $\text{Hom}_I(\Gamma, K_n)$. The image of its embedding in $\text{ind}(\Gamma \setminus I)^{\{1,\ldots,n\}}$ has the additional condition that for each vertex in I, there is some element of $\{1, \ldots, n\}$ that is not related to any of its neighbors in Γ. We have that, for $v \notin I$, B_v satisfies the same condition as A_v above. For $v \in I$, B_v also satisfies the condition that if $x \in B_v$ and $y \leq x$, then $y \in B_v$. Hence,

$$\text{Hom}_I(\Gamma, K_n) \approx_G \bigcap_{v \in V_\Gamma} \bigcup_{j=1}^{n} \prod_{i=1}^{n} \begin{cases} |\Delta B_v|, & i = j \\ |\Delta \, \text{ind}(\Gamma \setminus I)|, & i \neq j \end{cases}$$

Thus, using the G–homeomorphism from the hypothesis on each coordinate in the product, we obtain that $\text{Hom}(\Gamma, K_n) \approx_G \text{Hom}_I(\Gamma, K_n)$. $\square$

Chapter 5

Equivariant Csorba Conjecture

5.1 The Stiefel Manifold $V_{n-1,2}$ and $\mathrm{Hom}(C_5, K_n)$

The Stiefel manifold $V_{n,k}$ is the space of ordered, orthonormal k–frames in the Euclidean space $\mathbb{R}^n$ [J77]. Explicitly, we define $V_{n,k} := \{(x_1, x_2, \ldots, x_k) \in S^{n-1} \times S^{n-1} \times \ldots \times S^{n-1} \,|\, x_i \cdot x_j = 0 \text{ for all } i \neq j\}$.

In particular, we consider $V_{n-1,2}$, the space of ordered, orthonormal 2–frames in $\mathbb{R}^{n-1}$. The orthogonal group $O_2(\mathbb{R})$ acts on $V_{n-1,2}$ with the quotient space being the Grassmannian $Gr_{n-1,2}$. O_2 is the semi-direct product of rotations SO_2 with any reflection. Two natural reflections to consider are (i) $(x, y) \mapsto (x, -y)$ and (ii) $(x, y) \mapsto (y, x)$. Since any two reflections are conjugate via a rotation, these give equivalent actions on $V_{n-1,2}$. An explicit G–homeomorphism $V_{n-1,2} \to V_{n-1,2}$ interchanging the actions (i) and (ii) is the map $(x, y) \mapsto \frac{1}{\sqrt{2}}(x + y, x - y)$.

On the combinatorial side, we consider the multimorphism cycle complex $\mathrm{Hom}(C_5, K_n)$. Recall that C_5 has vertices $\{1, 2, 3, 4, 5\}$ and edges $\{i, i + 1\}$ for $1 \leq i < 5$ and $\{1, n\}$. There is a G–action on the cycle C_5: -1 acts by the reflection $i \mapsto 6 - i$ for all $i \in V_{C_5}$. This induces an involution on $\mathrm{Hom}(C_5, K_n)$.

In his thesis [C05], Csorba showed that, for small n, $\mathrm{Hom}(C_5, K_n)$ is G–homeomorphic to the Stiefel manifold $V_{n-1,2}$ with the involution (i) and conjectured that this was true for all n. That the two are nonequivariantly homeomorphic was proven by C. Schultz, who further proved that that $\mathrm{Hom}(C_5, K_n)$ is equivariantly homotopy equivalent to $V_{n-1,2}$ [S08], again using action (i). Using the equivalent G–action (ii) on $V_{n-1,2}$, we give a proof of the equivariant version of Csorba's conjecture.

Theorem 5.1 (Equivariant Csorba Conjecture)**.** *Let $G = \{\pm 1\}$ act on $V_{n-1,2}$ via the involution $(x, y) \mapsto (y, x)$. Then $\mathrm{Hom}(C_5, K_n)$ is G–homeomorphic to $V_{n-1,2}$.*

Both of the involutions on $V_{n-1,2}$ extend to all of $S^{n-2} \times S^{n-2}$. The reflection (i) has no fixed points in $S^{n-2} \times S^{n-2}$, while (ii) fixes every point in the diagonal subspace $\{(x, x) \in S^{n-2} \times S^{n-2}\}$.

The set $N = \{(x, y) \in S^{n-2} \times S^{n-2} \,|\, x \cdot y \geq 0\}$ is a regular neighborhood of the diagonal with boundary exactly $V_{n-1,2}$. Thus, if G is the group $\{\pm 1\}$ with the nontrivial element acting by the reflection $(x, y) \mapsto (y, x)$, the diagonal is G–invariant, and we have that N is a G–regular neighborhood of the diagonal.

