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A GENERALIZATION OF CONVEXITY

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A GENERALIZATION OF CONVEXITY

CHAPTER 0

INTRODUCTION

Convex sets were first studied systematically by Brunn [2], in 1887. There has been interest recently in the study of generalizations of convexity, the majority of these being algebraic or topological in nature. Several of these are mentioned in [5]. It is the author's opinion that while such examples are useful in studying the structure of convex sets, they tend to lead one away from the geometric intuition that convexity offers.

In this paper, we study a generalization of convexity where one does not require the join of each pair of points in the set to lie in that set, but, instead, one requires some subset of each $m \geq 2$ points to determine joins which belong to the set. This concept is but a special case of an even more generalized convexity proposed by J.E. Allen [1].

The precise definition of our generalization of convexity appears in the next chapter along with several necessary basic set theoretic, algebraic, and topological properties.

A Helly order for one class of generalized convex sets is determined and several finite convex covering theorems are proved. By a <u>convex covering</u> of a set S we mean a family of convex sets whose union is S.

In Chapter II, we characterize the kernel of a certain family of m-convex sets, answer a conjecture posed by Danzer, Grünbaum, and Klee, prove a generalized Helly theorem, and introduce the notion of local nonconvexity. The later concept leads us to several representation theorems for m-convex sets.

Valentine's theorem states that a closed, connected 3-convex set in E^2 is the union of three or fewer closed convex sets. Guay has extended this result in showing that a closed 4-convex set in E^2, which is not simply-connected, is the union of five or fewer closed convex sets. In Chapter III, we show that any closed 4-convex set in E^2 is the union of nine or fewer closed convex sets.

Except for one or two symbols, the notation used in this paper is consistent with that used by Valentine [18].

CHAPTER I

(M,N) CONVEXITY

The results of this chapter apply generally to subsets of a linear topological space E, as defined in such sources as Kelley and Namioka [11], while others will apply only to finite dimensional spaces, denoted E^d (d = dimension). Some of the more combinatorial results will apply even to subsets of a vector space over an ordered field. The segment, or join, between two points x and y in E is the set of all points in E of the form $\alpha x + (1-\alpha)y$, where $0 \leq \alpha \leq 1$, denoted xy. In order to simplify later notation, we let the symbol C_m stand for the number of combinations of m things taken two at a time. That is, $C_m = m(m-1)/2$, $m \geq 1$. Familiarity with the basic properties of convex sets, as found in [18], is assumed. In this chapter the basic combinatorial, set theoretic, and linear properties of (m,n) convex sets will be developed, the Helly order for the family of (3,2) convex sets in the plane will be discussed, and several convex covering theorems for (m,n) convex sets will be derived.

1.1. DEFINITION: A set S is said to be (m,n) convex provided $|S| \geq m$ and if for each m distinct points of S at least n of the possible C_m joins between these m points are contained in S. (It is understood that m and n are non-negative integers, with $0 \leq n \leq C_m$, and $m \geq 2$). A set is said to be <u>exactly</u> (m,n) convex iff it is (m,n) convex but not (m,n + 1) convex (a simple combinatorial argument shows that for $n > 0$ this is equivalent to saying that a set is exactly (m,n) convex iff it is (m,n) convex but neither (m - 1,n) nor (m,n + 1) convex). An (m,1) convex set is referred to simply as an <u>m-convex</u> set, or a set having <u>property P^m</u>. An <u>exactly</u> <u>m-convex</u> set is one which is m-convex but not (m - 1)-convex. As in Kay and Guay [9], we make the convention that no nonempty set is 1-convex. Thus, a convex set having more than one point is exactly 2-convex.

In considering the preceding definition, we find that (2,1) convexity is ordinary convexity, and more generally any (m,C_m) convex set for $m > 2$ is convex. It is a straight-foward application of the definition of (m,n) convexity to show that if S is (m,n) convex, then S is also (m,k) convex for $0 < k \leq n$, and therefore m-convex.

<u>1.2. PROPOSITION</u>: If S is an (m,n) convex set with $n > C_{m-1}$, then S is connected.

<u>Proof</u>. Suppose that S is not connected; then it has at least two components, say A and B. Choose any m - 1 points in A and a point in B. But there exists at most C_{m-1} joins between these m points, and thus $n \leq C_{m-1}$, a contradiction. ◊

By considering a set consisting of a convex set and an isolated point, we can see that the bound in 1.2 is best possible.

The next proposition shows that the bound used in 1.2 is also large enough to ensure convexity for a closed (m,n) convex set, and thus 1.2 becomes a corollary. For convenience, $\hat{r}$ will stand for the set $\{1,2,\ldots,r\}$, where r is any natural number.

1.3. PROPOSITION: If S is a closed (m,n) convex set for which $n > C_{m-1}$, then S is convex.

Proof. This result will be immediate if it is established that for any integer $m \geq 2$ a closed $(m+1, C_m + 1)$ convex set is $(m, C_{m-1} + 1)$ convex. Let S be a closed $(m+1, C_m + 1)$ convex set in E, and select $x_1,\ldots,x_m$ any m points in S. Suppose that there are not more than C_{m-1} joins determined by these m points. Let y be any other point in S, and suppose there are r joins of the type yx_i, for $i \in \hat{m}$, in S. We have $r + C_{m-1} \geq C_m + 1$ by hypothesis, which implies that $r \geq m$. Therefore yx_i is in S, for all $i \in \hat{m}$ and for any y in $S \setminus \{x_1,\ldots,x_m\}$. Since S is closed and connected (by 1.2) and we may take y in an arbitrary neighborhood of x_i, we have $x_i x_j$ in S for any i and j in $\hat{m}$, a contradiction. ◊

If S is a closed (m,n) convex set which is also connected, then Kay has shown that if $n > \frac{1}{4}(m-1)^2$, then S is convex. It can be seen that this result is best possible by considering two intersecting lines.

In an (m,n) convex set, if $n > C_{m-1}$ then in the proof of 1.3 we see a relationship between the given (m,n) convexity of a closed set and a lower order convexity for the same set. In the direction of higher order convexity for an (m,n) convex set we offer the following proposition.

<u>1.4. PROPOSITION</u>: An (m,n) convex set is $(m + k,n + k)$ convex, where $n > 0$ and k is any natural number.

<u>Proof</u>. Consider any $m + k$ points $p_1,\ldots,p_{m+k}$ in S. Among $p_1,\ldots,p_m$ there are at least n joins. Suppose that $p_i p_j$ is one of them, where $1 \leq i < j \leq m$. The points in the set $\{p_1,\ldots,p_{m+1}\}\setminus\{p_i\}$, determine n joins, none of them being the join $p_i p_j$. Let $p_r p_s$ denote one of these joins. Now there are at least n joins among the m points $\{p_1,\ldots,p_{m+2}\}\setminus\{p_i,p_r\}$, none of these joins being $p_i p_j$ nor $p_r p_s$. Continuing this process, we obtain $n + k$ joins between the given $m + k$ points in S, and we have shown that S is $(m + k,n + k)$ convex. ◊

The necessity of the restriction $n > 0$ in 1.4 is evident when one considers the set M consisting of four isolated points: M is $(3,0)$ convex and also $(4,0)$ convex, but not $(4,1)$ convex. The subset in E^2 defined by $m - 1$ segments emanating from a single point is an example of a connected set which is $(m,1)$ convex and $(m + k,k + 1)$ convex but not $(m + k,k + 2)$ convex for $k = 1,\ldots,m - 1$. This shows that 1.4 is best possible for the case when $n = 1$.

It is not hard to construct examples to convince oneself that the set of (m,n) convex sets, in E, for fixed m and n is not closed under intersection, union, set difference, complementation, or cross product in E X E. This is to be expected, since even convex sets in general are not closed under union, set difference, or complementation.

However, certain set theoretic properties of (m,n) convex sets are true. In fact, as a consequence of the definition of (m,n) convexity, we have that the union of k disjoint (m,n) convex sets is exactly (k(m - 1) + 1,n) convex.

The next result together with Zorn's lemma will be used later to establish the existance of certain maximal m-convex subsets of a set.

1.5. PROPOSITION: The union of the members of a family of (m,n) convex sets which is directed by $\supset$ (the union of any two members is contained in some third) is an (m,n) convex set.

Proof. Let $\mathfrak{F} = \{C_\alpha : \alpha \in A\}$ be such a family and consider $\bigcup\{C_\alpha : \alpha \in A\} = B$. Select any m points in B, say $p_1, \ldots, p_m$. Suppose p_i is in C_{α_i}, for $i \in \hat{m}$. By induction there is a set C_β such that $C_{\alpha_i} \subset C_\beta$ for all $i \in \hat{m}$. Therefore $p_1, \ldots, p_m$ are in C_β. Now C_β is (m,n) convex and hence the p_i determine at least n joins in C_β; since $C_\beta \subset B$, they determine at least n joins in B. Thus B is an (m,n) convex set. ◊

The most singular difference between general (m,n) convex sets and convex sets is closure under intersection. A combinatorial result may be stated, where the underlying assumption is that the intersection under consideration contains at least m points. Here the square brackets will denote the greatest integer function. An easy preliminary result is that the intersection of two (m,n) convex sets is (m,1) convex if $n \geq \left[\frac{1}{2}C_m\right] + 1$. By considering the two (3,2) convex sets X and Y indicated in Figure 1.1, we see that this result is best possible (dashed lines indicate the deletion of boundary points).

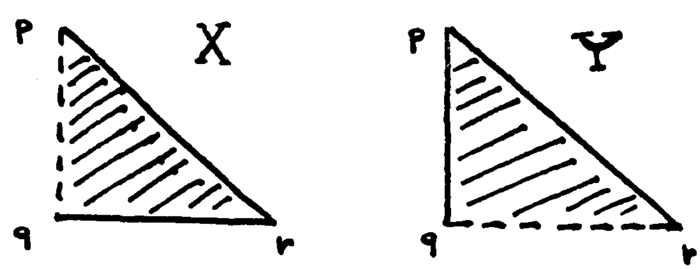


Figure 1.1

More generally, we have:

1.6 PROPOSITION: For each integer $k \geq 2$, the intersection of k (m,n) convex sets is (m,1) convex provided $n \geq \left[C_m(k-1)/k\right] + 1$.

Proof. Let $C = A_1 \cap A_2 \cap \cdots \cap A_k$, where A_i for $i \in \hat{k}$ is an (m,n) convex set with $n \geq \left[C_m(k-1)/k\right] + 1$. Choose any m distinct points in C. It is obvious that among these m points in A_i we can be missing at most $C_m - (\left[C_m(k-1)/k\right] + 1)$ joins. Regarding this as a matrix, with a column for each set A_i and a row for each of the possible C_m joins, labeling these joins consecutively from 1 to C_m, we put a one in the a_{ij}-th position if the i-th join is in the set A_j and zero otherwise. We need to show that if we put at most $C_m - (\left[C_m(k-1)/k\right] + 1)$ zeros arbitrarily in each column, then there is still one row free of zeros, or equivalently that,

$$k(C_m - [C_m(k-1)/k] - 1) < C_m \le k(C_m - [C_m(k-1)/k]). \quad (1.1)$$

However, this inequality is an immediate consequence of a property of the greatest integer function, namely, $[x] \le x < [x] + 1$. (The value on the right of (1.1) shows that our bound is best possible). Therefore, under the hypothesis given, the intersection of k (m,n) convex sets is (m,1) convex. ◊

To establish several basic algebraic properties of (m,n) convex sets we recall the well known result that if A and B are nonempty subsets of E and α and β are scalars then $\text{conv}(\alpha A + \beta B) = \alpha(\text{conv } A) + \beta(\text{conv } B)$, where conv A denotes the convex hull of A. This result implies that the scalar multiple of a convex set is convex and the sum of two convex sets is convex. If we are careful with the value for n, we have some idea what the sum of two (m,n) convex sets is like. It is straightfoward to show that if A and B are (m,n) convex sets with $n > [\frac{1}{2}C_m] + 1$, then the sum $A + B = \{a + b : a \in A, b \in B\}$ is (m,1) convex. However, if we wish to conclude that the sum is (m,n) convex for general values of m and n, then it is sufficient to assume that one of the summands be convex, as the following result shows.

<u>1.7. PROPOSITION</u>: If C is convex and S is (m,n) convex, then for any two scalars α and β, $\alpha C + \beta S$ is (m,n) convex.

<u>Proof</u>. Let $A = \alpha C + \beta S$. Choose any m distinct points in A and denote them by $a_k = \alpha c_k + \beta s_k$, where $c_k \in C$, $s_k \in S$, and $k \in \hat{m}$. If $s_i s_j$ is one of the guaranteed joins in S, then

$a_i a_j \subset A$. Since for $0 \leq \gamma \leq 1$ we have

$$\begin{aligned}\gamma a_i + (1-\gamma)a_j &= \gamma(\alpha c_i + \beta s_i) + (1-\gamma)(\alpha c_j + \beta s_j) \\ &= \alpha(\gamma c_i + (1-\gamma)c_j) + \beta(\gamma s_i + (1-\gamma)s_j) \\ &\in \alpha C + \beta S.\end{aligned}$$

Since we have at least n joins in S, we must have at least n joins in A. Hence, A is (m,n) convex. ◊

An immediate consequence of 1.7 is that the translate of an (m,n) convex set is (m,n) convex. This fact together with the next result shows that in any real vector space, (m,n) convexity is an affine invariant.

<u>1.8. PROPOSITION</u>: If S is an (m,n) convex set in a real vector space V and T is a linear transformation over V, then T(S) is (m,n) convex.

<u>Proof</u>. Let $y_1,\ldots,y_m$ be any m distinct points in T(S). There exist m distinct points $x_1,\ldots,x_m$ in S such that $y_i = T(x_i)$, for $i \in \hat{m}$. Since S is (m,n) convex there are at least n joins among the points $x_1,\ldots,x_m$. Suppose one of them is $x_i x_j$, where $1 \leq i < j \leq m$. Now for $0 \leq \alpha \leq 1$, we have

$$\begin{aligned}T(\alpha x_i + (1-\alpha)x_j) &= \alpha T(x_i) + (1-\alpha)T(x_j) \\ &= \alpha y_i + (1-\alpha)y_j.\end{aligned}$$

That is, $y_i y_j$ is contained in T(S). Hence, since there are at least n joins in S, there will be at least n joins in T(S), among the corresponding points. Thus T(S) is (m,n) convex. ◊

Using the techniques of the last two propositions it can be shown that the Cartesian product of a convex set and an (m,n) convex set is (m,n) convex, and the intersection of a convex set with an (m,n) convex set is (m,n) convex.

Several topological properties of (m,n) convex sets, listed below, will be useful in establishing later results. All of these properties are straightfoward for convex sets. Recall that a set S is <u>polygonally connected</u> if for any x and y in S there exists a finite set $x_1, \ldots, x_m$ of points in S, such that $xx_1, x_1x_2, \ldots, x_{m-1}x_m, x_my$ are contained in S. Let cl S denote the closure of S in E. The following definition will introduce another concept which will be useful.

<u>1.9. DEFINITION</u>: A set S is said to be <u>relatively</u> (m,n) <u>convex with respect to a set</u> T if for each m points $x_1, \ldots, x_m$ in S there exist n joins x_ix_j determined by these points such that for each such $i \neq j$ the open segment $(x_ix_j) = x_ix_j \setminus \{x_i, x_j\}$ is a subset of T. (We use the term <u>absolute</u> (m,n) convexity to designate ordinary (m,n) convexity of a subset of T independent of T, and <u>relative convexity</u> for relative (2,1) convexity.) S is said to be <u>exactly</u> (m,n) convex with respect to T iff it is (m,n) convex but not (m,n + 1) convex with respect to T, and S is <u>exactly</u> m-convex with respect to T iff it is m-convex but not (m - 1)-convex with respect to T. (Again, we assume that no nonempty subset of T is 1-convex relative to T.)

Observe that if a set S is (m,n) convex it is (m,n) convex relative to any set T containing it, and (m,n) convexity for subsets of a convex set T is equivalent to relative (m,n) convexity with respect to T.

<u>1.10. PROPOSITION</u>: If S is an (m,n) convex set in E, then cl S is (m,n) convex with respect to cl S.

<u>Proof</u>. Select any m points $x_1,\ldots,x_m$ in cl S, and suppose that $x_ix_j \not\subset$ cl S for at least $C_m - n + 1$ pairs (i,j), where $i < j$. Let $U_j(x_i)$ and $U_i(x_j)$ be neighborhoods of x_i and x_j respectfully with the property that for $u \in U_j(x_i)$ and $v \in U_i(x_j)$ $uv \not\subset S$. Let $U_i = \bigcap_j U_j(x_i)$, where j is such that $x_ix_j \not\subset$ cl S. Now from the construction of the U_i, if y_i is a point in $S \cap U_i$, $i \in \hat{m}$, then $y_iy_j \not\subset S$. Hence, for at least $C_m - n + 1$ pairs (i,j) where $i < j$, $y_iy_j \not\subset S$, contradicting the (m,n) convexity of S. $\Diamond$

It is natural to ask, if S is an (m,n) convex set in E, whether the topological interior of S is (m,n) convex. At this writing, however, a proof of the conjecture has not been found. The truth of the conjecture is, of course, well known for convex sets.

Several useful concepts are now introduced.

<u>1.11. DEFINITION</u>: For any point x in $S \subset E$, let $S_x = \{y \in S: xy \subset S\}$. S_x is called the <u>x star of S</u>. The <u>kernel</u> of a set S, denoted by ker S, is defined as the set $\{z \in S: S_z = S\}$. A set S in E is called <u>starshaped</u> if there exists a point x in S with the property that $S_x = S$. A set S is called <u>locally starshaped</u> iff each point x in S lies in some neighborhood whose intersection with S is starshaped with respect to x.

<u>1.12. DEFINITION</u>: For any point x in $S \subset E$, let $S^x = \{y \in S:$

$xy \not\subset S\}$. S^x is called the x anti-star of S. (Note that if S is closed then S^x is relatively open for any $x \in S$ and if S is m-convex then S^x is (m - 1)-convex with respect to S).

It is shown in [9] that every closed m-convex set is locally starshaped and that in a finite dimensional linear space every connected m-convex set is polygonally connected. Since every (m,n) convex set is k-convex for some $k \geq 2$, we have both of these results valid for (m,n) convex sets.

We frequently have occasion to deal with exactly (m,n) convex sets. One may generate such sets by using the following constructive proposition.

1.13. PROPOSITION: Given the nonnegative integers $m \geq 2$, n,r, and k such that:

i) $0 \leq n \leq C_m$,

ii) r is the least nonnegative integer such that n + r is in the set $\{C_{s+1}: s = 1,2,\ldots\}$,

iii) $k = \frac{1}{2}(2m - 1 - \sqrt{8(r + n) + 1})$,

then the regular (m - k)-gon (interior included), with r adjacent open sides removed, together with k isolated points is an exactly (m,n) convex set.

Proof. Consider the regular (m - k)-gon M with the k isolated points as described above. To show that this set is exactly (m,n) convex, we must prove it is (m,n) convex and obtain m distinct points which determine exactly n joins in this set. Choose the k isolated points and the m - k vertices of M. The only joins in the set determined by these points corresponds to the C_{m-k} joins between the vertices of M, minus the r deleted open sides. Hence, the number of joins in the set determined by these m points is $C_{m-k} - r = \frac{1}{2}(m - k)(m - k - 1) - r = n$ (by use of iii). By changing the choice of the m points it is obvious that the number of joins in M increases. Thus, m arbitrary points determine at least n joins in the set, and some m determine no more than n. Therefore, the set described is exactly (m,n) convex.$\diamond$

In connection with the hypothesis of the theorem, it is desirable to show that such a choice of integers r and k satisfying (ii) and (iii) is always possible, and that $m - k \geq r$ (and that $m - k \geq 2$ if $r \leq 1$). Choose s the smallest integer such that $n \leq C_{s+1}$; then put $n + r = C_{s+1}$ (thus satisfying the choice of $r \geq 0$ in (ii)). Now we have

$$n + r = \tfrac{1}{2}s(s + 1)$$

or

$$8(n + r) + 1 = (2s + 1)^2.$$

It follows by the definition of k in (iii) that $k = m - s - 1$, and thus k is an integer. To show that $k \geq 0$, observe that our choice of s demands that since $n \leq C_m$, $s + 1 \leq m$. Finally, to show that $m - k \geq r$ (and ≥ 2) note that $s = m - k - 1$ and from the definition of s, $n \geq C_s + 1$, hence, we have

$$\begin{aligned} C_{s+1} &= C_s + s \\ n + r &= C_s + m - k - 1 \\ &\leq (n - 1) + m - k - 1 \end{aligned}$$

or, $r \leq m - k - 2.$

Hence, $m - k \geq r + 2$.

