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TWO-BOUNDARY CENTRALIZER ALGEBRAS FOR $\mathfrak{q}(n)$

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Contents

Abstract

We define the degenerate two boundary affine Hecke-Clifford algebra $\mathcal{H}_d$, and show it admits a well-defined $\mathfrak{q}(n)$-linear action on the tensor space $M \otimes N \otimes V^{\otimes d}$, where V is the natural module for $\mathfrak{q}(n)$, and M, N are arbitrary modules for $\mathfrak{q}(n)$, the Lie superalgebra of Type Q. When M and N are irreducible highest weight modules parametrized by a staircase partition and a single row, respectively, this action factors through a quotient of $\mathcal{H}_d$. Our second goal is to directly construct modules for this quotient, $\mathcal{H}_d^p$, using combinatorial tools such as shifted tableaux and the Bratteli graph. These modules belong to a family of modules which we call calibrated. Using the relations in $\mathcal{H}_d^p$, we also classifiy a specific class of calibrated modules. This result provides connection to a Schur-Weyl type duality: the irreducible summands of $M \otimes N \otimes V^{\otimes d}$ coincide with the combinatorial construction.

Chapter 1

Introduction

In the early twentieth century, Schur studied the actions of the general linear group, $GL(V)$, and the symmetric group on a tensor space $V^{\otimes d}$. The two actions fully centralize each other. This is now known as Schur-Weyl duality and it provides a powerful link between the representation theories of these two groups. For example, the finite dimensional irreducible representations of $GL(V)$ which occur as summands of $V^{\otimes d}$ are in bijection with the finite dimensional irreducible representations of the symmetric group, whenever $\dim V \geq d$. Schur-Weyl duality continues to be studied and has been generalized to many other settings.

For example, following Schur's work, given a finite dimensional module M and the natural module V for the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, Arakawa-Suzuki [1] studied an action of the affine Hecke algebra $\mathcal{H}^{\text{aff}}$ on $M \otimes V^{\otimes d}$ which centralizes the action of $\mathfrak{gl}_n(\mathbb{C})$. As a result, there exists a family of functors from the category of finite-dimensional $\mathfrak{gl}_n(\mathbb{C})$-modules to the category of finite-dimensional $\mathcal{H}^{\text{aff}}$-modules.

An analogue of $\mathcal{H}^{\text{aff}}$, called the degenerate two-boundary braid algebra $\mathcal{G}_d$, was studied by Daugherty [6]. The quantum version of this algebra has a diagrammatic presentation in [7] . In particular, given any two finite dimensional $\mathfrak{gl}_n(\mathbb{C})$-modules M, N, there exists a well defined action of $\mathcal{G}_d$ on the tensor module $M \otimes N \otimes V^{\otimes d}$. As a special case, one can choose M, N to be simple modules parametrized by rectangular Young diagrams as explained in Section 2.4 of this thesis. Daugherty then defined a quotient $\mathcal{H}_d^{\text{ext}}$ of $\mathcal{G}_d$, under extra relations dependent on the Young diagrams. It follows that the action of $\mathcal{G}_d$ satisfies the extra relations and factors through the quotient $\mathcal{H}_d^{\text{ext}}$. Moreover, Daugherty constructed irreducible $\mathcal{H}_d^{\text{ext}}$-modules using combinatorial tools, and showed that irreducible summands of $M \otimes N \otimes V^{\otimes d}$ are isomorphic to these combinatorially constructed modules.

A super (i.e. $\mathbb{Z}_2$-graded) analogue of the above result was developed by the author in [18]. This was done by studying the connection between the representation theory of $\mathcal{H}_d^{\text{ext}}$ and the representation theory of the general linear Lie superalgebras $\mathfrak{gl}_{n|m}(\mathbb{C})$. Similar to the representation theory of $\mathfrak{gl}_n(\mathbb{C})$, polynomial $\mathfrak{gl}_{n|m}(\mathbb{C})$-representations are also controlled by combinatorial tools, in the sense that characters of polynomial representations are given by hook Schur functions.

In particular, given finite dimensional supermodules M, N for the general linear Lie superalgebra $\mathfrak{gl}_{n|m}(\mathbb{C})$, $M \otimes N \otimes V^{\otimes d}$ is naturally a module for $\mathfrak{gl}_{n|m}(\mathbb{C})$. An action of $\mathcal{G}_d$ on $M \otimes N \otimes V^{\otimes d}$ can be defined using a certain Casimir element, and this action commutes with the action of $\mathfrak{gl}_{n|m}(\mathbb{C})$. Moreover, when M and N are irreducible representations whose highest weights are given by rectangular Young diagrams, the defining relations for $\mathcal{H}_d^{\text{ext}}$, as a quotient of $\mathcal{G}_d$, are also satisfied, and this induces a further action of $\mathcal{H}_d^{\text{ext}}$. We therefore recover the irreducible $\mathcal{H}_d^{\text{ext}}$-modules in [6] by studying irreducible summands of $M \otimes N \otimes V^{\otimes d}$ as right $\mathcal{H}_d^{\text{ext}}$-modules, and they coincide with the combinatorial construction in [6].

On the other hand, Hill-Kujawa-Sussan [10] studied the Type Q version of the construction in Arakawa-Suzuki [1]. In particular, the Type Q analogue of the affine Hecke algebra is the affine Hecke-Clifford algebra $\mathcal{H}_{\mathrm{Cl}}^{\mathrm{aff}}$, whose underlying vector space is the tensor product between $\mathcal{H}^{\mathrm{aff}}$ and the Clifford algebra. Given any finite dimensional module M for the Type Q Lie superalgebra $\mathfrak{q}(n)$, there is a well defined action of $\mathcal{H}_{\mathrm{Cl}}^{\mathrm{aff}}$ on $M \otimes V^{\otimes d}$, which commutes with the $\mathfrak{q}(n)$-action. Similar to [1], there is a family of functors from the category of finite-dimensional $\mathfrak{q}(n)$-modules to the category of finite-dimensional $\mathcal{H}_{\mathrm{Cl}}^{\mathrm{aff}}$-modules. In addition, Hill-Kujawa-Sussan constructed $\mathcal{H}_{\mathrm{Cl}}^{\mathrm{aff}}$-modules using combinatorial tools such as shifted Young tableaux.

This thesis is a generalization of Hill-Kujawa-Sussan [10] to the two boundary case, or alternatively, a generalization of [6] and [18] to Type Q. In Section 3.1, we first define the degenerate two boundary affine Hecke-Clifford algebra $\mathcal{H}_d$ in the spirit of the two boundary Hecke algebra $\mathcal{G}_d$. Similar to the case of $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{gl}_{n|m}(\mathbb{C})$, there is an odd Casimir element $\Omega \in \mathfrak{q}(n) \otimes \mathfrak{q}(n)$ which induces a $\mathfrak{q}(n)$-linear action on $M \otimes N$ for any $\mathfrak{q}(n)$-modules M and N. We have our first main result in Theorem 3.7:

Theorem 1.1. *Let $d \geq 0$ and M, N be arbitrary modules for $\mathfrak{q}(n)$. There is a well-defined algebra homomorphism $\mathcal{H}_d \to \mathrm{End}_{\mathfrak{q}(n)}(M \otimes N \otimes V^{\otimes d})$.*

We then focus our study on the case when M and N are polynomial modules. By definition these are modules which occur as direct summands of $V^{\otimes e}$ for some $e \geq 0$. Polynomial modules are semisimple as explained in Section 2.3, and they are closed under tensor product. According to [17], irreducible polynomial modules of $\mathfrak{q}(n)$ are parametrized by strict partitions defined in Section 2.4, and their tensor product decomposes into polynomial representations in a fashion controlled by combinatorics of the Schur P-function [14]. We further choose M and N so that the irreducible summands occuring in the decomposition of $M \otimes N$ have the smallest possible multiplicity. By a combinatorial result developed by Bessenrodt [2], this is true when $M = L(\alpha)$ is parametrized by a staircase shape α and $N = L(\beta)$ is parametrized by a single row β. In this case any irreducible summand occurs with multiplicity 2 (see Section 2.4). A similar result is true for $W \otimes V$ when W is any simple polynomial module and N is the natural module. These multiplicity results allow us to describe summands of $M \otimes N \otimes V^{\otimes d}$ using the Bratteli graph introduced in Section 5.1 .

If β has p total boxes, we define a quotient $\mathcal{H}_d^p$ of $\mathcal{H}_d$ under extra relations involving n and p. A key formula in Proposition 4.6 establishes a connection between Ω and certain even central elements in $\mathfrak{q}(n)$, whose actions on a simple polynomial module can be easily calculated. Using this formula, we show the above action factors through the quotient and we have the following result in Theorem 4.11.

Theorem 1.2. *There is a well defined action*

$$\rho : \mathcal{H}_d^p \to \mathrm{End}_{\mathfrak{q}(n)}(L(\alpha) \otimes L(\beta) \otimes V^{\otimes d}).$$

Similar to Daugherty and Hill-Kujawa-Sussan, we also construct $\mathcal{H}_d^p$-modules in Section 5.2 using combinatorial rules. In Lemma 2.3, we reformulated the decomposition formula for Schur P-fuctions given by Stembridge [14], and define a Bratteli graph $\Gamma_{\alpha,\beta}$ whose vertices are strict partitions and directed edges are defined representation theoretically. Let λ be a fixed vertex in this graph, and denote Γ^λ to be the set of all paths from α to λ. Let $f : \Gamma^\lambda \to \mathbb{C}^\times$ be any function with the condition (5.2.1), such as the one in Definition 5.10. We define a vector space $\mathcal{D}_f^\lambda$, whose basis is given in terms of paths in the Bratteli graph and the generators of $\mathcal{H}_d^p$ acts by explicit formulas. In particular, the graph and these formulas depend on α and β, which are determined by the parameters n and p in the defining relations of $\mathcal{H}_d^p$. We have the following result in Theorem 5.12 and Theorem 5.22:

Theorem 1.3. *The vector space $\mathcal{D}_f^\lambda$ admits a well-defined action of $\mathcal{H}_d^p$, and is irreducible.*

Following the spirit of Hill-Kujawa-Sussan [10] and Wan [16], we study $\mathcal{H}_d^p$-modules on which certain polynomial generators act by eigenvalues. We then classify a specific family of these $\mathcal{H}_d^p$-modules, where the eigenvalues are assumed to be given by the combinatorial data in the Bratteli graph. In particular, for each path T and integer i, the eigenvalue $\kappa_T(i)$ is defined in Lemma 5.4. We then focus on the case when the parameter n in the defining relations of $\mathcal{H}_d^p$, is even. Equivalently, when the number of nonzero rows in α (and hence λ) is even, we have the following result in Proposition 6.1 and Theorem 6.2, which shows that these eigenvalues determine the simple modules defined above.

Theorem 1.4. *When the number of nonzero rows in α is even, given an $\mathcal{H}_d^p$-module $\mathcal{W}^\lambda$, if it admits a free Cl_{d+1}-module structure $\mathcal{W}^\lambda = \bigoplus_{T\in\Gamma^\lambda} Cl_{d+1}v_T$, where each v_T is homogeneous and $z_i.v_T = \kappa_T(i)v_T$, $0 \le i \le d$, then $\mathcal{W}^\lambda \simeq \mathcal{D}_f^\lambda$ for some f.*

In the case when n is odd, we expect a similar result to hold due to the theory of taking Clifford twists of a category as mentioned in [4, 9].

Similar to Schur-Weyl duality, a $\mathfrak{q}(n)$-version of the double centralizer theorem in [17] states that $L(\alpha)\otimes L(\beta)\otimes V^{\otimes d}$ decomposes into bimodules for $\mathfrak{q}(n)$ and its centralizer, $\mathcal{Z}_d = \mathrm{End}_{\mathfrak{q}(n)}(L(\alpha)\otimes L(\beta)\otimes V^{\otimes d})$. Since the image

$\rho(\mathcal{H}_d^p)$ in Theorem 1.2 is a subalgebra of $\mathcal{Z}_d$, any irreducible $\mathcal{Z}_d$-summand $\mathcal{L}^\lambda$ of $L(\alpha) \otimes L(\beta) \otimes V^{\otimes d}$ admits a module struture for $\rho(\mathcal{H}_d^p)$. In Theorem 6.5 and Theorem **??**, we show we can recover the combinatorially constructed irreducible modules $\mathcal{D}_f^\lambda$ via this restriction:

Theorem 1.5. *When the number of nonzero rows in α is even, there exists an isomorphism* $\operatorname{Res}^{\mathcal{Z}_d}_{\rho(\mathcal{H}_d^p)} \mathcal{L}^\lambda \simeq \mathcal{D}_f^\lambda$ *for some choice of f.*

Chapter 2

Preliminaries

2.1. Lie Superalgebras

The study of Lie algebras dates back to the ninteeth century. The Lie algebra $\mathfrak{gl}(n)$ is a noncommutative, nonassociative algebra consisting of n-by-n matrices, whose algebra operation is given by the commutator backet. In more generality, Lie algebras are vector spaces imposed with a bilinear map, called the Lie bracket, which is antisymmetric and satisfies the Jacobi identity. Mathematicians such as Cartan and Killing studied its structure using Cartan subalgebras, root space decomposition and the Killing form, and they were able to classify semisimple Lie algebras. Their representation theory was later studied using highest weight theory.

This thesis is based on the "super" analogue of the above theory, the representation theory of Lie superalgebras. In [11], Kac studied such algebras $\mathfrak{g}$ that are $\mathbb{C}$-vector spaces with a $\mathbb{Z}_2$ - grading:

$$\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}.$$

We shall call $\mathfrak{g}_{\overline{0}}$ the even part of $\mathfrak{g}$ and $\mathfrak{g}_{\overline{1}}$ the odd part of $\mathfrak{g}$. An element is homogenous if it is purely in $\mathfrak{g}_{\overline{0}}$ or $\mathfrak{g}_{\overline{1}}$. We denote by $\overline{x} \in \mathbb{Z}_2$ the degree of a homogeneous element $x \in \mathfrak{g}$.

It has an algebra operation called the Lie superbracket:

$$[-,-]:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$$

and the bracket satisfied the following:

1) It is compatible with the grading, i.e. $[\mathfrak{g}_{\bar{i}},\mathfrak{g}_{\bar{j}}]\subset\mathfrak{g}_{\overline{i+j}}$.

2) It has "super" antisymmetry: $[x,y]=-(-1)^{\overline{x}\cdot\overline{y}}[y,x]$.

3) It satisfies the super version of the Jacobi identity: if $x\in\mathfrak{g}_{\overline{x}}$, $y\in\mathfrak{g}_{\overline{y}}$, $z\in\mathfrak{g}_{\overline{z}}$:

$$(-1)^{\overline{x}\cdot\overline{y}}[[x,y],z]+(-1)^{\overline{y}\cdot\overline{z}}[[y,z],x]+(-1)^{\overline{z}\cdot\overline{x}}[[z,x],y]=0.$$

A Lie superalgebra homomorphism $\rho:\mathfrak{g}\to\mathfrak{g}'$ between two Lie superalgebras $\mathfrak{g}$ and $\mathfrak{g}'$, is a linear map which preserves the Lie superbracket and grading, i.e. $\rho(\mathfrak{g}_{\bar{i}})\subset\mathfrak{g}'_{\bar{i}}, i=0,1$, and

$$[\rho(x),\rho(y)]=\rho([x,y]),\forall x,y\in\mathfrak{g}.$$

A representation or module V for the Lie superalgebra $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space V, with a Lie superalgebra homomorphism

$$\phi:\mathfrak{g}\to\mathrm{End}_{\mathbb{C}}(V)$$

where $\mathrm{End}_{\mathbb{C}}(V)$ has a Lie superalgebra structure given by the super commutator $[f,g]:=f\circ g-(-1)^{\overline{f}\cdot\overline{g}}g\circ f$, for homogeneous elements $f,g\in\mathrm{End}(V)$. Here, composition is as linear maps, and the grading on $\mathrm{End}(V)$ is defined as follows: $f\in\mathrm{End}(V)$ is even if $f(V_{\bar{0}})\subset V_{\bar{0}}$ and $f(V_{\bar{1}})\subset V_{\bar{1}}$, and it is odd if $f(V_{\bar{0}})\subset V_{\bar{1}}$

and $f(V_{\bar{1}}) \subset V_{\bar{0}}$. For $x \in \mathfrak{g}$ and $v \in V$, we will sometimes denote $\phi(x)(v)$ as $x.v$.

In this thesis we shall only be concerned with graded representations. In particular, a representation is irreducible if it has no proper graded subrepresentations.

Given two modules M, N for a Lie superalgebra $\mathfrak{g}$, a homogeneous module homomorphism is a linear map $\phi : M \to N$ which is either even (i.e. preserves the grading) or odd (i.e. reverses the grading,) with $\overline{\phi}$ defined according to its parity, such that

$$\phi(x.m) = (-1)^{\overline{\phi}\cdot\overline{x}} x.\phi(m).$$

We denote by $\mathrm{End}_{\mathfrak{g}}(M, N)$ the space spanned by homogeneous module homomorphisms from M to N, and $\mathrm{End}_{\mathfrak{g}}(M) = \mathrm{End}_{\mathfrak{g}}(M, M)$.

2.2. $\mathfrak{q}(n)$ and its representations

We will use [17] as reference throughout the rest of the chapter. The Lie superalgebra of type Q, denoted by $\mathfrak{q}(n)$, is defined as follows:

$$\mathfrak{q} = \mathfrak{q}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} | A, B \in \mathrm{Mat}_{n,n}(\mathbb{C}) \right\},$$

where $\mathrm{Mat}_{n,n}(\mathbb{C})$ is the set of n by n matrices with entries in $\mathbb{C}$. The $\mathbb{Z}_2$ -grading is given via

$$\mathfrak{q}_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} | A \in \mathrm{Mat}_{n,n}(\mathbb{C}) \right\},$$
$$\mathfrak{q}_{\bar{1}} = \left\{ \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} | B \in \mathrm{Mat}_{n,n}(\mathbb{C}) \right\}.$$

The Lie superbracket is the super commutator bracket, similar to that defined in the previous section,

$$[x, y] = xy - (-1)^{\bar{x} \cdot \bar{y}} yx,$$

for homogeneous $x \in \mathfrak{q}_{\bar{x}}$, $y \in \mathfrak{q}_{\bar{y}}$, and the multiplication is as matrices. It can be checked that all the axioms in the previous section are satisfied.

For future convenience let us compute the superbracket on a basis. Let $E_{ij} \in \mathrm{Mat}_{n,n}(\mathbb{C})$ be the matrix unit with a single 1 in the i, j-position and 0's elsewhere. Let

$$e_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & E_{ij} \end{pmatrix}, \tag{2.2.1}$$

$$f_{ij} = \begin{pmatrix} 0 & E_{ij} \\ E_{ij} & 0 \end{pmatrix}, \tag{2.2.2}$$

then $\{e_{ij}, f_{ij}\}_{1 \le i,j \le n}$ constitutes a homogeneous basis of $\mathfrak{q}$. It can be checked that the superbracket is according to the following formulas:

$$[e_{ij}, e_{pq}] = \delta_{jp}e_{iq} - \delta_{iq}e_{pj} \tag{2.2.3}$$

$$[e_{ij}, f_{pq}] = \delta_{jp}f_{iq} - \delta_{iq}f_{pj} \tag{2.2.4}$$

$$[f_{ij}, e_{pq}] = \delta_{jp}f_{iq} - \delta_{iq}f_{pj} \tag{2.2.5}$$

$$[f_{ij}, f_{pq}] = \delta_{jp}e_{iq} + \delta_{iq}e_{pj} \tag{2.2.6}$$

We will only check the last identity, and the rest of the calculations are similar:

$$\begin{aligned}
[f_{ij}, f_{pq}] &= \left[\begin{pmatrix} 0 & E_{ij} \\ E_{ij} & 0 \end{pmatrix}, \begin{pmatrix} 0 & E_{pq} \\ E_{pq} & 0 \end{pmatrix}\right] \\
&= \begin{pmatrix} E_{ij}E_{pq} & 0 \\ 0 & E_{ij}E_{pq} \end{pmatrix} + \begin{pmatrix} E_{pq}E_{ij} & 0 \\ 0 & E_{pq}E_{ij} \end{pmatrix} \\
&= \delta_{jp}e_{iq} + \delta_{iq}e_{pj}
\end{aligned}$$

We will also denote e_{ii} by h_i and f_{ii} by $\overline{h}_i$.

Define

$$\mathfrak{h}_{\overline{0}} = \mathbb{C} - \text{span}\{h_i\}_{1 \leq i \leq n}.$$

Similarly, define

$$\mathfrak{h}_{\overline{1}} = \mathbb{C} - \text{span}\{\overline{h_i}\}_{1 \leq i \leq n}.$$

The Cartan subalgebra of $\mathfrak{q}(n)$ is the Lie supersubalgebra

$$\mathfrak{h} = \mathfrak{h}_{\overline{0}} \oplus \mathfrak{h}_{\overline{1}}.$$

Using equations (2.2.3) – (2.2.6), one can check $\mathfrak{h}$ is a Lie subsuperalgebra. For $\alpha \in \mathfrak{h}_{\overline{0}}^*$, define the root space

$$\mathfrak{q}_\alpha = \{x \in \mathfrak{q} \mid [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}_{\overline{0}}\}.$$

Let $\Delta = \{\alpha \in \mathfrak{h}_{\overline{0}}^*/\{0\}, \mathfrak{q}_\alpha \neq 0\}$. Then $\Delta = \{\epsilon_i - \epsilon_j\}_{1 \leq i,j \leq n}$ where

$$\epsilon_i : \mathfrak{h}_{\overline{0}} \to \mathbb{C}, \qquad \epsilon_i(d) = d_i \tag{2.2.7}$$

and d_i is the i-th diagonal entry in $d \in \mathfrak{h}_{\overline{0}}$. Define the set of positive roots $\Delta^+ = \{\epsilon_i - \epsilon_j\}_{1 \leq i < j \leq n}$ and the set of negative roots $\Delta^- = \{\epsilon_i - \epsilon_j\}_{1 \leq j < i \leq n}$. Define $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{q}_\alpha$ and $\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{q}_\alpha$. Unlike semisimple Lie algebras, the root spaces $\mathfrak{q}_\alpha$ are two-dimensional, which suggests that the representation theory for $\mathfrak{q}(n)$ will be rather different than the classical case.