The strategy employed in Schultz's proof and in our proof is to find a regular neighborhood of the diagonal in a triangulation of $S^{n-2} \times S^{n-2}$ whose boundary is $\mathrm{Hom}(C_5, K_n)$. Since the diagonal is G–invariant under action (ii), we are able to find a neighborhood that is G–regular. Therefore, for the rest of this chapter, G will be $\{\pm 1\}$ acting on $S^{n-2} \times S^{n-2}$ (and hence also $V_{n-1,2}$) by the involution (ii).

Vital in both proofs is selecting a G–invariant independent set I of vertices in C_5 and passing to a restricted multimorphism complex $\mathrm{Hom}_I(C_5, K_n)$. Where Schultz uses the set $\{2, 4\}$, we use $\{3\}$. We may pass to the restricted complex

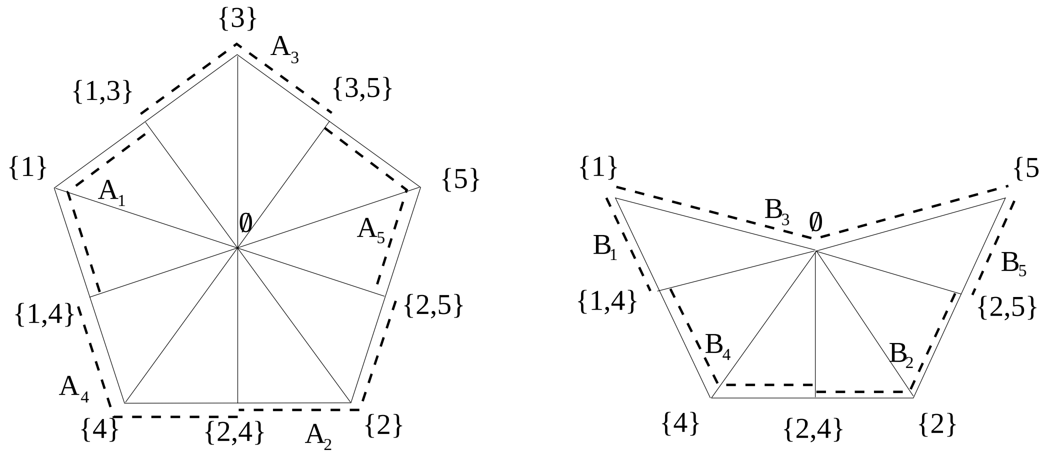

Figure 5.1: $|\Delta\,\mathrm{ind}(C_5)|$ and $|\Delta\,\mathrm{ind}(C_5 \setminus \{3\})|$

by showing that $\{3\}$ meets the conditions of 4.4.

Proposition 5.2. $\mathrm{Hom}_{\{3\}}(C_5, K_n)$ *is G–homeomorphic to* $\mathrm{Hom}(C_5, K_n)$.

Proof. We show the requirements of 4.4 are satisfied. It is easily seen from Figure 5.1. Let $K = \Delta\,\mathrm{ind}(C_5)$ and $L = \Delta\,\mathrm{ind}(C_5 \setminus \{3\})$. We first construct the required G–homeomorphism from $|\partial K|$ to $|\partial L|$. Triangulate $|L|$ by choosing a derived G–subdivision of $\mathrm{ind}(C_5 \setminus \{3\})$ near $\{1\}$ and $\{5\}$; cone $\{1, 5\}$ with the vertex $\emptyset$. We have now a triangulation of $|\partial L|$. We cone it with the point $x = \frac{1}{2}\{2, 4\} + \frac{1}{2}\emptyset \in |L|$. This gives a new G–triangulation of $|L|$; call it L'. There is a simplicial map from K to L' given by mapping $\{1\}$ and $\{5\}$ to the two new derived vertices of $\partial L'$, the vertices $\{1, 3\}$ and $\{3, 5\}$ to $\{1\}$ and $\{5\}$ respectively, the vertex $\{3\}$ to $\emptyset$, $\emptyset$ to x and the remaining vertices to themselves. This simplicial map induces a G–homeomorphism from $|K|$ to $|L'|$, and it is easily seen that each $|\Delta A_v|$ is mapped to $|\Delta B_v|$. The theorem now follows from 4.4. $\square$