Krasnossel'skiĭ's theorem states that if S is a compact, connected set in a normed linear space of dimension n and for each set of n + 1 points $x_1, \ldots, x_{n+1}$ in S there is at least one point y in S such that yx_i is contained in S for i in $\widehat{n+1}$, then S is starshaped. A condition that would guarantee a subset of E^d to be the union of at most two starshaped sets was given by Koch and Marr [12]. For m-convexity it is easy to show that every m-convex set is the union of m - 1 or fewer starshaped sets, as in [6]. Given an (m,n) convex set, it too can be represented as a finite union of starshaped subsets. In 1.14 we not only get a bound, but we also get Guay's result for the case n = 1.

1.14. PROPOSITION: If S is an (m,n) convex set in a linear space with $k = C_m - n$, let r be determined by $C_r \leq k < C_{r+1}$ for $k \geq 1$. Then S is the union of r or fewer starshaped sets.

Before proceeding with the proof, let us establish a lemma. For convenience, we adopt the terminology that a subset $V = \{v_1, \ldots, v_t\}$, of a set S is visually independent relative to S if for all i and j such that $1 \leq i < j \leq t$, $v_i v_j \not\subset S$. We say that a point x can see a point y via S iff the open segment (xy) belongs to S.

1.15. LEMMA: An $(m, C_m - 1)$ convex set S is the union of two starshaped sets. If $m > 3$, then S is the union of two convex sets.

<u>Proof</u>. Let S be an $(m, C_m - 1)$ convex set, $m > 2$ (the conclusion of the lemma is false if $m = 2$). Consider the case where $m = 3$. If for any two points x and y in S, we have $xy \subset S$, then S is convex, and the result follows. Suppose that there exists x and y in S such that $xy \not\subset S$. For any other point z in S, we have $xz \subset S$, for otherwise, the set $\{x,y,z\}$ would consist of three points in S with only one join in S. Hence $S = S_x \cup \{y\}$, and the lemma is true for this case.

Let $m > 3$, and suppose that S is not convex. Hence, there exist u and v in S such that $uv \not\subset S$. Suppose that there are points w and z in S such that $wz \not\subset S$, where (w,z) and (u,v) are distinct pairs. Consider the case where the four points are distinct. If we choose u,v,w,z, and $m - 4$ other points in S, then we have m points in S with at most $C_m - 2$ joins between them, a contradiction of the $(m, C_m - 1)$ convexity of S. If $w = u$ or v, we get a similar contradiction by considering u,v,z, and $m - 3$ other points in S. Similarly if $z = u$ or v. Hence, given a pair (w,z) of points in S distinct from (u,v), $wz \subset S$. For any x in S, therefore, $xu \subset S$ (obviously, $uv \subset \mathrm{bd}\, S$). Hence $S = S_u \cup \{v\}$. From the fact that uv is the only join not contained in S, S_u is convex, thus completing the proof. ◊

<u>Proof of 1.14</u>. Suppose that r is determined by $C_r \leq k < C_{r+1}$. Assume $k \geq 1$. There cannot exist $r + 1$

visually independent points in S, for otherwise any other $m - r - 1$ points of S determine with these $r + 1$ points a set of m points in S missing at least $C_{r+1} > k$ joins in S, a contradiction to the $(m, C_m - k)$ convexity of S. Therefore, assuming S is not convex there exists a largest positive integer t with $2 \leq t \leq r$, such that there exists a set of t visually independent points in S. Let $p_1, \ldots, p_t$ be such a set. It is a straightfoward application of the maximality of t to see that for any other point x in S, we must have $xp_i \subset S$, for at least one value of i in $\hat{t}$. Therefore $S = S_{p_1} \cup \cdots \cup S_{p_t}$, and the proof is complete. ◊

It was pointed out in 1.15 that $(m, C_m - 1)$ convex sets are expressible as the union of two convex sets. Sets of this type are also starshaped, since by 1.2 they are connected. In fact, if S is a (m,n) convex set in E^d with $n > C_{m-1}$ one can show that int S, core S, and lin S are all convex (see Valentine [17, p. 11]).

One useful description of (m,n) convex sets, and most difficult to obtain, is in terms of finite unions of convex sets. For general (m,n) convex sets with n sufficiently large such characterizations are easy to obtain. For example, if S is closed and (m,n) convex with $n > C_{m-1}$, then S is convex (by 1.3). However, if S is not closed, then S is still representable as a finite union of convex sets in some cases. Prior to characterizing these (m,n) convex sets with $n > C_{m-1}$, we have the next result, which exhibits a strong topological property characteristic of such sets.

<u>1.16. PROPOSITION</u>: Let A be an (m,n) convex set with $n > C_{m-1}$. If points x,y, and z in A are such that xy and xz lie in A, then $\mathrm{int}(\mathrm{conv}\{x,y,z\}) \subset A$, where the interior is taken relative to the plane of x,y, and z.

<u>Proof</u>. Since there is nothing to prove if x,y, and z are collinear, assume they are not. Choose w in (xy) and u in (xz) and suppose that there is a v in (wu) such that v is not in A. It is clear from the (m,n) convexity of A that there can be at most a finite number of points in $(wu) \cap A$. So choose $m - 1$ points in $(wu) \cap C(A)$, where $C(A)$ denotes the complement of A relative to E, say $q_1, \ldots, q_{m-1}$. Let $z_i = zq_i \cap (xw)$, for $1 \leq i \leq m - 1$. Now $z, z_1, \ldots, z_{m-1}$ is a set of m points in A determining at most $C_m - (m - 1) = C_{m-1}$ joins in A, a contradiction. Hence, there cannot exist such a v in $(wu) \cap C(A)$ and it follows that $\mathrm{int}(\mathrm{conv}\{x,y,z\}) \subset A$. ◊

It should be mentioned here that in stating 1.16 for $(3,2)$ convex sets, $(3,2)$ convexity <u>implies</u> that $xy \cup xz \subset A$. It should also be pointed out that since the rather large lower bound on n $(n > C_{m-1})$ implies that the closure (and therefore the interior) of S is convex, the nonconvexity characteristics of such a set are derived from properties of the boundary.

<u>1.17. PROPOSITION</u>: A planar, bounded (m,n) convex set

S, with $n > C_{m-1}$, may be expressed as the union of k convex sets, where $k \leq \frac{1}{2}(1 + \sqrt{8m - 15})$. The result is best possible.

Proof. We shall make use of a well-known theorem of graph theory: If G is any graph without circuits (that is, a tree), then the vertices of G can be colored with two colors.

The proposition is trivial for all cases except when int $S \neq \emptyset$, and it readily follows that int S is convex and cl S = cl(int S). Suppose p and q are points in bd S such that $pq \not\subset S$. Then $pq \subset$ bd S. Since S is bounded, let $J = xy$ be the maximal segment in cl S containing pq. Thus, $xy \not\subset S$ and hence $xy \subset$ bd S. Since xy contains no infinite subset of S (by the $(m, C_{m-1} + 1)$ convexity of S), then $xy \cap S$ consists of a finite set of points, say

$$x_1 = p,\ x_2 = q,\ x_3,\ \ldots,\ x_r, \qquad r \geq 2.$$

Clearly, $r < m$, for otherwise S contains m points none of whose joins belong to S. Choose $m - r$ distinct points $x_{r+1}, \ldots, x_m$ from int S and consider $x_1, \ldots, x_m$. These points determine no more than $C_m - C_r$ joins belonging to S, so $C_{m-1} + 1 \leq n \leq C_m - C_r$. The inequality $r \leq \frac{1}{2}(1 + \sqrt{8m - 15})$ follows. Since this argument applies to all the maximal segments $J_1, J_2, \ldots$ lying in bd S and containing points not in S we may let $k \leq \frac{1}{2}(1 + \sqrt{8m - 15})$ be the maximal cardinality of the sets $J_i \cap S$, $i \geq 1$.

If $T = \text{bd } S \setminus \bigcup_{i \geq 1} (J_i)$, where (J_i) denotes the open segment J_i, let $A_1, A_2, \ldots$ denote the components of T; since it lies in the boundary of a convex set, each component is either a single point or an arc. There are two cases.

<u>Case 1</u>: At least one component A_i is an arc, or there exist infinitely many components A_i. If $E(A_i)$ denotes the endpoints of A_i, define the graph $G(T)$ having as vertex set $V = S \cap [E(A_1) \cup E(A_2) \cup \cdots]$ and as edge set those pairs (x,y) in $V \times V$ such that $xy \not\subset S$. Suppose $G(T)$ contains a circuit $x_1, \ldots, x_{n+1} = x_1$, with (x_i, x_{i+1}), $1 \leq i \leq n$, edges in $G(T)$. In this case, the points $x_1, \ldots, x_{n+1} = x_1$ lie in bd S and determine the joins x_1x_2, x_2x_3, ..., x_nx_1 not in S. Clearly, bd S $= \bigcup_{i=1}^{n} x_ix_{i+1}$. But then T could have at most n components and none of them is an arc, a contradiction. Hence, $G(T)$ is a tree and can be colored with two colors. Therefore, $V = V_1 \cup V_2$, where V_1 and V_2 are the vertices of empty subgraphs of $G(T)$. Define $O(A_i) = A_i \setminus E(A_i)$ for $i \geq 1$ and consider the set

$$C_j = V_j \cup (\text{int } S) \cup \left[\bigcup_{i \geq 1} O(A_i)\right], \qquad j = 1,2.$$

We show that each C_j is convex. Let $x \in C_j$ and $y \in C_j$. If $z \in xy \cap \text{int } S$, then since each interior point of S can see bd S via int S, it follows that $(xy) = (xz] \cup [zy) \subset \text{int } S$. Thus, assume $xy \subset \text{bd } S$. Then if $xy \subset S$, xy lies in one of the components A_i of T, and $(xy) \subset O(A_i)$ or $xy \subset C_j$. Finally, if $xy \not\subset S$ then xy belongs to one of the segments J_i and $\{x,y\} \cap \left[\bigcup_{i\geq 1} O(A_i)\right] = \emptyset$ implies $\{x,y\} \subset V_j$ or $xy \subset S$, a contradiction.

It remains to consider the points of $(J_i) \cap S$. For convenience, let $J_1 \cap S$ have maximal cardinality k among the $J_i \cap S$, and suppose $(J_1) \cap S = \{x_1, \ldots, x_{k-2}\}$. There is an onto mapping $f_r : (J_1) \cap S \rightarrow (J_r) \cap S$ for each $r \geq 1$, so define

$$C_{j+2} = \text{conv}(\bigcup_{r\geq 1} f_r(x_j)), \qquad j = 1,2,\ldots,k-2.$$

Since $xy \not\subset S$ implies $xy \subset \text{bd } S$ for x and y in C_{j+2}, it follows easily that C_{j+2} is a convex subset of S. Thus, $S = \bigcup_{j=1}^{k}(C_j \cap S)$.

Case 2. There exists finitely many components $A_1,\ldots,A_s$ and all components are singletons. It follows that there are finitely many maximal segments $J_1,\ldots,J_t$ and $\text{bd } S = \bigcup_{i=1}^{t} J_i$. Thus, we may suppose that $J_i \cap S = \{x_{i1},\ldots,x_{ik}\}$, that x_{i1} and x_{ik} are the endpoints of J_i $(1 \leq i \leq t)$, $x_{ik} = x_{i+1,1}$ $(1 \leq i \leq t-1)$, and $x_{tk} = x_{11}$. (If $J_i \cap S$ has cardinality less than k simply choose arbitrary points on J_i to define the x_{ij}'s.) We have two subcases:

Case 2.1: t even. Let $t = 2r$ and define the sets

$$C_1 = \text{int } S \cup \{x_{11},x_{31},x_{51},\ldots,x_{2r-1,1}\},$$
$$C_2 = \text{int } S \cup \{x_{21},x_{41},x_{61},\ldots,x_{2r,1}\},$$
$$C_{j+1} = \text{int } S \cup \{x_{1j},x_{2j},x_{3j},\ldots,x_{2r,j}\}, \qquad j = 2,\ldots,k-1.$$

It is clear that $C_j \cap S$ is a convex subset of S, and $S = \bigcup_{j=1}^{k}(C_j \cap S)$.

Case 2.2: t odd. Let $t = 2r + 1$. Assuming $k \geq 3$, define

$$C_1 = \text{int } S \cup \{x_{11},x_{22},x_{41},x_{61},\ldots,x_{2r-2,1},x_{2r,1}\},$$
$$C_2 = \text{int } S \cup \{x_{12},x_{23},x_{32},x_{42},\ldots,x_{2r,2},x_{2r+1,2}\},$$
$$\ldots$$
$$C_j = \text{int } S \cup \{x_{1j},x_{2,j+1},x_{3j},x_{4j},\ldots,x_{2r,j},x_{2r+1,j}\}, \qquad 2 \leq j \leq k-2,$$
$$\ldots$$
$$C_{k-1} = \text{int } S \cup \{x_{1,k-1},x_{31},x_{4,k-1},x_{5,k-1},\ldots,x_{2r+1,k-1}\},$$
$$C_k = \text{int } S \cup \{x_{21},x_{3,k-1},x_{51},x_{71},\ldots,x_{2r-1,1},x_{2r+1,1}\}.$$

Again each $C_j \cap S$ is a convex subset of S, and $S = \bigcup_{j=1}^{k}(C_j \cap S)$.

Finally, if $k = 2$ then S consists of a convex polygon and interior, having vertices $x_1,\ldots,x_{2r+1}$, with the open sides $(x_i x_{i+1})$ removed. Here, let $C_1 = \{x_1\}$, $C_2 = \text{int } S \cup \{x_2, x_4, x_6, \ldots, x_{2r}\}$, $C_3 = \text{int } S \cup \{x_3, x_5, x_7, \ldots, x_{2r+1}\}$, and $S = \bigcup_{j=1}^{3}(C_j \cap S)$. It remains to show that in this case, unless the cardinality of $S \cap \{x_1, \ldots, x_{2r+1}\}$ is less than 3,

$$3 \leq \tfrac{1}{2}(1 + \sqrt{8m - 15}).$$

But it is clear that regardless of the value of $r \geq 1$, S cannot be $(3, C_2 + 1) = (3,2)$ nor $(4, C_3 + 1) = (4,4)$ convex; hence, $m \geq 5$ and we have $3 = \frac{1}{2}(1 + \sqrt{25}) \leq \frac{1}{2}(1 + \sqrt{8m - 15})$. The result is best possible as the obvious example shows. $\Diamond$

1.18. COROLLARY: A planar, bounded (3,2) convex set is the union of two convex sets.

1.19. COROLLARY: A planar, bounded (4,4) convex set is the union of two convex sets.

Eduard Helly discovered a theorem in 1913 concerning the intersection of convex sets. The first published proof of this important theorem was given by Radon in 1921. For future reference, we state the theorem.

1.20. HELLY'S THEOREM: Suppose that $\mathcal{C}$ is a family of at least $d + 1$ convex sets in E^d, and $\mathcal{C}$ is finite or each members of $\mathcal{C}$ is compact. Then if each $d + 1$ members of $\mathcal{C}$ have a common point, then there is a point common to all members of $\mathcal{C}$.

For a compendium on Helly's theorem and its applications, see the excellent paper by Danzer, Grünbaum, and Klee [4]. In that paper several generalizations of Helly's theorem are mentioned. A useful concept in deriving such theorems is the following definition.

1.21. DEFINITION: Let $\mathfrak{F}$ be a family of sets in E^d. $\mathfrak{F}$ is said to have Helly order n, if n is the smallest cardinal number such that for each finite subfamily $\mathfrak{A}$ of $\mathfrak{F}$ a nonempty intersection of any combination of n sets in $\mathfrak{A}$ implies a nonempty intersection of all sets in $\mathfrak{A}$.

Helly's theorem states that the Helly order of a finite or compact family of convex sets in E^d is $d + 1$. It is an interesting but somewhat difficult problem to determine the Helly order of the family of (m,n) convex sets in E^d for general m and n. We restrict ourselves in this paper to the special cases $m = 3$ and $n = 1,2$. A series of lemmas will lead us directly not only to the finiteness of the Helly order for the family of (3,2) convex sets in E^2, but to an exact value for it. This development will reveal the Helly order for the family of $(m,C_m - 1)$ convex sets in E^2 since it is easy to show that $(m,C_m - 1)$ convexity implies (3,2) convexity. We have already shown that a (3,2) convex set is in general a $(3 + k,2 + k)$ convex set $(k \geq 0)$. However, due to the strong topological properties of a (3,2) convex set, we get the following result.

1.22. PROPOSITION: A (3,2) convex set B in a linear topological space E is $(m,C_m - [m/2])$ convex, for $m > 2$.

Proof. Let $p_1,\ldots,p_m$ be any m distinct points in B, where $m > 2$. Note that (p_ip_j) and (p_ip_k) cannot both have points in common with C(B), for otherwise B would not be

(3,2) convex. We can therefore have at most $[m/2]$ open segments joining the given m points having a nonempty intersection with C(B) (for example, $(p_{i_1}p_{i_2}),\ldots,(p_{i_{r-1}}p_{i_r})$, where $(i_1,\ldots,i_r)$ is some element of order $[m/2]$ in $S([m/2])$, the permutation group on $[m/2]$ objects). Hence in B, we have at least $C_m - [m/2]$ joins between the given m points. Thus, B is $(m, C_m - [m/2])$ convex. ◊

An extremely useful result is the following, which is an extension of 1.16 in the plane.

1.23. LEMMA: Given x,y,z, and w in a planar (3,2) convex set A, then int(conv{x,y,z,w}) is a subset of A.

Proof. If one of the four points lies in the convex hull of the other three, then the result follows immediately from 1.16 and the application of (3,2) convexity. Consider the case where no one point is in the convex hull of the other three. That is, we have the four points determining a convex quadrilateral. By the previous result, A is (4,4) convex. If one of the four guaranteed joins is a diagonal, then it follows from 1.16 that int(conv{x,y,z,w}) is a subset of A. Suppose on the contrary that the diagonals, say xz and yw, are not subsets of A. Thus, the boundary of conv{x,y,z,w} is in A. Again by 1.16, we have every point of conv{x,y,z,w} in A, except possibly $v = xz \cap yw$. But if v is not in A, by considering the set {x,z,s}, where s is an element of $xv \cup vz$, we get a contradiction of the (3,2) convexity of A. Therefore

$\operatorname{int}(\operatorname{conv}\{x,y,z,w\})$ is a subset of A. ◇

The next lemma is the main tool in establishing the finite Helly order for the family of (3,2) convex sets in E^2. It is an interesting result in itself.

<u>1.24. LEMMA</u>: Any five (3,2) convex sets in E^2 each four of which intersect have nonempty intersection.

<u>Proof</u>. Let A_i, for $i \in \hat{5}$, be five (3,2) convex sets in E^2, each four of which intersect. Denote by p_i the point guaranteed in $\bigcap_{j=1\ j\neq i}^{5} A_j$. If at least one of the five points is in the interior of the convex hull of the other four, then that point is in all five sets by 1.23. So consider the three remaining cases where no one of the five points is in the interior of the convex hull of the other four.