The tensor algebra $T(\mathfrak{q})$ is the associative superalgebra

$$T(\mathfrak{q}) \simeq \mathfrak{q} \oplus (\mathfrak{q} \otimes \mathfrak{q}) \oplus (\mathfrak{q} \otimes \mathfrak{q} \otimes \mathfrak{q}) \oplus \cdots,$$

where multiplication is given by concatenation and the $\mathbb{Z}_2$ - grading is defined as $\overline{x_1 \otimes \cdots \otimes x_d} = \overline{x_1} + \cdots + \overline{x_d}$. The universal enveloping superalgebra $U(\mathfrak{q})$ is the associative superalgebra that is the quotient of $T(\mathfrak{q})$ under the relations $[x, y] = xy - (-1)^{\overline{x} \cdot \overline{y}} yx$, for homogeneous elements $x, y \in \mathfrak{q}$.

Just as for Lie algebras, a $\mathfrak{q}$-module M can be regarded as a $U(\mathfrak{q})$-module.

Vice versa, any $U(\mathfrak{q})$-module is a $\mathfrak{q}$-module based on the usual imbedding $\mathfrak{q} \subset U(\mathfrak{q})$ into the one tensor subspace. Hence we will not distinguish between a $\mathfrak{q}$-module and a $U(\mathfrak{q})$-module.

There is a map of superalgebras:

$$\begin{aligned} \Delta : U(\mathfrak{q}) &\to U(\mathfrak{q}) \otimes U(\mathfrak{q}) \\ x &\mapsto x \otimes 1 + 1 \otimes x, \qquad \forall x \in \mathfrak{g} \end{aligned}$$

where $U(\mathfrak{q}) \otimes U(\mathfrak{q})$ is a superalgebra whose the multiplication obeys

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = (-1)^{\overline{x_2} \cdot \overline{y_1}} x_1 x_2 \otimes y_1 y_2.$$

For $U(\mathfrak{q})$-supermodules M and N, $U(\mathfrak{q}) \otimes U(\mathfrak{q})$ acts on $M \otimes N$ via the following: for $x \otimes y \in U(\mathfrak{q}) \otimes U(\mathfrak{q})$ and $m \in M$, $n \in N$,

$$(x \otimes y).(m \otimes n) = (-1)^{\overline{y} \cdot \overline{m}} (x.m) \otimes (y.n).$$

Therefore we can define the action of $U(\mathfrak{q})$ on $M \otimes N$ as

$$\begin{aligned} x.(m \otimes n) = \Delta(x).(m \otimes n) &= (x \otimes 1 + 1 \otimes x).(m \otimes n) \\ &= (x.m) \otimes n + (-1)^{\overline{x} \cdot \overline{m}} m \otimes (x.n) \end{aligned}$$

for all homogenous $x \in \mathfrak{q}, m \in M, n \in M$. One can check the above satisfy the axiom $(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta$, therefore, tensor products of three modules or beyond can be defined as taking successive tensor products: $M \otimes N \otimes P = (M \otimes N) \otimes P \simeq M \otimes (N \otimes P)$.

As mentioned in [17, Section 2.1.6], a class of irreducible modules for

$\mathfrak{q}(n)$ can be constructed using Verma modules, similar to the theory for the Lie algebra $\mathfrak{gl}(n)$ or the general linear Lie superalgebra $\mathfrak{gl}(n|m)$. Recall $\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} \mathfrak{q}_\alpha$ and $\mathfrak{n}^- = \bigoplus_{\alpha\in\Delta^-} \mathfrak{q}_\alpha$. It is known that $U(\mathfrak{q})$ has a triangular decomposition $U(\mathfrak{q}) \simeq U(\mathfrak{n}^+) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^-)$. Recall $\epsilon_i \in \mathfrak{h}^*$ in (2.2.7). We identify the set $P = \bigoplus_{i=1}^{n} \mathbb{Z}\epsilon_i$ with $\mathbb{Z}^n$ via

$$\lambda = \lambda_1\epsilon_1 + \cdots \lambda_n\epsilon_n \mapsto (\lambda_1, \ldots, \lambda_n). \tag{2.2.8}$$

For $\lambda \in P$, define a 1-dimensional $\mathfrak{h}_{\bar{0}}$-module spanned by v_λ, where $h_i.v_\lambda = \lambda_i v_\lambda$. The induction $C(\lambda) = U(\mathfrak{h}) \otimes_{U(\mathfrak{h}_{\bar{0}})} \mathbb{C}v_\lambda$ is an $\mathfrak{h}$-module, and $C(\lambda)$ has a unique irreducible quotient $E(\lambda)$.

Let $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$. With a slight abusive use of notion let $E(\lambda)$ be the $\mathfrak{b}$-module where $\mathfrak{n}^+$ acts as zero. Define the Verma module $M(\lambda) = U(\mathfrak{q}) \otimes_{U(\mathfrak{b})} E(\lambda)$ to be the Verma module associate to the sequence λ. By [8], $M(\lambda)$ has a unique irreducible quotient $L(\lambda)$ for $\lambda \in \mathbb{Z}^n$. Moreover, if $\exists \mu \in \mathfrak{h}^*$, $w_\mu \in L(\lambda)$ such that $h_i w_\mu = \mu_i w_\mu$, $1 \le i \le n$, we will call w_μ a vector of weight μ, and one can impose a partial order on the set of weights in $L(\lambda)$. By construction, λ is highest according to this partial order, and if w_λ is a vector of weight λ, $\mathfrak{n}^+.w_\lambda = 0$. We will call an element such as w_λ a highest weight vector of weight λ.

Define $V = \mathbb{C}^{2n}$ to be the set of column vectors of height $2n$ with complex entries, with a $\mathbb{Z}_2$-grading given by

$$V_{\bar{0}} = \left\{ \begin{pmatrix} u \\ 0 \end{pmatrix} \mid \ u \in \mathbb{C}^n \right\} \qquad V_{\bar{1}} = \left\{ \begin{pmatrix} 0 \\ w \end{pmatrix} \mid \ w \in \mathbb{C}^n \right\}.$$

The $\mathfrak{q}(n)$-action is given by matrix multiplication on the left, and the action is compatible with the grading. The module V corresponds to the highest weight simple module $L(\epsilon_1)$, with $\epsilon_1 = (1, 0, \ldots, 0)$ from the above construction.

By [17, Lemma 3.4], similar to the Type A Lie algebras and superalgebras, there is a super version of Schur's Lemma. As a result, one can define a simple module to be of Type M or Type Q based on its $\mathfrak{q}(n)$-endomorphism space.

Lemma 2.1 (Super Schur's Lemma)**.** *Let W, U be two simple modules for $\mathfrak{q}(n)$, then*

$$\dim \operatorname{End}_{\mathfrak{q}(n)}(W, U) = \begin{cases} 0 & \textit{if } W \not\simeq U \\ 1 & \textit{if } W \simeq U \textit{ is of Type M} \\ 2 & \textit{if } W \simeq U \textit{ is of Type Q} \end{cases}$$

In the latter case, $\operatorname{End}_{\mathfrak{q}(n)}(W)$ has a basis $\{1, c\}$, where c is an odd $\mathfrak{q}(n)$-endomorphism. When $W = V$ is the natural representation defined above, c can be taken as left multiplication by

$$c = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{2.2.9}$$

where I_n is the $n \times n$ identity matrix. In particular, it can be checked that $[c, x] = 0$ for all elements $x \in \mathfrak{q}(n)$.

Choose an ordered basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of V, where e_i is the vector of height $2n$ with 1 in the i-th entry and 0 elsewhere, and f_i is the vector of height $2n$ with 1 in the $(n+i)$-th entry and 0 elsewhere. Observe that $c(e_i) = f_i$, $c(f_i) = -e_i$ where $c \in \operatorname{End}_{\mathfrak{q}}(V)$ is the map defined above. For notation purposes let w be either symbol e or f, then $e_{pq}(w_i) = \delta_{iq} w_p$, $f_{pq}(w_i) = (-1)^{\overline{w_i}} \delta_{iq}(c(w_p))$.

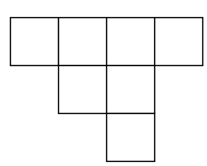
For any $\mathfrak{q}(n)$-module W, define ΠW to be the module with the same underlying vector space and opposite grading, i.e. $v \in \Pi W$ is even if and only if v is odd in W, and vice versa. The action of $\mathfrak{q}(n)$ on ΠW is defined as $x.w = (-1)^{\overline{x}} x.w$, for homogeneous $x \in \mathfrak{q}(n)$ and $w \in W$.

2.3. Combinatorics of Shifted Diagrams

Let us now introduce shifted diagrams (or equivalently, strict partitions). Let

$$\begin{aligned} \mathbb{Z}^n_{++} = \{\lambda = (\lambda_1, \dots, \lambda_n) \mid \lambda_i \in \mathbb{Z}_{\geq 0}, \lambda_i \geq \lambda_{i+1}, \\ \lambda_i = \lambda_{i+1} \text{ only if } \lambda_i = 0, \forall 1 \leq i \leq n-1\} \end{aligned} \tag{2.3.1}$$

be the set of positive, dominant integral weights whose entries are strictly decreasing. The *shifted diagram* associated to $\lambda \in \mathbb{Z}^n_{++}$ consists of rows of boxes, with λ_1 boxes in the first row, λ_2 boxes in the second row, ... , where the beginning of the i-th show is the i-th column. By *length* $\ell(\lambda)$ of λ, we mean the number of rows in the diagram of λ. When giving the entries of a strict partition λ, we sometimes omit the repeating zeros at the end. For example, the shifted diagram associated to the weight $(4, 2, 1)$ is:



(2.3.2)

and the length of $(4, 2, 1)$ is 3. For a box b in a diagram, define the *content* $c(b)$ of the box b to be

$$c(b) = \text{col}(b) - \text{row}(b),$$

where col(b) is the index of the column in which b lies and row(b) is the index of the row. Notice one can intepret $c(b)$ as the index of the diagonal in which b sits. In Example (2.3.2), the content for each box is as follows:

0	1	2	3
	0	1	
		0	

We say a strict partition γ is contained in another strict partition λ, if $\gamma_i \leq \lambda_i$ for all i. For strict partitions γ contained in λ, Stembridge [14] defined a *semistandard* tableau of shape λ/γ to be a filling of the boxes that are in λ but not in γ, with entries from an ordered set $1' < 1 < 2' < 2 < \cdots$, such that the numbers are weakly increasing along each row and each column, and each i' (or primed integers) occur at most once along each row and each i (regular integer) occur at most once along each column. Given a semistandard tableau T of shape λ/γ with entries from $\{1, 1', \ldots, t, t'\}$, let the *multiplicity* $\mu(T) \in \mathbb{Z}^{\mathbb{N}}$ be $(\mu_1, \mu_2, \ldots, \mu_t, 0, 0, \ldots)$, where μ_1 is the total number of 1's and $(1')$'s, μ_2 is the total number of 2's and $(2')$'s, etc.

Example. The following is a semistandard tableau of shape λ/μ, where $\lambda = (5, 3, 2, 1)$, $\mu = (3, 1)$, and the multiplicity of the fillings is $(4, 2, 1, 0, 0 \ldots)$:

			1	$1'$
		$1'$	2	
		$1'$	$2'$	
			3	

When $\gamma = (0, 0, \ldots)$ is the empty partition, the definition above specializes to tableaux of shape λ. In this case, define the Schur Q-function associated to

λ to be the following formal sum:

$$Q_\lambda(x) = \sum_{\mu \in \mathbb{Z}^{\mathbb{N}}} \# \left\{ \begin{matrix} \text{semistandard tableaux of} \\ \text{shape } \lambda \text{ and fillings with} \\ \text{multiplicity } \mu \end{matrix} \right\} x^\mu. \tag{2.3.3}$$

Here, $x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_t^{\mu_t}$, when $\mu = (\mu_1, \ldots, \mu_t, 0, 0, \ldots)$, and $\#$ denotes the cardinality of a set. The Schur P-function is

$$P_\lambda(x) = 2^{-\ell(\lambda)} Q_\lambda(x)$$

where $\ell(\lambda)$ denotes the number of nonzero rows in λ as defined before.

Denote by $P_\lambda(x_1, \ldots, x_n)$ the truncated polynomial obtained from P_λ by setting $x_i = 0$ for $i > n$. Notice $P_\lambda(x_1, \ldots, x_n)$ can be defined using (2.3.3) by only counting semistandard tableaux with entries up to n.

Also observe that when $\ell(\lambda) = n$, the largest entry in a tableau of shape λ must be at least n in the partial order, using the following pattern in the boxes with content 0 and 1:

$$\begin{array}{|c|c|c|cc} \hline 1' & 1 & \ldots & & \\ \hline & 2' & 2 & \ldots & \\ \cline{2-5} & & 3' & 3 & \ldots \\ \cline{3-5} \end{array}$$

For any other taleaux, entries in these boxes must be equal or larger, because of the nonrepeating condition. Therefore, there are no semistandard tableaux of shape λ with entries up to n, when $\ell(\lambda) > n$. In other words, all terms in P_λ must contain some x_i for $i > n$, hence $P_\lambda(x_1, \ldots, x_n) = 0$ when $\ell(\lambda) > n$.

In [14, Section 8], it is shown that multiplication of Schur P-functions obeys the following decomposition

$$P_\lambda P_\mu = \sum_\gamma f^\gamma_{\lambda,\mu} P_\gamma.$$

We evaluate the above equation at $x_i = 0$ for $i \geq n+1$:

$$P_\lambda(x_1,\ldots,x_n)P_\mu(x_1,\ldots,x_n) = \sum_\gamma f^\gamma_{\lambda,\mu} P_\gamma(x_1,\ldots,x_n) \tag{2.3.4}$$

Because $P_\gamma(x_1,\ldots,x_n) = 0$ when $\ell(\gamma) > n$, the remaining discussion will be applied to the coefficient $f^\gamma_{\lambda,\mu}$ when $\ell(\gamma) \leq n$.

The coefficients $f^\gamma_{\lambda,\mu}$ can be computed combinatorially as follows. Similar to the definition of semistandard shifted tableax of shape λ, a semistandard shifted tableaux of skew shape γ/λ, is a filling of boxes in γ that are not in λ, with integers $1, 2, \ldots$ and $1', 2', \ldots$ under the order $1' < 1 < 2' < 2 < \cdots$, subject to the semistandard condition introduced in the beginning of this section. To compute $f^\gamma_{\lambda,\mu}$ we also need to define a *lattice* condition: in particular, let $w = w_1 \cdots w_t$ be the word obtained by reading the entries in a tableaux from the bottom row to the top row, and from left to right. Denote by # the number of times an entry appears. For any i, define the occurrence $m_i(k)$ at step k as follows:

1) $1 \leq k \leq t$,

$$m_i(k) = \#i \text{ in } w_{t-k+1}, \ldots, w_t$$

2) $t+1 \leq k \leq 2t$,

$$m_i(k) = m_i(k-t) + \#i' \text{ in } w_1, \ldots, w_{k-t}$$

With the convention $m_i(0) = 0$. In other words, the function $m_i(k)$ is counting regular integers backward from the end of w to the beginning, and counting primed integers forward from the beginning of w to the end. Call w a lattice word if the following is true

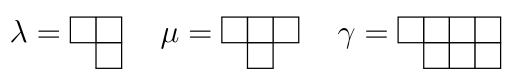
1) When $1 \leq k \leq t$, if $m_i(k) = m_{i+1}(k)$, then $w_{t-k} \neq i+1, (i+1)'$

2) When $t+1 \leq k \leq 2t$, if $m_i(k) = m_{i+1}(k)$, then $w_{k-t+1} \neq i, (i+1)'$

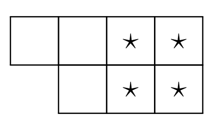
Define the absolute value $|i| = |i'| = i$ for $1 \leq i \leq n$. In [14], the Stembridge states the analogue of the Littlewood-Richardson rule for $f^{\gamma}_{\lambda,\mu}$ in (2.3.4) as follows.

Theorem 2.2. *[14, Theorem 8.3]* $f^{\gamma}_{\lambda,\mu}$ *is equal to the number of semistandard tableaux of shape* γ/μ *such that the word* w *is a lattice word, and for any* i*, the first letter in* w *with absolute value* i *is a regular integer.*

Example. Let us compute $f^{\gamma}_{\lambda,\mu}$, where



All semistandard shifted tableaux of shape γ/λ whose fillings have multiplicity given by μ, are in the form of the following, where $\star$ is representing one of $1, 1', 2, 2'$.



The square can be filled with the following posibilities:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1' & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1' & 2' \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1' \\ \hline 1' & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1' \\ \hline 1' & 2' \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1' & 2' \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2' \\ \hline 1' & 2' \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2' \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1' \\ \hline 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1' \\ \hline 2 & 2' \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 2' \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2' \\ \hline 2 & 2' \\ \hline \end{array}$$

However, only the following satisfies the Stembridge rule, and therefore $f^{\gamma}_{\lambda,\mu} = 1$:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2' \\ \hline \end{array}$$

The others fail for various reasons:

$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1' & 2 \\ \hline \end{array}$: the associated word $1'211$ has $1'$ before all other 1.

$\begin{array}{|c|c|}\hline 1 & 1' \\ \hline 2 & 2 \\ \hline \end{array}$: yellow box violates the lattice condition (1).

$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2' \\ \hline \end{array}$: yellow box violates the lattice condition (2).

Using the above result, Bessenrodt [2] classified all the cases of pairs of partitions (λ, μ) such that $f^{\gamma}_{\lambda,\mu}$ is either 0 or 1. Among them, two cases are important for us and they are as follows:

1) $\lambda = (t, t-1, \ldots, 1)$, which we call a staircase shape, and $\mu = (s)$.

2) λ is arbitrary and μ is a single box.

For Case 2, it is immediate that $f^{\gamma}_{\lambda,\mu} = 1$ if and only if γ can be obtained from adding a single box to λ. We reformulate Theorem 2.2 for Case 1:

Lemma 2.3. *When* $\lambda = (t, t-1, \ldots, 1, 0, 0, \ldots)$, $\mu = (s)$, $f^{\gamma}_{\lambda,\mu} = 1$ *if and only* *if γ is of the form*

$$\begin{aligned}\gamma &= \lambda + (s_0, 1, 1, \ldots, 1)\\ &= (t + s_0, t, t-1, \ldots, j+1, j, j-2, j-3, \ldots, 1)\end{aligned}$$

Where $j \geq 2$, $s_0 + t - j + 1 = s$. In other words, γ is obtained by pasting an upside down L-shaped diagram to the right of λ.

Proof. By the previous theorem, if $f^{\gamma}_{\lambda,\mu} \neq 0$ if and only if there exists a semistandard tableau of shape γ with content s, satisfying the above conditions. Specifically, boxes in γ that are not in λ must be filled with 1 or $1'$. Since the semistandard condition implies that no square can be filled with 1 or $1'$, any such tableau must be in the mentioned form. On the other hand, given any such partition $\lambda + (s_0, 1, 1, \ldots, 1)$, one can fill the "L" shape using the following pattern:

$1'$	1	$\cdots$	1	1
$1'$				
$1'$				
$\vdots$				
$1'$				
1				

Since only one integer is used, the lattice condition is easily satisfied and the first letter in w is also unprimed. □

2.4. Polynomial Representations and Characters

For this section we shall focus on finite dimensional $\mathfrak{q}(n)$-modules. For a module W and $\lambda \in \mathfrak{h}^*$, let $W_\lambda = \{w \in W | h.w = \lambda(h)w, \forall h \in \mathfrak{h}\}$ be the λ-weight space in W. Recall λ_i is the i-th entry in λ as in (2.2.8). If $W \simeq \bigoplus_\lambda W_\lambda$, define the set of weights to be $\{\lambda \mid W_\lambda \neq 0\}$, and the character of W to be the formal sum

$$\mathrm{Ch}(W) = \sum_{\lambda \in \mathfrak{h}} \dim W_\lambda \cdot x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}.$$

One can also check the following is true

$$\mathrm{Ch}(W_1 \otimes W_2) = \mathrm{Ch}(W_1) \cdot \mathrm{Ch}(W_2),$$
$$\mathrm{Ch}(W_1 \oplus W_2) = \mathrm{Ch}(W_1) + \mathrm{Ch}(W_2).$$

Recall that V denotes the natural representation for $\mathfrak{q}(n)$ from Section 2.2. Summands of $V^{\otimes d}$ for an arbitrary d are called *polynomial representations*. By [17, Theorem 3.49], $V^{\otimes d}$ is semisimple, therefore polynomial representations are semisimple. Furthermore, if M is a summand of $V^{\otimes d_1}$ and N is a summand of $V^{\otimes d_2}$, then $M \otimes N$ is a summand of $V^{\otimes (d_1+d_2)}$, and therefore is again polynomial. Therefore the full subcategory of polynomial representations is closed under tensor product. By [17, Theorem 3.49], any polynomial representation W is isomorphic to some $L(\lambda)$ defined in Section 2.2, where λ is the unique weight in W that is highest under the partial order mentioned in Section 2.2, and λ is a strict partition. By [17, Theorem 3.48], the connection between characters and Schur P-functions is given as follows:

$$\mathrm{Ch}(L(\lambda)) = 2^{\lfloor \frac{\ell(\lambda)+1}{2} \rfloor} P_\lambda(x_1, \ldots, x_n) \tag{2.4.1}$$

For $\lambda, \mu \in \mathbb{Z}^n_{++}$, $L(\lambda) \otimes L(\mu)$ is polynomial and hence semisimple. In particular,

$$L(\lambda) \otimes L(\mu) \simeq \bigoplus_{\lambda \in \mathbb{Z}^n_{++}} L(\gamma)^{\oplus c^{\gamma}_{\lambda,\mu}} \tag{2.4.2}$$

for nonnegative integers $c^{\gamma}_{\lambda,\mu}$, where $\mathbb{Z}^n_{++}$ was defined in (2.3.1). Equations (2.4.1) and (2.4.2) yield

$$2^{\lfloor \frac{\ell(\lambda)+1}{2} \rfloor + \lfloor \frac{\ell(\mu)+1}{2} \rfloor} P_{\lambda}(x_1, \ldots, x_n) P_{\mu}(x_1, \ldots, x_n) = \sum c^{\gamma}_{\lambda,\mu} 2^{\lfloor \frac{\ell(\gamma)+1}{2} \rfloor} P_{\gamma}(x_1, \ldots, x_n).$$

The coefficients $c^{\gamma}_{\lambda,\mu}$ are therefore related to the Littlewood-Richardson type coefficients $f^{\gamma}_{\lambda,\mu}$ in (2.3.4). In particular, $c^{\gamma}_{\lambda,\mu} = 0$ if and only if $f^{\gamma}_{\lambda,\mu} = 0$, and the set of partitions γ occuring in the direct sum obeys the combinatorial rule by Stembridge.