Now we may turn our attention to $\mathrm{Hom}_{\{3\}}(C_5, K_n)$. Let P_4 be the path of length 4 starting at 1 and ending at 5 (i.e., the graph obtained from C_5 by deleting the edge $\{1,5\}$). Then $\mathrm{Hom}_{\{3\}}(C_5, K_n) \subset \mathrm{Hom}_{\{3\}}(P_4, K_n) \subset \mathrm{Hom}(P_4 \setminus \{3\}, K_n)$. These inclusions are cellular and equivariant with respect to the involution induced from $i \mapsto 6 - i$ for $1 \le i \le 5$. Note that $P_4 \setminus \{3\}$ consists of two disjoint copies of K_2, the edges $\{1,2\}$ and $\{4,5\}$, which are interchanged by the G–action. We see then that $\mathrm{Hom}(P_4 \setminus \{3\}, K_n)$ is naturally identified with $\mathrm{Hom}(K_2, K_n) \times \mathrm{Hom}(K_2, K_n)$, which we know by 4.3 is homeomorphic to $S^{n-2} \times S^{n-2}$. Finally, we have the graph morphism $P_4 \setminus \{3\} \to K_2$ which sends 1 and 5 to $+$ and 2 and 4 to $-$ inducing the diagonal embedding of the sphere $\mathrm{Hom}(K_2, K_n)$ in $\mathrm{Hom}(P_4 \setminus \{3\}, K_n)$ as the fixed point set of the involution (but not as a subcomplex).

We will show that $\mathrm{Hom}_{\{3\}}(P_4, K_n)$ is a G–regular neighborhood of the diagonal in $\mathrm{Hom}(P_4 \setminus \{3\}, K_n) = \mathrm{Hom}(K_2, K_n) \times \mathrm{Hom}(K_2, K_n)$ with boundary $\mathrm{Hom}_{\{3\}}(C_5, K_n)$. This will prove the conjecture.

We describe these face posets concretely: All are comprised of four-tuples of nonempty subsets A, B, C, D of $\{1, \ldots, n\}$ satisfying further conditions. For any cell ϕ, A, B, C, D are $\phi(1), \phi(2), \phi(5)$, and $\phi(4)$ respectively. In $\mathrm{Hom}(P_4 \setminus \{3\}, K_n)$ we have only that $A \cap B = \emptyset = C \cap D$. In $\mathrm{Hom}_{\{3\}}(P_4, K_n)$ we also have that $B \cup D \neq \{1, \ldots, n\}$, and $\mathrm{Hom}_{\{3\}}(C_5, K_n)$ has the further restriction that $A \cap C = \emptyset$. The diagonal $\mathrm{Hom}(K_2, K_n)$ has $A = C$ and $B = D$ (in addition to $A \cap B = \emptyset$). We prefer to denote these cells as arrays

$$\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

to remind us of their locations in terms of the vertices of the pentagon C_5, and the involution simply interchanges the rows. Such an array should not be thought of as a 2×2 matrix.

Before proceeding, we will show that $S^{n-2} \times S^{n-2}$ is a G–manifold. This will require the following fact.

Proposition 5.3. *Let $G = \{\pm 1\}$ act on a join of two spheres $S^m * S^m$ with the nontrivial element -1 acting by interchanging the two spheres. Then $S^m * S^m$ is a $(2m+1)$–dimensional G–sphere.*

Proof. Consider first the case $m = 0$. Let $S^0 = \{-1, 1\}$. A point in the join $S^0 * S^0$ looks like $tv + (1-t)w$ where $0 \leq t \leq 1$, $v \in \{(1,0), (-1,0)\}$, and $w \in \{(0,1), (0,-1)\}$. We map $S^0 * S^0$ to S^1 be sending a point $tv + (1-t)w$ to $\sqrt{t}v + \sqrt{1-t}w$. The involution fixes the subspace of $\mathbb{R}^2$ spanned by the vector $(1,1)$ and acts nontrivially on the orthogonal complement, the subspace spanned by $(1,-1)$. We see then that this S^1 is a G–sphere $S(\rho)$ where ρ is the direct sum of the trivial representation and the sign representation of G.

Now any join of sphere $S^m * S^m$ can be rewritten as the join $(S^0 * S^0) * (S^0 * S^0) * \ldots (S^0 * S^0)$ of $m+1$ copies of $(S^0 * S^0)$, where each pair of 0–spheres is interchanged by the involution. Each joined pair is a 1–dimensional G–sphere $S(\rho)$ as above, so $S^m * S^m$ is G–homeomorphic to a join of $m+1$ copies of $S(\rho)$, which is itself G–homeomorphic to the $(2m-1)$–dimensional sphere $S((m+1)\rho)$.