<u>Case 1</u>. No three points are collinear, then the five points are vertices of a convex pentagon. Let $e_{ij} = p_ip_j$, $r_5 = e_{13} \cap e_{24}$, and r_i, for $i \in \hat{4}$, defined similarly. Let $T = \operatorname{conv}(\bigcup_{i=1}^{5} r_i)$. Since $\operatorname{int}(\operatorname{conv}\{p_i : i \in \hat{5},\ i \neq j\})$ is a subset of A_j, by 1.23, we have int T a nonempty subset of A_j, for $j \in \hat{5}$. Hence, int T is a subset of $\bigcap_{i=1}^{5} A_i$.

<u>Case 2</u>. Exactly three of the points are collinear. We may assume without loss of generality that p_3, p_4, and p_5 are collinear, with $p_4 \in p_3p_5$ (see Figure 1.2). Define the segments e_{ij} and the points r_k as before, and let $L = \operatorname{conv}\{r_3, r_4, r_5, p_4\}$. Since $\operatorname{int}(\operatorname{conv}\{p_i : i \quad 5,\ i \neq j\})$ is a subset of A_j, by 1.23, it follows that int L is a nonempty subset of $\bigcap_{i=1}^{5} A_i$.

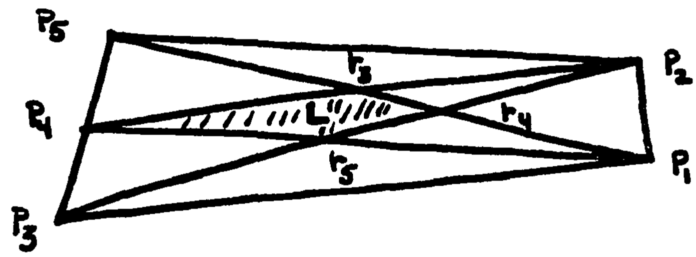


Figure 1.2

Case 3. Exactly four of the points are collinear. Assume that p_1, p_2, p_3, and p_4 are the four collinear points, taking the order indicated in Figure 1.3.

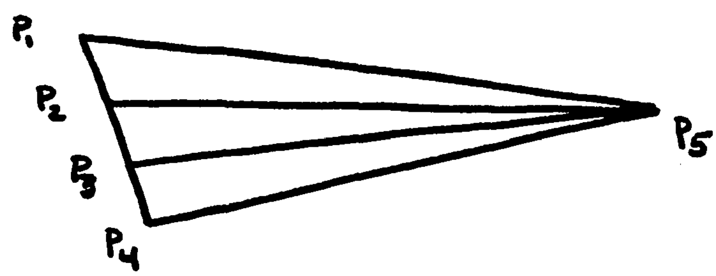


Figure 1.3

Now p_1, p_3, and p_4 are in A_2, and it follows that $p_1p_4 \subset A_2$; for, the existance of a single point of $C(A_2)$ on p_1p_4 denies the (3,2) convexity of A_2. Hence $p_2 \in A_2$ and therefore $p_2 \in \bigcap_{i=1}^{5} A_i$. ◊

It is straightfoward to see that if x,y,z, and w are any four distinct points in $\bigcap_{i=1}^{r} A_i$, where for all $i \in \hat{r}$ A_i is a (3,2) convex set in E^2, then $\mathrm{int}(\mathrm{conv}\{x,y,z,w\})$ is a subset of $\bigcap_{i=1}^{r} A_i$. It follows that if A_1, A_2, A_3, and A_4 are four (3,2) convex sets in E^2, and $A_5 = \bigcap_{i=1}^{s} B_i$, where B_i for $i \in \hat{s}$ is a planar (3,2) convex set, and if each four of the sets A_1, A_2, A_3, A_4, and A_5 have a point in common, then there is a point common to all five sets (simply apply the argument of the preceding lemma in each of the cases regarding $p_1,\ldots,p_5$ to A_1,A_2,A_3,A_4, and any one of the sets B_i).

1.25. LEMMA: Given $n - 1$ (3,2) convex sets in E^2 $A_1,\ldots,A_{n-1}$, and A_n a finite intersection of planar (3,2) convex sets, if each four of the sets A_i, for $i \in \hat{n}$, have a common point, then $\bigcap_{i=1}^{n} A_i \neq \emptyset$.

Proof. The conclusion is true if $n = 5$, by the preceding observation. Suppose the lemma is true for $n = k$. Consider the $k + 1$ sets $A_1,\ldots,A_{k+1}$, where each A_i is (3,2) convex for $i \in \hat{k}$, and A_{k+1} is a finite intersection of (3,2) convex sets in the plane, such that each four of $A_1,\ldots,A_{k+1}$ have nonempty intersection. Let $B_k = A_k \cap A_{k+1}$. $A_1,\ldots,A_{k-1},B_k$ is a collection of k sets the first $k - 1$ of which are (3,2) convex, and the k-th, a finite intersection of (3,2) convex sets. Each four of the sets in the collection $A_1,\ldots,A_{k-1},B_k$ have a common point. For, consider A_i,A_j,A_m and B_k. Each four of the sets A_i,A_j,A_m,A_k, and A_{k+1} have a common point by hypothesis. Hence, A_i,A_j,A_m, and B_k have a point in common by the observation preceding this lemma. Therefore, by the induction hypothesis $\bigcap_{i=1}^{k+1} A_i \neq \emptyset$. $\diamond$

1.26. LEMMA: Given n (3,2) convex sets in E^2 each four of which intersect, then they all have a common point, where $n \geq 5$.

Proof. If the number of sets is five, then the result is already true by 1.24. Suppose it is true when $n = k \geq 5$.

Let $A_1, \ldots, A_{k+1}$ be k + 1 (3,2) convex sets in E^2 each four of which have a common point. Consider the collection $A_1, \ldots, A_{k-1}, B_k$, where $B_k = A_k \cap A_{k+1}$. Each four of the sets have a point in common. For, consider A_i, A_j, A_m, and B_k; each four of the sets A_i, A_j, A_m, A_k, and A_{k+1} have a point in common, and by 1.24 the intersection of these five sets is nonempty. Thus A_i, A_j, A_m, and B_k have a common point. We now have k - 1 (3,2) convex sets $A_1, \ldots, A_{k-1}$, and B_k, where B_k is the intersection of two (3,2) convex sets, each four of which have a point in common. By 1.25, we have $(\bigcap_{i=1}^{k-1} A_i) \cap B_k \neq \emptyset$. Thus, $\bigcap_{i=1}^{k+1} A_i \neq \emptyset$. ◊

<u>1.27. PROPOSITION</u>: The Helly order of the family of (3,2) convex sets in E^2 is four.

<u>Proof</u>. From the previous lemma we have proved that the Helly order for the family of (3,2) convex sets in the plane is no greater than four. The following example is offered to show that the bound used in 1.24 is best possible, and that the Helly order is exactly four.

<u>Example</u>. With the usual coordinatization of E^2, let x = (0,0), y = (1,0), z = (0,1), and w = (1,1). Take A as the interior of the triangle formed by x,y, and z, including the sides xy and xz, B the interior of the triangle formed by x,y, and w including the sides xy and yw, C the interior of the triangle formed by y,z, and w including the sides yw and wz, and , finally, D the interior of the triangle formed by x,z, and w together with the sides xz and zw.

We have four (3,2) convex sets A,B,C, and D in E^2 each three of which have a point in common yet the intersection of all four sets is empty. $\Diamond$

It would be interesting to know if 1.27 generalizes to E^d. That is, if $\mathcal{C}$ is a finite family of at least $d + 2$ (3,2) convex sets in E^d each $d + 2$ of which have a common point, then is there a common point for all the members of $\mathcal{C}$?

CHAPTER II

M-CONVEXITY

It is interesting to specialize the concept of (m,n) convexity to (m,1) convexity in order to discover the more basic properties of such sets (every (m,n) convex set is (k,1) convex, or k-convex, for some $k \geq 2$). In this chapter, therefore, we turn our attention to m-convexity. We will characterize the kernel of a certain family of m-convex sets, give a negative answer to a conjecture of Danzer, Grünbaum, and Klee concerning the Helly order of 3-convex sets, and introduce the notion of local nonconvexity, which will lead us to several convex covering theorems for m-convex sets. In the process, we prove a generalized Helly theorem.

The concept of the kernel of a set was introduced by Brunn [3], when he showed that in E^d the kernel of any set is convex, and is closed iff the original set is closed. Toranzos [16] formulated in another connection a previously unpublished result which has been common knowledge in the theory of convexity for some time, namely that the kernel of a set is the intersection of all its maximal convex subsets. In connection with this, Hare and

Kenelly [7] have shown that the intersection of the maximal starshaped subsets of a compact, simply-connected, planar set is starshaped or empty. For m-convex sets in E^d, we obtain the following results. First, we observe that a straightfoward application of the proof of 1.5 implies that the union of a chain of relatively m-convex subsets of a set is relatively m-convex with respect to that set.

2.1. PROPOSITION: For each relatively r-convex subset T in S, where S is any set containing at least k visually independent points, there exists a maximal closed subset of S which is exactly k-convex with respect to S and contains T, where $2 \leq r \leq k$.

Proof. If T is a relatively r-convex subset of S it will be an exactly s-convex subset of S relative to S for some $s \leq r$. Let $x_1,\ldots,x_k$ be a set of k visually independent points in S. Inductively, consider the sets $T_o = T$, $T_1 = T \cup x_1, T_2 = T \cup x_1 \cup x_2, \ldots, T_k = T \cup \{\bigcup_{i=1}^{k} x_i\}$. At least one of these sets, say T_i, must be exactly k-convex relative to S since T_o is relatively exactly s-convex, T_k is relatively exactly t-convex for some $t \geq k$, and the addition of a point in S to each T_j does not increase the order of the relative, exact m-convexity of T_j by more than one. By Zorn's lemma, there is a maximal subset M of S containing $T_i \supset T$ which is k-convex relative to S. But since T_i is exactly k-convex relative to S, it contains

$k - 1$ points $y_1, \ldots, y_{k-1}$ which are visually independent relative to S. Since M contains $y_1, \ldots, y_{k-1}$, M itself is exactly k-convex relative to S. ◊

<u>2.2. COROLLARY</u>: If T is any convex subset of an exactly m-convex set S, then there exists for each k, $2 \leq k \leq m$, a maximal subset of S containing T which is exactly k-convex relative to S.

We shall need the following concept for subsequent results:

<u>2.3. DEFINITION</u>: $T \subset S$ is said to be <u>weakly relatively convex</u> with respect to S iff for each two points x and y of T such that $(xy) \subset S$, then $xy \subset T$.

Thus, any set is weakly relatively convex with respect to itself while it need not be <u>convex</u> relative to itself (see 1.9). The convexity of a set implies both weak relative convexity and relative convexity with respect to any set containing it, but, unfortunately for the terminology, a relatively convex subset of even a convex set need not be weakly relatively convex. Moreover, it is not necessarily true that a maximal, absolutely k-convex subset of an m-convex set S be weakly convex relative to S, as the following example shows. The set S illustrated in Figure 2.1 consists of two squares (interiors included) and two line segments. S is 5-convex, but the subset T consisting of U, pq, and rs is a maximal 4-convex subset of S which is not weakly convex relative to S. For, consider the points x and y, as shown in the figure. (This example also shows that relative convexity does not imply weak relative convexity.)

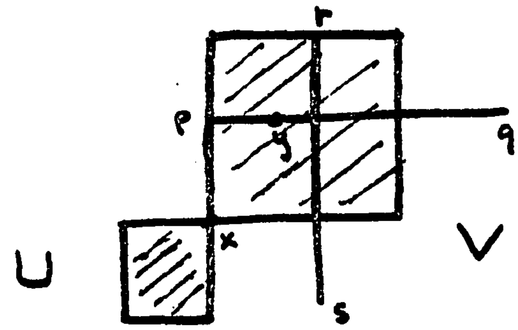


Figure 2.1

For k-convex subsets which do satisfy weak relative convexity we can establish a positive result.

<u>2.4. PROPOSITION</u>: If S is any set, then the intersection of any collection of (absolutely) k-convex subsets of S (k fixed, $k \geq 2$) which are weakly convex relative to S, where the intersection contains at least k points, is k-convex.

<u>Proof</u>. Let $M = \bigcap \{S_i : i \in I\}$, where each S_i is a k-convex subset of S which is weakly convex relative to S. Choose any k distinct points in M $x_1, \ldots, x_k$. Now each x_j, for $j \in \hat{k}$, is in S_i for all i in I. If for some s and t in $\hat{k}$ and u in I $x_s x_t \subset S_u$, then $x_s x_t$ is in S, since $S_u \subset S$. Hence, $x_s x_t$ is in S_i for all i in I by the weak relative convexity of S_i. Therefore $x_s x_t$ is in M. Since S_u is k-convex, it must contain at least one join $x_s x_t$ determined by these k points. Hence, M contains a join determined by the k given points, and thus M is k-convex. ◊

2.5. REMARK: It would be interesting to obtain a direct analogue to the Hare-Kenelly result mentioned earlier, that is, to establish that the intersection of the maximal k-convex subsets of a closed, simply-connected, planar set is k-convex. This assertion remains a conjecture at this time, however.

It is easy to show that the kernel of any m-convex set T is contained in any maximal subset R of T which is k-convex relative to T, $2 \leq k \leq m$. For if $x \in (\ker T) \setminus R$, then $\{x\} \cup R$ is clearly k-convex relative to T and contains R properly, denying the maximal property of R. A slightly different result is possible when T is not required to be m-convex.

2.6. DEFINITION: The join of x and A is the set $xA = \{\alpha x + (1 - \alpha)a : a \in A, 0 \leq \alpha \leq 1\}$. This is sometimes referred to as the cone over A with vertex x.

2.7. PROPOSITION: If R is any maximal absolutely (relatively) k-convex subset of T, then $\ker T \subset R$.

Proof. We prove this only for absolute k-convexity; the proof for relative k-convexity is similar. Suppose that there is an element x in ker T which is not in R, where R is a maximal k-convex subset of T. Hence, R is a proper subset of xR. Moreover, xR is k-convex. For, if we are to select any k points $p_1, \ldots, p_k$ in xR, then there exist points $x_i \in R$, for $i \in \hat{k}$, such that $p_i \in xx_i$. There is an i and j in $\hat{k}$ such that $x_i x_j$ is in R, since R is k-convex. Hence, $p_i p_j \subset \mathrm{conv}\{x, x_i, x_j\} \subset xR$. But this contradicts the fact that R is a maximal k-convex subset of T. Therefore, ker $T \subset R$. ◊

2.8. COROLLARY: The kernel of any set T is contained in the intersection of all maximal absolutely (relatively) k-convex subsets of T.

The next result contains Toranzos's theorem in the special case k = 2.

2.9. PROPOSITION: Suppose T is any set with the property that for some integer $k \geq 2$ and for any $x \in T \setminus \ker T$, T^x has at least k - 1 points which are visually independent relative to T. Then ker T is the intersection of all the maximal subsets of T which are exactly k-convex relative to T.

Proof. Let ker T = K and consider x any element of $T \setminus K$. By hypothesis, T^x contains k - 1 points $x_1, \ldots, x_{k-1}$ in T^x visually independent relative to T (if k = 2, simply choose any point x_1 in T^x). The set $S = x_1K \cup \cdots \cup x_{k-1}K$ is the union of k - 1 convex subsets of T and hence is relatively k-convex. It is easy to show that S is also exactly k-convex relative to T. There exists a maximal subset M of T containing S which is exactly k-convex relative to T. The point x cannot be an element of M since $x, x_1, \ldots, x_{k-1}$ are visually independent relative to T. Therefore x cannot be in the intersection of all maximal k-convex subsets of T. Hence, the intersection of the maximal k-convex subsets of T is a subset of ker T. By 2.8 the proposition is established. ◊

2.10. COROLLARY: If T is any m-convex set, with k a positive integer $2 \leq k \leq m - 1$, and T has the property that T^x for $x \in T \setminus \ker T$ is exactly r-convex for some $r \geq k$, then ker T is the intersection of all maximal, relatively exactly k-convex subsets of T.

Proof. Straightfoward, since an exactly r-convex set relative to T for some $r \geq k$ has at least $k - 1$ visually independent points. ◊

Note that, in the event ker T is not the intersection of all the maximal k-convex subsets of T, then it cannot have the property of T assumed in the theorem. A simple example of this is shown in the figure below.

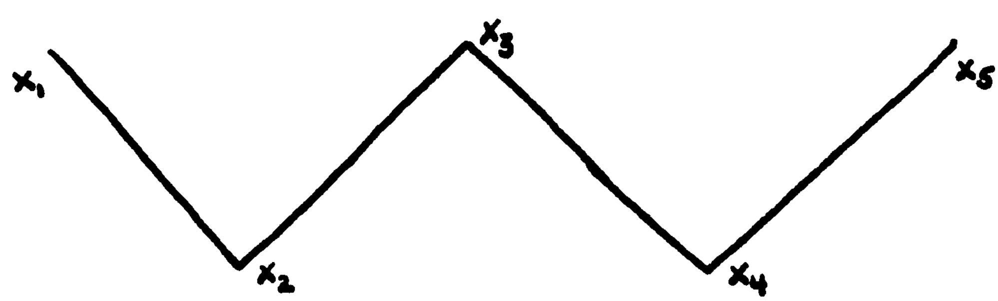


Figure 2.2

Here, $\ker T = \emptyset$, but the intersection of all maximal 4-convex subsets of T is the point x_3. Thus, at least one anti-star T^x for $x \in T \setminus \ker T = T$ is 3-convex (T^{x_3} is obviously that set, and the only one). This observation shows that the plausible conjecture

$$\ker T = \bigcap_{M \in T} M, \tag{2.1}$$

where the intersection is taken over all maximal relatively k-convex subsets, is false even for m-convex sets, $m \geq k + 1$; some condition similar to that given in the corollary is needed. A more interesting counterexample to (2.1) is indicated in the next figure. This set is compact, simply-connected, and 4-convex, but the kernel

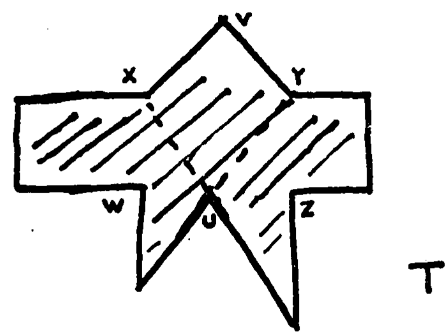


Figure 2.3

is not obtained by intersecting maximal k-convex sets for any $k > 2$ (that is, $k = 3$). Here, $\ker T = \operatorname{conv}\{x,y,u\}$, but $\bigcap_{M \subset T} M$ (M = maximal 3-convex subsets) $= \operatorname{conv}\{w,x,y,z\}$. Moreover, note that T^v is convex relative to T. On the positive side, Figure 2.4 shows an example of a set T in E^2 which satisfies the property required in 2.9 for each $k \geq 2$; using complex notation, T consists of a small square B centered at the origin, and the union of the cones of the points z^n_{2j} and z^n_{2j+1} over B, $n = 1,2,\ldots,$ and $j = 0,1,2,3$, where $z^n_{2j} = \exp(\pi j/2 - \alpha + \alpha/n - \alpha/n^2)i$ and $z^n_{2j+1} = \exp(\pi j/2 + \alpha - \alpha/n + \alpha/n^2)i$, with α chosen so that z^1_j ($j = 0,\ldots,7$) are the points of intersection of the sides of B and the unit circle $|z| = 1$. Here, $\ker T = B$, and, according to 2.9, B is obtained by intersecting all maximal, relatively exactly k-convex subsets of T, for each value of k.