In the special case where λ is a staircase of length n, all partitions γ that occur in the decomposition contain λ, and must have length exactly n, and $\ell(\lambda) = \ell(\gamma) = n$ for all occuring γ. If in addition, $\ell(\mu) = 1$, the above equation reduces to

$$2P_{\lambda}(x_1, \ldots, x_n) P_{\mu}(x_1, \ldots, x_n) = \sum c^{\gamma}_{\lambda,\mu} P_{\gamma}(x_1, \ldots, x_n), \tag{2.4.3}$$

therefore $c^{\gamma}_{\lambda,\mu} = 2f^{\gamma}_{\lambda,\mu}$.

Chapter 3

The Degenerate Two Boundary Affine Hecke-Clifford Algebra

3.1. Presentations

In [6], Daugherty studied the degenerate two boundary Hecke algebra $\mathcal{G}$ and its representations using certain representations of the general linear Lie algebra $\mathfrak{gl}(n)$. In particular, $\mathcal{G}$ is generated by three polynomial rings and $\mathbb{C}\Sigma_d$, where Σ_d is the symmetric group on d letters. On the other hand, Hill-Kujawa-Sussan [10] studied the degenerate affine Hecke-Clifford algebra $\mathcal{H}_{Cl}^{\mathrm{aff}}$, and its representations using certain representations of the Lie superalgebra $\mathfrak{q}(n)$. To study the two boundary version of the work of Hill-Kujawa-Sussan, or equivalently, the Type Q version of Daugherty, we need to define the two boundary analogue of $\mathcal{H}_{\mathrm{Cl}}^{\mathrm{aff}}$.

Definition 3.1. *Define the degenerate two boundary affine Hecke-Clifford algebra $\mathcal{H}_d$ to be the associative algebra over $\mathbb{C}$ generated by generators $\tilde{x}_1, \ldots, \tilde{x}_d$, $\tilde{y}_1, \ldots, \tilde{y}_d$, $\tilde{z}_0, \ldots, \tilde{z}_d$, $c_1, \ldots, c_d$, $s_1, \ldots, s_d$, where $\tilde{x}_i$, $\tilde{y}_i$, $\tilde{z}_i$, c_i are odd, s_i is even for all i, subject to the following relations:*

(Sergeev relations)

$$s_i^2 = 1 \qquad 1 \le i \le d$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad 1 \le i \le d-1$$

$$s_i s_j = s_j s_i \qquad |i-j| > 1$$

$$c_i^2 = -1 \qquad 1 \le i \le d$$

$$c_i c_j = -c_j c_i \qquad i \ne j$$

$$s_i c_j = c_j s_i \qquad j \ne i, i+1$$

$$s_i c_j = c_{j+1} s_i$$

$$s_i c_{j+1} = c_j s_i$$

(Hecke relations)

$$s_i \tilde{x}_i = \tilde{x}_{i+1} s_i + c_i - c_{i+1} \qquad (1 \le i \le d-1)$$

$$s_i \tilde{y}_i = \tilde{y}_{i+1} s_i + c_i - c_{i+1} \qquad (1 \le i \le d-1)$$

$$s_i \tilde{z}_i = \tilde{z}_{i+1} s_i + c_i - c_{i+1} \qquad (1 \le i \le d-1)$$

(commuting relations)

$$\tilde{x}_j s_i = s_j \tilde{x}_j$$

$$\tilde{y}_j s_i = s_j \tilde{y}_j$$

$$\tilde{z}_j s_i = s_i \tilde{z}_j$$

$$(j \ne i, i+1)$$

(near commuting relations)

$$s_i(\tilde{x}_i + \tilde{x}_{i+1}) = (\tilde{x}_i + \tilde{x}_{i+1})s_i + 2(c_i - c_{i+1})$$
$$s_i(\tilde{y}_i + \tilde{y}_{i+1}) = (\tilde{y}_i + \tilde{y}_{i+1})s_i + 2(c_i - c_{i+1})$$
$$s_i(\tilde{z}_i + \tilde{z}_{i+1}) = (\tilde{z}_i + \tilde{z}_{i+1})s_i + 2(c_i - c_{i+1})$$
$$(1 \le i \le d-1)$$

(anticommuting relations)

$$c_i\tilde{x}_j = -\tilde{x}_j c_i$$
$$c_i\tilde{y}_j = -\tilde{y}_j c_i$$
$$c_i\tilde{z}_j = -\tilde{z}_j c_i$$
$$(1 \le i, j \le d)$$
$$\tilde{x}_i\tilde{x}_j = -\tilde{x}_j\tilde{x}_i$$
$$\tilde{y}_i\tilde{y}_j = -\tilde{y}_j\tilde{y}_i$$
$$\tilde{z}_i\tilde{z}_j = -\tilde{z}_j\tilde{z}_i$$
$$(i \ne j)$$

(further relations)

$$\tilde{z}_i \tilde{x}_j = -\tilde{x}_j \tilde{z}_i$$

$$\tilde{z}_i \tilde{y}_j = -\tilde{y}_j \tilde{z}_i$$

$$(j < i)$$

$$(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_i + \tilde{x}_i)\tilde{x}_i = -\tilde{x}_i(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_i + \tilde{x}_i)$$

$$(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_i - \tilde{y}_i)\tilde{y}_i = -\tilde{y}_i(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_i - \tilde{y}_i)$$

$$(1 \leq i \leq d)$$

$$\tilde{x}_i + \tilde{y}_i = \tilde{z}_i - \sum_{1 \leq j \leq i-1} (c_j - c_i) t_{j,i}$$

$$(1 \leq i \leq d)$$

Here, $t_{j,i} = s_j s_{j+1} \cdots s_{i-1} s_i s_{i-1} \cdots s_{j+1} s_j$ corresponds to the symmetric group element (ij) in cycle notation.

To emphasize the similarity to the Type A setting, the list of relations above are introduced in a similar pattern as the degenerate two boundary braid group in [6], with some slight differences in "further relations". However, most of the generators and relations above are redundant, and we aim to obtain a shorted presentation for later purposes. In particular, we first introduce a few lemmas to show some relations can be obtained by others, for an intermediate presentation in Corollary 3.4, and a more reduced presentation with odd polynomial generators in Theorem 3.5. The final goal is a reduced presentation with even polynomial generators in Proposition 3.9.

Proposition 3.2. *The "near commuting relations" can be obtained from the "Hecke relations", and "further relations" can be reduced to the following*

(shortened further relations)

$$\tilde{z}_2\tilde{x}_1 = -\tilde{x}_1\tilde{z}_2$$
$$\tilde{z}_2\tilde{y}_1 = -\tilde{y}_1\tilde{z}_2$$
$$(\tilde{z}_0 - \tilde{z}_1 + \tilde{x}_1)\tilde{x}_1 = -\tilde{x}_1(\tilde{z}_0 - \tilde{z}_1 + \tilde{x}_1)$$
$$(\tilde{z}_0 + \tilde{z}_1 - \tilde{y}_1)\tilde{y}_1 = -\tilde{y}_1(\tilde{z}_0 + \tilde{z}_1 - \tilde{y}_1)$$
$$\tilde{x}_1 + \tilde{y}_1 = \tilde{z}_1$$

Proof. 1) We claim the "near commuting relations" are redundent: the "Hecke relations" are equivalent to

$$s_i\tilde{x}_i = \tilde{x}_{i+1}s_i + (c_i - c_{i+1})$$
$$\tilde{x}_i s_i = s_i\tilde{x}_{i+1} - (c_i - c_{i+1})$$

By combining the two equations one obtains the "near commuting relations" for x_i.

2) The relation

$$\tilde{z}_i\tilde{x}_j = -\tilde{x}_j\tilde{z}_i \qquad (i > j)$$

can be obtained from

$$\tilde{z}_i\tilde{x}_1 = -\tilde{x}_1\tilde{z}_i \qquad (2 \leq i \leq d)$$

and other relations.

First, the second relation above imples $s_i x_i s_i = x_{i+1} + (c_i - c_{i+1})s_i$. Suppose the claim is true for all $j \leq j_0$. When $j = j_0 + 1$, $\forall i > j_0 + 1$:

$$\begin{aligned}
&\tilde{z}_i\tilde{x}_{j_0+1} + \tilde{x}_{j_0+1}\tilde{z}_i \\
&= \tilde{z}_i(s_{j_0}\tilde{x}_{j_0}s_{j_0} - (c_{j_0} - c_{j_0+1})s_{j_0}) + (s_{j_0}\tilde{x}_{j_0}s_{j_0} - (c_{j_0} - c_{j_0+1})s_{j_0})\tilde{z}_i \\
&= s_{j_0}(\tilde{z}_i\tilde{x}_{j_0} + \tilde{x}_{j_0}\tilde{z}_i)s_{j_0} - (\tilde{z}_i(c_{j_0} - c_{j_0+1})s_{j_0}) + (c_{j_0} - c_{j_0+1})s_{j_0})\tilde{z}_i) \\
&= 0
\end{aligned}$$

The last equality holds because of the induction hypothesis and the fact that z_i anticommutes with $(c_{j_0} - c_{j_0+1})$ but commutes with s_{j_0}. Similarly, the relation

$$\tilde{z}_i\tilde{y}_j = -\tilde{y}_j\tilde{z}_i \qquad (i > j)$$

can also be obtained from other relations.

3) The relation $\tilde{z}_i\tilde{x}_1 = -\tilde{x}_1\tilde{z}_i(i \geq 2)$ can be obtained from $\tilde{z}_2\tilde{x}_1 = -\tilde{x}_1\tilde{z}_2$ and the Hecke relations.

Assume the claim is true for all $i \leq k$.

$$\begin{aligned}
&\tilde{z}_{k+1}\tilde{x}_1 + \tilde{x}_1\tilde{z}_{k+1} \\
&= (s_k\tilde{z}_k s_k - (c_k - c_{k+1})s_k)\tilde{x}_1 + \tilde{x}_1(s_k\tilde{z}_k s_k - (c_k - c_{k+1})s_k) \\
&= s_k(\tilde{z}_k\tilde{x}_1 + \tilde{x}_1\tilde{z}_k)s_k = 0
\end{aligned}$$

Similarly, $\tilde{z}_i\tilde{y}_1 = -\tilde{y}_1\tilde{z}_i(i \geq 2)$ can be obtained from $\tilde{z}_2\tilde{y}_1 = -\tilde{y}_1\tilde{z}_2$ and the Hecke relations.

4) The relation

$$(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_i + \tilde{x}_i)\tilde{x}_i = -\tilde{x}_i(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_i + \tilde{x}_i)$$

can be obtained from

$$(\tilde{z}_0 - \tilde{z}_1 + \tilde{x}_1)\tilde{x}_1 = -\tilde{x}_1(\tilde{z}_0 - \tilde{z}_1 + \tilde{x}_1)$$

and other relations. Assume the claim is true for integers $i \leq k$.

$$\begin{aligned}
&(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_{k+1} + \tilde{x}_{k+1})\tilde{x}_{k+1} \\
&\quad + \tilde{x}_{k+1}(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_{k+1} + \tilde{x}_{k+1}) \\
&= (\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_{k+1} + \tilde{x}_{k+1})(s_k\tilde{x}_k s_k - (c_k - c_{k+1})s_k) + \\
&\quad (s_k\tilde{x}_k s_k - (c_k - c_{k+1})s_k)(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_{k+1} + \tilde{x}_{k+1}) \\
&= s_k(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_{k+1})\tilde{x}_k s_k + s_k\tilde{x}_k(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_{k+1})s_k \\
&\quad + 2(c_k - c_{k+1})\tilde{x}_k s_k - 2s_k\tilde{x}_k(c_k - c_{k+1}) + \tilde{x}_{k+1}s_k\tilde{x}_k s_k + s_k\tilde{x}_k s_k\tilde{x}_{k+1} \\
&\quad - (\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_{k+1})(c_k - c_{k+1})s_k \\
&\quad - (c_k - c_{k+1})s_k(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_{k+1}) \\
&\quad - (c_k - c_{k+1})s_k\tilde{x}_{k+1} - \tilde{x}_{k+1}(c_k - c_{k+1})s_k
\end{aligned}$$

The calculation is too long in one setting, therefore let us break up the pieces:

$$\begin{aligned}
&s_k(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_{k+1})\tilde{x}_k s_k + s_k\tilde{x}_k(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_{k+1})s_k \\
&= s_k(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_k + \tilde{x}_k)\tilde{x}_k s_k + s_k\tilde{x}_k(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_k + \tilde{x}_k)s_k \\
&\quad - s_k(\tilde{x}_k + \tilde{z}_{k+1})\tilde{x}_k s_k - s_k\tilde{x}_k(\tilde{x}_k + \tilde{z}_{k+1})s_k \\
&= -2s_k\tilde{x}_k^2 s_k
\end{aligned}$$

by the induction hypothesis.

$$\begin{aligned}
&\tilde{x}_{k+1}s_k\tilde{x}_k s_k + s_k\tilde{x}_k s_k\tilde{x}_{k+1}\\
&= (s_k\tilde{x}_k - (c_k - c_{k+1}))\tilde{x}_k s_k + s_k\tilde{x}_k(\tilde{x}_k s_k + (c_k - c_{k+1}))\\
&= 2s_k\tilde{x}_k^2 s_k - (c_k - c_{k+1})\tilde{x}_k s_k + s_k\tilde{x}_k(c_k - c_{k+1})
\end{aligned}$$

$$\begin{aligned}
&- (\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_{k+1})(c_k - c_{k+1})s_k\\
&\quad - (c_k - c_{k+1})s_k(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_{k+1})\\
&= (c_k - c_{k+1})(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_{k+1})s_k\\
&\quad - (c_k - c_{k+1})(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_{k+1})s_k + 2(c_k - c_{k+1})^2\\
&= 2(c_k - c_{k+1})^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_{k+1} + \tilde{x}_{k+1})\tilde{x}_{k+1}\\
&\quad + \tilde{x}_{k+1}(\tilde{z}_0 - \tilde{z}_1 - \tilde{z}_2 - \cdots - \tilde{z}_{k+1} + \tilde{x}_{k+1})\\
&= 2(c_k - c_{k+1})\tilde{x}_k s_k - 2s_k\tilde{x}_k(c_k - c_{k+1}) - (c_k - c_{k+1})s_k\tilde{x}_{k+1} - \tilde{x}_{k+1}(c_k - c_{k+1})s_k\\
&\quad - (c_k - c_{k+1})\tilde{x}_k s_k + s_k\tilde{x}_k(c_k - c_{k+1}) + 2(c_k - c_{k+1})^2\\
&= (c_k - c_{k+1})\tilde{x}_k s_k - s_k\tilde{x}_k(c_k - c_{k+1}) - (c_k - c_{k+1})(\tilde{x}_k s_k + (c_k - c_{k+1}))\\
&\quad + (s_k\tilde{x}_k - (c_k - c_{k+1}))(c_k - c_{k+1}) + 2(c_k - c_{k+1})^2 = 0
\end{aligned}$$

The calculations for the y_i version of the relations are similar yet slightly different. Again assume the y-portion of the claim is true for all $i \leq k$.

$$
\begin{aligned}
&(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_{k+1} - \tilde{y}_{k+1})\tilde{y}_{k+1} \\
&\quad + \tilde{y}_{k+1}(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_{k+1} - \tilde{y}_{k+1}) \\
&= (\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_{k+1} - \tilde{y}_{k+1})(s_k\tilde{y}_k s_k - (c_k - c_{k+1})s_k) \\
&\quad + (s_k\tilde{y}_k s_k - (c_k - c_{k+1})s_k)(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_{k+1} - \tilde{y}_{k+1}) \\
&= s_k(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_{k+1})\tilde{y}_k s_k + s_k\tilde{y}_k(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_{k+1})s_k \\
&\quad - 2(c_k - c_{k+1})\tilde{y}_k s_k + 2s_k\tilde{y}_k(c_k - c_{k+1}) \\
&\quad - \tilde{y}_{k+1}(s_k\tilde{y}_k s_k - (c_k - c_{k+1})s_k) - (s_k\tilde{y}_k s_k - (c_k - c_{k+1})s_k)\tilde{y}_{k+1} \\
&\quad - (\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_{k+1})(c_k - c_{k+1})s_k \\
&\quad - (c_k - c_{k+1})s_k(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_{k+1})
\end{aligned}
$$

Calculate each piece individually:

$$
\begin{aligned}
&s_k(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_{k+1})\tilde{y}_k s_k + s_k\tilde{y}_k(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_{k+1})s_k \\
&= s_k(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_k - \tilde{y}_k)\tilde{y}_k s_k + s_k\tilde{y}_k(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_k - \tilde{y}_k)s_k \\
&\quad + s_k(\tilde{z}_{k+1} + \tilde{y}_k)\tilde{y}_k s_k + s_k\tilde{y}_k(\tilde{z}_{k+1} + \tilde{y}_k)s_k \\
&= 2s_k\tilde{y}_k^2 s_k
\end{aligned}
$$

$$
\begin{aligned}
&- \tilde{y}_{k+1}(s_k\tilde{y}_k s_k - (c_k - c_{k+1})s_k) - (s_k\tilde{y}_k s_k - (c_k - c_{k+1})s_k)\tilde{y}_{k+1} \\
= \quad & -(s_k\tilde{y}_k - (c_k - c_{k+1}))\tilde{y}_k s_k + \tilde{y}_{k+1}(c_k - c_{k+1})s_k \\
&- s_k\tilde{y}_k(\tilde{y}_k s_k + (c_k - c_{k+1})) + (c_k - c_{k+1})s_k\tilde{y}_{k+1} \\
= & -2s_k\tilde{y}_k^2 s_k + \tilde{y}_{k+1}(c_k - c_{k+1})s_k + (c_k - c_{k+1})s_k\tilde{y}_{k+1} \\
&+ (c_k - c_{k+1})\tilde{y}_k s_k - s_k\tilde{y}_k(c_k - c_{k+1})
\end{aligned}
$$

$$
\begin{aligned}
&- (\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_{k+1})(c_k - c_{k+1})s_k \\
&- (c_k - c_{k+1})s_k(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_{k+1}) \\
= & (c_k - c_{k+1})(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_{k+1})s_k - (c_k - c_{k+1})(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_{k+1})s_k \\
&- 2(c_k - c_{k+1})^2 = -2(c_k - c_{k+1})^2
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_{k+1} - \tilde{y}_{k+1})\tilde{y}_{k+1} \\
&+ \tilde{y}_{k+1}(\tilde{z}_0 + \tilde{z}_1 + \tilde{z}_2 + \cdots + \tilde{z}_{k+1} - \tilde{y}_{k+1}) \\
= & \tilde{y}_{k+1}(c_k - c_{k+1})s_k + (c_k - c_{k+1})s_k\tilde{y}_{k+1} \\
&+ (c_k - c_{k+1})\tilde{y}_k s_k - s_k\tilde{y}_k(c_k - c_{k+1}) \\
&- 2(c_k - c_{k+1})\tilde{y}_k s_k + 2s_k\tilde{y}_k(c_k - c_{k+1}) \\
&- 2(c_k - c_{k+1})^2 \\
= & -(c_k - c_{k+1})(s_k\tilde{y}_k - (c_k - c_{k+1})) + (c_k - c_{k+1})(\tilde{y}_k s_k + (c_k - c_{k+1})) \\
&- (c_k - c_{k+1})\tilde{y}_k s_k + s_k\tilde{y}_k(c_k - c_{k+1}) \\
&- 2(c_k - c_{k+1})^2 = 0
\end{aligned}
$$

5) The relations

$$\tilde{x}_i + \tilde{y}_i = \tilde{z}_i - \sum_{1 \leq j \leq i-1} (c_j - c_i) t_{j,i} \qquad (1 \leq i \leq d)$$

can be obtained from $\tilde{x}_1 + \tilde{y}_1 = \tilde{z}_1$ and other relations.