To check that the links of vertices in $S^m * S^m$ are G–spheres and finish the proof, we need an admissible G–triangulation. Let L be the triangulation $S^0 * \ldots * S^0$ of a single S^m, and we let K be the barycentric subdivision of the triangulation of $L * L = (S^0 * S^0) * (S^0 * S^0) * \ldots (S^0 * S^0)$. A vertex v in K^0 corresponds to a simplex of the original triangulation $L * L$. If v corresponds to a simplex not in the diagonal (i.e., whose coordinates in at least one of the $S^0 * S^0$ pairs differ), then v has trivial stabilizer in G, and the condition is just that $\mathrm{lnk}_K(v)$

is a $(2m-2)$–sphere, which we have automatically since $S^m * S^m$ is a manifold. If v corresponds to a simplex $\sigma = \sigma_1 \cup \sigma_2$ in the diagonal (where σ_1 and σ_2 are the sets of vertices from the first and second S^m respectively with $(-1)\sigma_1 = \sigma_2$), then $\mathrm{lnk}_K(v)$ is isomorphic to the join of $\Delta F(K)_{<\sigma}$ with $\Delta F(K)_{>\sigma}$.

The poset $F(K)_{<\sigma}$ is simply $F(\partial\sigma)$. However, $\partial\sigma = \sigma_1 * \partial\sigma_2 \cup \partial\sigma_1 * \sigma_2$, the union of two disks whose intersection is the (by induction) G–sphere $\partial\sigma_1 * \partial\sigma_2$. Since σ_1 and σ_2 are switched by the involution, we have that $|\partial\sigma|$ is G–homeomorphic to $|S^0 * \partial\sigma_1 * \partial\sigma_2|$, with the new S^0 a sphere in the sign representation of G. Therefore, $\Delta F(K)_{<\sigma}$ is a G–sphere.

On the other hand, the poset $F(K)_{>\sigma}$ is isomorphic to $F(\mathrm{lnk}_L(\sigma_1) * \mathrm{lnk}_L(\sigma_2))$. Since σ was on the diagonal, σ_1 and σ_2 are the same simplex in L, and $\mathrm{lnk}_L(\sigma_1) * \mathrm{lnk}_L(\sigma_2)$ is a join of two identical spheres which are interchanged by the G–action. Thus, $\Delta F(K)_{>\sigma}$ is also a G–sphere by induction.

Since both $\Delta F(K)_{<\sigma}$ and $\Delta F(K)_{>\sigma}$ are G–spheres and their triangulations are admissible, their join, $\mathrm{lnk}_K(v)$, is a G–sphere. □

Proposition 5.4. *Let $G = \{\pm 1\}$ act on $S^m \times S^m$ with -1 interchanging the spheres. Then $S^m \times S^m$ is a $2m$–dimensional G–manifold.*

Proof. A regular cellular G–structure for $S^m \times S^m$ is obtained by again taking L to be the join of $m+1$ copies of S^0 to triangulate S^m and then letting K be the face poset $L \times L$. Then ΔK is an admissible G–triangulation of $S^m \times S^m$, and we need only check that the link of a vertex v is a G_v–sphere. A vertex v of ΔK corresponds to a cell (σ, τ), where σ and τ are simplices of L. If $\sigma \neq \tau$, the stabilizer G_v is trivial, and the link is a sphere since $S^m \times S^m$ is a manifold, so there is nothing more to check. If $\sigma = \tau$, $G_v = G$ and $\mathrm{lnk}_{\Delta K}(v)$ is the join $\Delta K_{<(\sigma,\sigma)} * \Delta K_{>(\sigma,\sigma)}$.

An element of the lower link is obtained by deleting a proper subset from at least one of the two copies of σ. Thus, $K_{<(\sigma,\sigma)}$ is isomorphic to the poset $F(\partial\sigma * \partial\sigma)^{op}$ (where P^{op} is the dual poset to P). Since ΔP^{op} is isomorphic to ΔP for any poset P, we have that $|\Delta K_{<(\sigma,\sigma)}|$ is G–homeomorphic to $|\partial\sigma * \partial\sigma|$, a G–sphere by 5.3.

The upper link of (σ,σ) in K is isomorphic to $F(\mathrm{lnk}_L(\sigma) * \mathrm{lnk}_L(\sigma))$. Since L is a combinatorial sphere, $\mathrm{lnk}_L(\sigma)$ is a sphere, so we have that $\Delta F(\mathrm{lnk}_L(\sigma) * \mathrm{lnk}_L(\sigma))$ is a G–sphere, again by 5.3.