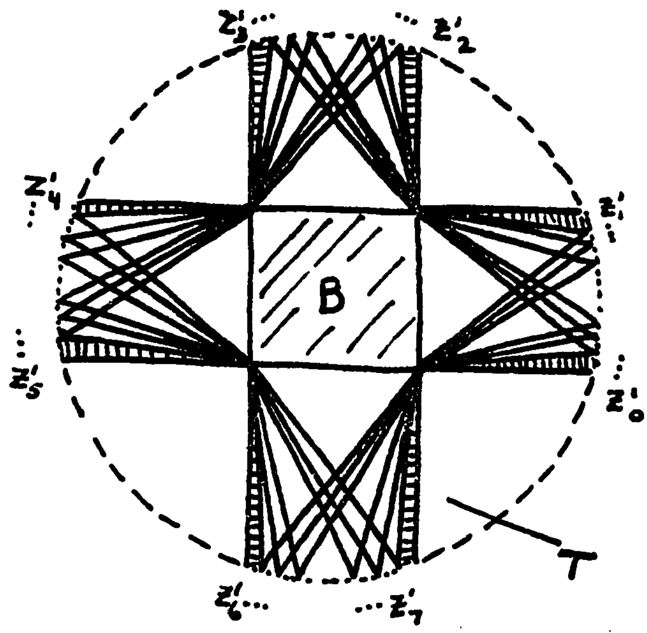


Figure 2.4

Danzer, Grünbaum, and Klee have conjectured [4] that the family of all 3-convex subsets of E^d has finite Helly order. The next result gives a negative answer to this conjecture.

2.11. PROPOSITION: The Helly order of the family of closed, connected, planar 3-convex sets is infinite.

Proof. It suffices to exhibit a set of k closed, connected, planar 3-convex sets each k - 1 of which have a point in common but with all k of them having empty intersection, for each even integer $k \geq 4$.

Let $z_1 = (1,0), z_2, \dots, z_k$ be the k-th roots of unity. Let $k = 2m$ and consider for $4 \leq i \leq k - 1$ the following sets.

$$A_1 = \text{conv}\{z_2, z_3, \dots, z_{m+1}\} \cup \text{conv}\{z_{m+1}, \dots, z_k\},$$

$$A_2 = \operatorname{conv}\{z_3, z_4, \dots, z_{m+2}\} \cup \operatorname{conv}\{z_{m+2}, \dots, z_k, z_1\},$$

$$A_3 = \operatorname{conv}\{z_4, z_5, \dots, z_{m+3}\} \cup \operatorname{conv}\{z_{m+3}, \dots, z_k, z_1, z_2\},$$

$$\dots$$

$$A_i = \operatorname{conv}\{z_{i+1}, \dots, z_{m+i}\} \cup \operatorname{conv}\{z_{m+i}, \dots, z_k, z_1, \dots, z_{i-1}\},$$

$$\dots$$

$$A_k = \operatorname{conv}\{z_1, z_2, \dots, z_m\} \cup \operatorname{conv}\{z_m, \dots, z_{k-1}\}.$$

By construction, we have $z_i \in \bigcap_{j \neq i} A_j$. Hence, the intersection of any k - 1 of the given sets is nonempty. Let $c = (0,0)$, every point in $\operatorname{conv}\{z_1, \dots, z_k\}$ must, for some j, lie in $B_j = \operatorname{int}(\operatorname{conv}\{c, z_{j-1}, z_j, z_{j+1}\}) \cup \{c\}$. Since B_j is a subset of the complement of A_j, we have $\bigcap_{i=1}^{k} A_i = \emptyset$. $\Diamond$

Thus, a family of sets each of which is closed, connected, and is the union of three or fewer convex sets need not have finite Helly order. The difficulty lies in the fact that the intersection of members of such a family may be more complicated in structure than the members themselves. In [5], Grünbaum and Motzkin considered a modified situation in which $\mathcal{F}$ consists of sets which are expressible as the union of at most n distinct compact convex sets, and which also have the property that the intersection of n or fewer members of $\mathcal{F}$ can be expressed as the union of at most n disjoint compact convex sets. In E^d, let $D_{i,d}$ denote the collection of those sets which can be expressed as the union of at most i disjoint compact convex sets. Grünbaum and Motzkin were able to establish for the case $i = 2$ that if $\mathcal{F}$ is a family of sets in $D_{i,d}$ such that any $i(d + 1)$ members have nonempty intersection, and for

$r \leq i$ the intersection of any r members of $\mathcal{F}$ is a member of $D_{i,d}$, then $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$. Larman in [14] has extended this result for the case when $i = 3$.

A different way of obtaining a finite Helly order for a family of sets in E^d each of which is the union of k or fewer convex sets is to require that the intersection of members of the family be in the family. One may also require that each set in the family be a special union of k-convex sets. Turning our attention in that direction, we can obtain a generalization of Helly's theorem in E^d.

2.12. DEFINITION: A k-isolated set is a set consisting of a convex set and k or fewer isolated points, for k a non-negative integer.

Since convex sets are 0-isolated sets, the next result reduces to a form of Helly's theorem when $k = 0$.

2.13. PROPOSITION: The Helly order of the family of k-isolated sets in E^d is no greater than $(d + 1)(k + 1)$.

Proof. We shall prove the inductive proposition for each integer $r \geq (d + 1)(k + 1) + 1$: If $\{S_i : i \in \hat{r}\}$ is a family of r k-isolated sets in E^d each $r - 1$ of which have nonempty intersection, then all r sets have nonempty intersection. It is obvious that this will then imply the desired result since by mathematical induction it follows that each family $\{S_i : i \in \hat{r}\}$ of r k-isolated sets in E^d each $(d + 1)(k + 1)$ of which intersect have nonempty intersection. Assume that $\{S_i : i \in \hat{r}\}$ is a family of r k-isolated sets in E^d each $r - 1$ of which have nonempty intersection, $r \geq (d + 1)(k + 1) + 1$. Let the set of isolated points of $S_1, \ldots, S_r$ be $p_1, \ldots, p_s$; thus, each S_i has the form

$$S_i = C_i \cup \{p_{i_1}, \ldots, p_{i_{t(i)}}\},$$

where each C_i is convex and $t(i) \leq k$. Choose q_j in $\bigcap_{i \in \hat{r}, i \neq j} S_i$ for each $j \in \hat{r}$ (since each $r - 1$ of the sets S_i have nonempty intersection), and put $T = \{q_1, \ldots, q_r\}$. Note that for any $u \in \hat{r}$ S_u contains $r - 1$ of the q_j's. Hence, each C_u must contain at least $r - k - 1$ of the q_j's (of the $r - 1$ q_j's in S_u at most k can belong to $S_u \setminus C_u$). Now consider any two sets S_u and S_v. Letting A denote the cardinality of A and applying the inclusion-exclusion formula $|A \cap B| = |A| + |B| - |A \cup B|$, it follows that $|(C_u \cap C_v) \cap T| \geq (r - k - 1) + (r - k - 1) - r = r - 2(k + 1)$. Continuing inductively, each intersection of the form $C_{i_1} \cap \cdots \cap C_{i_{d+1}}$ contains at least $r - (d + 1)(k + 1) \geq 1$ of the q_j's. Hence, $\{C_i : i \in \hat{r}\}$ is a family of convex sets in E^d each $d + 1$ of which have a common point. By Helly's theorem, $\bigcap_{i=1}^{r} C_i \neq \emptyset$. Therefore $\bigcap_{i=1}^{r} S_i \neq \emptyset$. $\Diamond$

We now turn our attention to the concept of m-convexity as a tool in characterizing sets which are the union of finitely many convex sets. The following example, due to Kay, shows that if one attempts to use m-convexity as the only criterion then the restriction to closed sets is necessary.

<u>2.14. EXAMPLE:</u> Let E^2 be identified with the complex

plane and let C be the unit circle $|z| = 1$, with $z_n = e^{-\pi i/2^n}$ for $n = 0,1,2,\ldots$. Let P be the infinite sided polygon which circumscribes C, touching C at precisely the points $1, e^{-\pi i/2}$, and z_n for n even. The set S is then defined as the set of points on and inside P with those z_n deleted for which n is odd. It can then be shown that S is 4-convex but is not the union of any finite number of convex sets (see [9]).

Many of the convex covering theorems for m-convex sets have been obtained by imposing conditions on certain subsets of S. For example, if one requires that the kernel of a compact m-convex set be empty, then the compact m-convex set is the union of finitely many compact $(m - 1)$-convex sets. Another useful concept for us is the following:

2.15. DEFINITION: A set T is said to be locally convex at a point p in T if there exists a neighborhood N of p such that $T \cap N$ is relatively convex in T. If a set is locally convex at every point, it is said to be locally convex. A point q of T is a point of local nonconvexity (or lnc point) if T is not locally convex at q.

It is clear that q is an lnc point of T iff it is a limit point of a pair of nets $\{x_i : i \in D\}$ and $\{y_i : i \in D\}$ in T such that for every $i \in D$ the join $x_i y_i \not\subset T$. Knowledge of the set of lnc points of a set is useful in determining properties of the set. In [6] it is proved that if $S \subset E^d$ and the set Q of lnc points of S consists of a single point, then S is starshaped from Q. A representation theorem appearing in the same paper states that if $|Q| = 1$ and S is m-convex, then S is the union of $m - 1$ or fewer convex sets. Stamey and Marr [15] have shown that if S is a bounded 3-convex set with $|Q| > 1$ and a point $q \in (\ker S) \cap (\operatorname{bd} S)$ and S is locally convex at q, then S can be expressed as the union of two closed convex sets.

For the sake of completeness, and to give an indication of the importance of the concept of local convexity, we state Tietze's theorem. A proof may be found in [18].

2.16. TIETZE'S THEOREM: A closed, connected set in a linear topological space which is locally convex is convex.

Kay and Guay [10] have recently generalized Tietze's theorem by showing that if the set Q of lnc points of a closed set T in a linear topological space has finite cardinality $n > 0$ and $T \setminus Q$ is connected, then T is planar and is the union of $n + 1$ or fewer convex sets.

A result due to Valentine [17] states that if S is a closed, connected, planar 3-convex set, then S is the union of three or fewer closed convex sets. Guay, in his thesis, was able to extract the essence of Valentine's proof and establish a result we shall make use of later. From now on, Q denotes the lnc points of S and K denotes the kernel of S.

2.17. GUAY'S THEOREM: Let $S \subset E^2$ be closed, connected, and have at least two points of local nonconvexity. If

$Q \subset K$, then S may be expressed as the union of three or fewer closed convex sets.

By considering the five pointed star, one may see that for both Valentine and Guay's results the number three is best possible. Two representation theorems follow directly from Guay's theorem. We introduce the notation $T_A = \{x \in T : xa \subset T \text{ for all } a \in A\}$.

2.18. COROLLARY: If S has the property that $Q = \bigcup_{i=1}^{n} Q_i$ and $S = \bigcup_{i=1}^{n} S_{Q_i}$, then S is the union of 3n or fewer closed convex sets.

2.19. COROLLARY: An (m,n) convex set S with $n > C_{m-2}$ is the union of three or fewer closed convex sets.

Proof. We need to show that if $n > C_{m-2}$, then $Q \subset K$. Suppose that $q \in Q \setminus K$, and let x be a point in S such that $qx \not\subset S$. Take a sequence $\{x_i\}$ of points in S with the property that $\lim_{i \to \infty} x_i = x$. In addition, there exist two sequences $\{y_i\}$ and $\{z_i\}$ such that $\lim_{i\to\infty} y_i = \lim_{i\to\infty} z_i = q$ and $y_i z_i \not\subset S$, for all i. There exists a positive integer i_o such that for $i > i_o$ $qx_i \not\subset S$. Hence, there exists a j_o with the property that y_{j_o} and z_{j_o}, together with any m - 1 elements of $\{x_i : i > i_o\}$, form a set of m points in S with at most C_{m-2} joins, a contradiction. Hence, Q is a subset of K. $\Diamond$

2.20. DEFINITION: A set T is called an L_n set if every pair of points in T can be joined by a polygonal arc in T consisting of at most n segments.

In [8], Horn and Valentine characterize properties of L_2 sets in the plane. It is straightfoward to see that in a linear space every connected m-convex set is an L_{2m-3} set, assuming the set is polygonally connected (a result obtained in [9] for finite dimensions). For if $P = x_0x_1 \cup x_1x_2 \cup \cdots \cup x_{n-1}x_n$ ($x_0 = x$ and $x_n = y$) is a polygonal arc in S joining x and y such that the number of sides is minimal among all such paths joining x and y, and $n > 2m - 2$, then x_{2i} for $i \in \widehat{m-1}$ is a set of m visually independent points in S, a contradiction. It can also be shown that any closed m-convex set is an L_{m-1} set.

<u>2.21. PROPOSITION</u>: Every m-convex set T which is an exactly L_{2m-3} set (an L_{2m-3} set which is not an L_{2m-4} set) can be expressed as the union of $2m - 3$ convex sets.

<u>Proof</u>. Let x and y be points in T such that the minimal number of sides of any polygonal arc joining x and y is $2m - 3$, and let P be such an arc, with the vertices of P denoted by $x = x_0, x_1, \ldots, x_{2m-3} = y$. Denote by L_i the set of all points z in T with the property that the minimal number of sides of a polygonal arc joining x and z is i, for $i \in \widehat{2m-3}$. It is clear that $T = \bigcup_{i=1}^{2m-3} L_i \cup \{x\}$. Now each L_i is convex; for otherwise if there exist p and q in L_i with $pq \not\subset T$, then by considering $\{x = x_0, x_2, \ldots, x_{2k-2}, p, q, x_{2k+2}, \ldots, x_{2m-2}\}$ (even subscripts) if $i = 2k$, or $\{x_1, x_3, \ldots, x_{2k-3}, p, q, x_{2k+1}, \ldots, x_{2m-3}\}$ (odd subscripts) if $i = 2k - 1$, we see that since P was a minimal polygonal

arc joining x to y, in either case we have a set of m visually independent points in T, a contradiction of the m-convexity of T. Also $L_1 \cup \{x\}$ is convex since x can see every point in L_1 and L_1 itself is convex. Thus, T is the union of 2m - 3 convex sets. ◊

Valentine has shown [19] that knowledge of Q in certain cases implies polygonal connectedness. He proves that if T is a closed, connected set in E^d with $Q = \bigcup_{i=1}^{n} Q_i$, where Q_i is relatively convex, connected, and closed for all $i \in \hat{n}$, then T is an L_{2n+1} set. As a corollary, he shows that a closed, connected set in E^d with $|Q| = n$ is an L_{n+1} set.

2.22. LEMMA: Any closed, connected, m-convex set is locally starshaped.

Proof. Let $x \in S$ and suppose no such neighborhood of x exists. There is a net $N = \{x_n : n \in D\} \subset S$ converging to x such that $xx_n \not\subset S$ for frequently many x_n. Let x_{n_1} be an element of the net such that $xx_{n_1} \not\subset S$. There exists a neighborhood U_{n_1} about x such that x_{n_1} cannot see any point in that neighborhood via S, since S is closed. Let x_{n_2} be any point in $N \cap U_{n_1}$ such that $xx_{n_2} \not\subset S$. Thus, $x_{n_1}x_{n_2} \not\subset S$. There is a neighborhood U_{n_2} of x such that $U_{n_2} \subset U_{n_1}$ and x_{n_2} cannot see any point in U_{n_2} via S. Select any point in $N \cap U_{n_2}$, say x_{n_3}, then $\{x, x_{n_1}, x_{n_2}, x_{n_3}\}$ forms a visually independent set with respect to S. There exists a neighborhood U_{n_3} of x such that x_{n_3} cannot see any point of U_{n_3} via S and $U_{n_3} \subset U_{n_2}$. Continuing this process, we obtain a contradiction of the m-convexity of S. $\Diamond$

A conjecture of Kay [9] that a closed, m-convex set in E^n is the union of finitely many convex sets has been established for several special cases, but the conjecture for more general sets remains. We develop here a few tools which might be useful toward establishing the conjecture in E^2, which we also use in case of 4-convexity in the following chapter.

First let S be a closed, m-convex set in E^2, and let Q be the set of all lnc points of S. We use the notation $H = \text{conv } Q$, and $\{W_i : i \in I\}$ will denote the collection of connected components of $S \setminus H$ (for the m-convex sets we shall consider, H will be a subset of S). Note that if $H \subset S$, then $Q \subset \text{bd } H$; for otherwise, there exists an lnc point $q \in \text{int } H \subset \text{int } S$, denying the obvious property $q \in \text{bd } S$ for all $q \in Q$.

By m-convexity there can be at most finitely many one-dimensional components W_i (each such component must be a segment or ray, and thus, for all but possibly one other component, no point in W_i can see via S <u>any</u> point of <u>any</u> other component). The remaining components have mutually disjoint interiors. Hence, I is countable, and we shall assume I consists of a subset of the positive integers.

For convenience, we shall now assume that S is compact. This will simplify many of the arguments, although many of these results can be established without that assumption.

A <u>simply-connected</u> subset of E^2 is a set whose complement contains no bounded component. We establish the following result:

<u>2.23. LEMMA</u>: If S is a connected, compact, m-convex subset of E^2 with conv $Q \subset S$, then S is simply-connected.

<u>Proof</u>. With H = conv Q, suppose G is a bounded component of $E^2 \setminus S$, and let $g \in G$. Since $g \notin H$ and H is compact there is a line ℓ strongly separating g and H, and let the closed half-plane determined by ℓ not containing H be denoted by F. Let $\{Z_j : j \in J\}$, denote the closures of the components of $F \cap S$. The m-convexity of S implies that there can be only finitely many components Z_j, so we may assume without loss of generality that $J = \{1,2,\ldots,k\}$. Each Z_j is a compact, connected subset of F; we can show further that Z_j is locally convex, and therefore convex by Tietze's theorem. For, let $x \in Z_j$. Since $x \notin H$, there exists a convex neighborhood U of x deviod of points of H, and if $\{y_n\}$ and $\{z_n\}$, n = 1,2,..., are sequences in $U \cap Z_j$ converging to x such that $y_n z_n \not\subset Z_j$ then since $y_n z_n \subset F$, we have $y_n z_n \not\subset S$, for otherwise points of $y_n z_n$ belong to different components of $F \cap S$. Then x is an lnc point of S proving that $x \in H$, a contradiction. Thus, Z_j is convex for each j.

It is an obvious (easily proved) property of a compact convex subset of a half-space that its complement relative to that half-space is connected. Hence, for each j $F \setminus Z_j$ is connected. Suppose it has been proved that

$F \setminus (Z_1 \cup Z_2 \cup \cdots \cup Z_j)$ is an open, connected subset of F. Consider $F \setminus (Z_1 \cup Z_2 \cup \cdots \cup Z_j \cup Z_{j+1})$, which will be shown to be connected (it is obviously open in F). There is a sufficiently small circular neighborhood V of 0 such that $Z_{j+1} + V$ is disjoint from $Z_1 \cup Z_2 \cup \cdots \cup Z_j$. Since $(Z_{j+1} + V) \setminus Z_{j+1}$ is open and connected (the proof is basically the argument for the connectedness of the boundary of a compact convex set), then $(Z_{j+1} + V) \setminus Z_{j+1}$ is polygonally connected. Let x and y be any two points of $F \setminus (Z_1 \cup Z_2 \cup \cdots \cup Z_{j+1}) \subset F \setminus (Z_1 \cup Z_2 \cup \cdots \cup Z_j)$. There is a polygonal arc P with consecutive vertices $x = x_o, x_1, \ldots, x_n = y$ in $F \setminus (Z_1 \cup Z_2 \cup \cdots \cup Z_j)$ joining x and y. If P is disjoint from Z_{j+1} then $P \subset F \setminus (Z_1 \cup Z_2 \cup \cdots \cup Z_{j+1})$ and we are done. Otherwise P cuts Z_{j+1}, and without loss of generality (by the convexity of Z_{j+1}) we may assume that $P \cap Z_{j+1} = x_i x_{i+1}$. But x_i and $x_{i+1} \in$ bd Z_{j+1} so there exist points x_i' and x_{i+1}' in $P \cap ((Z_{j+1} + V) \setminus Z_{j+1})$, and there is a polygonal arc in $(Z_{j+1} + V) \setminus Z_{j+1}$ joining x_i' and x_{i+1}', say, with consecutive vertices $x_i' = y_o, y_1, \ldots, y_r = x_{i+1}'$. Thus, $x = x_o, x_1, \ldots, x_{i-1}, y_o, y_1, \ldots, y_r, x_{i+2}, \ldots, x_n = y$ are the consecutive vertices of a polygonal arc P' in $F \setminus (Z_1 \cup Z_2 \cup \cdots \cup Z_{j+1})$ joining x and y, so the latter is a connected open subset of F. This proves, by mathematical induction, that $F \setminus \bigcup_{j=1}^{k} Z_j$ is connected. Since $g \in F \cap G$ and G is a maximal connected subset of $E^2 \setminus S$, G contains $F \setminus \bigcup_{j=1}^{k} Z_j$, denying the boundedness of G. Hence, S is simply connected. ◇

In the proof of 2.23 the situation arose where a certain convex set (the half-plane F) disjoint from Q met S. It was then shown that any lnc point of a component of $F \cap S$ is an lnc point of S. The contradiction thereby establishes the local convexity of each component of $F \cap S$, and since these components were closed and connected Tietze's theorem implies they are convex. This situation is of sufficient generality and occurs frequently, so we cite a corresponding lemma, the obvious proof of which will be omitted.