Assume the claim is true for $i \leq k$.

$$\begin{aligned}
& \tilde{x}_{k+1} + \tilde{y}_{k+1} \\
= & s_k \tilde{x}_k s_k + s_k \tilde{y}_k s_k - 2(c_k - c_{k+1}) s_k \\
= & s_k (\tilde{z}_k - \sum_{1 \leq j \leq k-1} (c_j - c_k) t_{j,k}) s_k - 2(c_k - c_{k+1}) s_k \\
= & \tilde{z}_{k+1} - s_k (\sum_{1 \leq j \leq k-1} (c_j - c_k) t_{j,k}) s_k - (c_k - c_{k+1}) s_k \\
= & \tilde{z}_{k+1} - (\sum_{1 \leq j \leq k-1} (c_j - c_{k+1}) s_k t_{j,k} s_k) - (c_k - c_{k+1}) s_k \\
= & \tilde{z}_{k+1} - (\sum_{1 \leq j \leq k-1} (c_j - c_{k+1}) t_{j,k+1}) - (c_k - c_{k+1}) s_k \\
= & \tilde{z}_{k+1} - (\sum_{1 \leq j \leq k} (c_j - c_{k+1}) t_{j,k+1})
\end{aligned}$$

□

Proposition 3.3. *The "commuting relations" can be obtained by "Hecke relations" and the following*

$$\tilde{x}_1 s_j = s_j \tilde{x}_1 \qquad (2 \leq j \leq d)$$

$$\tilde{y}_1 s_j = s_j \tilde{y}_1 \qquad (2 \leq j \leq d)$$

$$\tilde{z}_i s_j = s_j \tilde{z}_i \qquad i \neq j, j+1$$

and the "anticommuting relations" can be obtained by "Hecke relations" by

$$c_i\tilde{x}_1 = -\tilde{x}_1 c_i \qquad (2 \le i \le d)$$

$$c_i\tilde{y}_1 = -\tilde{y}_1 c_i \qquad (2 \le i \le d)$$

$$c_i\tilde{z}_j = -\tilde{z}_j c_i \qquad (1 \le i, j \le d)$$

$$\tilde{x}_1(s_1\tilde{x}_1 s_1 - (c_1 - c_2)s_1) = -(s_1\tilde{x}_1 s_1 - (c_1 - c_2)s_1)\tilde{x}_1$$

$$\tilde{y}_1(s_1\tilde{y}_1 s_1 - (c_1 - c_2)s_1) = -(s_1\tilde{y}_1 s_1 - (c_1 - c_2)s_1)\tilde{y}_1$$

$$\tilde{z}_0(s_1\tilde{z}_0 s_1 - (c_1 - c_2)s_1) = -(s_1\tilde{z}_0 s_1 - (c_1 - c_2)s_1)\tilde{z}_0$$

Proof. 1) First claim that $\tilde{x}_i s_j = s_j \tilde{x}_i$, $i \neq j, j+1$ is the result of $\tilde{x}_1 s_j = s_j \tilde{x}_1$ and other relations. Suppose the statement is true for all $i \le k$, $j \neq i, i-1$. When $k \neq j-1, j, j+1$,

$$\begin{aligned}
&\tilde{x}_{k+1}s_j - s_j\tilde{x}_{k+1} \\
&= (s_k\tilde{x}_k s_k - (c_k - c_{k+1})s_k)s_j - s_j(s_k\tilde{x}_k s_k - (c_k - c_{k+1})s_k) \\
&= s_k\tilde{x}_k s_j s_k - s_k s_j\tilde{x}_k s_k - (c_k - c_{k+1})s_k s_j - s_j(c_k - c_{k+1})s_k \\
&= -(c_k - c_{k+1})s_k s_j + (c_k - c_{k+1})s_j s_k = 0
\end{aligned}$$

When $k = j + 1$,

$$
\begin{aligned}
&\tilde{x}_{k+1}s_j - s_j\tilde{x}_{k+1} \\
&= (s_k\tilde{x}_k s_k - (c_k - c_{k+1})s_k)s_{k-1} - s_{k-1}(s_k\tilde{x}_k s_k - (c_k - c_{k+1})s_k) \\
&= (s_k(s_{k-1}\tilde{x}_{k-1}s_{k-1} - (c_{k-1} - c_k)s_{k-1})s_k - (c_k - c_{k+1})s_k)s_{k-1} \\
&\quad - s_{k-1}(s_k(s_{k-1}\tilde{x}_{k-1}s_{k-1} - (c_{k-1} - c_k)s_{k-1})s_k - (c_k - c_{k+1})s_k) \\
&= s_k s_{k-1}\tilde{x}_{k-1}s_k s_{k-1}s_k - s_k(c_{k-1} - c_k)s_{k-1}s_k s_{k-1} - (c_k - c_{k+1})s_k s_{k-1} \\
&\quad - s_k s_{k-1}s_k\tilde{x}_{k-1}s_{k-1}s_k + s_{k-1}s_k(c_{k-1} - c_k)s_{k-1}s_k + s_{k-1}(c_k - c_{k+1})s_k \\
&= -s_k(c_{k-1} - c_k)s_k s_{k-1}s_k - (c_k - c_{k+1})s_k s_{k-1} \\
&\quad + s_{k-1}(c_{k-1} - c_{k+1})s_k s_{k-1}s_k + (c_{k-1} - c_{k+1})s_{k-1}s_k \\
&= -(c_{k-1} - c_{k+1})s_{k-1}s_k - (c_k - c_{k+1})s_k s_{k-1} \\
&\quad + s_{k-1}(c_{k-1} - c_{k+1})s_{k-1}s_k s_{k-1} + (c_{k-1} - c_{k+1})s_{k-1}s_k \\
&= -(c_k - c_{k+1})s_k s_{k-1} + (c_k - c_{k+1})s_k s_{k-1} = 0
\end{aligned}
$$

Similarly, the relations $\tilde{y}_i s_j = s_j\tilde{y}_i$, $(i \neq j, j+1)$ can also be obtained from other relations.

2) Next we claim that the relations $\tilde{x}_i c_j = -c_j\tilde{x}_i$ can be obtained from $\tilde{x}_1 c_j = -c_j\tilde{x}_1$ and other relations. Assume that the statement is true for $i \leq k$.

$$
\begin{aligned}
&\tilde{x}_{k+1}c_j + c_j\tilde{x}_{k+1} \\
&= (s_k\tilde{x}_k s_k - (c_k - c_{k+1})s_k)c_j + c_j(s_k\tilde{x}_k s_k - (c_k - c_{k+1})s_k) \\
&= -(c_k - c_{k+1})s_k c_j - c_j(c_k - c_{k+1})s_k
\end{aligned}
$$

The above quantity is 0 if $j \neq k, k+1$. When $j = k$,

$$\begin{aligned} & \tilde{x}_{k+1} c_j + c_j \tilde{x}_{k+1} \\ &= -(c_k - c_{k+1}) s_k c_k - c_k (c_k - c_{k+1}) s_k \\ &= -(c_k - c_{k+1}) c_{k+1} s_k - c_k (c_k - c_{k+1}) s_k = 0 \end{aligned}$$

Similarly when $j = k+1$,

$$\begin{aligned} & \tilde{x}_{k+1} c_j + c_j \tilde{x}_{k+1} \\ &= -(c_k - c_{k+1}) s_k c_{k+1} - c_{k+1} (c_k - c_{k+1}) s_k \\ &= -(c_k - c_{k+1}) c_k s_k - c_{k+1} (c_k - c_{k+1}) s_k = 0 \end{aligned}$$

The y-version of the claim is true by a similar argument.

3) Claim that $\tilde{x}_i \tilde{x}_j = -\tilde{x}_j \tilde{x}_i$, $(i \neq j)$ can be obtained from

$$\overline{x}_1 (s_1 \overline{x}_1 s_1 - (c_1 - c_2) s_1) = -(s_1 \overline{x}_1 s_1 - (c_1 - c_2) s_1) \tilde{x}_1$$

(or equivalently $\tilde{x}_1 \tilde{x}_2 = -\tilde{x}_2 \tilde{x}_1$) and other relations. Assume the claim is true for all $i < j \leq k$. For $i < k$,

$$\begin{aligned} & \tilde{x}_i \tilde{x}_{k+1} + \tilde{x}_{k+1} \tilde{x}_i \\ &= \tilde{x}_i (s_k \tilde{x}_k s_k - (c_k - c_{k+1}) s_k) + (s_k \tilde{x}_k s_k - (c_k - c_{k+1}) s_k) \tilde{x}_i \\ &= s_k (\tilde{x}_i \tilde{x}_k + \tilde{x}_k \tilde{x}_i) s_k - \tilde{x}_i (c_k - c_{k+1}) s_k) - (c_k - c_{k+1}) s_k \tilde{x}_i = 0 \end{aligned}$$

When $i = k$,

$$
\begin{aligned}
&\tilde{x}_k\tilde{x}_{k+1} + \tilde{x}_{k+1}\tilde{x}_k \\
&= (s_{k-1}\tilde{x}_{k-1}s_{k-1} - (c_{k-1} - c_k)s_{k-1})\tilde{x}_{k+1} + \tilde{x}_{k+1}(s_{k-1}\tilde{x}_{k-1}s_{k-1} - (c_{k-1} - c_k)s_{k-1}) \\
&= (s_{k-1}\tilde{x}_{k-1}\tilde{x}_{k+1}s_{k-1} + s_{k-1}\tilde{x}_{k+1}\tilde{x}_{k-1}s_{k-1} \\
&\quad - (c_{k-1} - c_k)s_{k-1}\tilde{x}_{k+1} - \tilde{x}_{k+1}(c_{k-1} - c_k)s_{k-1} = 0
\end{aligned}
$$

The y and z-version of the claim is also true by a similar calculation. □

These two lemmas lead to a shortened presentation, which is an intermediate version towards the final presentation.

Corollary 3.4. *The algebra $\mathcal{H}_d$ is isomorphic to the algebra generated by $\tilde{x}_1$, $\tilde{y}_1$, $\tilde{z}_0, \ldots, \tilde{z}_d$, $c_1, \ldots, c_d$, $s_1, \ldots, s_d$, where $\tilde{x}_1$, $\tilde{y}_1$, $\tilde{z}_i$, c_i are odd, s_i is even for all i, subject to the "Sergeev relations" and the following relations: (Hecke relations)*

$$s_i\tilde{z}_i = \tilde{z}_{i+1}s_i + c_i - c_{i+1} \qquad (0 \le i \le d-1)$$

(commuting relations)

$$
\begin{aligned}
\tilde{x}_1 s_i &= s_i\tilde{x}_1 && 2 \le i \le d \\
\tilde{y}_1 s_i &= s_i\tilde{y}_1 && 2 \le i \le d \\
\tilde{z}_j s_i &= s_i\tilde{z}_j && (j \ne i, i+1)
\end{aligned}
$$

(anticommuting relations)

$$c_i\tilde{x}_1 = -\tilde{x}_1 c_i \qquad (2 \le i \le d)$$

$$c_i\tilde{y}_1 = -\tilde{y}_1 c_i \qquad (2 \le i \le d)$$

$$c_i\tilde{z}_j = -\tilde{z}_j c_i \qquad (1 \le i, j \le d)$$

$$\tilde{x}_1(s_1\tilde{x}_1 s_1 - (c_1 - c_2)s_1) = -(s_1\tilde{x}_1 s_1 - (c_1 - c_2)s_1)\tilde{x}_1$$

$$\tilde{y}_1(s_1\tilde{y}_1 s_1 - (c_1 - c_2)s_1) = -(s_1\tilde{y}_1 s_1 - (c_1 - c_2)s_1)\tilde{y}_1$$

$$\tilde{z}_0(s_1\tilde{z}_0 s_1 - (c_1 - c_2)s_1) = -(s_1\tilde{z}_0 s_1 - (c_1 - c_2)s_1)\tilde{z}_0$$

(shortened further relations)

$$\tilde{z}_2\tilde{x}_1 = -\tilde{x}_1\tilde{z}_2$$

$$\tilde{z}_2\tilde{y}_1 = -\tilde{y}_1\tilde{z}_2$$

$$(\tilde{z}_0 - \tilde{z}_1 + \tilde{x}_1)\tilde{x}_1 = -\tilde{x}_1(\tilde{z}_0 - \tilde{z}_1 + \tilde{x}_1)$$

$$(\tilde{z}_0 + \tilde{z}_1 - \tilde{y}_1)\tilde{y}_1 = -\tilde{y}_1(\tilde{z}_0 + \tilde{z}_1 - \tilde{y}_1)$$

$$\tilde{x}_1 + \tilde{y}_1 = \tilde{z}_1$$

Proof. Let $\mathcal{H}'_d$ be the algebra given by the above presentation, and define a map $\phi : \mathcal{H}_d \to \mathcal{H}'_d$, where $\tilde{x}_1$, $\tilde{y}_1$, $\tilde{z}_i$, c_i, s_i get mapped to the generators with the same name, and $\phi(\tilde{x}_i)$ is defined recursively as

$$\phi(\tilde{x}_{i+1}) = s_i\tilde{x}_i s_i - (c_i - c_{i+1})s_i$$

(with the slight abusive notation that symbols on the right hand of the equation represent elements in $\mathcal{H}'_d$.) Then all relations except the Hecke relations in $\mathcal{H}_d$ are already the relations in $\mathcal{H}'_d$, and the Hecke relations are satisifed by the definition of ϕ. The inverse of ϕ is given by the map taking all generators to their counterparts in $\mathcal{H}_d$ with the same name. □

We aim to obtain a reduced version of the presentation with even generators in Section 3.3, and the following is an odd version of the main presentation, where the polynomial generators are still odd.

Theorem 3.5 (Reduced presentation with odd polynomial generators)**.** *The algebra $\mathcal{H}_d$ is isomorphic to the algebra generated by $\tilde{x}_1$, $\tilde{z}_0, \dots, \tilde{z}_d$, $c_1, \dots, c_d$, $s_1, \dots, s_d$, where $\tilde{x}_1$, $\tilde{z}_i$, c_i are odd, s_i is even for all i, subject to the "Sergeev relations", and the following relations:*

(Hecke relations)

$$s_i\tilde{z}_i = \tilde{z}_{i+1}s_i + c_i - c_{i+1} \qquad (0 \leq i \leq d-1)$$

(commuting relations)

$$\tilde{x}_1 s_i = s_i \tilde{x}_1 \qquad (2 \leq i \leq n)$$

$$\tilde{z}_j s_i = s_i \tilde{z}_j \qquad (j \neq i, i+1)$$

(anticommuting relations)

$$c_i \tilde{x}_1 = -\tilde{x}_1 c_i \qquad 2 \leq i \leq d$$

$$c_i \tilde{z}_j = -\tilde{z}_j c_i \qquad (1 \leq i, j \leq d)$$

$$\tilde{x}_1(s_1\tilde{x}_1 s_1 - (c_1 - c_2)s_1) = -(s_1\tilde{x}_1 s_1 - (c_1 - c_2)s_1)\tilde{x}_1$$

$$\tilde{z}_0(s_1\tilde{z}_0 s_1 - (c_1 - c_2)s_1) = -(s_1\tilde{z}_0 s_1 - (c_1 - c_2)s_1)\tilde{z}_0$$

(further relation)

$$\tilde{z}_2\tilde{x}_1 = -\tilde{x}_1\tilde{z}_2$$
$$(\tilde{z}_0 - \tilde{z}_1 + \tilde{x}_1)\tilde{x}_1 = -\tilde{x}_1(\tilde{z}_0 - \tilde{z}_1 + \tilde{x}_1)$$

Proof. Let $\mathcal{H}'_d$ be the algebra defined in Corollary 3.4 (the third presentation,) and let $\mathcal{H}''_d$ be the algebra defined by the presentation in this proposition. Define $\phi : \mathcal{H}'_d \to \mathcal{H}''_d$ to be the map taking all generators except $\tilde{y}_1$ to their counterparts under the same name, and $\phi(\tilde{y}_1) = \tilde{z}_1 - \tilde{x}_1$. The inverse ψ is given by taking all generators of $\mathcal{H}''_d$ to their counterparts in $\mathcal{H}'_d$. The map ψ is a homomorphism since the relations in $\mathcal{H}''_d$ are already relations in $\mathcal{H}'_d$. To check ϕ is a homomorphism, it is enough to show that the relations

$$\tilde{y}_1 s_i = s_i \tilde{y}_1 \qquad 2 \leq i \leq d$$
$$c_i\tilde{y}_1 = -\tilde{y}_1 c_i \qquad (2 \leq i \leq d)$$
$$\tilde{z}_i\tilde{y}_1 = -\tilde{y}_1\tilde{z}_i \qquad (2 \leq i \leq d)$$
$$\tilde{y}_1(s_1\tilde{y}_1 s_1 - (c_1 - c_2)s_1) = -(s_1\tilde{y}_1 s_1 - (c_1 - c_2)s_1)\tilde{y}_1$$
$$(\tilde{z}_0 + \tilde{z}_1 - \tilde{y}_1)\tilde{y}_1 = -\tilde{y}_1(\tilde{z}_0 + \tilde{z}_1 - \tilde{y}_1)$$

are satisfied. Equivalently, it is enough to show the following relations are satisfied in $\mathcal{H}_d''$

$$(\tilde{z}_1 - \tilde{x}_1)s_i = s_i(\tilde{z}_1 - \tilde{x}_1) \qquad 2 \leq i \leq d$$

$$c_i(\tilde{z}_1 - \tilde{x}_1) = -(\tilde{z}_1 - \tilde{x}_1)c_i \qquad (2 \leq i \leq d)$$

$$\tilde{z}_i(\tilde{z}_1 - \tilde{x}_1) = -(\tilde{z}_1 - \tilde{x}_1)\tilde{z}_i \qquad (2 \leq i \leq d)$$

$$(\tilde{z}_1 - \tilde{x}_1)(s_1(\tilde{z}_1 - \tilde{x}_1)s_1 - (c_1 - c_2)s_1) = -(s_1(\tilde{z}_1 - \tilde{x}_1)s_1 - (c_1 - c_2)s_1)(\tilde{z}_1 - \tilde{x}_1)$$

$$(\tilde{z}_0 + \tilde{z}_1 - (\tilde{z}_1 - \tilde{x}_1))(\tilde{z}_1 - \tilde{x}_1) = -(\tilde{z}_1 - \tilde{x}_1)(\tilde{z}_0 + \tilde{z}_1 - (\tilde{z}_1 - \tilde{x}_1))$$

The first three relations hold via a straighforward calculation. In the following calculations, let $\tilde{x}_2 = s_1\tilde{x}_1s_1 - (c_1 - c_2)s_1$. The second to last relation becomes

$$\begin{aligned}
&(\tilde{z}_1 - \tilde{x}_1)(s_1(\tilde{z}_1 - \tilde{x}_1)s_1 - (c_1 - c_2)s_1) \\
&\quad + (s_1(\tilde{z}_1 - \tilde{x}_1)s_1 - (c_1 - c_2)s_1)(\tilde{z}_1 - \tilde{x}_1) \\
&= \tilde{z}_1\tilde{z}_2 + \tilde{z}_2\tilde{z}_1 - \tilde{z}_1s_1\tilde{x}_1s_1 - s_1\tilde{x}_1s_1\tilde{z}_1 \\
&\quad - \tilde{x}_1\tilde{z}_2 - \tilde{z}_2\tilde{x}_1 + \tilde{x}_1s_1\tilde{x}_1s_1 + s_1\tilde{x}_1s_1\tilde{x}_1 \\
&= -(s_1\tilde{z}_2 - (c_1 - c_2))\tilde{x}_1s_1 - s_1\tilde{x}_1(\tilde{z}_2s_1 + (c_1 - c_2)) \\
&\quad + (s_1\tilde{x}_2 - (c_1 - c_2))\tilde{x}_1s_1 + s_1\tilde{x}_1(\tilde{x}_2s_1 + (c_1 - c_2)) = 0
\end{aligned}$$

The last relation becomes

$$\begin{aligned}
&(\tilde{z}_0 + \tilde{z}_1 - (\tilde{z}_1 - \tilde{x}_1))(\tilde{z}_1 - \tilde{x}_1) + (\tilde{z}_1 - \tilde{x}_1)(\tilde{z}_0 + \tilde{z}_1 - (\tilde{z}_1 - \tilde{x}_1)) \\
&= (\tilde{z}_0 + \tilde{x}_1)(\tilde{z}_1 - \tilde{x}_1) + (\tilde{z}_1 - \tilde{x}_1)(\tilde{z}_0 + \tilde{x}_1) \\
&= \tilde{x}_1(\tilde{z}_1 - \tilde{x}_1) - \tilde{z}_0\tilde{x}_1 + \tilde{z}_0\tilde{z}_1 + (\tilde{z}_1 - \tilde{x}_1)\tilde{x}_1 + \tilde{z}_1\tilde{z}_0 - \tilde{x}_1\tilde{z}_0 \\
&= \tilde{x}_1(\tilde{z}_1 - \tilde{x}_1 - \tilde{z}_0) + (\tilde{z}_1 - \tilde{x}_1 - \tilde{z}_0)\tilde{x}_1 = 0
\end{aligned}$$

Hence all the relations in $\mathcal{H}'_d$ are satisfied and $\mathcal{H}'_d \simeq \mathcal{H}''_d$. □

3.2. An Action of $\mathcal{H}_d$

The Sergeev algebra $\mathcal{S}(d)$ is the algebra generated by $c_1, \ldots, c_d, s_1, \ldots, s_{d-1}$ subject to the "Sergeev relations" in Definition 3.1. Therefore, there is a homomorphism $\mathcal{S}(d) \to \mathcal{H}_d$ of algebras, mapping the generators c_i and s_i to generators with the same name. In [12], Sergeev defined the following action of $\mathcal{S}(d)$ on $V^{\otimes d}$.

Theorem 3.6. *[12, Theorem 3] There is a well defined surjection of superalgebras*

$$\begin{aligned} \mathcal{S}(d) &\to \mathrm{End}_{\mathfrak{q}(n)}(V^{\otimes d}) \\ c_i &\mapsto 1^{\otimes(i-1)} \otimes c \otimes 1^{\otimes d-i} \\ s_i &\mapsto 1^{\otimes(i-1)} \otimes \sigma \otimes 1^{\otimes d-i-1} \end{aligned}$$

where c is the odd map in $\mathrm{End}(V)$ *as mentioned in Section 2.2, and σ is the signed permutation on $V^{\otimes 2}$:*

$$\sigma.(w_1 \otimes w_2) = (-1)^{\overline{w_1}\cdot\overline{w_2}} w_2 \otimes w_1.$$

Fix arbitrary finite-dimensional $\mathfrak{q}(n)$-modules M and N. To generalize the work of [6] and [18] to the case of $\mathfrak{q}(n)$, as well as to generalize the work of [10] to the two boundary case, we will define an action of the algebra $\mathcal{H}_d$ on $M \otimes N \otimes V^{\otimes d}$. Recall $\{e_{ij}\}_{1\le i,j\le n}$ and $\{f_{ij}\}_{1\le i,j\le n}$ defined in Equations (2.2.1) and (2.2.2) constitute a homoegeneous basis of $\mathfrak{q}(n)$. Define the following odd element in $\mathfrak{q} \otimes \mathfrak{q}$:

$$\Omega = \sum_{1 \leq i,j \leq n} e_{ij} \otimes f_{ji} - \sum_{1 \leq i,j \leq n} f_{ij} \otimes e_{ji}.$$

Note for any modules M and N, Ω has a natural action on $M \otimes N$, following the definition in Section 2.2. Based on the fact that $\overline{e_{ji}} = \overline{0}$ and $\overline{f_{ji}} = \overline{1}$, we have

$$\Omega.(m \otimes n) = \sum_{1 \leq i,j \leq n} (-1)^{\overline{m}} (e_{ij}.m) \otimes (f_{ji}.n) - \sum_{1 \leq i,j \leq n} (f_{ij}.m) \otimes (e_{ji}.n).$$

Use the imbedding $\iota_{M,N} : U(\mathfrak{q}) \otimes U(\mathfrak{q}) \to U(\mathfrak{q})^{\otimes(d+2)}$, $x \otimes y \mapsto x \otimes y \otimes 1^{\otimes d}$, and denote $\Omega_{M,N} = \iota(\Omega)$. It follows that $\Omega_{M,N}$ acts on $M \otimes N \otimes V^{\otimes}$ by only acting Ω on the M and N tensor factors. Similarly, denote $\iota_{M,i}$, $\iota_{N,i}$ and $\iota_{i,j}$ to be the imbeddings $U(\mathfrak{q}) \otimes U(\mathfrak{q}) \to U(\mathfrak{q})^{\otimes(d+2)}$ defined as following:

$$\begin{aligned} \iota_{M,i} &: x \otimes y \mapsto x \otimes 1 \otimes 1^{\otimes(i-1)} \otimes y \otimes 1^{\otimes(d-i)} \qquad (3.2.1) \\ \iota_{N,i} &: x \otimes y \mapsto 1 \otimes x \otimes 1^{\otimes(i-1)} \otimes y \otimes 1^{\otimes(d-i)} \\ \iota_{i,j} &: x \otimes y \mapsto 1 \otimes 1 \otimes 1^{\otimes(i-1)} \otimes x \otimes 1^{\otimes(j-i-1)} \otimes y \otimes 1^{(d-j)}. \end{aligned}$$

That is, $\iota_{M,i}$ imbeds the tensor factors x and y in the corresponding positions of M and the i-th copy of V, and similarly for $\iota_{N,i}$ and $\iota_{M,N}$.