Thus $\mathrm{lnk}_{\Delta K}(v)$ is the join of two combinatorial G–spheres with admissible triangulations, so ΔK is a combinatorial G–manifold with dimension $2m$ (since we already know that, nonequivariantly, $S^m \times S^m$ is a $2m$–manifold). $\square$

5.2 Proof of the Conjecture

In this section, G is the group $\{\pm 1\}$. We show that $\mathrm{Hom}_{\{3\}}(P_4, K_n)$ is a G–regular neighborhood of the diagonal $\mathrm{Hom}(K_2, K_n)$ in $\mathrm{Hom}(P_4 \setminus \{3\}, K_n)$ using the collapsing criterion, i.e., we show that it is a manifold of the correct dimension, that it (simplicially) G–collapses to the diagonal, and that its boundary is $\mathrm{Hom}_{\{3\}}(C_5, K_n)$. For simplicity, we represent elements of the posets in question as arrays whose entries A, B, C, D are nonempty subsets of $\{1, \ldots, n\}$.

Define

$$M := \{\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad | \quad A \cap B = \emptyset, \quad C \cap D = \emptyset\}$$

$$K := \{\phi \in M \quad | \quad B \cup D \neq \{1, \ldots, n\}\}$$

$$L := \{\phi \in K \quad | \quad A \cap C = \emptyset\}$$

$$S := \{\phi \in K \quad | \quad A = C, \quad B = D\}$$

We reiterate that M, K, and L are the face posets of the G–regular cell complexes $\mathrm{Hom}(P_4 \setminus \{3\}, K_n)$, $\mathrm{Hom}_{\{3\}}(P_4, K_n)$, and $\mathrm{Hom}_{\{3\}}(C_5, K_n)$ respectively, and S that of the diagonal $\mathrm{Hom}(K_2, K_n)$ in $\mathrm{Hom}(P_4 \setminus \{3\}, K_n) = \mathrm{Hom}(K_2 \cup K_2, K_n)$. By passing to order complexes, we obtain that ΔS and ΔL are full G–subcomplexes of ΔK, which is a full G–subcomplex of ΔM, and they are all admissible. Our goal is to show that $|\Delta K|$ is a G–regular neighborhood of $|\Delta S|$ whose boundary is $|\Delta L|$.

Proposition 5.5. *ΔK is a $(2n-4)$–manifold with boundary ΔL.*

Proof. We show that the link of an element of K is a sphere or a disk of dimension $(2n-5)$. For any $\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K$, $\mathrm{lnk}_{\Delta K}(\phi) = \Delta K_{<\phi} * \Delta K_{>\phi}$. For any $\phi \in K$, we obtain an element of its lower link by deleting proper subsets from each of A, B, C, and D, at least one of which is nonempty. Therefore, $K_{<\phi}$ is isomorphic to the face poset of $\partial\Delta A * \partial\Delta B * \partial\Delta C * \partial\Delta D$, yielding that $\Delta K_{<\phi}$ is a combinatorial sphere of dimension $|A| + |B| + |C| + |D| - 5$. (Recall that, if A is an unordered set, ΔA is the full simplex having A as its vertex set, whereas, if P is a poset, ΔP is its order complex.)

When $\phi \in K \setminus L$, we show that $\Delta K_{>\phi}$ is a sphere of dimension $2n - |A| - |B| - |C| - |D| - 1$, yielding that $\mathrm{lnk}_{\Delta K}(\phi)$ is a sphere of dimension $2n - 5$. For any $\phi' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in M$ such that $\phi' > \phi$, we have that

$$\emptyset \neq A \cap C \subseteq A' \cap C' \subseteq (B' \cup D')^c$$

so that $\phi' \in K$. Thus, to obtain an element of the upper link of ϕ, any element of the complement $(A \cup B)^c$ can be added to either A or B, but not to both, and similarly for elements of $(C \cup D)^c$. As a consequence, we have that $K_{>\phi}$ is isomorphic to the face poset of $*_{i=1}^{m} S^0$ where $m = |(A \cup B)^c| + |(C \cup D)^c|$, and therefore $\Delta K_{>\phi}$ is a sphere of dimension $2n - |A| - |B| - |C| - |D| - 1$ as claimed.

In the case where $\phi \in L$, we claim that $\Delta K_{>\phi}$ is a disk of dimension $2n - |A| - |B| - |C| - |D| - 1$, meaning that $\mathrm{lnk}_{\Delta K}(\phi)$ is a disk of dimension $2n - 5$. This will finish the proof that ΔK is a manifold and ΔL is its boundary. To see that $K_{>\phi}$ is the face poset of a subcomplex of a join of spheres, we consider the various types of elements that we can add to one or more of A, B, C, and D to obtain a larger element of K.