2.24. LEMMA: If S is any closed set in E^d, with Q the set of lnc points and C any closed convex set disjoint from Q, then any component of $C \cap S$ is convex.

At this point we also state the classical Caratheodory theorem for E^2, which will be used frequently.

2.25. LEMMA: If $x \in \operatorname{conv} S$ there exist points y, z, and w in S such that $x \in \operatorname{conv}\{y, z, w\}$.

We prove a result which will be used later to extend any convex covering of S of the form $S = \bigcup_{j=1}^{n} C_j$, C_j convex, when $|Q| < \infty$, to the case $|Q| = \infty$. The proof uses the concept of the Hausdorff limit and a theorem of C. Kuratowski [13] (Theorem VIII, p. 246) which states that any sequence of subsets of a second countable topological space contains a topologically convergent subsequence. From the definition of the Hausdorff limit, it follows that if the sequence consists of convex sets, then the set to which the sequence converges is convex. We have then, the following lemma, phrased in the context in which it will be used.

2.26. LEMMA: Each subsequence of convex sets in E^d contains a subsequence which converges to a closed convex set.

If $x_{i_1}, x_{i_2}, \ldots, x_{i_n}, \ldots$ represents a subsequence Y of $x_1, x_2, \ldots, x_n, \ldots$, we write $Y = \{x_j : j \in I'\}$, where $I' = \{i_1, i_2, \ldots, i_n, \ldots\}$ (I itself will denote the set of positive integers).

2.27. LEMMA (KAY): If $S = cl(\bigcup_{i \in I} S_i)$ and for each $i \in I$ S_i is the union of m convex sets and $S_i \subset S_{i+1}$, then S is the union of m convex sets.

Proof. Let $S_i = \bigcup_{j=1}^{m} C_{ij}$, where C_{ij} is convex for each i and j. Apply 2.26 to $\{C_{i1} : i \in I\}$. There exists a convex set C_1 (perhaps empty) and a subsequence $\{C_{i1} : i \in I_1\}$ such that $\lim_{i \in I_1} C_{i1} = C_1$. Consider $\{S_i : i \in I_1\}$ and the corresponding $\{C_{i2} : i \in I_1\}$. Apply 2.26 once again to $\{C_{i2} : i \in I_1\}$. There exists a subsequence $\{C_{i2} : i \in I_2\}$, $I_2 \subset I_1$ and a convex set C_2 such that $\lim_{i \in I_2} C_{i2} = C_2$. Assume that C_k and I_k have been defined and apply 2.26 to $\{C_{i,k+1} : i \in I_k\}$. There exists a convex set C_{k+1} and a subsequence $I_{k+1} \subset I_k$ such that $\lim_{i \in I_{k+1}} C_{i,k+1} = C_{k+1}$. Hence $C_1, C_2, \ldots, C_m$ and I_m may be defined. Since $\{S_i : i \in I\}$ is a nondecreasing family, it is clear that $S = cl(\bigcup_{i \in I_m} S_i)$; also $\lim_{i \in I_m} C_{ij} = C_j$ for each $j \in \hat{m}$. We claim that $S = \bigcup_{j=1}^{m} C_j$. Let $x \in \bigcup_{j=1}^{m} C_j$. Hence, for some j, $x \in C_j$. There exists a sequence $\{y_k : k \in I_m\}$ of elements of $C_{kj} \subset S_k$ converging to x. Hence x is a limit point of S, and since S is closed, $x \in S$. On the other hand, if $x \in S$ there exists a j_o such that x is contained in infinitely many C_{ij_o}, $i \in I_m$. Therefore $x \in \lim_{i \in I_m} C_{ij_o} = C_{j_o}$, which implies that $x \in \bigcup_{j=1}^{m} C_j$. $\Diamond$

A similar proposition may be established for a nonincreasing sequence $\{S_i : i \in I\}$ $(S_i \supset S_{i+1})$; the set $S = \bigcap_{i \in I} \mathrm{cl}\ S_i$ is the union of m convex sets if each S_i is so expressible.

We continue the study of the structure of closed m-convex sets in E^2 in a sequence of results. The hypothesis that S is compact, connected and conv $Q \subset S$ (again we write H = conv Q) will be carried throughout.

<u>2.28. LEMMA</u>: Each component W of $S \setminus H$ has at least one member of Q in its closure.

<u>Proof</u>. Certainly there exists $x \in \mathrm{cl}\ W \cap \mathrm{bd}\ H$ (by connectedness of S). If $x \in Q$, we are finished. Otherwise, since H is closed (S is compact, so Q and thus conv Q is compact), $x \in H$. By 2.25 there exist points q_1, q_2, and q_3 in Q such that $x \in \mathrm{conv}\{q_1, q_2, q_3\}$. Since $x \notin \mathrm{int}\ H$, $x \in \mathrm{bd\ conv}\{q_1, q_2, q_3\}$ and hence, $x \in (q_1q_2)$, say. Consider the maximal subsegment x_1x_2 of q_1q_2 containing x and belonging to cl W. Again we are finished unless $x_1x_2 \subset (q_1q_2) \setminus Q$. Hence, in that case, a disk D centered at x_1 exists such that $D \cap S$ is convex, and if $D \cap q_1q_2 = y_1y_2$ and $y_3 \in D \cap W$ then $\mathrm{conv}\{y_1, y_2, y_3\} \subset D \cap S \subset S$, so it follows that $x_1x_2 \cup y_1y_2 \subset$

$q_1q_2 \cap \text{cl } W$, denying the maximal property of x_1x_2 as a subset of $q_1q_2 \cap \text{cl } W$. Hence, $x_1x_2 = q_1q_2$ proving that q_1 and q_2 belong to cl W. ◇

2.29. LEMMA: If H is two-dimensional, each component W of $S \setminus H$ contains at most two elements of Q in its closure.

Proof. We borrow a consequence of the Jordan closed curve theorem for E^2: If A_1, A_2, and A_3 are arcs having only endpoints x and y in common, then for some i = 1,2, or 3 the open arc $A_i \setminus \{x,y\}$ lies in the interior of the simple closed curve formed by A_{i+1} and A_{i+2} (cyclic indexing understood). Suppose q_1, q_2, and q_3 are points of Q in cl W. Since it is obvious that cl W contains exactly one point in Q if W is one-dimensional, we may assume W is two-dimensional. It then follows that cl W = cl int W; since cl W is polygonally connected it can be easily proved that int W is also polygonally connected. Hence, since S is locally starshaped, there exist points x_1, x_2, and x_3 in int W such that for i = 1,2,3, $x_iq_i \subset S$, with the x_i chosen sufficiently close to make x_1q_1, x_2q_2, and x_3q_3 pairwise disjoint. It follows that points q_i' exist in Q such that $[x_iq_i') \subset W$, i = 1,2,3, and hence $[x_iq_i') \subset \text{int } W$ (we are using here the local convexity of W); for convenience, we drop the primes. Since int W is connected there are polygonal arcs $P_1 \subset \text{int } W$ and $P_2 \subset \text{int } W$ joining the respective pairs (x_2,x_3) and (x_3,x_1), such that $P_1 \setminus \{x_3\}$ and $P_2 \setminus \{x_3\}$ are disjoint from each other and from q_ix_i, i = 1,2,3. But H is two-dimensional, compact, and convex, so bd H is a simple closed curve and q_1, q_2, and q_3 seperate bd H into three arcs B_1, B_2, and B_3, with q_i and q_{i+1} the endpoints of B_{i+2} for $i = 1,2,3$. Choose $y \in B_3$ distinct from q_1 and q_2, separating B_3 into two subarcs B_{31} and B_{32}, with $q_i \in B_{3i}$, $i = 1,2$. By convexity of H, $yq_3 \subset H$. Hence, the arcs $A_1 = B_{31} \cup (q_1x_1) \cup P_2$, $A_2 = B_{32} \cup (q_2x_2) \cup P_1$, and $A_3 = yq_3 \cup q_3x_3$ have only the endpoints x_3 and y in common. Denoting the interior region determined by the simple closed curve C by I(C), the above-mentioned consequence of the Jordan curve theorem implies that for some i,

$$A_i \setminus \{x_3, y\} \subset I(A_{i+1} \cup A_{i+2}).$$

But $q_i \in A_i \setminus \{x_3, y\}$ and by the simple connectedness of S, $I(A_{i+1} \cup A_{i+2}) \subset \text{int } S$. That is, $q_i \in \text{int } S$, which is impossible, thus establishing the desired result. ◊

That the above proof necessarily breaks down if H is not two-dimensional is easily shown by examples, such as that illustrated in the figure below (S is a 6-convex set with $H \subset S$, but W is a component of $S \setminus H$ with <u>all</u> <u>of</u> <u>Q</u> in its closure):

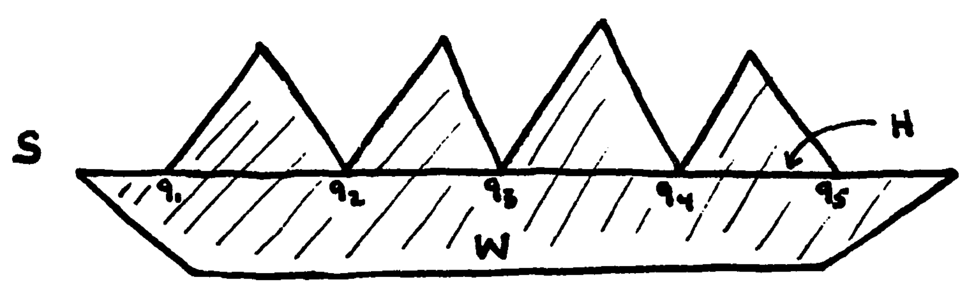


Figure 2.5

2.30. LEMMA: If H is two-dimensional, then the closure of each component W of $S \setminus H$ has at most two lnc points.

Proof. Let $x \in \mathrm{cl}\ W$, and suppose $x \notin Q$. Since Q is closed there exists a convex neighborhood U of x disjoint from Q, and by 2.24 any component of $U \cap S$ is convex. Thus $U \cap \mathrm{cl}\ W$ is locally convex at x. It follows that if x is an lnc point of cl W then $x \in Q$. Hence, by 2.29 cl W has at most two lnc points. ◊

In the preceding lemma we find that, unless cl W is convex, $S' \equiv \mathrm{cl}\ W$ is a set similar to S in that it is a closed, m-convex subset of E^2 with H', the convex hull of the set of lnc points of S', a subset of S'. But in this case, H' is a subset of a line. We then turn our attention to the case when the set H associated with S is a subset of a line, since in that case the problem of covering S by finitely many convex sets can be completely solved.

First, we shall need several concepts involving two-dimensional compact, convex subsets of E^2. If C is such a set, bd C is a simple closed curve (homeomorphic to a circle) and, as such, permits a cyclic ordering of its points. With x_0 any point of bd C, this ordering induces a linear ordering $>$ on any arc on bd C containing x_0 as an interior point. Thus, if A is such an arc, we may consider the two subarcs

$$A^+_{x_0} = \{x \in A : x > x_0\}, \quad A^-_{x_0} = \{x \in A : x < x_0\}.$$

For each $x \in A$, define $R(x, x_0)$ as the ray consisting of the set

$$\{(1 - \lambda)x + \lambda x_0 : \lambda \geq 0,\ \lambda \text{ real}\}.$$

As x tends to x_o from one side it is well known that $R(x,x_o)$ assumes a limiting position, which we can denote by $\lim R(x,x_o)$ (this is also a topological limit). Thus, the <u>one-sided</u> <u>support</u> <u>rays</u> of C at x_o,

$$R^+_{x_o} = \lim_{x \to x_o} R(x,x_o),\ x \in A^+_{x_o},$$

and

$$R^-_{x_o} = \lim_{x \to x_o} R(x,x_o),\ x \in A^-_{x_o},$$

each exist. Note that the lines containing $R^+_{x_o}$ and $R^-_{x_o}$ are ordinary lines of support of C at x_o. Define further the open half-planes $G^+_{x_o}$ and $G^-_{x_o}$ determined by the support lines containing $R^+_{x_o}$ and $R^-_{x_o}$, respectively, and not containing C (thus, $C \subset E^2 \setminus G^+_{x_o}$).

The following result will be used quite frequently from this point on.

<u>2.31. LEMMA</u>: If C is a two-dimensional convex subset of the plane and an arc $A \subset \text{bd}\ C$ which contains a point x_o bd C in its interior is ordered by <, the open half-planes $G^+_{x_o}$ and $G^-_{x_o}$ determined by the one-sided support rays $R^+_{x_o}$ and $R^-_{x_o}$ not containing C have the property that given compact subsets $M_1 \subset G^-_{x_o}$ and $M_2 \subset G^+_{x_o}$ there exist points $x_i \in A$, $i = 1,2$, such that $x_1 < x_o < x_2$ and for any point $u \in M_i$, $ux_i \cap C = \{x_i\}$.

<u>Proof</u>. It is only necessary to prove the desired property for M_1 (see Figure 2.6). For each $x \in A$ define the open half-plane F_x determined by $R(x,x_o)$ not containing bd $C \setminus A$, and let F'_x denote the open half-plane whose edge is a support line of C parallel to the edge of F_x, with $F'_x \subset F_x$. Elementary properties of convex sets enable one to prove

the topological limit

$$\lim_{x \to x_o} F_x = \lim_{x \to x_o} F'_x = \text{cl } G^-_{x_o}, \quad x \in A^-_{x_o}.$$

Now we show that for some $x < x_o$, $M_1 \subset F'_x$. First, for any $y \in M_1$, suppose some sequence $\{x_n\}$ of points on bd C, with $x_1 < x_2 < \cdots < x_n < \cdots$, and converging to x_o exists such that $y \notin F'_{x_n}$. Let U be a circular neighborhood of y of radius $r > 0$ such that $U \subset G^-_{x_o}$ (since $G^-_{x_o}$ is open and $M_1 \subset G^-_{x_o}$). Since $y \in \lim F'_{x_n}$ there is an n_o such that for $n > n_o$ U meets F'_{x_n}; since $y \notin F'_{x_n}$ there is a circular neighborhood $V_n \subset U$ with center z_n of radius r/2 deviod of points of F'_{x_n}. We may assume without loss of generality that $\lim z_n = z \in U$, with V the circular neighborhood about z of radius r/2, and that for all n sufficiently large, $V_n \cap V$ contains a fixed circular neighborhood V' of radius r/3. But $V' \subset U \subset G^-_{x_o}$ so for all n sufficiently large F'_{x_n} meets V' and hence V_n, a contradiction. Thus, given $y \in M_1$ there is an $x < x_o$ such that $y \in F'_u$ for $x < u < x_o$. Suppose $M_1 \not\subset F'_x$ for all $x < x_o$. Then we may choose a sequence $x_n \to x_o$ such that $x_1 < x_2 < \cdots < x_n < \cdots$, and $y_n \in M_1$ such that $y_n \notin F'_{x_n}$. By compactness of M_1 we may assume $y_n \to y \in M_1$. But $y \in F'_{x_n}$ for all sufficiently large n as was proved, and if $U \subset F'_{x_n}$ is a neighborhood of y then some $y_n \in U$ or $y_n \in F'_{x_n}$, a contradiction. Hence, for some $x < x_o$, $M_1 \subset F'_x$. If L_x is the edge of F'_x it is a support line of C and meets C in some point $p \leq x_o$. If $p \neq x_o$, then set $x_1 = p < x_o$; if $p = x_o$, then by definition of F_x, $L_x \supset R(x, x_o)$ and hence $x \in L_x$, and in this case set $x_1 = x < x_o$. In either case, since L_x is a line of support of C and $x_1 \in C$, with M_1 and C on opposite sides of L_x, we have $ux_1 \cap C = \{x_1\}$ for each $u \in M_1$. ◊

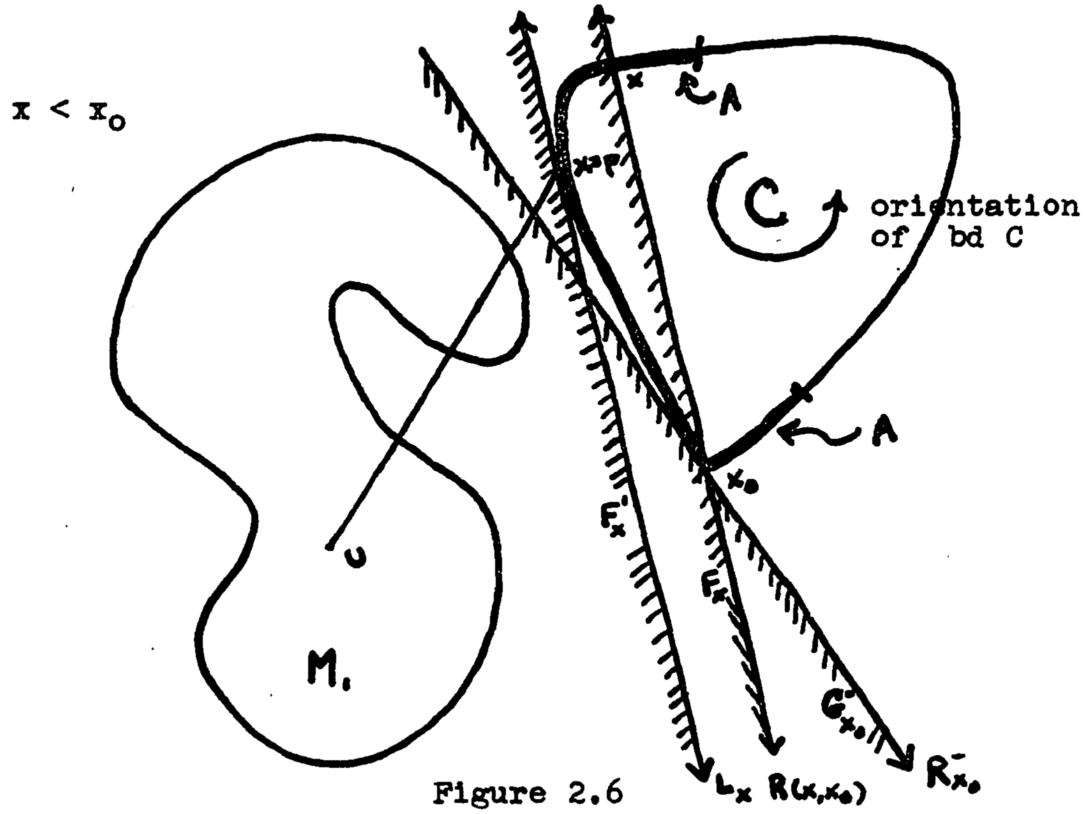


Figure 2.6

The figure below illustrates the fact that 2.31 does not follow if M_1 is merely closed.