Define $\Omega_{M,i} = \iota_{M,i}(\Omega)$, $\Omega_{N,i} = \iota_{N,i}(\Omega)$, $\Omega_{i,j} = \iota_{i,j}(\Omega)$, and let

$$\begin{aligned} \Omega_{M \otimes V^{\otimes i-1}, V} &= \Omega_{M,i} + \Omega_{1,i} + \cdots \Omega_{i-1,i} \\ \Omega_{N \otimes V^{\otimes i-1}, V} &= \Omega_{N,i} + \Omega_{1,i} + \cdots \Omega_{i-1,i} \\ \Omega_{M \otimes N \otimes V^{\otimes i-1}, V} &= \Omega_{M,i} + \Omega_{1,i} + \cdots \Omega_{i-1,i} + \Omega_{N,i} + \Omega_{1,i} + \cdots \Omega_{i-1,i} \end{aligned}$$

Notice that the effect of $\Omega_{M\otimes V^{\otimes i-1},V}$ acting on $M\otimes N\otimes V^{\otimes d}$ is the same as acting by Ω on the module $W\otimes V$, where $W=M\otimes V^{\otimes i-1}$, and similar is true for $\Omega_{N\otimes V^{\otimes i-1},V}$ and $\Omega_{M\otimes N\otimes V^{\otimes i-1},V}$.

One of the main results of this thesis is that one can define a $\mathcal{H}_d$-module structure on $M\otimes N\otimes V^{\otimes d}$ as follows.

Theorem 3.7. *For any finite dimensional $\mathfrak{q}(n)$-modules M and N, there is a well defined algebra map*

$$\begin{aligned}\rho:\mathcal{H}_d &\to \mathrm{End}_{\mathfrak{q}(n)}(M\otimes N\otimes V^{\otimes d})\\ \tilde{x}_i &\mapsto \Omega_{M\otimes V^{\otimes i-1},V},\\ \tilde{y}_i &\mapsto \Omega_{N\otimes V^{\otimes i-1},V},\\ \tilde{z}_i &\mapsto \Omega_{M\otimes N\otimes V^{\otimes i-1},V},\end{aligned}$$

and c_i and s_i act as specified in Theorem 3.6.

Proof. It is enough to show that all the relations in Theorem 3.5 are satisfied. Because of Theorem 3.6, the "Sergeev relations" are automatically satisfied. In addition, the "commuting relations" is a straightforward calculation to check. Therefore we shall focus on the remaining relations listed in Theorem 3.5.

1) Hecke relations

We will later give a presentation for $\mathcal{H}_d$ with even polynomial generators, and show that our version of the Hecke relations are equivalent to the even versions mentioned in [10]. As a consequence, since the Hecke relations in [10] are satisfied according to [10, Theorem 7.4.1], the Hecke relations in our definition will be satisfied. For completeness we also include the calculations here.

$$
\begin{aligned}
&s_i\tilde{x}_i - \tilde{x}_{i+1}s_i - c_i + c_{i+1} \\
&= s_i(\Omega_{M,i} + \Omega_{1,i}\cdots\Omega_{i-1,i}) - (\Omega_{M,i+1} + \Omega_{1,i+1}\cdots\Omega_{i,i+1})s_i - c_i + c_{i+1}
\end{aligned}
$$

We argue that

$$s_i\Omega_{j,i} = \Omega_{j,i+1}s_i.$$

For any $j \leq i-1$, $v_M \otimes v_N \otimes v_1 \otimes \cdots \otimes v_d \in M \otimes N \otimes V^{\otimes d}$,

$$
\begin{aligned}
&s_i\Omega_{j,i}(v_M \otimes v_N \otimes v_1 \otimes \cdots \otimes v_d) \\
&= (-1)^{\overline{v_M}+\cdots+\overline{v_{i-1}}}s_i \sum_{1\leq p,q\leq n} v_M \otimes \cdots \otimes e_{pq}v_j \otimes \cdots \otimes f_{qp}v_i \otimes v_{i+1} \otimes \cdots \otimes v_d \\
&\quad - (-1)^{\overline{v_M}+\cdots+\overline{v_{j-1}}}s_i \sum_{1\leq p,q\leq n} v_M \otimes \cdots \otimes f_{pq}v_j \otimes \cdots \otimes e_{qp}v_i \otimes v_{i+1} \otimes \cdots \otimes v_d \\
&= (-1)^{(\overline{v_i}+\overline{1})\overline{v_{i+1}}+\overline{v_M}+\cdots+\overline{v_{i-1}}} \sum_{1\leq p,q\leq n} v_M \otimes \cdots \otimes e_{pq}v_j \otimes \cdots \otimes v_{i+1} \otimes f_{qp}v_i \otimes \cdots \otimes v_d \\
&\quad - (-1)^{\overline{v_i}\cdot\overline{v_{i+1}}+\overline{v_M}+\cdots+\overline{v_{j-1}}} \sum_{1\leq p,q\leq n} v_M \otimes \cdots \otimes f_{pq}v_j \otimes \cdots \otimes v_{i+1} \otimes e_{qp}v_i \otimes \cdots \otimes v_d
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\Omega_{j,i+1}s_i(v_M \otimes v_N \otimes v_1 \otimes \cdots \otimes v_d) \\
&= (-1)^{\overline{v_{i+1}}\cdot\overline{v_i}}\Omega_{j,i+1}(v_M \otimes v_N \otimes v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_d) \\
&= (-1)^{\overline{v_{i+1}}\cdot\overline{v_i}+\overline{v_M}+\cdots+\overline{v_{i-1}}+\overline{v_{i+1}}} \sum_{1\leq p,q\leq n} v_M \otimes \cdots \otimes e_{pq}v_j \otimes \cdots \otimes v_{i+1} \otimes f_{qp}v_i \otimes \cdots \otimes v_d \\
&\quad + (-1)^{\overline{v_{i+1}}\cdot\overline{v_i}+\overline{v_M}+\cdots+\overline{v_{j-1}}} \sum_{1\leq p,q\leq n} v_M \otimes \cdots \otimes f_{pq}v_j \otimes \cdots \otimes v_{i+1} \otimes e_{qp}v_i \otimes \cdots \otimes v_d
\end{aligned}
$$

And the two expressions are equal as expected. Similarly,

$$s_i\Omega_{M,i} = \Omega_{M,i+1}s_i \tag{3.2.2}$$

We also check that

$$-\Omega_{i,i+1}s_i - c_i + c_{i+1} = 0. \tag{3.2.3}$$

Assume that v_i and v_{i+1} are basis vectors of V. Recall that for an ordered basis $\{e_1, \ldots, e_n, f_1, \ldots, f_1\}$ of V, where w is either symbol e or f, $E_{pq}w_i = \delta_{iq}w_p$, $f_{pq}w_i = (-1)^{\overline{w_i}}\delta_{iq}(c.w_p)$. Assume $v_i = w_a$, $v_{i+1} = u_b$, where w, u are symbols e or f. The following sums are over all $1 \leq p, q \leq n$.

$$\begin{aligned}
&(-\Omega_{i,i+1}s_i - c_i + c_{i+1}).(v_M \otimes \cdots \otimes v_{i-1} \otimes w_a \otimes u_b \otimes \cdots \otimes v_d)\\
=& -(-1)^{\overline{w_a}\cdot\overline{u_b}}\Omega_{i,i+1}(v_M \otimes \cdots \otimes v_{i-1} \otimes w_b \otimes u_a \otimes \cdots \otimes v_d)\\
&- (-1)^{\overline{v_M}+\cdots+\overline{v_{i-1}}}v_M \otimes \cdots \otimes v_{i-1} \otimes c.w_a \otimes u_b \otimes \cdots \otimes v_d\\
&+ (-1)^{\overline{v_M}+\cdots+\overline{v_{i-1}}+\overline{w_a}}v_M \otimes \cdots \otimes v_{i-1} \otimes w_a \otimes c.u_b \otimes \cdots \otimes v_d\\
=& -(-1)^{\overline{w_a}\cdot\overline{u_b}+\overline{v_M}+\cdots+\overline{v_{i-1}}+\overline{u_b}}\sum v_M \otimes \cdots \otimes v_{i-1} \otimes e_{pq}u_b \otimes f_{qp}w_a \otimes \cdots \otimes v_d\\
&+ (-1)^{\overline{w_a}\cdot\overline{u_b}+\overline{v_M}+\cdots+\overline{v_{i-1}}}\sum v_M \otimes \cdots \otimes v_{i-1} \otimes f_{pq}u_b \otimes e_{qp}w_a \otimes \cdots \otimes v_d\\
&- (-1)^{\overline{v_M}+\cdots+\overline{v_{i-1}}}v_M \otimes \cdots \otimes v_{i-1} \otimes c.w_a \otimes u_b \otimes \cdots \otimes v_d\\
&+ (-1)^{\overline{v_M}+\cdots+\overline{v_{i-1}}+\overline{w_a}}v_M \otimes \cdots \otimes v_{i-1} \otimes w_a \otimes c.u_b \otimes \cdots \otimes v_d\\
=& -(-1)^{\overline{w_a}\cdot\overline{u_b}+\overline{v_M}+\cdots+\overline{v_{i-1}}+\overline{u_b}+\overline{w_a}}v_M \otimes \cdots \otimes v_{i-1} \otimes u_a \otimes c.w_b \otimes \cdots \otimes v_d\\
&+ (-1)^{\overline{w_a}\cdot\overline{u_b}+\overline{v_M}+\cdots+\overline{v_{i-1}}+\overline{u_b}}v_M \otimes \cdots \otimes v_{i-1} \otimes c.u_a \otimes w_b \otimes \cdots \otimes v_d\\
&- (-1)^{\overline{v_M}+\cdots+\overline{v_{i-1}}}v_M \otimes \cdots \otimes v_{i-1} \otimes c.w_a \otimes u_b \otimes \cdots \otimes v_d\\
&+ (-1)^{\overline{v_M}+\cdots+\overline{v_{i-1}}+\overline{w_a}}v_M \otimes \cdots \otimes v_{i-1} \otimes w_a \otimes c.u_b \otimes \cdots \otimes v_d
\end{aligned}$$

The answer is zero if u and w are the same symbols. If u is e and w is f, $\overline{u_b} = \overline{0}$, $\overline{w_a} = \overline{1}$, $c.w_i = -u_i$ and $c.u_i = w_i$.

$$\begin{aligned}
&(-\Omega_{i,i+1}s_i - c_i + c_{i+1}).(v_M \otimes \cdots \otimes v_{i-1} \otimes w_a \otimes u_b \otimes \cdots \otimes v_d) \\
= &-(-1)^{\overline{v_M}+\cdots+\overline{v_{i-1}}} v_M \otimes \cdots \otimes v_{i-1} \otimes u_a \otimes u_b \otimes \cdots \otimes v_d \\
&+ (-1)^{\overline{v_M}+\cdots+\overline{v_{i-1}}} v_M \otimes \cdots \otimes v_{i-1} \otimes w_a \otimes w_b \otimes \cdots \otimes v_d \\
&- (-1)^{\overline{v_M}+\cdots+\overline{v_{i-1}}+1} v_M \otimes \cdots \otimes v_{i-1} \otimes u_a \otimes u_b \otimes \cdots \otimes v_d \\
&+ (-1)^{\overline{v_M}+\cdots+\overline{v_{i-1}}+\overline{1}} v_M \otimes \cdots \otimes v_{i-1} \otimes w_a \otimes w_b \otimes \cdots \otimes v_d = 0
\end{aligned}$$

On the other hand, if If u is f and w is e, $\overline{u_b} = \overline{1}$, $\overline{w_a} = \overline{0}$, $c.w_i = u_i$ and $c.u_i = -w_i$.

$$\begin{aligned}
&(-\Omega_{i,i+1}s_i - c_i + c_{i+1}).(v_M \otimes \cdots \otimes v_{i-1} \otimes w_a \otimes u_b \otimes \cdots \otimes v_d) \\
= &-(-1)^{\overline{v_M}+\cdots+\overline{v_{i-1}}+\overline{1}} v_M \otimes \cdots \otimes v_{i-1} \otimes u_a \otimes u_b \otimes \cdots \otimes v_d \\
&+ (-1)^{\overline{v_M}+\cdots+\overline{v_{i-1}}} v_M \otimes \cdots \otimes v_{i-1} \otimes w_a \otimes w_b \otimes \cdots \otimes v_d \\
&- (-1)^{\overline{v_M}+\cdots+\overline{v_{i-1}}} v_M \otimes \cdots \otimes v_{i-1} \otimes u_a \otimes u_b \otimes \cdots \otimes v_d \\
&+ (-1)^{\overline{v_M}+\cdots+\overline{v_{i-1}}+\overline{1}} v_M \otimes \cdots \otimes v_{i-1} \otimes w_a \otimes w_b \otimes \cdots \otimes v_d = 0
\end{aligned}$$

The Hecke relations for y_i and z_i can be checked similarly.

2) Anticommuting relations

Again, these relations are expected to hold as they are an odd version of the relations in [10, Section 3]. It is enough to check that $\Omega_{i,j}$ commutes with c_k, for any $1 \leq i, j \leq d$. The cases when $k \neq i, j$ are straightforward to check. When $i = k$, it is enough to check that $\Omega(c \otimes 1) = -(c \otimes 1) \otimes \Omega$ the calculations are similar to that in [10] (notice the Casimir tensor Ω in [10] is even.) The following sums are over all $1 \leq p, q \leq d$.

$$\begin{aligned}
\Omega(c \otimes 1) &= \sum (e_{pq} \otimes f_{qp})(1 \otimes c) - \sum (f_{pq} \otimes e_{qp})(1 \otimes c) \\
&= \sum (e_{pq} \otimes f_{qp}c) - \sum (f_{pq} \otimes e_{qp}c) \\
&= -\sum (e_{pq} \otimes cf_{qp}) - \sum (f_{pq} \otimes ce_{qp}) \\
&= -\sum (1 \otimes c)(e_{pq} \otimes f_{qp}) + \sum (1 \otimes c)(f_{pq} \otimes e_{qp}) \\
&= -(1 \otimes c)\Omega.
\end{aligned}$$

Similarly, $\Omega(c \otimes 1) = -(c \otimes 1) \otimes \Omega$ and therefore $\Omega_{i,j}c_j = -c_j\Omega_{i,j}$.

For the relations $\tilde{x}_1\tilde{x}_2 = -\tilde{x}_2\tilde{x}_1$ and $\tilde{z}_0\tilde{z}_1 = -\tilde{z}_1\tilde{z}_0$, we will first show $\Omega_{M,N}(\Omega_{M,1} + \Omega_{N,1}) = -(\Omega_{M,1} + \Omega_{N,1})\Omega_{M,N}$.

$$\begin{aligned}
&\Omega_{M,N}(\Omega_{M,1} + \Omega_{N,1}) + (\Omega_{M,1} + \Omega_{N,1})\Omega_{M,N} \\
&= \sum_{p,q} (e_{pq} \otimes f_{qp} \otimes 1 - f_{pq} \otimes e_{qp} \otimes 1) \sum_{i,j} (e_{ij} \otimes 1 \otimes f_{ji} - f_{ij} \otimes 1 \otimes e_{ji} \\
&\quad + 1 \otimes e_{ij} \otimes f_{ji} - 1 \otimes f_{ij} \otimes e_{ji}) + \sum_{i,j} (e_{ij} \otimes 1 \otimes f_{ji} - f_{ij} \otimes 1 \otimes e_{ji} \\
&\quad + 1 \otimes e_{ij} \otimes f_{ji} - 1 \otimes f_{ij} \otimes e_{ji}) \sum_{p,q} (e_{pq} \otimes f_{qp} \otimes 1 - f_{pq} \otimes e_{qp} \otimes 1) \\
&= \sum_{i,j,p,q} (e_{pq}e_{ij} - e_{ij}e_{pq}) \otimes f_{qp} \otimes f_{ji} + \sum_{i,j,p,q} (e_{pq}f_{ij} - f_{ij}e_{pq}) \otimes f_{qp} \otimes e_{ji} \\
&\quad + \sum_{i,j,p,q} e_{pq} \otimes (f_{qp}e_{ij} - e_{ij}f_{qp}) \otimes f_{ji} - \sum_{i,j,p,q} e_{pq} \otimes (f_{qp}f_{ij} + f_{ij}f_{qp}) \otimes e_{ji} \\
&\quad - \sum_{i,j,p,q} (f_{pq}e_{ij} - e_{ij}f_{pq}) \otimes e_{qp} \otimes f_{ji} + \sum_{i,j,p,q} (f_{pq}f_{ij} + f_{ij}f_{pq}) \otimes e_{qp} \otimes e_{ji} \\
&\quad - \sum_{i,j,p,q} f_{pq} \otimes (e_{qp}e_{ij} - e_{ij}e_{qp}) \otimes f_{ji} + \sum_{i,j,p,q} f_{pq} \otimes (e_{qp}f_{ij} - f_{ij}e_{qp}) \otimes e_{ji}
\end{aligned}$$

$$= \sum_{i,j,p,q} (\delta_{qi} e_{pj} - \delta_{jp} e_{iq}) \otimes f_{qp} \otimes f_{ji} + \sum_{i,j,p,q} (\delta_{qi} f_{pj} - \delta_{jp} f_{iq}) \otimes f_{qp} \otimes e_{ji}$$

$$+ \sum_{i,j,p,q} e_{pq} \otimes (\delta_{pi} f_{qj} - \delta_{jq} f_{ip}) \otimes f_{ji} - \sum_{i,j,p,q} e_{pq} \otimes (\delta_{pi} e_{qj} + \delta_{jq} e_{ip}) \otimes e_{ji}$$

$$- \sum_{i,j,p,q} (\delta_{qi} f_{pj} - \delta_{jp} f_{iq}) \otimes e_{qp} \otimes f_{ji} + \sum_{i,j,p,q} (\delta_{qi} e_{pj} + \delta_{jp} e_{iq}) \otimes e_{qp} \otimes e_{ji}$$

$$- \sum_{i,j,p,q} f_{pq} \otimes (\delta_{pi} e_{qj} - \delta_{jq} e_{ip}) \otimes f_{ji} + \sum_{i,j,p,q} f_{pq} \otimes (\delta_{pi} f_{qj} - \delta_{jq} f_{ip}) \otimes e_{ji}$$

$$= \sum_{j,p,q} e_{pj} \otimes f_{qp} \otimes f_{jq} - \sum_{i,p,q} e_{iq} \otimes f_{qp} \otimes f_{pi} + \sum_{j,p,q} f_{pj} \otimes f_{qp} \otimes e_{jq}$$

$$- \sum_{i,p,q} f_{iq} \otimes f_{qp} \otimes e_{pi} + \sum_{j,p,q} e_{pq} \otimes f_{qj} \otimes f_{jp} - \sum_{i,p,q} e_{pq} \otimes f_{ip} \otimes f_{qi}$$

$$- \sum_{j,p,q} e_{pq} \otimes e_{qj} \otimes e_{jp} - \sum_{i,p,q} e_{pq} \otimes e_{ip} \otimes e_{qi} - \sum_{j,p,q} f_{pj} \otimes e_{qp} \otimes f_{jq}$$

$$+ \sum_{i,p,q} f_{iq} \otimes e_{qp} \otimes f_{pi} + \sum_{j,p,q} e_{pj} \otimes e_{qp} \otimes e_{jq} + \sum_{i,p,q} e_{iq} \otimes e_{qp} \otimes e_{pi}$$

$$- \sum_{j,p,q} f_{pq} \otimes e_{qj} \otimes f_{jp} + \sum_{i,p,q} f_{pq} \otimes e_{ip} \otimes f_{qi} + \sum_{j,p,q} f_{pq} \otimes f_{qj} \otimes e_{jp}$$

$$- \sum_{i,p,q} f_{pq} \otimes f_{ip} \otimes e_{qi} = 0.$$

Therefore, $\tilde{z}_0 \tilde{z}_1 = -\tilde{z}_1 \tilde{z}_0$. Similarly, $\Omega_{M,1}(\Omega_{M,2}+\Omega_{1,2}) = -(\Omega_{M,2}+\Omega_{1,2})\Omega_{M,1}$, and $\tilde{x}_1 \tilde{x}_2 = -\tilde{x}_2 \tilde{x}_1$.

3) Further relations.