1. An element of $A \cap D$ cannot be added anywhere (while remaining in M). The same is true for elements of $B \cap D$ and $B \cap C$. Thus, these elements contribute nothing to the upper link.

2. An element of $B \setminus (C \cup D)$ can be added to C or to D; doing so will give us something in K, since the element was already in $B \cup D$. Thus, each of these elements contributes a copy of S^0 to the join of spheres. Similarly, each element of $D \setminus (A \cup B)$ contributes a copy of S^0 to the join.

3. An element of $A \setminus D$ can be added to C or to D, contributing a copy of $S^0 = \{\pm 1\}$ to the join with $+1$ indicating that the element was added to C and -1 indicating D. Similarly, an element of $C \setminus B$ can be added to A $(+1)$ or to B (-1). Adding elements of Type 3 to B or D could produce something not in K.

4. An element of $(A \cup B \cup C \cup D)^c$ can be added to A or B (but not both)

and, at the same time, to C or D (but not both). This contributes a copy of $S^1 = \{\pm 1\} * \{\pm 1\}$ (treated as a single coordinate) to the join of spheres with the $+1$'s corresponding to A and C and the -1's corresponding to B and D. As with Type 3, adding this type of element to B or D could yield something not in K.

To ensure that we remain in K, there must be an element of $(B \cup D)^c$ which is not added to $B \cup D$. In terms of coordinates, this means there must be at least one coordinate corresponding to Type 3 or 4 above that has no -1's.

Before proceeding, we define

$$F_{k,l} \subseteq (*_{i=1}^{k} S^1) * (*_{j=1}^{l} \{\pm 1\})$$

to be the subcomplex whose simplices have at least one coordinate from the join with no -1's. (Note that, as before, each copy of $S^1 = \{\pm 1\} * \{\pm 1\}$ is regarded as a single coordinate.) We will prove a lemma (5.6) stating that $F_{k,l}$ is a disk of dimension $2k + l - 1$.

Assuming Lemma 5.6 for now, since $K_{>\phi}$ is isomorphic to the face poset of $(*_{i=1}^{m} S^0) * F_{k,l}$ where $k = |(A \cup B \cup C \cup D)^c|$, $l = |A \setminus D| + |C \setminus B|$, and $m = |B \setminus (C \cup D)| + |D \setminus (A \cup B)|$, we have that $\Delta K_{>\phi}$ is a disk of dimension $2n - |A| - |B| - |C| - |D| - 1$ as we had claimed.

To see that this is the dimension, we simply verify the following calculation:

$$
\begin{aligned}
m-1+2k+l &= |B \setminus (C \cup D)| + |D \setminus (A \cup B)| \\
&\qquad + 2|(A \cup B \cup C \cup D)^c| + |A \setminus D| + |C \setminus B| - 1 \\
&= 2n - 2|A| - 2|B| - 2|C| - 2|D| \\
&\qquad + 2|A \cap D| + 2|B \cap C| + 2|B \cap D| \\
&\qquad + |B \setminus (C \cup D)| + |D \setminus (A \cup B)| \\
&\qquad + |A \setminus D| + |C \setminus B| - 1 \\
&= 2n - |A| - |B| - |C| - |D| - 1
\end{aligned}
$$

□

Lemma 5.6. *For $k, l \in \mathbb{N}$ such that $2k + l - 1 \geq 0$, $F_{k,l}$ is a disk of dimension $2k + l - 1$.*

Proof. We proceed by induction on the dimension, $2k+l-1$. In the initial case, $F_{0,1}$ has a single S^0 coordinate which must be $+1$, so it is a single point, i.e. a disk of dimension 0. To prove $F_{k,l}$ is a disk, we will show that it is a $(2k+l-1)$–manifold, show it collapses to a vertex, and then apply Corollary 3.28. There are four types of vertices whose links we need to consider:

1. $+1$ coming from one of the k S^1 coordinates has as its link $F_{k-1,l+1}$, a $(2k + l - 2)$–disk by induction.

2. -1 coming from one of the S^1 coordinates has as its link $S^0 * F_{k-1,l}$, a $(2k + l - 2)$–disk.

3. $+1$ coming from one of the l $\{\pm 1\}$ coordinates has as its link $*_{i=1}^{2k+l-1} S^0$, a $(2k + l - 2)$–sphere.

4. -1 coming from one of the $\{\pm 1\}$ coordinates has as its link $F_{k,l-1}$, a $(2k+l-2)$–disk.