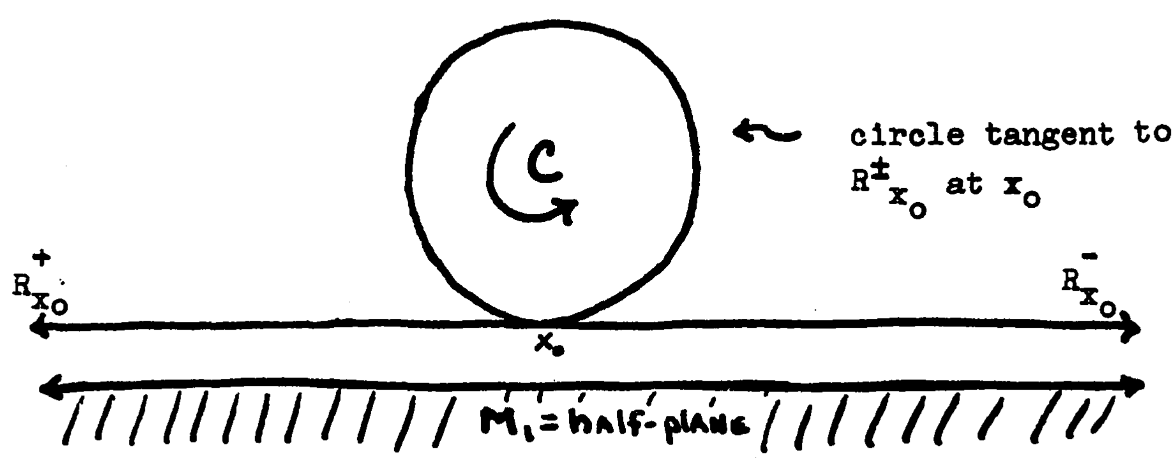


Figure 2.7

2.32. PROPOSITION: If S is a compact, m-convex subset of E^2, with $H = \text{conv } Q \subset S$ and H is one-dimensional or consists of a single point, then S is the union of m - 1 convex sets.

Proof. Consider the components of $S \setminus L$, where L is a line containing H. The m-convexity of S implies there are at most m - 1 of these on each side of L. If W is any such component, let L_t denote a line parallel to L and at a distance t from it, F_t the closed half-plane determined by L_t disjoint from L, and put $W_t = W \cap F_t$. Since for each $t > 0$ W_t is a component of $F_t \cap S$ and F_t is disjoint from Q, by 2.24 W_t is convex and hence, W is convex. Thus, at this point it has been proved that S is the union of $2(m - 1) + 1 = 2m - 1$ or fewer convex sets. To finish the proof we shall use induction on m.

Two simple cases must be ruled out first: When one or more of the components of $S \setminus H$ (1) are one-dimensional, or (2) contain only one point of L in their closure. For (1), suppose W is a one-dimensional component of $S \setminus L$, and let L' be the line containing W, with W' the component of $S \cap L'$ containing W. Then W' is convex and it is clear that $\text{cl}(S \setminus W')$ is $(m - 1)$-convex. Hence, by the induction hypothesis $\text{cl}(S \setminus W') \cup W' = S$ is the union of $(m - 2) + 1 = m - 1$ convex sets. For (2), suppose W is a component of $S \setminus L$ such that $\text{cl } W \cap L = \{x\}$. By (1) we may assume that W is two-dimensional. Since W is convex, $\text{int } W \neq \emptyset$ and there exists a circular disk $U \subset W$. Let $x_1, x_2, \ldots, x_k$ be any $k \geq 2$ points of $(S \setminus W) \setminus L$. Since there are only finitely many lines passing through x and the points $x_1, x_2, \ldots, x_k$ there is obviously a point $x_0 \in U$ not on any of these lines. Hence, for each $i = 1, \ldots, k$, $x \notin x_0x_i$. If x_i lies on the same side of L as x_0 then $x_0x_i \subset S$ implies $x_i \in W$ (since $x_0x_i \cap L = \emptyset$ and W is a component of $S \setminus L$ containing x_0), a contradiction. If x_i lies on the opposite side of L as x_0 then x_0x_i meets L in a point $y \neq x$ and if $x_0x_i \subset S$ it follows that $(yx_0] \subset W$ or $y \in \mathrm{cl}\ W$, a contradiction of $\mathrm{cl}\ W \cap L = \{x\}$. Finally, if for some i and j, $1 \leq i < j \leq k$, $x_ix_j \subset S$ but $x_ix_j \not\subset S \setminus W$ then there exists a $z \in x_ix_j \cap W$ and hence, by similar reasoning either x_iz or x_jz belongs to W, a contradiction. Thus, if $x_1, x_2, \ldots, x_k$ are visually independent via $S \setminus W$ then they are visually independent via S. By m-convexity of S, and since $x_0x_i \not\subset S$ for all i, $k \leq m - 2$ and hence any $m - 1$ points of $S \setminus W \setminus L$ are visually dependent via $S \setminus W$. But $\mathrm{cl}\ (S \setminus W) \setminus L = S \setminus W$, since it may be assumed that no component of $S \setminus H$ lies on L (by (1) above). Therefore, $S \setminus W$ is a closed, $(m - 1)$-convex set and by the induction hypothesis, $(S \setminus W) \cup W = S$ is the union of $(m - 2) + 1 = m - 1$ convex sets.

Thus, it may be assumed that each component $\mathrm{cl}\ W$ of $S \setminus L$ is a compact two-dimensional convex set which meets L in a nontrivial segment xy. We may then designate the components W and the corresponding segments in the order in which they occur on L by

$$W_1, x_1y_1; W_2, x_2y_2; \ldots; W_r, x_ry_r,$$

where $W_1, \ldots, W_r$ are those components on one side of L, with

$W_1 \cap L = x_1y_1$, and

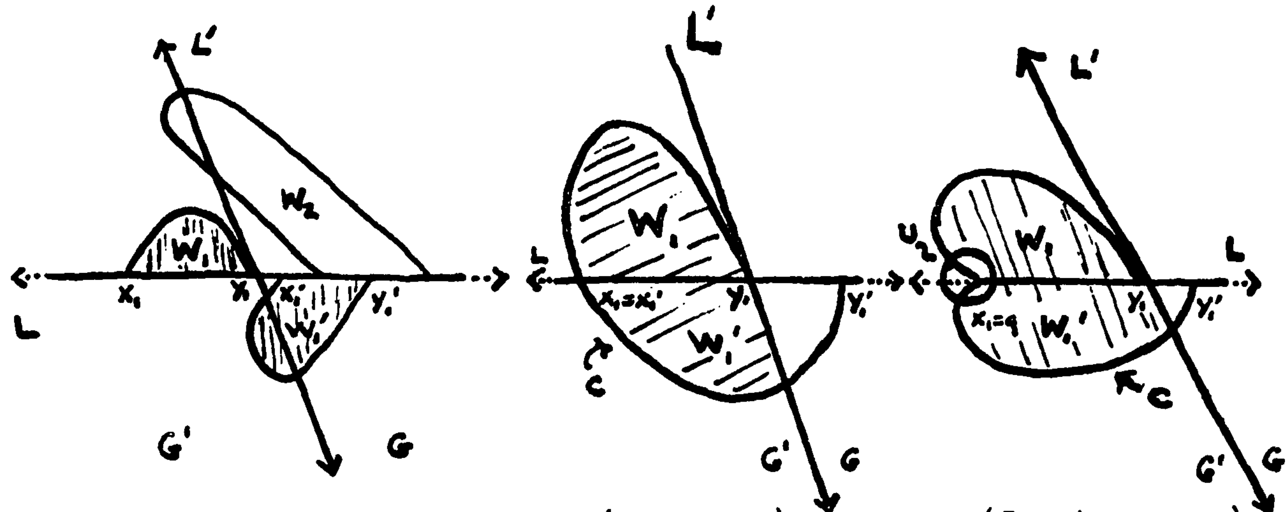
$$W_1', x_1'y_1'; W_2', x_2'y_2'; \ldots; W_s', x_s'y_s',$$

where $W_1', \ldots, W_s'$ are those components on the other side of L and $W_i' \cap L = x_i'y_i'$. Thus, if $<$ denotes the natural ordering on L, we may assume that $x_1 < y_1 \leq x_2 < y_2 \leq \cdots \leq x_r < y_r$ and $x_1' < y_1' \leq x_2' < y_2' \leq \cdots \leq x_s' < y_s'$. But for notation we have $y_1 \leq y_1'$, and either (1) $y_1 \leq x_1'$, or (2) $y_1 > x_1'$. Figure 2.8 illustrates the various cases in the following argument.

Case 1: ($y_1 \leq x_1'$) Let $<$ induce an orientation on bd(cl W_1) and let A be any arc on bd(cl W_1) containing x_1y_1 in its interior. As previously defined, let $R^+_{y_1}$ be the one-sided ray at y_1 and $G^+_{y_1} = G$ the open half-plane determined by $R^+_{y_1}$ and not containing cl W_1. If L' is the line containing $R^+_{y_1}$ consider $z_1, z_2, \ldots, z_{m-1}$ any $m - 1$ points in $S \setminus$cl $W_1 \setminus L'$, where $z_1, z_2, \ldots, z_k$ lie in G and $z_{k+1}, \ldots, z_{m-1}$ lie in the opposite open half-plane G' of G. Applying 2.31, with M = $\{z_1, z_2, \ldots, z_k\}$, there is a point $z_o \in$ bd(cl $W_1) \subset$ cl W_1 such that $z_o > y_1$ and $z_oz_i \cap$ cl $W_1 = \{z_o\}$ for $i = 1,2,\ldots,k$. It follows that $z_o \notin L$; thus $z_oz_i \not\subset S$, for otherwise, z_oz_i meets L in a point $w \notin x_1y_1$ with $z_ow \subset$ cl W_1, a contradiction. For $i = k + 1, \ldots, m - 1$, we note that z_i must either belong to a component W_u different from W_1 or a component W_v'. In the former case $z_oz_i \not\subset S$ or else $z_i \in$ cl W_1, and in the latter, z_oz_i meets L at a point $w < y_1$. But $w \in x_v'y_v'$ and hence $x_v' < y_1$ for some v, a contradiction that $x_v' \geq x_1' \geq y_1$. Hence, z_o cannot see z_i via S, $i > 0$, and hence, for some

$1 \leq i < j \leq m - 1$, $z_i z_j \subset S$ by the m-convexity of S. It remains to show that $z_i z_j \subset cl(S \setminus cl\ W_1)$. But if $z_i z_j \not\subset cl(S \setminus cl\ W_1)$ then there is some point $z \in z_i z_j$ not in $cl(S \setminus cl\ W_1)$; hence, $z \notin S \setminus cl\ W_1$ so $z \in cl\ W_1$. It follows that $z \neq y_1$ for $y_1 \in cl(S \setminus cl\ W_1)$. One of the points z_i or z_j, say z_i, must belong to G' and, since z_i cannot belong to any component W_u, $z_i z$ meets $x_1 y_1$ at a point $w < y_1$, producing a contradiction similar to one observed previously. Hence, $z_i z_j \subset cl(S \setminus cl\ W_1)$ and it follows that $cl(S \setminus cl\ W_1 \setminus L') = cl(S \setminus cl\ W_1)$ is $(m - 1)$-convex. By the induction hypothesis $S = cl(S \setminus cl\ W_1) \cup cl\ W_1$ is the union of $(m - 2) + 1 = m - 1$ convex sets.

Case 2: $(y_1 > x_1')$ Again define the line L' containing $R^+_{y_1}$ and the open half-planes G and G' determined by L'. Consider the closed, connected set $C = cl\ W_1 \cup cl(W_1' \cap G')$. If C has no lnc points then C is convex by Tietze's theorem, and an argument similar to that given in the preceding case shows that $cl(S \setminus C)$ is $(m - 1)$-convex. Thus, $S = cl(S \setminus C) \cup C$ is the union of $(m - 2) + 1 = m - 1$ convex sets. Otherwise, C has an lnc point q, and it is clear that $q = x_1$ or $q = x_1'$ which implies $q < y_1$ and $q \in Q$. Let $z_1, z_2, \ldots, z_{m-2}$ be any $m - 2$ points of $S \setminus cl\ W_1 \setminus cl\ W_1' \setminus L$. Then no z_i can see q via S since, otherwise, $z_i \in W_1$ or $z_i \in W_1'$. Hence, since S is closed, there exists a neighborhood U of q such that no point of U can see any z_i via S; if z_{m-1} and z_m are points of U such that $z_{m-1} z_m \not\subset S$ then by m-convexity there is an i, j with $1 \leq i < j \leq m - 2$ and $z_i z_j \subset S$, and it is obvious again



(C convex) (C not convex)

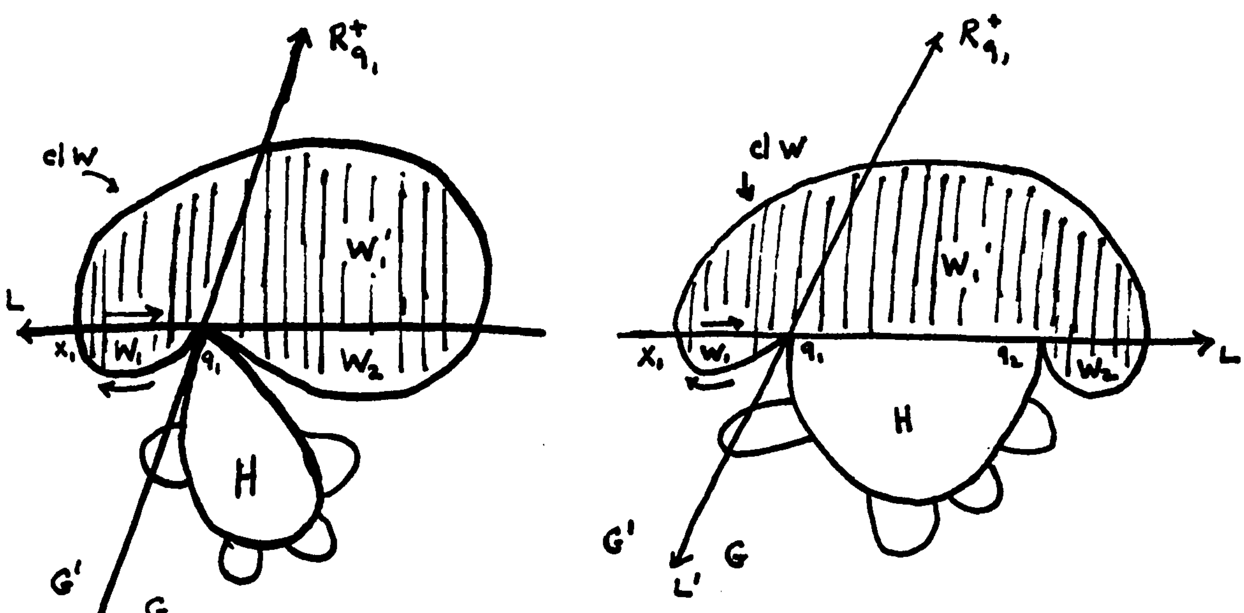
Case 1. $y_1 \le x_1'$ Case 2. $y_1 > x_1'$

Figure 2.8

that $z_i z_j$ cannot meet cl $W_1 \cup$ cl W_2. Thus, $\mathrm{cl}(S \setminus \mathrm{cl}\ W_1 \setminus \mathrm{cl}\ W_1' \setminus L) = \mathrm{cl}(S \setminus \mathrm{cl}\ W_1 \setminus \mathrm{cl}\ W_1')$ is $(m-2)$-convex. By the induction hypothesis $S = \mathrm{cl}(S \setminus \mathrm{cl}\ W_1 \setminus \mathrm{cl}\ W_1') \cup \mathrm{cl}\ W_1 \cup \mathrm{cl}\ W_1'$ is the union of $(m-3)+1+1 = m-1$ convex sets, completing the proof. ◊

We note that the above proposition applies to any closed m-convex set S having only one or two lnc points, thus providing the same result that appears in [10]. (The proof of this result given in [10] differs considerably from the one presented here.) In particular, it also shows that the closure of any nonconvex component W of $S \setminus H$, where H is two-dimensional, is the union of either <u>two</u> or <u>three</u> convex sets (by 2.30 and certain observations). It is clear that, in our handling of the problem of proving that a closed m-convex set S is the union of finitely many convex sets when $H \subset S$ and H is two-dimensional, we need to distinguish between the two cases: (1) The closure of some component of $S\setminus H$ is not convex, or (2) the closures of all components of $S\setminus H$ are convex. We turn our attention to the first of these cases.

Suppose the component W of $S\setminus H$ is such that cl W is not convex. Then by 2.30, cl W has at most two lnc points q_1 and q_2, and by the proof of that lemma, q_1 and q_2 belong to Q. Thus, cl W has either one lnc point q_1 or two distinct lnc points q_1 and q_2 belonging to $Q\subset H$. In either case there is a line L through q_1 such that $W\setminus L$ has a component W_1' on one side of L and precisely two components W_1 and W_2 on the other side, with W_1' and H on the opposite side of L (see Figure 2.9). (We may take L to be the line determined by q_1 and q_2 in the latter case, and in the former, if $x\in$ cl W and $y\in$ cl W such that $xy\not\subset$ cl W and $q_1\notin xy$, choose L any line through q_1 not passing through x or y.)



(cl W has one lnc point)

(cl W has two lnc points)

Figure 2.9

As in the proof of the preceding theorem, cl W_1, cl W_2, and cl W_1' are each convex sets, and since cl W is necessarily two-dimensional then cl W_1 is two-dimensional, and since $W \setminus q_1$ is connected cl W_1 must meet L in some point $x_1 \neq q_1$; if < orders the points on L, we may assume $x_1 < q_1$. Taking A any arc on bd(cl W_1) containing $x_1 q_1$ in its interior we may define $R^+_{q_1}$ as before and let L' be the line containing $R^+_{q_1}$. We note that since H and W_1 lie on the same side of L, H and W_1 lie on opposite sides of L' (otherwise, it could be shown that W is not maximal as a connected subset of $S \setminus H$). Thus, it follows that any point $x > q_1$ in $A \subset$ bd(cl W_1) is in bd S. Now it follows, just as in a previous argument, that if G and G' are the two open half-planes determined by L' with $W_1 \subset$ cl G', then cl $W \cap$ cl G' = C is convex and $\mathrm{cl}(S \setminus C)$ is $(m - 1)$-convex. Thus, our problem would be solved by the inductive hypothesis in this case, since $S = \mathrm{cl}(S \setminus C) \cup C$.

Collecting a number of situations in which S can be decomposed into a convex set and an $(m - 1)$-convex set (by use of previous arguments) we have

2.33. PROPOSITION: If S is any compact m-convex subset of E^2 such that $H \subset S$ and H is two-dimensional, then S is the union of a convex set and a compact $(m - 1)$-convex set provided there exists a component W of $S \setminus H$ such that either

(a) W is one-dimensional,

(b) cl W is convex and contains only one point of H, or

(c) cl W is not convex.

Thus, we turn to case (2) mentioned above and to the cases not covered by 2.32 and 2.33. That is, we assume that for a compact, m-convex set S, H is two-dimensional and the closure of each component of $S \setminus H$ is a two-dimensional convex set, meeting H in at least two distinct points. Thus, if W is a component of $S \setminus H$ and cl $W \cap H = xy$ it is clear that x and y are points in Q. (However, it is not true that if $W_1, W_2, \ldots, W_i, \ldots$ are the components of $S \setminus H$ then all points of Q belong to $\bigcup_{i=1}$ cl W_i. A counterexample is provided by the infinite-sided polygon and interior S illustrated in the figure below, which is 3-convex since it is the union of 2 convex sets, has the properties being discussed, but the point $q \in Q$ shown does not belong to cl W_i for any i.)