Since

$$\tilde{z}_2 \tilde{x}_1 + \tilde{x}_1 \tilde{z}_2 = (\Omega_{M,2} + \Omega_{N,2} + \Omega_{1,2})\Omega_{M,1} + \Omega_{M,1}(\Omega_{M,2} + \Omega_{N,2} + \Omega_{1,2})$$

On one hand, $(\Omega_{M,2} + \Omega_{1,2})\Omega_{M,1} + \Omega_{M,1}(\Omega_{M,2} + \Omega_{1,2}) = 0$ was shown earlier,

and

$$\begin{aligned}
&\Omega_{N,2}\Omega_{M,1}\\
&=\sum_{i,j}(1\otimes e_{ij}\otimes 1\otimes f_{ji}-1\otimes f_{ij}\otimes 1\otimes f_{ji})\\
&\quad\cdot\sum_{pq}(e_{pq}\otimes 1\otimes f_{qp}\otimes 1-f_{pq}\otimes 1\otimes e_{qp}\otimes 1)\\
&=\sum_{i,j,p,q}(-e_{pq}\otimes e_{ij}\otimes f_{qp}\otimes f_{ji}+f_{pq}\otimes e_{ij}\otimes e_{qp}\otimes f_{ji}\\
&\quad-e_{pq}\otimes f_{ij}\otimes e_{qp}\otimes f_{ji}-f_{pq}\otimes f_{ij}\otimes e_{qp}\otimes f_{ji})=-\Omega_{M,1}\Omega_{N,2}.
\end{aligned}$$

Lastly,

$$\begin{aligned}
&(\tilde{z}_0-\tilde{z}_1+\tilde{x}_1)\tilde{x}_1+\tilde{x}_1(\tilde{z}_0-\tilde{z}_1+\tilde{x}_1)\\
&=(\Omega_{M,N}-\Omega_{N,1})\Omega_{M,1}+\Omega_{M,1}(\Omega_{M,N}-\Omega_{N,1}),
\end{aligned}$$

where

$$\begin{aligned}
&\Omega_{M,N}\Omega_{M,1}+\Omega_{M,1}\Omega_{M,N}\\
&=\sum_{i,j}(e_{ij}\otimes f_{ji}\otimes 1-f_{ij}\otimes e_{ji}\otimes 1)\sum_{p,q}(e_{pq}\otimes 1\otimes f_{qp}-f_{pq}\otimes 1\otimes e_{qp})\\
&\quad+\sum_{p,q}(e_{pq}\otimes 1\otimes f_{qp}-f_{pq}\otimes 1\otimes e_{qp})\sum_{i,j}(e_{ij}\otimes f_{ji}\otimes 1-f_{ij}\otimes e_{ji}\otimes 1)\\
&=\sum_{i,j,p,q}((e_{ij}e_{pq}-e_{pq}e_{ij})\otimes f_{ji}\otimes f_{qp}+(e_{ij}f_{pq}-f_{pq}e_{ij})\otimes f_{ji}\otimes e_{qp}\\
&\quad-(f_{ij}e_{pq}-e_{pq}f_{ij})\otimes e_{ji}\otimes f_{qp}+(f_{ij}f_{pq}+f_{pq}f_{ij})\otimes e_{ji}\otimes f_{qp})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j,p,q} ((\delta_{jp}e_{iq} - \delta_{iq}e_{pj}) \otimes f_{ji} \otimes f_{qp} + (\delta_{jp}f_{iq} - \delta_{iq}f_{pj}) \otimes f_{ji} \otimes e_{qp} \\
&\qquad - (\delta_{jp}f_{iq} - \delta_{iq}f_{pj}) \otimes e_{ji} \otimes f_{qp} + (\delta_{ip}e_{iq} + \delta_{iq}e_{pj}) \otimes e_{ji} \otimes f_{qp}) \\
&= \sum_{i,p,q} e_{iq} \otimes f_{pi} \otimes f_{qp} - \sum_{j,p,q} e_{pj} \otimes f_{jq} \otimes f_{qp} + \sum_{i,p,q} f_{iq} \otimes f_{pi} \otimes e_{qp} \\
&\qquad - \sum_{j,p,q} f_{pj} \otimes f_{jq} \otimes e_{qp} - \sum_{i,p,q} f_{iq} \otimes e_{pi} \otimes f_{qp} + \sum_{j,p,q} f_{pj} \otimes e_{jq} \otimes f_{qp} \\
&\qquad + \sum_{j,p,q} e_{iq} \otimes e_{jp} \otimes f_{qp} + \sum_{j,p,q} e_{pj} \otimes e_{jq} \otimes f_{qp},
\end{aligned}
$$

and

$$
\begin{aligned}
&\Omega_{N,1}\Omega_{M,1} + \Omega_{M,1}\Omega_{N,1} \\
&= \sum_{i,j}(1 \otimes e_{ij} \otimes f_{ji} - 1 \otimes f_{ij} \otimes e_{ji}) \sum_{pq}(e_{pq} \otimes 1 \otimes f_{qp} - f_{pq} \otimes 1 \otimes e_{qp}) \\
&\quad + \sum_{pq}(e_{pq} \otimes 1 \otimes f_{qp} - f_{pq} \otimes 1 \otimes e_{qp}) \sum_{i,j}(1 \otimes e_{ij} \otimes f_{ji} - 1 \otimes f_{ij} \otimes e_{ji}) \\
&= \sum_{i,j,p,q} (e_{pq} \otimes e_{ij} \otimes (f_{ji}f_{qp} + f_{qp}f_{ji}) + f_{pq} \otimes e_{ij} \otimes (f_{ji}e_{qp} - e_{qp}f_{ji}) \\
&\quad - e_{pq} \otimes f_{ij} \otimes (e_{ji}f_{qp} - f_{qp}e_{ji}) - f_{pq} \otimes f_{ij} \otimes (e_{ji}e_{qp} - e_{qp}e_{ji})) \\
&= \sum_{i,j,p,q} (e_{pq} \otimes e_{ij} \otimes (\delta_{iq}e_{jp} + \delta_{jp}e_{qi}) + f_{pq} \otimes e_{ij} \otimes (\delta_{iq}f_{jp} - \delta_{jp}f_{qi}) \\
&\quad - e_{pq} \otimes f_{ij} \otimes (\delta_{iq}f_{jp} - \delta_{jp}f_{qi}) - f_{pq} \otimes f_{ij} \otimes (\delta_{iq}e_{jp} - \delta_{jp}e_{qi})) \\
&= \sum_{j,p,q} e_{pq} \otimes e_{qj} \otimes e_{jp} + \sum_{i,p,q} e_{pq} \otimes e_{ip} \otimes e_{qi} + \sum_{j,p,q} f_{pq} \otimes e_{qj} \otimes f_{jp} \\
&\quad - \sum_{i,p,q} f_{pq} \otimes e_{ip} \otimes f_{qi} - \sum_{j,p,q} e_{pq} \otimes f_{qj} \otimes f_{jp} + \sum_{i,p,q} e_{pq} \otimes f_{ip} \otimes f_{qi} \\
&\quad - \sum_{j,p,q} f_{pq} \otimes f_{qj} \otimes e_{jp} + \sum_{i,p,q} f_{pq} \otimes f_{ip} \otimes e_{qi}.
\end{aligned}
$$

By comparing the two expressions one obtains

$$(\Omega_{M,N} - \Omega_{N,1})\Omega_{M,1} + \Omega_{M,1}(\Omega_{M,N} - \Omega_{N,1}) = 0.$$

Therefore we have checked all the relations. □

3.3. When M or N Admits An Odd Endomorphism

Given two associative superalgebras A and B, define their tensor product $A \otimes B$ to be the algebra whose product is given by

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{\overline{b_1} \cdot \overline{a_2}}(a_1 a_2) \otimes (b_1 b_2)$$

where $a_1, a_2 \in A$, $b_1, b_2 \in B$ are homogeneous elements.

When N is a $\mathfrak{q}(n)$-module with a distinguished odd map $c \in \mathrm{End}_{\mathfrak{q}(n)}(N)$ and $c^2 = -1$ (for example, one can take N to be a simple module of Type Q), we define an enlarged Hecke-Clifford algebra $\mathcal{H}_d^N = \mathcal{H}_d \otimes \mathrm{Cl}_1$, where the Clifford generator of Cl_1 is denoted by c_0.

Notice that $\mathcal{H}_d^N$ has a presentation given by generators and relations. It is generated by all generators mentioned in Section 3.1 for $\mathcal{H}_d$, together with an extra generator c_0, subject to all defining relations for $\mathcal{H}_d$ and the following relations:

$$
\begin{aligned}
c_0^2 &= -1 \\
c_0 c_i &= -c_i c_0 && 1 \le i \le d \\
c_0 \tilde{x}_i &= -\tilde{x}_i c_0 && 1 \le i \le d \\
c_0 \tilde{y}_i &= -\tilde{y}_i c_0 && 1 \le i \le d \\
c_0 \tilde{z}_i &= -\tilde{z}_i c_0 && 0 \le i \le d \\
c_0 s_i &= s_i c_0 && 1 \le i \le d-1
\end{aligned}
$$

The following result shows that $\mathcal{H}_d^N$ also acts on $M \otimes N \otimes V^{\otimes d}$, when N admits a distinguished odd $\mathfrak{q}(n)$-endomorphism which squares to -1.

Theorem 3.8. *There is a well defined superalgebra homomorphism*

$$
\begin{aligned}
\rho^N : \mathcal{H}_d^N &\to \mathrm{End}_{\mathfrak{q}(n)}(M \otimes N \otimes V^{\otimes d}) \\
c_0 &\mapsto 1 \otimes c \otimes 1^{\otimes d}
\end{aligned}
$$

where c is the distinguished odd map in $\mathrm{End}(N)$*, and all other generators act as those specified in Theorem 3.7.*

Proof. Notice $\Omega_{M,1}(1 \otimes c \otimes 1) = -(1 \otimes c \otimes 1)\Omega_{M,1}$. By an earlier calculation, $\Omega_{N,1}(1 \otimes c \otimes 1) = -(1 \otimes c \otimes 1)\Omega_{N,1}$, therefore the additional relations $\tilde{x}_1 c_0 = -c_0 \tilde{x}_1$, $\tilde{y}_1 c_0 = -c_0 \tilde{y}_1$, $\tilde{z}_0 c_0 = -c_0 \tilde{z}_0$, $\tilde{z}_1 c_0 = -c_0 \tilde{z}_1$ holds. Other relations hold by a similar argument. □

With the addition of c_0, the element $\tilde{z}_0 c_0$ is even. We now give the final presentation we will be using in later chapters.

Proposition 3.9 (Main presentation)**.** *The algebra $\mathcal{H}_d^N$ is isomorphic to the algebra generated by x_1, $z_0, \dots, z_d$, $c_0, c_1, \dots, c_d$, $s_1, \dots, s_d$, where c_i is odd, s_i, x_1, z_i are even for all i, subject to the "Sergeev relations", and the following relations:*

(Hecke relations)

$$s_i z_i = z_{i+1} s_i - 1 + c_i c_{i+1} \qquad (1 \leq i \leq d-1)$$

(commuting relations)

$$x_1 s_i = s_i x_1 \qquad (2 \leq i \leq n)$$

$$z_j s_i = s_i z_j \qquad (j \neq i, i+1)$$

$$x_1(s_1 x_1 s_1 + (1 - c_1 c_2) s_1) = (s_1 x_1 s_1 + (1 - c_1 c_2) s_1) x_1$$

$$c_i x_1 = x_1 c_i \qquad (2 \leq i \leq d)$$

$$c_i z_j = z_j c_i \qquad (i \neq j)$$

(anticommuting relations)

$$c_1 x_1 = -x_1 c_1 \qquad 2 \leq i \leq d$$

$$c_i z_i = -z_i c_i \qquad (1 \leq i \leq d)$$

$$z_0 z_1 = z_1 z_0$$

(further relation)

$$z_2 x_1 = x_1 z_2$$

$$(z_0 c_0 c_1 + z_1 - x_1) x_1 = -x_1 (z_0 c_0 c_1 + z_1 - x_1)$$

Proof. Let $\mathcal{H}_d$ be the algebra with presentation given in Proposition 3.5, and $\mathcal{H}'_d$ the algebra defined in this proposition. Define $\phi : \mathcal{H}_d \to \mathcal{H}'_d$ via $\phi(\overline{x}_1) = -x_1 c_1$, $\phi(\tilde{z}_i) = -z_i c_i$, and other generators get mapped to the elements in $\mathcal{H}'_d$ with the same name. Define $\psi : \mathcal{H}'_d \to \mathcal{H}_d$ via $\psi(x_1) = \tilde{x}_1 c_1$, $\psi(z_i) = \tilde{z}_i c_i$, and other generators get mapped to elements in $\mathcal{H}_d$ with the same name. ϕ and ψ are inverses of each other, it is enough to show their are both homomorphism. In fact, the relations above are image of the relations in Proposition 3.5 via the map ϕ. In particular, when $i \neq j$,

$$
\begin{aligned}
&\phi(c_i \tilde{z}_j + \tilde{z}_j c_i) \\
&= -c_i z_j c_j - z_j c_j c_i \\
&= (-c_i z_j + z_j c_i) c_j
\end{aligned}
$$

when $i = j$,

$$
\begin{aligned}
&\phi(c_i \tilde{z}_i + \tilde{z}_i c_i) \\
&= -c_i z_i c_i - z_i c_i c_i \\
&= -(c_i z_i + z_i c_i) c_i
\end{aligned}
$$

For the Hecke relations,

$$\begin{aligned}
&\phi(s_i\tilde{z}_i - \tilde{z}_{i+1}s_i - (c_i - c_{i+1}))\\
&= -s_iz_ic_i + z_{i+1}c_{i+1}s_i - (c_i - c_{i+1})\\
&= -s_iz_ic_i + z_{i+1}s_ic_i - (c_i - c_{i+1})\\
&= (-s_iz_i + z_{i+1}s_i + (c_i - c_{i+1})c_i)c_i\\
&= (-s_iz_i + z_{i+1}s_i - (1 - c_ic_{i+1}))c_i
\end{aligned}$$

and the relation $x_1(s_1x_1s_1 + (1 - c_1c_2)s_1) = (s_1x_1s_1 + (1 - c_1c_2)s_1)x_1$ can be checked using $x_2 = s_1x_1s_1 + (1 - c_1c_2)s_1$ and $x_1x_2 = x_2x_1$, similar to the calculations for z_i. For the "further relations",

$$\begin{aligned}
&\phi(\tilde{z}_2\tilde{x}_1 + \tilde{x}_1\tilde{z}_2)\\
&= z_2c_2x_1c_1 + x_1c_1z_2c_2\\
&= z_2x_1c_2c_1 + x_1z_2c_1c_2\\
&= (z_2x_1 - x_1z_2)c_2c_1
\end{aligned}$$

and

$$\begin{aligned}
&\phi((\tilde{z}_0 - \tilde{z}_1 + \tilde{x}_1)\tilde{x}_1 + \tilde{x}_1(\tilde{z}_0 - \tilde{z}_1 + \tilde{x}_1))\\
&= (z_0c_0 - z_1c_1 + x_1c_1)x_1c_1 + x_1c_1(z_0c_0 - z_1c_1 + x_1c_1)\\
&= (-z_0c_0c_1 - z_1 + x_1)c_1x_1c_1 + x_1(-z_0c_0c_1 + z_1c_1^2 - x_1c_1^2)\\
&= (-z_0c_0c_1 - z_1 + x_1)x_1 + x_1(-z_0c_0c_1 - z_1 + x_1)\\
&= -(z_0c_0c_1 + z_1 - x_1)x_1 - x_1(z_0c_0c_1 + z_1 - x_1)
\end{aligned}$$

$\square$

Similarly, when both M and N are modules with odd maps $c_M \in \mathrm{End}_{\mathfrak{q}(\mathrm{n})}(\mathrm{M})$ and $c_N \in \mathrm{End}_{\mathfrak{q}(\mathrm{n})}(\mathrm{N})$, under the condition that $c_M^2 = c_N^2 = -1$, we can further define an enlarged version $\mathcal{H}_d^{MN} = \mathcal{H}_d \otimes \mathrm{Cl}_1 \otimes \mathrm{Cl}_1$. Let c_M and c_0 be the Clifford generators in each factor Cl_1.

Theorem 3.10. *There is a well defined homomorphism*

$$\rho^{MN} : \mathcal{H}_d^{MN} \to \mathrm{End}_{\mathfrak{q}(n)}(M \otimes N \otimes V^{\otimes d})$$
$$c_M \mapsto c \otimes 1 \otimes 1^{\otimes d}$$

where c is the odd map in $\mathrm{End}_{\mathfrak{q}(n)}(M)$, and all other generators act as those specified in the previous theorem.

3.4. The Degenerate Affine Hecke-Clifford Algebra

In [10], Hill-Kujawa-Sussan defined the degenerate affine Hecke-Clifford algebra $\mathcal{H}_d^{\mathrm{aff}}$ to be the algebra generated by $z_1, \ldots, z_d$, $c_1, \ldots, c_d$, $s_1, \ldots, s_d$ subject to the "Sergeev relations" in Definition 3.1 and the following relations

$$z_i z_j = z_j z_i \qquad (1 \leq i, j \leq d)$$
$$c_i z_j = z_j c_i \qquad (i \neq j)$$
$$c_i z_i = z_i c_i$$
$$s_i z_i = z_i s_i - 1 + c_i c_{i+1}$$

When $V = \mathbb{C}^{2n}$ is the natural representation for $\mathfrak{q}(n)$, elements in $\mathrm{End}(V)$ can be written as $2n \times 2n$ matrices. Recall $e_{ij}, f_{ij} \in \mathfrak{q}(n)$ are defined in (2.2.1) and 2.2.2, and the map $c \in \mathrm{End}_{\mathfrak{q}(n)}(V)$ can be written as in (2.2.9). Let $\overline{e}_{ij}, \overline{f}_{ij} \in \mathrm{End}(V)$ be given via the following matrices:

$$\overline{e}_{ij} = f_{ij} \circ c = \begin{bmatrix} E_{ij} & 0 \\ 0 & -E_{ij} \end{bmatrix} \qquad \overline{f}_{ij} = e_{ij} \circ c = \begin{bmatrix} 0 & -E_{ij} \\ E_{ij} & 0 \end{bmatrix}$$

Hill-Kujawa-Sussan [10, Section 7.2] defined an even Casimir element

$$\begin{aligned} \Omega^{\overline{0}} &= \sum_{1\leq i,j\leq n} e_{ij} \otimes \overline{e}_{ji} - \sum_{1\leq i,j\leq n} f_{ij} \otimes \overline{f}_{ji} \\ &= \sum_{1\leq i,j\leq n} e_{ij} \otimes (f_{ji} \circ c) - \sum_{1\leq i,j\leq n} f_{ij} \otimes (e_{ji} \circ c) \\ &= \Omega \circ (1 \otimes c) \in \mathfrak{q}(n) \otimes \mathrm{End(V)} \end{aligned}$$

We can use $\Omega^{\overline{0}}$ to define elements $\Omega^{\overline{0}}_{M,i} \in \mathfrak{q}(n) \otimes \mathrm{End}(V)^{\otimes d} = \iota_{M,i}(\Omega^{\overline{0}})$ where $\iota_{M,i}$ is the imbedding defined in (3.2.1).

Hill-Kujawa-Sussan then defined an action of $\mathcal{H}_d^{\mathrm{aff}}$ on $M \otimes V^{\otimes d}$:

Theorem 3.11. *[10, Theorem 7.4.1] Let M be a $\mathfrak{q}(n)$-supermodule. There is an algebra homomorphism*

$$\mathcal{H}_d^{\mathrm{aff}} \to \mathrm{End}_{\mathfrak{q}(n)}(M \otimes V^{\otimes d})$$

where the s_i and c_i act as generators for the Sergeev algebra in Theorem 3.6, and z_i acts as

$$\Omega^{\overline{0}}_{M,i} + \sum_{1\leq j<i} (c_j - c_i) s_{ji}$$

where $s_{ji} = s_j s_{j+1} \cdots s_{i-2} s_{i-1} s_{i-2} \cdots s_{j+1} s_j$ is the transposition that inter-

changes i and j.

Recall that $\mathcal{S}(d)$ is the Sergeev algebra on d letters. The action of $\mathcal{H}_d$ on $M \otimes N \otimes V^{\otimes d}$ specializes to the action of $\mathcal{S}(d)$ and $\mathcal{H}_d^{\mathrm{aff}}$ mentioned above, in the following sense.

Proposition 3.12. *1) There exists a surjective homomorphism $\phi_1 : \mathcal{H}_d \to \mathcal{S}(d)$. When $M = N = \mathbb{C}$ is the trivial $\mathfrak{q}(n)$-module, the action of $\mathcal{H}_d$ on $M \otimes N \otimes V^{\otimes d}$ factors through the quotient $\mathcal{S}(d)$ and induces the action introduced in Theorem 3.6.*

2) There exists a surjective homomorphism $\phi_2 : \mathcal{H}_d \to \mathcal{H}_d^{\mathrm{aff}}$. When $M = \mathbb{C}$ is the trivial $\mathfrak{q}(n)$-module, the action of $\mathcal{H}_d$ on $M \otimes N \otimes V^{\otimes d}$ factors through the quotient $\mathcal{H}_d^{\mathrm{aff}}$ and induces the action introduced in Theorem 3.11. A similar result is true for a different surjection $\phi_3 : \mathcal{H}_d \to \mathcal{H}_d^{\mathrm{aff}}$ and $N = \mathbb{C}$.

Proof. In Theorem 3.9 we gave a reduced presentation of $\mathcal{H}_d^N$ with even polynomial generators. In the general case when M and N are arbitrary finite dimensional $\mathfrak{q}(n)$-modules, there is a similar presentation of $\mathcal{H}_d$ with even polynomial generators $x_1, z_1, \ldots, z_d$ and odd polynomial generator $\tilde{z}_0$, by omitting the generator c_0 in $\mathcal{H}_d^N$ and modifying the relations involving z_0. Using this presentation, there is a homomorphism $\phi_1 : \mathcal{H}_d \to \mathcal{S}(d)$, given by $x_1 \mapsto 0$, $\tilde{z}_0 \mapsto 0$,$z_i \mapsto 0 (1 \leq i \leq d)$, where s_i and c_i get mapped to generators with the same name. This map is well-defined, since all relations regarding x_1, $\tilde{z}_0$ or z_i are satisfied after setting $x_1 = \tilde{z}_0 = z_i = 0$, and the remaining relations are the Sergeev relations in $\mathcal{S}(d)$. Moreover, the kernel of this map is generated by $x_1, z_0, \ldots, z_d$. In the special case when $M = N = \mathbb{C}$ is the trivial representation on which $\mathfrak{q}(n)$ acts as zero, the action of $x_1, z_0, \ldots, z_d$ is therefore 0, and the action of $\mathcal{H}_d$ factors through the quotient, where the generators $c_1, \ldots, c_d$ and

$s_1, \ldots, s_{d-1}$ act as introduced in Theorem 3.6.

When $M = \mathbb{C}$, there is a homomorphism $\phi_2 : \mathcal{H}_d \to \mathcal{H}_d^{\text{aff}}$ given by $x_1 \mapsto 0, \tilde{z}_0 \mapsto 0$, where the kernel is generated by x_1 and $\tilde{z}_0$. The action of $\mathcal{H}_d$ also factors through the quotient when N is the trivial representation. Moreover, x_i is defined recursively as $x_{i+1} = s_i x_i s_i + (1 - c_i c_{i+1}) s_i$. In Equation (3.2.2) we showed that

$$\Omega_{M,i+1} = s_i \Omega_{M,i} s_i$$

When x_{i-1} acts as z_{i-1} in Theorem 3.11, x_i acts as

$$\begin{aligned} & s_i(\Omega_{M,i} c_i + \sum_{1 \le j < i} (1 - c_j c_i) s_{ji}) s_i + (1 - c_i c_{i+1}) s_i \\ &= (s_i \Omega_{M,i} s_i) c_{i+1} + \sum_{1 \le j < i} (1 - c_j c_{i+1}) s_{j,i+1} + (1 - c_i c_{i+1}) s_{i,i+1} \\ &= \Omega_{M,i+1} c_{i+1} + \sum_{1 \le j < i+1} (1 - c_j c_{i+1}) s_{j,i+1} \end{aligned}$$

which agrees with the action of z_i.