Now we will define a matching on $F_{k,l}$. First, we order the coordinates. In each S^1 coordinate, we also choose one of the two copies of $\{\pm 1\}$ to be distinguished. Associate each simplex in $F_{k,l}$ with the simplex obtained by inserting or removing $+1$ to or from the first coordinate lacking a -1 (in the distinguished copy of $\{\pm 1\}$ in the case the first such coordinate is S^1). Doing this does not change which coordinate is the first without a -1, so the pairing is well-defined. Every simplex is paired ($\emptyset$ is paired with the vertex with a $+1$ in the first coordinate and nothing in any other coordinate), so if there are no cycles in this matching, $F_{k,l}$ collapses to a point.

Suppose there were a cycle. It would have to be of the form:

$$\sigma_0 \lessdot \tau_0 \gtrdot \sigma_1 \lessdot \tau_1 \gtrdot \sigma_2 \lessdot \ldots \lessdot \tau_{s-1} \gtrdot \sigma_s = \sigma_0$$

where each σ_i is paired with τ_i. Also, for $1 \leq i \leq s$, σ_i must be τ_{i-1} minus a vertex v_i. Therefore, since this is a cycle, there must be a j such that $\tau_j = \sigma_j \cup \{v_i\}$. For this to be possible, v_i must be a $+1$. Thus, all of the simplices in the cycle must have all the same -1 coordinates, but if that is the case, the vertex to be added in any $\sigma_i \lessdot \tau_i$ pair is always the same, and v_i must be the same for every i. This is a contradiction. Therefore, there are no cycles, and we have a Morse matching with a single critical simplex. $\square$

Proposition 5.7. *ΔK simplicially G–collapses to ΔS.*

Proof. The collapsing will occur in three steps. Define

$$K_1 := \{\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K \quad | \quad A \cap C \neq \emptyset\}$$

$$K_2 := \{\phi \in K_1 \quad | \quad A = C\}$$

First, we collapse ΔK to ΔK_1. Let σ be a chain of the form

$$\phi_0 < \phi_1 < \ldots < \phi_{m-1} < \phi_m$$

in ΔK where $\phi_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$. If $A_0 \cap C_0 = \emptyset$, we want to pair σ with another chain for which that is also true. Find the last k such that $A_k \cap C_k = \emptyset$. Then compare B_k and D_k to B_m and D_m. If $B_k = B_m$ and $D_k = D_m$, pair σ with the chain obtained by adding to (or deleting from) the end of σ the element $\begin{pmatrix} A_m \cup (B_m \cup D_m)^c & B_m \\ C_m \cup (B_m \cup D_m)^c & D_m \end{pmatrix}$. Otherwise, find the first $l > k$ where $B_l \neq B_k$ or $D_l \neq D_k$. Now pair σ with the chain obtained by inserting (or removing if it equals X_{l-1}) $\begin{pmatrix} A_l & B_{l-1} \\ C_l & D_{l-1} \end{pmatrix}$ before ϕ_l. Nowhere are we inserting or deleting elements with $A \cap C = \emptyset$, so the selection of k is not affected. In the second case, we are inserting or deleting an element with $B_{l-1} = B_k$ and $D_{l-1} = D_k$, so the selection of l is not affected. Therefore, the matching is well-defined. The critical simplices are exactly those where $A_0 \cap C_0 \neq \emptyset$, forming ΔK_1, a subcomplex. Therefore, if there are no cycles, we have a collapsing from ΔK to ΔK_1. Also, the pairings are chosen equivariantly, so we will have a G–collapse.

Suppose we have a cycle

$$\sigma_0 \lessdot \tau_0 \gtrdot \sigma_1 \lessdot \tau_1 \gtrdot \sigma_2 \lessdot \ldots \lessdot \tau_{s-1} \gtrdot \sigma_s = \sigma_0$$

Again, for $1 \leq i \leq s$, σ_i is obtained from τ_{i-1} by deleting an element ψ_i, so there must be a pair $\sigma_j \lessdot \tau_j = \sigma_j \cup \{\psi_i\}$ coming from our matching. Therefore, $\psi_i \in K_1$ for all i, which means that all the simplices in our cycle have all of the same elements with $A \cap C = \emptyset$. Thus, they all have the same ϕ_k, so B_k and D_k are fixed and we know that every ψ_i has them as its second column. As a result, the elements after ϕ_k that have $B \neq B_k$ or $D \neq D_k$ are not changing as we move through the cycle, implying that ψ_i is the same for all i. This is a contradiction, so our matching has no cycles. This proves that ΔK G–collapses to ΔK_1.