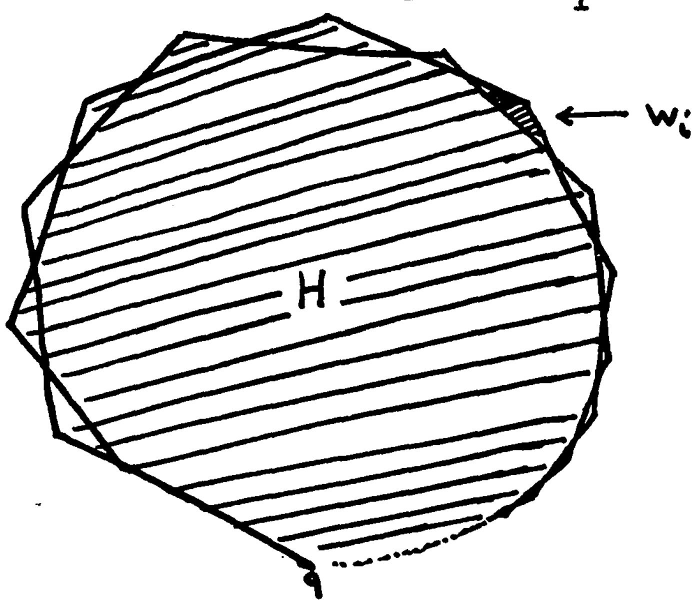


Figure 2.10

But owing to 2.27 if we consider the sets $S_1 = H \cup W_1$, $S_2 = H \cup W_1 \cup W_2, \ldots, S_i = H \cup (\bigcup_{j=1}^{i} W_j)$, then $S = \bigcup S_i$, so it suffices to consider each set S_i. If $H_i = \text{conv } Q_i$, where

Q_1 is the set of lnc points of S_1, then there are only finitely many lnc points, and only finitely many components in $S_1 \setminus H_1$. S_1 is obviously m-convex, so this means we have only to consider sets having finitely many lnc points.

It is clear that any result giving a bound to the number of convex sets decomposing a compact m-convex set can also be obtained for closed sets by applying 2.27. Thus, to solve the finite convex covering problem for closed m-convex sets in E^2 with $H = \text{conv } Q \subset S$, it suffices to consider sets S having the following properties (in addition to $H \subset S$):

(1) S is compact.

(2) Q is finite and there are finitely many components $W_1, W_2, \ldots, W_n$ of $S \setminus H$.

(3) H is two-dimensional.

(4) Each set cl W_i is convex and two-dimensional.

(5) For each i, cl $W_i \cap H = q_i q_i'$, where q_i and q_i' are distinct lnc points.

For convenience, such sets will be referred to as <u>type W*</u> (W-star).

<u>2.34. REMARK</u>: It was proved in [10] that if such a set has n lnc points then it is the union of n + 1 or fewer convex sets. However, this result is not relevant to the present situation as the example of the infinite-sided polygon and interior given before emphatically shows.

The next two results will enable us to make other assumptions later.

<u>2.35. LEMMA</u>: If S is a compact m-convex set with $H \subset S$, then

for any $x \in S$, S_x is also m-convex.

Proof. Since S is simply-connected, if $y_1 \in S_x$ and $y_2 \in S_x$ with $y_1y_2 \subset S$, then $y_1x \cup xy_2 \cup y_1y_2 \subset S$ implies $\text{conv}\{x, y_1, y_2\} \subset S$. Hence, for $u \in y_1y_2$, $xu \subset S$ and $u \in S_x$. Therefore, $y_1y_2 \subset S_x$. If $y_1, y_2, \ldots, y_m$ be any m points of S_x, then by the m-convexity of S $y_iy_j \subset S$ for some $1 \leq i < j \leq m$. Thus, by the preceding argument, $y_iy_j \subset S_x$ and it follows that S_x is m-convex. The fact that S_x is compact is a consequence compactness of S. ◊

2.36. LEMMA: If S is any closed m-convex set the anti-star S^x is (m - 1)-convex relative to S for any $x \in S$. If $x = q \in Q$, then S^q is (m - 2)-convex relative to S.

Proof. Since S^x is the set of all points of S which do not see x via S, then obviously, the m-convexity of S implies that any m - 1 points of S^x must be visually dependent via S. If $x = q \in Q$, suppose $y_1, \ldots, y_k$ are any k points of S^x which are visually independent via S. There is a neighborhood U such that if $u \in U$, $uy_i \not\subset S$ for all i (since S is closed). In particular, there exist points y_{k+1} and y_{k+2} in U such that $y_{k+1}y_{k+2} \not\subset S$. Hence, $y_1, \ldots, y_{k+2}$ are k + 2 visually independent points. By m-convexity, $k + 2 \leq m - 1$ and $k \leq m - 3$. Hence, S^q is (m - 2)-convex, relative to S. ◊

It is not known whether an m-convex set of type W* for values of $m \geq 5$ is the union of even a finite number of convex sets. The following result "localizes" the problem; Q' will denote the set of limit points of lnc points. Note that $Q' \subset Q$.

2.37. PROPOSITION: A necessary and sufficient condition for a compact m-convex set S in E^2 to be the union of finitely many closed convex sets is that for each $q \in Q' \cap K$ there is a neighborhood N of q such that cl N is the union of finitely many closed convex sets.

Proof. The necessity is obvious. For the sufficiency, we apply induction on m. The theorem is obvious if m = 2. Each member q' of $Q' \cap K$ by hypothesis has a neighborhood N(q') such that cl N(q') is the union of finitely many closed convex sets. For $q \in (Q \cap K) \setminus Q'$, since q is not a limit point of Q there exists a convex neighborhood N(q) devoid of points of $Q \setminus \{q\}$. Then cl N(q) is a compact m-convex set in E^2 having only one lnc point, namely q, and thus by 2.32, N(q) is the union of m - 1 closed convex sets. For $q \in Q \setminus K$, there is a point x(q) and a convex neighborhood N(q) which cannot see x(q). Then N(q) is (m - 1)-convex, so by the induction hypothesis cl N(q) is the union of finitely many closed convex sets. Finally, for $x \in S \setminus Q$, by definition of local convexity, there exists a convex neighborhood $N(x) \subset S$. Thus, for each $x \in S$, N(x) is a neighborhood of x whose closure is a finite union of closed convex sets. Since S is compact, there is a finite subcover $N(x_1), \ldots, N(x_n)$ of S, which proves that S itself is the union of finitely many closed, convex sets. ◊

CHAPTER III

4-CONVEXITY

It will be established that a closed, simply-connected 4-convex subset of E^2 is the union of 9 or fewer convex sets. It is not known whether the bound on the number of convex sets is best; it is highly probable that it is not. However, up to this time even this bound had not been established, in spite of attempts by several authors to do so. Guay's thesis includes results concerning convex coverings for a 4-convex set S when S has a cut point, $|Q \cap K| = 2$, $|Q| = 1$, $|Q \setminus K| \leq 1$, S is one-dimensional at some point not in Q, or K is one-dimensional. (As before, K denotes the kernel of S, Q stands for the set of lnc points of S, and H = conv Q.) In the cases where $|Q \cap K| = 2$ or $|Q| = 1$, Guay proved that S may be expressed as the union of three or fewer closed convex sets, and in the remaining cases, S is the union of four or fewer closed convex sets. Guay's main result was that a closed 4-convex set in E^2 which is <u>not</u> simply-connected is the union of five or fewer convex sets. (This result is best possible as illustrated in Figure 3.1; the set S indicated there is compact, 4-convex and not simply-connected, but it is not the union of any four convex sets.) Establishing a best bound for the remaining case, when S is simply connected, would complete the finite convex covering problem for closed, connected, 4-convex subsets of E^2.

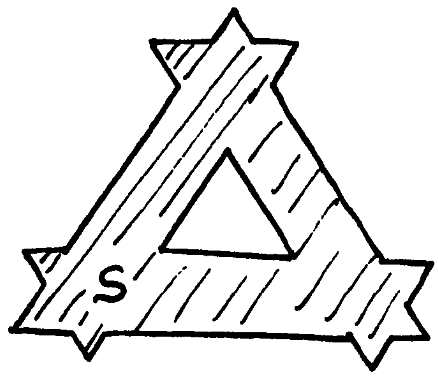


Figure 3.1

The following preliminary result reverses a previous one, namely 2.23, in the case of 4-convexity.

3.1. LEMMA: For a closed, connected 4-convex subset S of E^2, $H \subset S$ is equivalent to the simple-connectedness of S.

Proof. For compact, connected sets in E^2 2.23 implies the result that S is simply-connected if $H \subset S$, and this is clearly enough to establish that result for closed, connected sets. Conversly, suppose S is simply-connected, and let $x \in H$. By 2.25, there exist q_1, q_2, q_3 in Q such that $x \in \operatorname{conv}\{q_1, q_2, q_3\}$. Now if $q_1q_2 \not\subset S$ there exist neighborhoods U_1 and U_2 of q_1 and q_2 such that for $u_i \in U_i$, $i = 1,2$, $u_1u_2 \not\subset S$. But q_1 and q_2 are lnc points of S, so there exist points u_i and v_i in U_i such that $u_iv_i \not\subset S$, $i = 1,2$, and hence $\{u_1, v_1, u_2, v_2\}$ is a set of four visually independent points in S, denying 4-convexity. Hence, $q_1q_2 \subset S$, and in the same manner, $q_2q_3 \subset S$ and $q_1q_3 \subset S$. By simple-connectedness, conv$\{q_1, q_2, q_3\} \subset S$ and $x \in S$. Therefore, $H \subset S$. ◊

Thus, is S is a closed, simply-connected, 4-convex subset of E^2, $H \subset S$; hence inside every disk S is a compact, simply-connected 4-convex set. By 2.27 we may then restrict our attention to compact, simply-connected 4-convex sets. All results on m-convexity established in the preceding chapter, therefore, apply here. As pointed out there, the problem has been reduced to the consideration of sets of type W* since Valentine's theorem may be applied to the 3-convexity arising from the use of 2.33.

If S is of type W*, suppose $\{W_i\}$ are the closures of the components of $S \setminus H$. Orient the boundary of H counterclockwise, thereby inducing a clockwise orientation of each bd W_i, $i = 1, \ldots, n$ (see figure below). Let A be any arc on bd W_i containing in its interior the two lnc points of S in cl W_i, and label those lnc points q_i and q_i', with $q_i < q_i'$. For convenience, we introduce the further

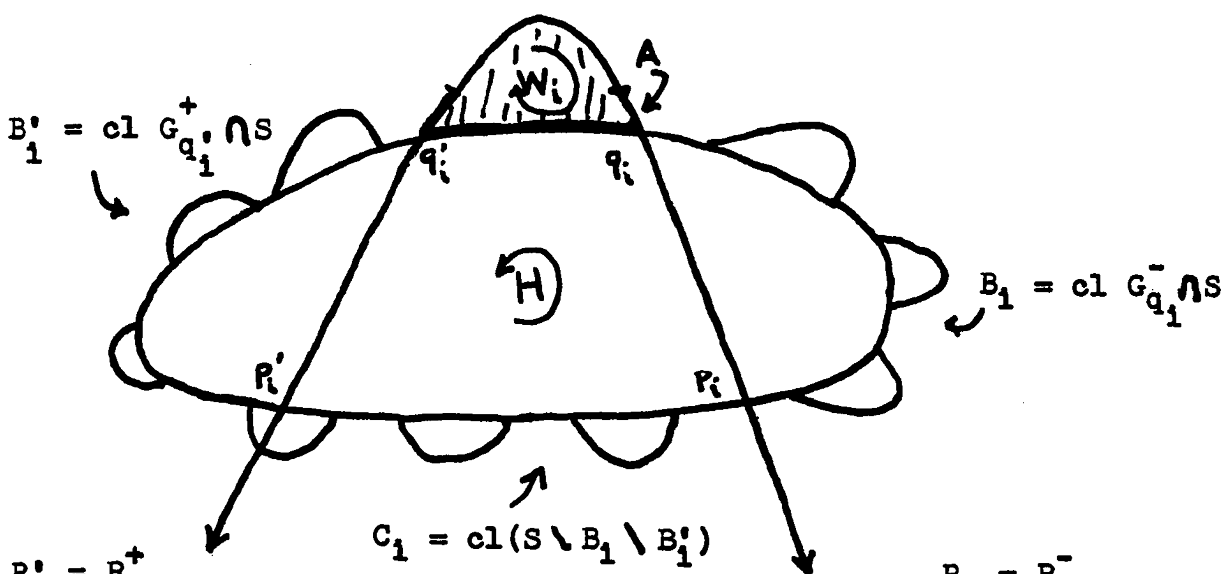


Figure 3.2

notation

$$R_i = R^-_{q_i}, \quad R'_i = R^+_{q'_i},$$

$$B_i = \mathrm{cl}\ G^-_{q_i} \cap S, \quad B'_i = \mathrm{cl}\ G^+_{q'_i} \cap S, \quad C_i = \mathrm{cl}(S \setminus B_i \setminus B'_i).$$

Essentially from 2.31 it follows that the sets B_i and B'_i are compact 3-convex subsets of S. Also, the set $W_i \cup (C_i \cap H)$ is convex, owing to its local convexity. For each i we let p_i and p'_i be the endpoints of the segments $R_i \cap H$ and $R'_i \cap H$ different from q_i and q'_i respectively. The following property of the components W_i is a key result to be used later.

<u>3.2. LEMMA</u>: The set $W_i \cup W_j$ is convex relative to S iff $q_iq'_i \subset C_j$ and $q_jq'_j \subset C_i$.

<u>Proof</u>. If $q_iq'_i \not\subset C_j$ then there exists a point $x \in q_iq'_i \setminus C_j$, which implies $x \in B_j$ or $x \in B'_j$. By 2.31 there exists a point $y \in \mathrm{bd}\ W_j \subset W_j$ such that $xy \not\subset S$. Hence, $W_i \cup W_j$ is not convex relative to S.

Conversely, assume $q_iq'_i \subset C_j$ and $q_jq'_j \subset C_i$. Since there is nothing to prove otherwise, assume $x \in W_i$ and $y \in W_j$. Since $q_iq'_i \subset H$ then $q_iq'_i \subset C_j \cap H$, and since $W_j \cup (C_j \cap H)$ is convex, for each $u \in q_iq'_i$, $uy \subset W_j \cup (C_j \cap H) \subset S$ (see Figure 3.3). Since $u \in W_i$ and W_i is convex, $xu \subset W_i \subset S$. Hence $xu \cup uy \subset S$. Choose $u \in q_iq'_i$ such that $xu \cup uy \subset S$ and $e(x,u) + e(u,y)$ is minimal, where e denotes the euclidean metric. Since $u \notin W_j$, uy cuts bd W_j at a point v. If $v \notin q_jq'_j$ then $v \notin H$, and hence there is a neighborhood U of v devoid of points of H. Then $uv \subset S$ implies there is a point on (uv) in another component

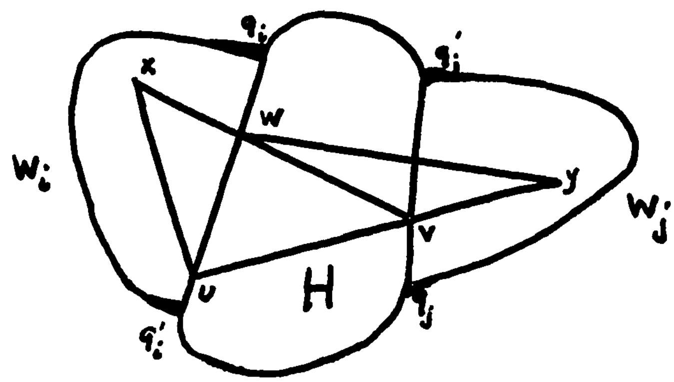


Figure 3.3

of $S \setminus H$, which is impossible. Hence $v \in q_j q_j' \subset C_i \cap H$ and $vx \subset W_i \cup (C_i \cap H) \subset S$. By the same reasoning as before there is a point $w \in vx \cap q_i q_i'$. Moreover, $xw \cup wy \subset W_i \cup W_j \cup (C_j \cap H) \subset S$. But

$$
\begin{aligned}
e(x,w) + e(w,y) &\leq e(x,w) + e(w,v) + e(v,y) \\
&= e(x,v) + e(v,y) \\
&\leq e(x,u) + e(u,v) + e(v,y) \\
&= e(x,u) + e(u,y).
\end{aligned}
$$

By the definition of u as a point on $q_i q_i'$, equallity prevails throughout, and $e(x,u) + e(u,v) = e(x,v)$. Thus $u \in xv$ and $v \in uy$, or x,u,v, and y are collinear. Hence $xy = xu \cup uv \cup vy \subset W_i \cup H \cup W_j \subset S$, so $W_i \cup W_j$ is relatively convex. ◊

We shall now consider a situation which will occur repeatedly throughout the remaining discussion. Suppose x_o is a point on bd H and that x_o lies in the kernel of S. As before, the removal of x_o from bd H results in a set which can be linearly ordered by <, with x_o as the least element. Using this ordering to produce the notation introduced earlier, we have $x_o \in C_i$ for all i, and if $x_o \notin q_i q_i'$, $p_i < q_i < q_i' < p_i'$

(see Figure 3.4), and it may be assumed that the sets W_i have been so labeled that $q_i < q_j$ whenever $i < j$. Moreover,

Figure 3.4

$(\text{int } B_i) \cap \text{bd } H$ consists of those points x on bd H such that $p_i < x < q_i$, and similarly for int B_i' (here, the interior is taken relative to S). Thus, we have

$$(\text{int } B_i) \cap \text{bd } H = \{x \in \text{bd } H;\ p_i < x < q_i\}, \text{ and}$$
$$(\text{int } B_i') \cap \text{bd } H = \{x \in \text{bd } H;\ q_i' < x < p_i'\}.$$

It is easy to verify the further relation

$$C_i \cap \text{bd } H = \{x \in \text{bd } H:\ q_i \leq x \leq q_i',\ x \leq p_i, \text{ or } x \geq p_i'\}.$$

Now consider any two sets W_i and W_j, for $i < j$. Then $q_i < q_i' \leq q_j < q_j'$ (see Figure 3.5). Suppose $x \in q_i q_i'$, and therefore $x \in \text{bd } H$ and $q_i \leq x \leq q_i'$. It follows that $q_i' \leq p_j$ implies $x \leq p_j$ or $x \in C_j$. Conversely, if $x \notin C_j$ then $x > p_j$ and therefore $p_j < q_i'$. Thus $q_i q_i' \subset C_j$ iff $q_i' \leq p_j$. In a similar fashion it can be proved that $q_j q_j' \subset C_i$ iff $p_i' \leq q_j$. In view of 3.2 this gives us

<u>3.3. LEMMA</u>: If $x_o \in \text{bd } H$ is a point in the kernel of S and $<$ is the linear order on bd H determined by x_o, with the points q_i ordered accordingly, then for any two integers

$i < j$ such that $x_0 \notin W_i \cup W_j$, W_i can see W_j via S iff both $q_i' \leq p_j$ and $p_i' \leq q_j$.