The other claim when $N = \mathbb{C}$ can be proven similarly, using the homomorphism $\phi_3 : \tilde{z}_0 \mapsto 0$ and $x_1 \mapsto z_1$.

□

We also have the following result:

Corollary 3.13. *There are imbeddings of algebras* $\mathcal{S}(d) \hookrightarrow \mathcal{H}_d$ *and* $\mathcal{H}_d^{\text{aff}} \hookrightarrow \mathcal{H}_d$.

Proof. The map $\psi_1 : \mathcal{S}(d) \to \mathcal{H}_d$ is given by mapping all Sergeev generators to elements with the same name. Since the Sergeev relations are included in the defining relations of $\mathcal{H}_d$, this map is well defined, and the composition

$\phi_1 \circ \psi = \mathrm{id}$ where ϕ_1 is the surjection in the previous proposition, and ψ_1 is injective.

Similarly, the map $\psi_2 : \mathcal{H}_d^{\mathrm{aff}} \to \mathcal{H}_d$ is given by $c_i \mapsto c_i$, $s_i \mapsto s_i$ and $z_i \mapsto x_i (1 \leq i \leq d)$. The image of all relations in $\mathcal{H}_d^{\mathrm{aff}}$ are also relations of $\mathcal{H}_d$, and $\phi_2 \circ \psi_2 = \mathrm{id}$, hence ψ_2 is injective. $\square$

Chapter 4

A Quotient of $\mathcal{H}_d$

4.1. The Quotient $\mathcal{H}_d^p$

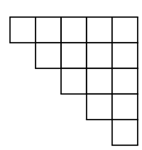
In this chapter we will define a quotient of $\mathcal{H}_d$ (and similarly of $\mathcal{H}_d^N$ and $\mathcal{H}_d^{MN}$.) These will be the Type Q equivalent of the extended degenerate affine Hecke algebra mentioned in [6]. The definition is dependent on specific choices of M and N. We will make the choices so the following is true:

1) Both M and N are polynomial, and therefore $M \otimes N \otimes V^{\otimes d}$ is semisimple.

2) The decomposition of $M \otimes N$ into simple modules with multiplicity 2. That is, each irreducible submodule of $M \otimes N$ occurs exactly twice in its isotypic component.

The decomposition of $M \otimes N \otimes V^{\otimes d}$ as $\mathfrak{q}(n)$-modules, and consequently the resulting $\mathcal{H}_d$-modules, can then be described combinatorially.

In particular, recall a strict partition λ is said to be a staircase shape if it is of the form $\lambda = (s, s-1, \ldots, 1)$. For example, when $s = 5$, the diagram is as follows:



Fix a positive integer n, and take $\alpha = (n, n-1, n-2, \ldots, 1)$, the staircase of length n. Let $M = L(\alpha)$ the associated highest weight irreducible $\mathfrak{q}(n)$-module defined in Section 2.2. In addition, fix a positive integer p and let

$\beta = (p, 0, 0, \ldots, 0)$, the partition with a single row of p boxes, and $N = L(\beta)$. By Lemma 1.44 in [17], for a strict partition λ, a simple module $L(\lambda)$ is of Type Q if and only if $\ell(\lambda)$ is odd. Therefore N is of Type Q, and M is of Type M if and only if n is even.

Define the quotient $\mathcal{H}_d^p$ by cases:

Definition 4.1. *1) When n is even, or equivalently $L(\alpha)$ is of Type M, define $\mathcal{H}_d^p$ to be a quotient of $\mathcal{H}_d^N$.*

2) When n is even, or equivalently $L(\alpha)$ is of Type Q, define $\mathcal{H}_d^p$ to be a quotient of $\mathcal{H}_d^{MN}$.

In both cases, define $\mathcal{H}_d^p$ to be the quotient of the corresponding degenerate affine Hecke-Clifford algebra under the following relations, which only depends on p and n.

$$\begin{aligned} \tilde{x}_1^2 - n(n+1) &= 0 \\ (\tilde{y}_1^2 - p(p+1))\tilde{y}_1^2 &= 0 \\ \tilde{x}_1\Phi^2 - \Phi^2\tilde{x}_1 &= 0. \end{aligned}$$

Here, we define $\Phi = z_0^2 - pz_1^2 + \frac{p(p+1)(p-1)}{2}$, an element in $\mathcal{H}_d$.

We also mention a presentation of $\mathcal{H}_d^p$ in the light of Proposition 3.9.

Corollary 4.2. *$\mathcal{H}_d^p$ is isomorphic to the algebra generated by $x_1, z_0, \ldots, z_d$, $c_0, \ldots, c_d, s_1, \ldots, s_d$ (and in the case when n is odd, an extra generator c_M),* *with the grading and relations in Proposition 3.9 and the extra relations*

$$x_1^2 = n(n+1)$$
$$(x_1 - z_1)^4 = p(p+1)(x_1 - z_1)^2.$$
$$x_1\Phi^2 = \Phi^2 x_1.$$

We will show these relations are satisfied by the actions ρ^N (in the case when n is even) or ρ^{MN} (in the case when n is odd) defined in the previous chapter, and the action factors through to give an action of $\mathcal{H}_d^p$. To show this we need some properties of the odd Casimir tensor Ω introduced in Section 3.2.

4.2. Casimir Elements

Sergeev [13] introduced certain central elements in $U(\mathfrak{q}(n))$. These elements are defined recursively as follows: $x_{ij}(1) = e_{ij}$, $x'_{ij}(1) = f_{ij}(1 \le i, j \le n)$, and

$$x_{ij}(m) = \sum_{s=1}^{n}(e_{is}x_{sj}(m-1) + (-1)^{m-1}f_{is}x'_{sj}(m-1))$$
$$x'_{ij}(m) = \sum_{s=1}^{n}(e_{is}x'_{sj}(m-1) + (-1)^{m-1}f_{is}x_{sj}(m-1))$$

The Sergeev's elements are

$$z_r = \sum_{i=1}^{n} x_{ii}(2r-1). \tag{4.2.1}$$

Brundan-Kleshchev calculated the action of z_i on a highest weight vector of weight $\lambda = (\lambda_1, \dots, \lambda_n)$:

Theorem 4.3. *[5, Lemma 8.4] Let M be a $\mathfrak{q}(n)$-modules and $v_\lambda \in M_\lambda$ be a* *vector annihilated by* e_{ij} *and* f_{ij}, $\forall 1 \leq i < j \leq n$, *then* $z_r.v_\lambda = z_r(\lambda)v$, *where*

$$z_r(\lambda) = \sum(-2)^{s+1}\lambda_{i_1}\cdots\lambda_{i_s}(\lambda_{i_1}^2 - \lambda_{i_1})^{a_1}\cdots(\lambda_{i_s}^2 - \lambda_{i_s})^{a_s}$$

and the sum is taken over all $1 \leq s \leq r$, $1 \leq i_1 < \cdots < i_s \leq n$, $a_i \in \mathbb{Z}_{\geq 0}$, $a_1 + \cdots + a_s = r - s$

Remark 4.4. *Left multiplication by the element* z_r *induces an even* $\mathfrak{q}(n)$*-endomorphism on* $L(\lambda)$*, and hence acts by a scalar according to Super Schur's Lemma mentioned in Lemma 2.1. Moreover, since* $v_\lambda \in L(\lambda)_\lambda$ *is a highest weight vector,* z_r *acts on* v_λ *by the above scalar. Hence for any* $v \in L(\lambda)$, $z_r.v = z_r(\lambda)v$.

Remark 4.5. *For future convenience the explicit formulas for elements of lower ranks are as follows*

$$\begin{aligned}
x_{ij}(2) &= \sum_{1\leq s\leq n}(e_{is}e_{sj} - f_{is}f_{sj}) \\
x'_{ij}(2) &= \sum_{1\leq s\leq n}(e_{is}f_{sj} - f_{is}e_{sj}) \\
x_{ij}(3) &= \sum_{1\leq s\leq n}(e_{is}x_{sj}(2) + f_{is}x'_{sj}(2)) \\
&= \sum_{1\leq s,k\leq n}(e_{is}e_{sk}e_{kj} - e_{is}f_{sk}f_{kj} + f_{is}e_{sk}f_{kj} - f_{is}f_{sk}e_{kj}) \\
z_1 &= \sum_{1\leq i\leq n} e_{ii} \\
z_2 &= \sum_{1\leq i\leq n} x_{ii}(3) \\
z_1(\lambda) &= \lambda_1 + \cdots \lambda_n \\
z_2(\lambda) &= (\lambda_1^3 + \cdots \lambda_n^3) - (\lambda_1 + \cdots + \lambda_n)^2
\end{aligned}$$

Theorem 4.3 along with the following proposition will allow us to compute the action of $z_0, \ldots, z_d$ on $L(\alpha) \otimes L(\beta) \otimes V^{\otimes d}$:

Proposition 4.6. *Let z_i be the elements defined in* (4.2.1)*. The following is true*

$$\Omega^2 = \frac{1}{3}(\Delta(z_2) - z_2 \otimes 1 - 1 \otimes z_2 + 2z_1 \otimes z_1).$$

Proof. First,

$$z_2 = \sum_{1 \leq i,s,k \leq n} (e_{is}e_{sk}e_{ki} - e_{is}f_{sk}f_{ki} + f_{is}e_{sk}f_{ki} - f_{is}f_{sk}e_{ki})$$

where

$$\Delta(\sum_{1 \leq i,s,k \leq n} e_{is}e_{sk}e_{ki})$$
$$= \sum_{1 \leq i,s,k \leq n} (e_{is} \otimes 1 + 1 \otimes e_{is})(e_{sk} \otimes 1 + 1 \otimes e_{sk})(e_{ki} \otimes 1 + 1 \otimes e_{ki})$$
$$= \sum_{1 \leq i,s,k \leq n} (1 \otimes e_{is}e_{sk}e_{ki} + e_{is} \otimes e_{sk}e_{ki} + e_{sk} \otimes e_{is}e_{ki} + e_{ki} \otimes e_{is}e_{sk} +$$
$$= e_{is}e_{sk} \otimes e_{ki} + e_{is}e_{ki} \otimes e_{sk} + e_{sk}e_{ki} \otimes e_{is} + e_{is}e_{sk}e_{ki} \otimes 1)$$

$$
\begin{aligned}
&-\Delta(\sum_{1\le i,s,k\le n} e_{is}f_{sk}f_{ki})\\
=&-\sum_{1\le i,s,k\le n}(e_{is}\otimes 1+1\otimes e_{is})(f_{sk}\otimes 1+1\otimes f_{sk})(f_{ki}\otimes 1+1\otimes f_{ki})\\
=&-\sum_{1\le i,s,k\le n}(e_{is}f_{sk}f_{ki}\otimes 1+e_{is}f_{sk}\otimes f_{ki}-e_{is}f_{ki}\otimes f_{sk}+f_{sk}f_{ki}\otimes e_{is}\\
&+e_{is}\otimes f_{sk}f_{ki}+f_{sk}\otimes e_{is}f_{ki}-f_{ki}\otimes e_{is}f_{sk}+1\otimes e_{is}f_{sk}f_{ki})
\end{aligned}
$$

$$
\begin{aligned}
&\Delta(\sum_{1\le i,s,k\le n} f_{is}e_{sk}f_{ki})\\
=&\sum_{1\le i,s,k\le n}(f_{is}\otimes 1+1\otimes f_{is})(e_{sk}\otimes 1+1\otimes e_{sk})(f_{ki}\otimes 1+1\otimes f_{ki})\\
=&\sum_{1\le i,s,k\le n}(f_{is}e_{sk}f_{ki}\otimes 1+f_{is}e_{sk}\otimes f_{ki}+f_{is}f_{ki}\otimes e_{sk}-e_{sk}f_{ki}\otimes f_{is}\\
&+f_{is}\otimes e_{sk}f_{ki}+e_{sk}\otimes f_{is}f_{ki}-f_{ki}\otimes f_{is}e_{sk}+1\otimes f_{is}e_{sk}f_{ki})
\end{aligned}
$$

$$
\begin{aligned}
&-\Delta(\sum_{1\le i,s,k\le n} f_{is}f_{sk}e_{ki}))\\
=&-\sum_{1\le i,s,k\le n}(f_{is}f_{sk}e_{ki}\otimes 1+f_{is}f_{sk}\otimes e_{ki}+f_{is}e_{ki}\otimes f_{sk}-f_{sk}e_{ki}\otimes f_{is}\\
&+f_{is}\otimes f_{sk}e_{ki}-f_{sk}\otimes f_{is}e_{ki}+e_{ki}\otimes f_{is}f_{sk}+1\otimes f_{is}f_{sk}e_{ki})
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&\Delta(z_2) - z_2 \otimes 1 - 1 \otimes z_2 \\
= &\sum_{1\le i,s,k\le n} (e_{is}e_{sk} \otimes e_{ki} + e_{is}e_{ki} \otimes e_{sk} + e_{sk}e_{ki} \otimes e_{is} \\
&- e_{is}f_{sk} \otimes f_{ki} + e_{is}f_{ki} \otimes f_{sk} - e_{sk}f_{ki} \otimes f_{is} \\
&+ f_{is}e_{sk} \otimes f_{ki} - f_{is}e_{ki} \otimes f_{sk} + f_{sk}e_{ki} \otimes f_{is} \\
&- f_{is}f_{sk} \otimes e_{ki} + f_{is}f_{ki} \otimes e_{sk} - f_{sk}f_{ki} \otimes e_{is} \\
&- f_{is} \otimes f_{sk}e_{ki} + f_{sk} \otimes f_{is}e_{ki} - f_{ki} \otimes f_{is}e_{sk} \\
&+ e_{is} \otimes e_{sk}e_{ki} + e_{sk} \otimes e_{is}e_{ki} + e_{ki} \otimes e_{is}e_{sk} \\
&+ f_{is} \otimes e_{sk}f_{ki} - f_{sk} \otimes e_{is}f_{ki} + f_{ki} \otimes e_{is}f_{sk} \\
&- e_{is} \otimes f_{sk}f_{ki} + e_{sk} \otimes f_{is}f_{ki} - e_{ki} \otimes f_{is}f_{sk})
\end{aligned}$$

On the other hand, let us compute Ω^2. All sums without indexing set are understood to be taken over $1 \leq i, j, p, q \leq n$.

$$
\begin{aligned}
&3\Omega^2 \\
&= \sum_{1\leq i,j\leq n} 3(e_{ij}\otimes f_{ji} - f_{ij}\otimes e_{ji}) \sum_{1\leq p,q\leq n} (e_{pq}\otimes f_{qp} - f_{pq}\otimes e_{qp}) \\
&= \sum 3(e_{ij}e_{pq}\otimes f_{ji}f_{qp} + e_{ij}f_{pq}\otimes f_{ji}e_{qp} - f_{ij}e_{pq}\otimes e_{ji}f_{qp} + f_{ij}f_{pq}\otimes e_{ji}e_{qp}) \\
&= \sum \frac{3}{2}(e_{ij}e_{pq}\otimes f_{ji}f_{qp} - e_{pq}e_{ij}\otimes f_{ji}f_{qp} + e_{pq}e_{ij}\otimes f_{ji}f_{qp} + e_{ij}e_{pq}\otimes f_{ji}f_{qp}) \\
&\quad + \sum 3(e_{ij}f_{pq}\otimes f_{ji}e_{qp} - f_{pq}e_{ij}\otimes f_{ji}e_{qp} + f_{pq}e_{ij}\otimes f_{ji}e_{qp} - f_{ij}e_{pq}\otimes e_{ji}f_{qp}) \\
&\quad + \sum \frac{3}{2}(f_{ij}f_{pq}\otimes e_{ji}e_{qp} + f_{pq}f_{ij}\otimes e_{ji}e_{qp} - f_{pq}f_{ij}\otimes e_{ji}e_{qp} + f_{ij}f_{pq}\otimes e_{ji}e_{qp}) \\
&= \sum \frac{3}{2}((\delta_{jp}e_{iq} - \delta_{iq}e_{pj}))\otimes f_{ji}f_{qp} + e_{pq}e_{ij}\otimes f_{ji}f_{qp} + e_{pq}e_{ij}\otimes f_{qp}f_{ji}) \\
&\quad + \sum 3((\delta_{jp}f_{iq} - \delta_{iq}f_{pj})\otimes f_{ji}e_{qp} + f_{pq}e_{ij}\otimes f_{ji}e_{qp} - f_{pq}e_{ij}\otimes e_{qp}f_{ji}) \\
&\quad + \sum \frac{3}{2}((\delta_{jp}e_{iq} + \delta_{iq}e_{pj})\otimes e_{ji}e_{qp} - f_{pq}f_{ij}\otimes e_{ji}e_{qp} + f_{pq}f_{ij}\otimes e_{qp}e_{ji}) \\
&= \sum_{i,p,q} \frac{3}{2}e_{iq}\otimes f_{pi}f_{qp} - \sum_{j,p,q} \frac{3}{2}e_{pj}\otimes f_{jq}f_{qp} + \sum \frac{3}{2}e_{pq}e_{ij}\otimes (\delta_{iq}e_{jp} + \delta_{jp}e_{qi}) \\
&\quad + \sum_{i,p,q} 3f_{iq}\otimes f_{pi}e_{qp} - \sum_{j,p,q} 3f_{pj}\otimes f_{jq}e_{qp} + \sum 3f_{pq}e_{ij}\otimes (\delta_{iq}f_{jp} - \delta_{jp}f_{qi}) \\
&\quad + \sum_{i,p,q} \frac{3}{2}e_{iq}\otimes e_{pi}e_{qp} + \sum_{j,p,q} \frac{3}{2}e_{pj}\otimes e_{jq}e_{qp} - \sum \frac{3}{2}f_{pq}f_{ij}\otimes (\delta_{iq}e_{jp} - \delta_{jp}e_{qi}) \\
&= \sum_{i,p,q} \frac{3}{2}e_{iq}\otimes f_{pi}f_{qp} - \sum_{j,p,q} \frac{3}{2}e_{pj}\otimes f_{jq}f_{qp} + \sum_{j,p,q} \frac{3}{2}e_{pq}e_{qj}\otimes e_{jp} + \sum_{i,p,q} \frac{3}{2}e_{pq}e_{ip}\otimes e_{qi} \\
&\quad + \sum_{i,p,q} 3f_{iq}\otimes f_{pi}e_{qp} - \sum_{j,p,q} 3f_{pj}\otimes f_{jq}e_{qp} + \sum_{j,p,q} 3f_{pq}e_{qj}\otimes f_{jp} - \sum_{i,p,q} 3f_{pq}e_{ip}\otimes f_{qi} \\
&\quad + \sum_{i,p,q} \frac{3}{2}e_{iq}\otimes e_{pi}e_{qp} + \sum_{j,p,q} \frac{3}{2}e_{pj}\otimes e_{jq}e_{qp} - \sum_{j,p,q} \frac{3}{2}f_{pq}f_{qj}\otimes e_{jp} + \sum_{i,p,q} \frac{3}{2}f_{pq}f_{ip}\otimes e_{qi}
\end{aligned}
$$

Comparing the two results,

$$
\begin{aligned}
&3\Omega^2 - (\Delta(z_2) - z_2 \otimes 1 - 1 \otimes z_2) \\
= &\sum_{i,p,q} \frac{1}{2}(e_{iq} \otimes f_{pi}f_{qp} + e_{iq} \otimes f_{qp}f_{pi}) + \sum_{j,p,q} \frac{1}{2}(e_{pq}e_{jp} \otimes e_{qj} - e_{jp}e_{pq} \otimes e_{qj}) \\
&+ \sum_{i,p,q} 2(f_{iq} \otimes f_{pi}e_{qp} - f_{iq} \otimes e_{qp}f_{pi}) - \sum_{j,p,q}(f_{pj} \otimes f_{jq}e_{qp} - f_{pj} \otimes e_{qp}f_{jq}) \\
&+ \sum_{j,p,q}(f_{pq}e_{qj} \otimes f_{jp} - e_{qj}f_{pq} \otimes f_{jp}) - \sum_{i,p,q} 2(f_{pq}e_{ip} \otimes f_{qi} - e_{ip}f_{pq} \otimes f_{qi}) \\
&+ \sum_{i,p,q} \frac{1}{2}(e_{iq} \otimes e_{pi}e_{qp} - e_{iq} \otimes e_{qp}e_{pi}) + \sum_{j,p,q} \frac{1}{2}(f_{pq}f_{qj} \otimes e_{jp} + f_{qj}f_{pq} \otimes e_{jp}) \\
= &\sum_{i,p,q} \frac{1}{2}e_{iq} \otimes (\delta_{iq}e_{pp} + e_{qi}) + \sum_{j,p,q} \frac{1}{2}(\delta_{qj}e_{pp} - e_{jq}) \otimes e_{qj} \\
&+ \sum_{i,p,q} 2f_{iq} \otimes (\delta_{iq}f_{pp} - f_{qi}) - \sum_{j,p,q} f_{pj} \otimes (f_{jp} - \delta_{pj}f_{qq}) \\
&+ \sum_{j,p,q}(f_{pj} - \delta_{pj}f_{qq}) \otimes f_{jp} - \sum_{i,p,q} 2(\delta_{qi}f_{pp} - f_{iq}) \otimes f_{qi} \\
&+ \sum_{i,p,q} \frac{1}{2}e_{iq} \otimes (\delta_{iq}e_{pp} - e_{ip}) + \sum_{j,p,q} \frac{1}{2}(e_{pj} + \delta_{pj}e_{qq}) \otimes e_{jp} \\
= &\,2\sum_{i,p} e_{ii} \otimes e_{pp} = 2(\sum_i e_{ii}) \otimes (\sum_p e_{pp}) = 2z_1 \otimes z_1
\end{aligned}
$$

□

4.3. $L(\alpha) \otimes L(\beta) \otimes V^{\otimes d}$ as an $\mathcal{H}_d^p$ -Module

Let $\mathbb{Z}_{++}^n$ be the set of strict partition with at most n rows, as defined in (2.3.1). Recall from Section 2.3, as a consequence of the Stembridge rule, for $\lambda \in \mathbb{Z}_{++}^n$, $L(\lambda) \otimes V \simeq \oplus_\gamma c_{\lambda,\epsilon_1}^\gamma L(\gamma)$ where the sum is over all $\gamma \in \mathbb{Z}_{++}^n$ that can be obtained from λ by adding a single box. Ω acts on $L(\lambda) \otimes V$ and consequently on the summand $L(\gamma)$.