The next two collapsings are proved by Lemma 2.6. For the first, we define $h_1 : K_1 \to K_1$ by $h_1(\phi) = \begin{pmatrix} A \cap C & B \\ A \cap C & D \end{pmatrix}$. This is an order-preserving G–poset map, and $h_1(\phi) \leq \phi$. The fixed point set of h_1 is exactly K_2, so 2.6 implies that ΔK_1 G–collapses to ΔK_2. For the second collapsing, we now define $h_2 : K_2 \to K_2$ by $h_2(\phi) = \begin{pmatrix} A & B \cup D \\ A & B \cup D \end{pmatrix}$. This is an order-preserving G–poset map, $h_2(\phi) \geq \phi$, and the fixed point set is S. Therefore, the same lemma implies that ΔK_2 G–collapses to ΔS. Hence, ΔK G–collapses to ΔS. □

Theorem 5.8. *$|\Delta K|$ is a G–regular neighborhood of $|\Delta S|$ with boundary $|\Delta L|$.*

Proof. G acts freely outside of $|\Delta S|$, so $\partial|\Delta K|$ is G–bicollarable in $|\Delta M|$. Now the theorem follows immediately from 3.30 (the collapsing criterion for G–regular neighborhoods) and Propositions 5.5 and 5.7. □

Now our main result 5.1 follows easily:

Theorem 5.9. $\mathrm{Hom}_{\{3\}}(P_4, K_n)$ *is a PL manifold with boundary* $\mathrm{Hom}_{\{3\}}(C_5, K_n)$, *equivariantly homeomorphic to* $N := \{(x, y) \in S^{n-2} \times S^{n-2} \mid x \cdot y \geq 0\}$, *where the involution on* N *interchanges* (x, y) *with* (y, x). *Hence,* $\mathrm{Hom}_{\{3\}}(C_5, K_n)$ *and* $\mathrm{Hom}(C_5, K_n)$ *are both equivariantly homeomorphic to* ∂N, *i.e. the Stiefel manifold* $V_{n-1,2}$.

Proof. It follows from 2.1 that we have $\mathrm{Hom}_{\{3\}}(P_4, K^n) \approx_G |\Delta K|$ with the subcomplex $\mathrm{Hom}_{\{3\}}(C_5, K^n) \approx_G |\Delta L|$. Because $|\Delta K|$ and N are both G–regular neighborhoods of the diagonal, they are equivariantly homeomorphic by 3.2. $\square$

5.3 Questions

The Stiefel manifold $V_{n-1,2}$ has a natural action of the orthogonal group O_2 (with the Grassmannian as the quotient). The equivariant homeomorphism above is with respect to a single reflection in O_2. The multimorphism complex $\mathrm{Hom}(C_5, K_n)$ does not have a combinatorial O_2–action; however, there is the induced action of the dihedral group D_5 (a subgroup of O_2) which is the group of symmetries of the cycle C_5. It seems natural to ask:

Question 5.10. *Is* $\mathrm{Hom}(C_5, K_n)$ *equivariantly homeomorphic to* $V_{n-1,2}$ *with respect to the action of the dihedral group* D_5*?*

Unfortunately, neither of the smaller restricted models $\mathrm{Hom}_{\{3\}}(C_5, K_n)$ or $\mathrm{Hom}_{\{2,4\}}(C_5, K_n)$ is D_5–invariant, so it seems that one needs to work with the full multimorphism complex $\mathrm{Hom}(C_5, K_n)$ which does not have an obvious D_5–equivariant embedding into $S^{n-2} \times S^{n-2}$. Also, a good (equivariant) combinatorial candidate for N is missing, which is the obstacle to applying the methodology above to answer this question positively.

Another interesting line of inquiry is finding similar combinatorial models for the other Stiefel manifolds $V_{n,k}$:

Question 5.11. *Do there exist multimorphism complexes or restricted multimorphism complexes equivariantly homeomorphic to the Stiefel manifolds $V_{n-1,k}$?*

There seems to be an obvious starting point: Let Γ be the disjoint union of k copies of K_2. Then $\mathrm{Hom}(\Gamma, K_n)$ is Σ_k–homeomorphic to the product of k copies of S^{n-2}, where the symmetric group Σ_k permutes the edges of Γ and the spheres in the product. Ideally, we would seek to find a larger graph Λ containing Γ as a subgraph such that $\mathrm{Hom}(\Lambda, K_n)$ sits inside $\mathrm{Hom}(\Gamma, K_n)$ as the Stiefel manifold $V_{n-1,k}$.

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