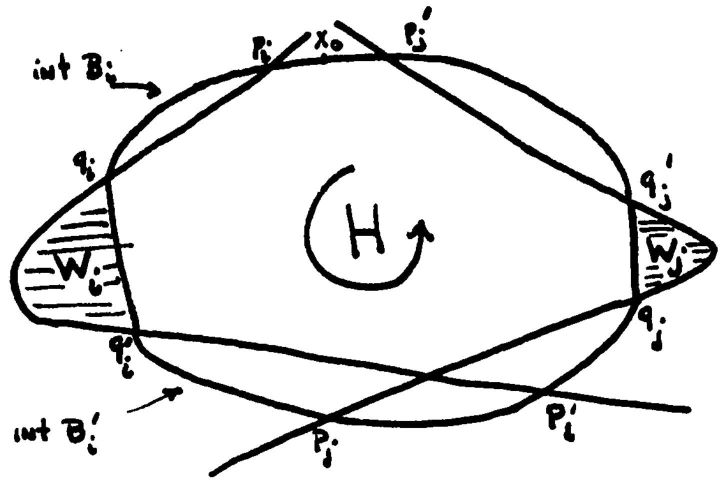


Figure 3.5

Another result which will be useful to us is the following:

3.4. LEMMA: If S is any closed, 4-convex subset of E^2 of type W*, and W_1 and W_2 are the closures of any two components of $S \setminus H$, let $\bar{B}_1$ be either one of the sets B_1 or B_1' and $\bar{B}_2$ either of B_2 or B_2', with $\bar{p}_1, \bar{q}_1$ and $\bar{p}_2, \bar{q}_2$ the corresponding endpoints of $\bar{B}_1 \cap \text{bd } H$ and $\bar{B}_2 \cap \text{bd } H$, respectively. If either

$$\bar{q}_1 \in \text{int } \bar{B}_2$$

or

$$\bar{q}_2 \in \text{int } \bar{B}_1,$$

then

$$\text{int } \bar{B}_1 \cap \text{int } \bar{B}_2 \cap Q = \emptyset.$$

Proof. Suppose $\bar{q}_1 \in \text{int } \bar{B}_2$ and that $q \in \text{int } \bar{B}_1 \cap \text{int } \bar{B}_2 \cap Q$ (the proof for the case $\bar{q}_2 \in \text{int } \bar{B}_1$ is similar). Let $\bar{R}_1$ and $\bar{R}_2$ denote the rays R_1 or R_1' and R_2 or R_2' corresponding to $\bar{q}_1$ and $\bar{q}_2$, respectively. Since $\bar{q}_1 \in \text{int } \bar{B}_2$ there exists a convex neighborhood U_1 of $\bar{q}_1$ such that $U_1 \subset \text{int } \bar{B}_2$. Since

q is on the opposite side of $\bar{R}_1$ as $W_1 \cap U_1$, 2.31 (with $C = W_1 \cap U_1$ and $M_1 = \{q\}$) implies the existance of a point $x_1 \varepsilon W_1 \cap U_1$ such that $x_1 q \not\subset S$. Hence both x_1 and q lie on the opposite side of $\bar{R}_2$ as W_2, so again applying 2.31 (with $C = W_2$ and $M_1 = \{x_1, q\}$) there exists a point $x_2 \varepsilon W_2$ such that $x_1 x_2 \not\subset S$ and $q x_2 \not\subset S$. Because S is closed there exists a neighborhood V of q such that x_1 and x_2 cannot see $v \varepsilon V$. Since q is an lnc point there exist points x_3, x_4 in V such that $x_3 x_4 \not\subset S$. But then $\{x_1, x_2, x_3, x_4\}$ would be a set of four visually independent points of S, denying 4-convexity. Hence, we conclude that int $\bar{B}_1 \cap$ int $\bar{B}_2 \cap Q = \emptyset$. $\Diamond$

We introduce one more concept which will be of use in the proof of the next theorem.

<u>3.5. DEFINITION</u>: If $\mathfrak{F} = \{W_i : i \varepsilon I\}$ is a family of closures of components of $S \setminus H$, then $\{\mathfrak{F}^1, \mathfrak{F}^2, \ldots, \mathfrak{F}^r\}$ is called a <u>convex partition of $\mathfrak{F}$ of order r</u> iff the sets $\mathfrak{F}^1, \mathfrak{F}^2, \ldots, \mathfrak{F}^r$ partition $\mathfrak{F}$ (they are pairwise disjoint and their union is $\mathfrak{F}$) and for each i the set $\bigcup\{W_j : W_j \varepsilon \mathfrak{F}^i\}$ is a relatively convex subset of S.

It is clear that it is pertinent to our problem to find a convex partition of finite order for the family $\mathfrak{F}$ of closures of components of $S \setminus H$, for if $D_i = \text{conv}\{W_j : W_j \varepsilon \mathfrak{F}^i\}$, $i = 1, 2, \ldots, r$ it would follow that $D_i \subseteq S$, and since

$$S = H \cup (\textstyle\bigcup_{i=1}^{r} D_i),$$

S would be the union of $r + 1$ convex sets.

3.6. THEOREM: If S is a closed 4-convex set in E^2 which has at least one lnc point in the kernel of S, then S is the union of 8 or fewer convex sets.

Proof. By previous observations, we may assume that S is of type W*. By hypothesis, $S = S_q$ for some point $q \in Q$. Let $<$ be the linear ordering on bd H induced by q, and, as before, assume that the lnc points occur in the order

$$q \leq q_1 < q_1' \leq q_2 < q_2' \leq \cdots < q_{n-1}' \leq q_n,$$

with $q = q_n'$. Define inductively the integers $n_1, n_2, \ldots, n_k$ as follows: Let n_1 be the largest integer such that the family

$$\mathcal{F}_1 = \{W_i : 1 \leq i < n_1\}$$

has a convex partition $\{\mathcal{F}_1^1, \mathcal{F}_1^2, \mathcal{F}_1^3\}$ of order 3. Let n_2 be the largest integer such that the family

$$\mathcal{F}_2 = \{W_i : 1 \leq i < n_2\} \setminus \{W_{n_1}\}$$

has a convex partition $\{\mathcal{F}_2^1, \mathcal{F}_2^2, \mathcal{F}_2^3\}$ of order 3. In general, having defined $n_1, n_2, \ldots, n_j$, define n_{j+1} as the largest integer such that the family

$$\mathcal{F}_{j+1} = \{W_i : 1 \leq i < n_{j+1}\} \setminus \{W_{n_1}, W_{n_2}, \ldots, W_{n_j}\}$$

has a convex partition $\{\mathcal{F}_{j+1}^1, \mathcal{F}_{j+1}^2, \mathcal{F}_{j+1}^3\}$ of order 3. Since there are only finitely many sets W_i, the process ends in a finite number of steps and we let n_k denote the last such integer.

We shall prove first that for each $n_j < n$ there exist integers $r < s < t$ in the set $\hat{n}_j \setminus \{n_1, \ldots, n_j\}$ such that W_{n_j} cannot see W_r, W_s, or W_t via S (that is, there is a point in W_{n_j} which cannot see via S some point in W_i, for $i = r, s, t$).

Suppose on the contrary, that given such integers r, s, and t, W_{n_j} can see at least one of W_r, W_s, or W_t. Choose the three largest integers $r < s < t$ in the set $\hat{n}_j \setminus \{n_1, \ldots, n_j\}$; then W_{n_j} can see W_u via S for either $u = r$, $u = s$, or $u = t$. If W_{n_j} can see W_t, we may assume $W_t \varepsilon \mathfrak{F}^1_j$, and consider any other set $W_u \varepsilon \mathfrak{F}^1_j$. Since W_u can also see W_t, then by 3.3 $q'_u \leq p_t$, $p'_u \leq q_t$, $q'_t \leq p_{n_j}$, and $p'_t \leq q_{n_j}$. Therefore, $q'_u < q'_t \leq p_{n_j}$ and $p'_u \leq q_t < q_{n_j}$ so that W_u can see W_{n_j}. That is, W_{n_j} can see all the members of $\mathfrak{F}^1_j$ via S. If W_{n_j} cannot see W_t then we have the cases (1) W_{n_j} can see W_s and (2) W_{n_j} cannot see W_s and therefore sees W_r via S. In case (1), assume $W_s \varepsilon \mathfrak{F}^1_j$. At most one W_u for $u < s$ exists such that W_u cannot see W_{n_j}, for if $u < v$ and both W_u and W_v cannot see W_s then, since $q'_u < q'_v < q'_s \leq p_{n_j}$, we must have both $p'_u > q_s$ and $p'_v > q_s$ so that

$$q'_u < q'_v \leq q_r < q_s < p'_u$$

and

$$q'_v < q_s < p'_v$$

which implies that $q_s \varepsilon \operatorname{int} B'_u \cap \operatorname{int} B'_v$ and $q'_v \varepsilon \operatorname{int} B'_u$, contradicting 3.4. Suppose $W_u \varepsilon \mathfrak{F}^2_j$. Then W_{n_j} can see all other W_v for $v < s$ and hence, if $W_t \varepsilon \mathfrak{F}^1_j$, W_{n_j} can see all the members of $\mathfrak{F}^3_j$; if $W_t \notin \mathfrak{F}^1_j$ then W_t can see all the members of $\mathfrak{F}^1_j$. In case (2), our basic assumption regarding W_{n_j} implies that since W_{n_j} cannot see W_s nor W_t, it must see all W_u for $u \leq r$. Suppose $W_r \varepsilon \mathfrak{F}^1_j$. If both W_s and W_t are members of $\mathfrak{F}^1_j$ then W_{n_j} can see all the members of $\mathfrak{F}^2_j$, and if neither W_s nor W_t are members of $\mathfrak{F}^1_j$ then W_{n_j} can see all the members of $\mathfrak{F}^1_j$. If either $W_s \notin \mathfrak{F}^1_j$ or $W_t \notin \mathfrak{F}^1_j$, then we may assume W_s (or W_t) belongs to $\mathcal{F}_j^2$ and hence W_{n_j} can see all the members of $\mathcal{F}_j^3$. In all cases, our assumption has led us to the assertion that W_{n_j} can see all the members of $\mathcal{F}_j^i$ ($i = 1, 2$, or 3). But then it follows that $\{\mathcal{F}_j^i \cup \{W_{n_j}\}, \mathcal{F}_j^{i+1}, \mathcal{F}_j^{i+2}\}$ is a convex partition of order 3 for $\mathcal{F}_j \cup \{W_{n_j}\}$, denying the maximal property of n_j.

Therefore, given j there exist integers $r < s < t$ in the set $\hat{n}_j \setminus \{n_1, \ldots, n_j\}$ such that W_{n_j} cannot see W_r, W_s, nor W_t. The implication is now that $q_t \in \operatorname{int} B_{n_j}$. For, if $q_s' \leq p_{n_j}$ then $q_r' < q_s' \leq p_{n_j}$ and by 3.3, $p_r' > q_{n_j}$ and $p_s' > q_{n_j}$. Therefore,

$$q_r' < q_s' < q_t' \leq q_{n_j} < p_r'$$

and

$$q_s' < q_t' \leq q_{n_j} < p_s' ,$$

which implies that $q_t' \in \operatorname{int} B_r' \cap \operatorname{int} B_s'$ and $q_s' \in \operatorname{int} B_r'$, denying 3.4. Therefore,

$$p_{n_j} < q_s' \leq q_t < q_{n_j}$$

and hence, $q_t \in \operatorname{int} B_{n_j}$.

Now it can be proved that for each j such that $n_{j+3} < n$ W_{n_j} can see $W_{n_{j+3}}$. Assume otherwise, and that for some $n_{j+3} < n$ either (1) $p_{n_j}' > q_{n_{j+3}}$, or (2) $p_{n_j}' \leq q_{n_{j+3}}$ and, by 3.3, $q_{n_j}' > p_{n_{j+3}}$.

<u>Case 1</u>: $p_{n_j}' > q_{n_{j+3}}$. We consider the two subcases (1.1) $p_{n_{j+3}} \geq q_{n_j}'$ and (1.2) $p_{n_{j+3}} < q_{n_j}'$.

<u>Case 1.1</u>: $p_{n_{j+3}} \geq q_{n_j}'$. Let $t \neq n_i$ ($i = 1, \ldots, n_{j+3}$) be such that $q_t \in \operatorname{int} B_{n_{j+3}}$. Hence,

$$q_{n_j}' \leq p_{n_{j+3}} < q_t < q_{n_{j+3}} < p_{n_j}' ,$$

and since $q_t \neq q_{n_j}'$,

$$q'_{n_j} < q_t < q_{n_{j+3}} < p'_{n_j}.$$

Therefore, $q_t \in \text{int } B'_{n_j} \cap \text{int } B_{n_{j+3}}$ and $q_{n_{j+3}} \in \text{int } B'_{n_j}$, denying 3.4.

<u>Case 1.2</u>: $p_{n_{j+3}} < q'_{n_j}$. Here, we have

$$p_{n_{j+3}} < q'_{n_j} < q'_{n_{j+1}} \leq q_{n_{j+2}} < q_{n_{j+3}} < p'_{n_j}.$$

Hence, $q_{n_{j+2}} \in \text{int } B'_{n_j} \cap \text{int } B_{n_{j+3}}$ and $q_{n_{j+3}} \in \text{int } B'_{n_j}$, denying 3.4.

<u>Case 2</u>: $p'_{n_j} \leq q_{n_{j+3}}$ and $q'_{n_j} > p_{n_{j+3}}$. It follows that $p_{n_{j+2}} \geq q'_{n_j}$, for if $p_{n_{j+2}} < q'_{n_j}$ then

$$p_{n_{j+2}} < q'_{n_j} < q_{n_{j+2}}$$

and

$$p_{n_{j+3}} < q'_{n_j} < q_{n_{j+2}} < q_{n_{j+3}},$$

which implies $q'_{n_j} \in \text{int } B_{n_{j+2}} \cap \text{int } B_{n_{j+3}}$ and $q_{n_{j+2}} \in \text{int } B_{n_{j+3}}$, a contradiction. Hence $p_{n_{j+2}} \geq q'_{n_j} > p_{n_{j+3}}$. Let $t \neq n_i$ $(i = 1,\ldots,j+2)$ be such that $q_t \in \text{int } B_{n_{j+2}}$. Then

$$p_{n_{j+3}} < p_{n_{j+2}} < q_t < q_{n_{j+2}} < q_{n_{j+3}}$$

and therefore $q_t \in \text{int } B_{n_{j+2}} \cap \text{int } B_{n_{j+3}}$, with $q_{n_{j+2}} \in \text{int } B_{n_{j+3}}$.

Thus, the assumption that W_{n_j} cannot see $W_{n_{j+3}}$ via S has led in every case to a denial of 3.4. Therefore, we conclude that W_{n_j} can see $W_{n_{j+3}}$ for each j such that $n_j < n$. It then follows that each of the sets $\bigcup\{W_{n_j} : j \equiv r (\text{mod } 3), n_j < n\}$ for r = 0,1, and 2 (define $W_{n_0} = \emptyset$ for this purpose) is relatively convex in S. Since the convex hull of any relatively convex subset of S can easily be shown to lie in S by virtue of the simple-connectedness of S, define:

$D_1 = H$ $\qquad$ $D_3 = \text{conv}(\bigcup \mathfrak{F}^2_k)$

$D_2 = \text{conv}(\bigcup \mathfrak{F}^1_k)$ $\qquad$ $D_4 = \text{conv}(\bigcup \mathfrak{F}^3_k)$

$D_5 = \mathrm{conv}(\bigcup_{j\equiv 0} W_{n_j}),\ n_j < n$ $\quad D_7 = \mathrm{conv}(\bigcup_{j\equiv 2} W_{n_j}),\ n_j < n$

$D_6 = \mathrm{conv}(\bigcup_{j\equiv 1} W_{n_j}),\ n_j < n$ $\quad D_8 = W_n.$

It then follows that $S = \bigcup_{i=1}^{8} D_i$. ◊

<u>3.7. COROLLARY</u>: Any closed 4-convex subset S of E^2 is the union of 9 or fewer convex sets.

<u>Proof</u>. It is obvious that we may assume that S is connected; suppose first that S is simply-connected. If $q \in Q$ consider S^q and S_q. Then by 2.35 $S' = S_q$ is a closed, simply-connected 4-convex subset of E^2 with $q \in Q' \cap H'$, where Q' is the set of lnc points of S' and K' is the kernel. By 3.6, S_q is the union of 8 convex sets, say $D_1, \ldots, D_8$. By 2.36, S^q is relatively convex, and since S is simply-connected, $D_9 = \mathrm{conv}\, S^q$ is a convex subset of S. Then, $S = \bigcup_{i=1}^{9} D_i$. In the non-simply-connected case, Guay's result in [6] that S is the union of 5 or fewer convex sets may be invoked. ◊

We note in conclusion that our methods make short work of Valentine's theorem. For, if S is a closed 3-convex subset of E^2, it follows that $Q \subset K$ (since $qx \not\subset S$ for $q \in Q$ implies that x cannot see via S any point in some neighborhood U of q, there are two points x_2 and x_3 in U such that $x_2x_3 \not\subset S$ by virtue of q being an lnc point, contradicting 3-convexity). Hence, $H \subset S$ and we may consider the closures of the components $\{W_i : i \in I\}$ in $S \setminus H$. By 2.33, if one of the W_i is not convex then S is the union of two convex sets. As before, we need therefore only consider the case when S is of type W^*. Since $Q \subset K$, we select $q \in Q$ at random and let < order the points of bd H, as before. The previous results 3.3 and 3.4 still apply, so it may be easily proved that for each i, W_i can see W_{i+2} via S. For, if either $q_i' > p_{i+2}$ or $p_i' > q_{i+2}$ then either $p_{i+2} < q_i' \leq q_{i+1} < q_{i+2}$ or $q_i' \leq q_{i+1} < q_{i+2} < p_i'$ and either $q_i' \in \text{int } B_{i+2}$ or $q_{i+2} \in \text{int } B_i'$. But in either case it follows that an lnc point falls outside the kernel. Define $r = [n/2]$, where $n = |Q|$, and put

$$D_1' = \text{conv}(\bigcup_{i=1}^{r} W_{2i-1})$$

$$D_2' = \text{conv}(\bigcup_{i=1}^{r} W_{2i})$$

and, if n is odd,

$$D_3' = W_n.$$

It follows that each D_j' is a convex subset of S. Then let D_j denote any maximal convex subset of S containing D_j'. Since $Q \subset K$, $H = \text{conv } Q \subset \text{conv } K = K$. Recall that K is the intersection of all maximal convex subsets of S; then $K \subset D_j$ and therefore $H \subset D_j$. Therefore, $S = D_1 \cup D_2$ if n is even and $S = D_1 \cup D_2 \cup D_3$ if n is odd. That is, S is the union of 3 convex sets (2 if n is even), which is the substance of Valentine's theorem [17]. For, 2.27 extends this result to closed sets and to sets with $|Q| = \infty$ (where S is the union of two convex sets), as in Valentine's theorem.

LIST OF REFERENCES

1. J. E. Allen, Starlike and Inverse Starlike Sets, Ph.D. thesis, Oklahoma State University (1963).

2. H. Brunn, Über Ovale und Eiflächen (Inauguraldiss.), 1-42, Munich (1887).

3. ______, Über Kerneigebiete, Math. Ann., (1913), 436-440.

4. L. W. Danzer, B. Grünbaum, and V. Klee, Helly's theorem and its relatives, Proc. Symposia in Pure Mathematics, 7 (1963), 101-180.

5. B. Grünbaum and T. S. Motzkin, On components in some families of sets, Proc. Amer. Math. Soc., 12 (1967), 607-613.

6. M. Guay, Planar sets having Property P^n, Ph.D. thesis, Michigan State University (1967).

7. W. R. Hare and J. W. Kenelly, Intersections of Maximal Starshaped Sets, Proc. Amer. Math. Soc., 19 (1968), 1299-1302.

8. A. Horn and F. A. Valentine, Some Properties of L Sets in the Plane, Duke Math. J., 16 (1949), 131-140.

9. D. C. Kay and M. D. Guay, Convexity and a certain Property P_m, Israel Journal of Mathematics 8, Number 1, (1970), 39-52.

10. ______, On sets having finitely many points of Local Nonconvexity and Property P_m, to appear in the Israel Journal of Math.

11. J. L. Kelley and I. Namioka, Linear Topological Spaces, D. Van Nostrand Company, Inc., Princeton, New Jersey, (1963).

12. C. F. Koch and J. M. Marr, A characterization of unions of two starshaped sets, Proc. Amer. Soc., 17 (1966), 1341-1343.

13. C. Kuratowski, Topologie I, 2-nd ed. Monogr. Mat., 20, P. W. N. Warsaw, (1948).

14. D. G. Larman, Helly type properties of unions of convex sets, Mathematika 15 (1968), 53-59.

15. W. L. Stamey and J. M. Marr, Unions of two convex sets, Canadian J. of Math., 15 (1963), 152-156.

16. F. A. Toranzos, Radial functions of convex and starshaped bodies, Amer. Math. Monthly 74 (1967), 278-280.

17. F. A. Valentine, A three point convexity property, Pacific J. Math., 7 (1957) 1227-1235.

18. ______________, Convex Sets, McGraw-Hill Book Co., New York, (1964).

19. ______________, Local Convexity and L_n Sets, Proc. Amer. Math. Soc., 16 (1965), 1305-1310.