Recall that for a box b in a shifted Young diagram, we write $c(b)$ for the content of b, as defined in the beginning of Section 2.3. We compute the action of Ω^2 explicitly.

Lemma 4.7. *Let $L(\gamma)$ be an irreducible summand of $L(\lambda) \otimes V$. $\forall v \in L(\gamma)$, $\Omega^2.v = q(c(\gamma/\lambda))v$, where $q(t) := t(t+1)$ for any integer t, and $c(\gamma/\lambda)$ is the content of the distinct box in γ that is not in λ.*

Proof. By the previous proposition, Ω^2 acts as $\frac{1}{3}(\Delta(z_2) - z_2 \otimes 1 - 1 \otimes z_2 + 2z_1 \otimes z_1)$. Also recall that z_i acts on $L(\lambda)$ by the scalar $z_i(\lambda)$ as defined in the previous section, and $V = L(\epsilon_1)$. Assume $\gamma = \lambda + \epsilon_i$ for some $1 \leq i \leq n$. In particular, let $|\lambda| = \lambda_1 + \cdots \lambda_n$,

$$
\begin{aligned}
z_1(\lambda) &= |\lambda| \\
z_2(\lambda) &= \lambda_1^3 + \cdots \lambda_n^3 - |\lambda|^2 \\
z_1(\gamma) &= |\lambda| + 1 \\
z_2(\gamma) &= \lambda_1^3 + \cdots + (\lambda_i + 1)^2 + \cdots + \lambda_n^3 - (|\lambda| + 1)^2 \\
z_1(\epsilon_1) &= 1 \\
z_2(\epsilon_1) &= 0
\end{aligned}
$$

Therefore, Ω^2 acts as the scalar

$$
\begin{aligned}
&\frac{1}{3}(z_2(\gamma) - z_2(\lambda) - z_2(\epsilon_1) + 2z_1(\lambda)z_1(\epsilon_1)) \\
=& \frac{1}{3}((\lambda_1^3 + \cdots + (\lambda_i + 1)^2 + \cdots + \lambda_n^3 - (|\lambda| + 1)^2) \\
&- (\lambda_1^3 + \cdots \lambda_n^3 - |\lambda|^2)) + 2|\lambda| \\
=& \lambda_i(\lambda_i + 1) = q(c(\gamma/\lambda))
\end{aligned}
$$

and the last equality holds because the content of the first box in any row is zero, and the content of the added box is equal to the number of boxes in the i-th row of λ. $\square$

Recall for the fixed choice of n and p (and therefore α and β), in Lemma 2.3 we determined the set

$$\begin{aligned}\mathcal{P}_0(\alpha,\beta) &= \{\gamma \mid L(\gamma) \text{ is a direct summand of } L(\alpha)\otimes L(\beta)\}\\ &= \{\alpha+\tau \mid \tau = (s,1,1,\ldots,1), s+\ell(\tau) = p+1\}\end{aligned}$$

Here, $\ell(\tau)$ denotes the number of nonzero rows in τ.

Lemma 4.8. *Let $L(\gamma) \subset L(\alpha)\otimes L(\beta)$ be an irreducible summand, $\gamma = \alpha + (s,1,1,\ldots,1)$. Further let $m = \gamma_1$ be the number of boxes in the first row of γ, then $\forall v \in L(\gamma)$,*

$$\Omega^2.v = mp(m-p)v$$

Proof. Observe

$$\begin{aligned} z_1(\alpha) &= \frac{n(n+1)}{2}\\ z_2(\alpha) &= (1^3+2^3+\cdots+n^3)-(1+2+\cdots+n)^2 = 0\\ z_1(\beta) &= p\\ z_2(\beta) &= p^3-p^2\end{aligned}$$

On the other hand, notice

$$\begin{aligned}
\gamma &= \alpha + (s, 1, 1, \dots, 1) \\
&= (n+s, n, n-1, \dots, n-p+s+1, n-p+s-1, \dots, 1) \\
z_1(\gamma) &= \frac{n(n+1)}{2} + p \\
z_2(\gamma) &= (n+s)^3 + (1^3 + 2^3 + \cdots + n^3) \\
&\quad - (n-p+s)^3 - (\frac{n(n+1)}{2} + p)^2
\end{aligned}$$

Therefore, for any $v \in L(\gamma)$, Ω^2 acts by the scalar

$$\begin{aligned}
&\frac{1}{3}(z_2(\gamma) - z_2(\alpha) - z_2(\beta) + 2z_1(\alpha)z_1(\beta)) \\
=& \frac{1}{3}((n+s)^3 + (1^3 + 2^3 + \cdots + n^3) - (n-p+s)^3 \\
&- (\frac{n(n+1)}{2} + p)^2 - (p^3 - p^2) + 2 \cdot \frac{n(n+1)}{2} \cdot p) \\
=& \frac{1}{3}((n+s)^3 - (n-p+s)^3 - p^3) \\
=& \frac{1}{3}(m^3 - (m-p)^3 + p^3) \\
=& \frac{1}{3}(3m^2p - 3mp^2) \\
=& mp(m-p)
\end{aligned}$$

□

Using the above two lemmas, we can deduce the action of Ω on an irreducible summand.

Lemma 4.9. *1) For $L(\gamma)$ that is an irreducible summand of $L(\mu) \otimes V$, then $\forall \mu \in \{\lambda \in \mathbb{Z}^n_{\geq 0} | \lambda_1 > \lambda_2 > \cdots > \lambda_n > 0\}$, Ω acts on $L(\gamma)$ via the nonzeroscalar $\sqrt{c(b)(c(b)+1)}$ or $-\sqrt{c(b)(c(b)+1)}$, where $c(b)$ is the content of the distinct* *box b in γ but not in μ.*

2) For $L(\gamma)$ that is an irreducible summand of $L(\alpha) \otimes L(\beta)$, Ω acts on $L(\gamma)$ via the nonzeroscalar $\sqrt{mp(m-p)}$ or $-\sqrt{mp(m-p)}$, where m is the number of boxes in the first row of γ.

Proof. In Case 2), we showed in equation (2.4.3) that since $\ell(\beta) = 1$, the multiplicity of $L(\gamma)$ in $L(\alpha) \otimes L(\beta)$ is 2. Since Ω is central in $\Delta(U(\mathfrak{q}(n))) \subset U(\mathfrak{q}(n)) \otimes U(\mathfrak{q}(n))$, the action of Ω commutes with $\mathfrak{q}(n)$ on $L(\alpha) \otimes L(\beta)$. By Schur's Lemma, Ω acts on the isotypic component $L(\lambda)^{\oplus 2}$.

Since Ω^2 acts on $L(\alpha) \otimes L(\beta)$ via the scalar $a = mp(m-p)$, Ω has possible generalized eigenvalues $\sqrt{a}$ or $-\sqrt{a}$ on $L(\lambda)^{\oplus 2}$. Let U be the generalized $\sqrt{a}$-eigenspace of Ω. Since Ω commutes with $\Delta(U(\mathfrak{q}(n))$, U is a $\mathfrak{q}(n)$-submodule of $L(\lambda)^{\oplus 2}$.

If $U \simeq L(\lambda)$, then since Ω induces an even $\mathfrak{q}(n)$-modules homomorphism, Ω act by $\sqrt{a}$ on U by Super Schur's Lemma 2.1. Similarly, the generalized $(-\sqrt{a})$-eigenspace of Ω is also the $(-\sqrt{a})$-eigenspace, therefore Ω acts by scalars $\sqrt{a}, -\sqrt{a}$ on each $L(\lambda)$.

If $U \simeq L(\lambda)^{\oplus 2}$, let $t = \dim L(\lambda)$ and choose a basis so that Ω acts via an uppertriangular matrix. Let W, T be the two subspaces spanned by the first t and last t vectors in this basis. Since Ω acts invariantly on W, $W \simeq L(\lambda)$ and Ω acts on W via $\sqrt{a}$, Moreover, Ω induces a $\mathfrak{q}(n)$-modules homomorphism $U/W \to U/W$, therefore acts by the scalar $\sqrt{a}$ on $U/W \simeq L(\lambda)$. The matrix is as follows,

$$\begin{bmatrix} \sqrt{a} I_t & A \\ 0 & \sqrt{a} I_t \end{bmatrix}$$

By using $\Omega^2 = a \cdot I_{2t}$ and the fact that $a \neq 0$, we obtain that $A = 0$, therefore Ω acts on $L(\lambda)^{\oplus 2}$ via $\sqrt{\alpha}$. When $U = 0$, the same arguments applies to the generalized $(-\sqrt{\alpha})$-eigenspace and Ω acts on $L(\lambda)^{\oplus 2}$ via $-\sqrt{\alpha}$.

The argument is similar in case 1) as $\ell(\epsilon_1) = 1$. We now argue that the scalar α is never zero in either case. In case 1), $\alpha = c(b)(c(b)+1)$, and the added box has content 0 if and only if it sits at the beginning of each row, which is never true when the entries of μ are nonzero. In case 2), $\alpha = mp(m-p)$, which is only zero if $m = p$. However, using the combinatorial result in Lemma 2.3, if $\gamma = \alpha + \mu$, since $\ell(\mu) \leq n$, $p = |\mu| \leq (m-n) + (n-1) = m-1$, and α is never zero.

$\square$

Recall $\Phi = z_0^2 - pz_1^2 + \frac{p(p+1)(p-1)}{2}$ as in Definition 4.1. Since the elements z_0 and z_1 act only on the first three tensor factors of $L(\alpha) \otimes L(\beta) \otimes V^{\otimes d}$, we slightly abuse the notation Φ by considering it as an action on $L(\alpha) \otimes L(\beta) \otimes V$.

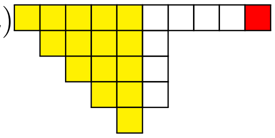
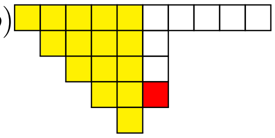
Lemma 4.10. *On any isotypic component of $L(\lambda)$ in the direct sum decomposition of $L(\alpha) \otimes L(\beta) \otimes V$, Φ^2 acts as a scalar which only depends on λ. Moreover, if λ only has only a single box with content n, the scalar is nonzero.*

Proof. We perform the decomposition as follows:

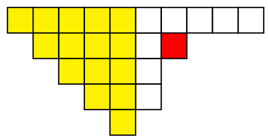
$$(L(\alpha) \otimes L(\beta)) \otimes V = \Big(\bigoplus_{\mu \in \mathcal{P}_0(\alpha,\beta)} L(\mu)\Big) \otimes V = \bigoplus_{\lambda} L(\lambda)$$

where λ ranges over all partitions such that $L(\lambda)$ is a summand of $L(\alpha) \otimes L(\beta) \otimes V$ In particular, each copy of $L(\lambda)$ occurs as a direct summand of $L(\mu) \otimes V$, for some $\mu \in \mathcal{P}_0(\alpha, \beta)$. Notice μ must be obtained by removing a box in λ. Recall in Lemma 2.3 we gave a cominatorial description of elements in $\mathcal{P}_0(\alpha, \beta)$. We list the two possibilities of the partition λ, where the yellow boxes highlight α, and the red box is the unique box in λ but not in μ:

1) The partition λ is of the form:

$a)$  $b)$ 

2) The partition λ is of the form:

 (4.3.1)

In case 2), if the red box is in any other position, the other boxes would not form a partition $\mathcal{P}_0(\alpha, \beta)$.

In all cases, let m be the number of boxes in the first row of λ. We calculate the action of Φ in each case

Case 1a):

By the previous lemma, z_0^2 acts as $(m-1)p(m-1-p)$, the red box has content $m-1$ therefore z_1^2 acts as $m(m-1)$, therefore Φ acts as

$$(m-1)p(m-1-p) - p(m-1)m + \frac{p(p+1)(p-1)}{2}$$
$$= p(p+1)(1-m+\frac{p-1}{2}) = p(p+1)(-m+\frac{p}{2}+\frac{1}{2})$$

Case 2a):

Similarly, z_0^2 acts as $mp(m-p)$, the red box has content $m-p-1$ and therefore z_1^2 acts as $(m-p)(m-p-1)$, Φ acts as

$$\begin{aligned} &mp(m-p) - p(m-p)(m-p-1) + \frac{p(p+1)(p-1)}{2} \\ &= p(p+1)(m-p+\frac{p-1}{2}) = p(p+1)(m-\frac{p}{2}-\frac{1}{2}) \end{aligned}$$

which is the opposite of the scalar above, therefore Φ^2 acts as a constant when λ is of the form in case 1). When λ only has a single box of content n, λ takes the form in Case 1), and the scalar is zero if and only if $m - \frac{p}{2} - \frac{1}{2} = 0$ or $2m = p+1$. Observe for any partition $\mu = \alpha + \gamma \in \mathcal{P}_0(\alpha, \beta)$, the combinatorial result in Lemma 2.3 implies that $\ell(\gamma) \leq n$, therefore $p \leq m$ and $p+1 < 2m$.

Case 2): The element z_0^2 acts as $mp(m-p)$ and the red box has content n, therefore Φ acts as

$$mp(m-p) - pn(n+1) + \frac{p(p-1)(p+1)}{2}$$

and Φ^2 acts as a constant.

On the other hand,

$$L(\alpha) \otimes L(\beta) \otimes V \simeq L(\alpha) \otimes V \otimes L(\beta) \simeq (L(\alpha + \epsilon_1)^{\otimes 2}) \otimes L(\beta)$$

and for any irreducible summand $L(\lambda) \subset (L(\alpha+\epsilon_1)^{\otimes 2}) \otimes V$, x_1 acts via a scalar by the previous lemma.

□

Theorem 4.11. *The action ρ^N (when n is even) and the action ρ^{MN} (when*

n is odd) satisfy the extra relations for $\mathcal{H}_d^p$, and induce a further action

$$\phi : \mathcal{H}_d^p \to \mathrm{End}_{\mathfrak{q}(n)}(L(\alpha) \otimes L(\beta) \otimes V^{\otimes d})$$

Proof. Since $L(\alpha) \otimes L(\beta) \otimes V^{\otimes d} \simeq L(\alpha) \otimes V^{\otimes d} \otimes L(\beta)$, and for any irreducible summand $L(\gamma) \subset L(\alpha) \otimes V$, Ω^2 acts on $L(\gamma)$ by $q(c(\gamma/\alpha))$ as shown in Lemma 4.7. Since $x_1^2 = \tilde{x}_1^2$ acts on the entry $M \otimes V$ and therefore on $L(\gamma)$, it also acts on $L(\gamma) \otimes V^{\otimes d-1} \otimes N$ by the same scalar. Since the only way to add a box to α is to add a box to the first row with content n, the action of $x_1^2 = \tilde{x}_1^2 = \Omega^2$ has to be the scalar $n(n+1)$.

The argument is similar for the action of y_1. Since $L(\alpha) \otimes L(\beta) \otimes V^{\otimes d} \simeq L(\beta) \otimes V^{\otimes d} \otimes L(\alpha)$, and for any irreducible summand $L(\gamma) \subset L(\beta) \otimes V$, Ω^2 acts on $L(\gamma)$ by $q(c(\gamma/\beta))$ as shown in Lemma 4.7. Since $y_1^2 = \tilde{y}_1^2$ acts on the entry $N \otimes V$ and therefore on $L(\gamma)$, it also acts on $L(\gamma) \otimes V^{\otimes d-1} \otimes M$ by the same scalar. There are two ways to add a box to β: by adding a box to the first row with content p or a box to the second row with content 0. Therefore the action of y_1^2 is either $p(p+1)$ or 0, satisfying the extra relation.

To see that the relation $x_1 \Phi^2 = \Phi^2 x_1$ is satisfied, we showed in Lemma 4.9 that $x_1 = \Omega_{M,1}$ acts invariantly on any irreducible summand of $L(\alpha) \otimes V$ and therefore any irreducible summand of $L(\alpha) \otimes L(\beta) \otimes V$, on the other hand, by Lemma 4.10, Φ^2 acts as a scalar on $L(\lambda) \subset L(\alpha) \otimes L(\beta) \otimes V$ which only depends on Λ, therefore the two actions commute.

□

Chapter 5

Construction of Calibrated Modules

5.1. The Bratteli Graph

The goal of this section is to construct a certain class of modules for $\mathcal{H}_d^p$ using combinatorial tools. We first introduce the Bratteli graph, the key combinatorial data which leads to the construction. Its connection to the module $M \otimes N \otimes V^{\otimes d}$ will be explained in the next few sections. Recall α is the staircase of height n, and $\beta = (p, 0, 0, \ldots, 0)$. For fixed n and p, define the Bratteli graph $\Gamma_{n,p}$ as follows

Definition 5.1. *The Bratteli graph $\Gamma_{n,p}$ associated to n, p is a directed graph, whose set of vertices is $\bigcup_{i=-1}^{\infty} \mathcal{P}_i(\alpha, \beta)$, a set strict partitions. Here,*

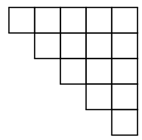
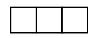
$$\begin{aligned}
\mathcal{P}_{-1}(\alpha, \beta) &= \{\alpha\} \\
\mathcal{P}_1(\alpha, \beta) &= \{\gamma \ \mid \ L(\gamma) \textit{ is a summand of } L(\alpha) \otimes L(\beta)\} \\
\mathcal{P}_i(\alpha, \beta) &= \{\gamma \ \mid \ L(\gamma) \textit{ is a summand of } L(\lambda) \otimes V, \\
&\qquad \textit{for some } \lambda \in \mathcal{P}_{i-1}(\alpha, \beta)\}, \quad i \geq 1
\end{aligned}$$

We allow no repetitions in the above definition, so that if a summand occurs with multiplicities, we only include it once in $\mathcal{P}_i(\alpha, \beta)$. Also, we call a partition $\mu \in \mathcal{P}_i(\alpha, \beta)$ a vertex at level i, and the combinatorial formula for $\mathcal{P}_i(\alpha, \beta)$ was given in Lemma 2.3 and the discussion prior to it.

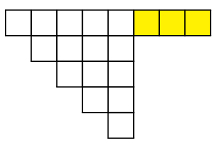
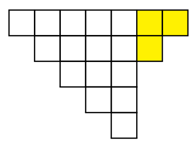
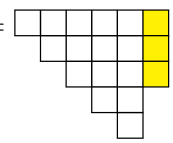
The directed edges in the Bratteli graph are defined as follows:

There is a directed edge from α *to each vertex at level* 0*. For* $i \geq 0$*, there is an edge from* $\lambda \in \mathcal{P}_i(\alpha, \beta)$ *to* $\gamma \in \mathcal{P}_{i+1}(\alpha, \beta)$ *if and only if* $L(\gamma)$ *is a summand of* $L(\lambda) \otimes V$*.*

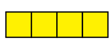
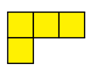
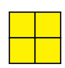
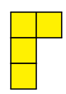
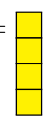
Example. When $n = 5$, $p = 3$, $d = 3$.

$\alpha =$  $\beta =$ 

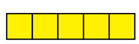
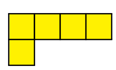
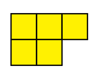
Partitions in $\mathcal{P}_0(\alpha, \beta)$:

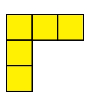
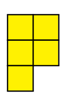
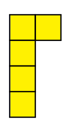
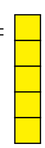
$A_1 =$  $A_2 =$  $A_3 =$ 

For the remaining partitions we omit the staircase portion α and only display the yellow portion. Partitions in $\mathcal{P}_1(\alpha, \beta)$:

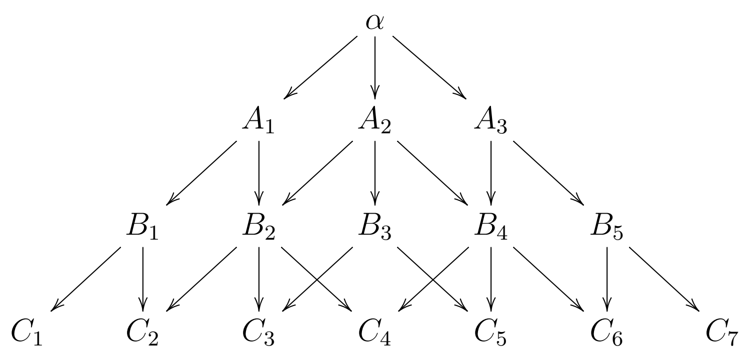
$B_1 =$  $B_2 =$  $B_3 =$  $B_4 =$  $B_5 =$ 

Partitions in $\mathcal{P}_2(\alpha, \beta)$:

$C_1 =$  $C_2 =$  $C_3 =$ 

$C_4 =$  $C_5 =$  $C_6 =$  $C_7 =$ 

The associated Bratteli graph starts with



We identify paths in $\Gamma_{n,p}$ and semistandard tableaux of a skew shape via the following definition.

Definition 5.2. *Fix $\lambda \in \mathcal{P}_d(\alpha, \beta)$, and define the set*

$$\Gamma^\lambda = \Gamma^\lambda_{n,p} = \{\textit{all paths } T \textit{ from } \alpha \textit{ to } \lambda\}.$$

Let $T \in \Gamma^\lambda$, we identify T with the sequence of vertices it travels through: $T = (T^{(0)}, T^{(1)}, \dots, T^{(d)} = \lambda)$, where $T^{(i)} \in \mathcal{P}_i(\alpha, \beta)$. By construction $T^{(i)}$ is a strict partition contained in $T^{(i+1)}$. Given a path T, we can construct a semistandard tableau of shape $\lambda/T^{(0)}$, where the distinct box in $T^{(i)}/T^{(i-1)}$ is filled with the integer i, $1 \le i \le d$.

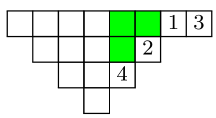
Since the unfilled boxes outline the shape of $T^{(0)} \in \mathcal{P}_0(\alpha, \beta)$, one has the following result

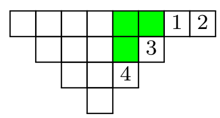
$$\{\text{paths from } \alpha \text{ to } \lambda\} \longleftrightarrow \left\{\begin{matrix}\text{semistandard tableau of shape } \lambda/T(0) \\ \text{with each integer 1,2,...,}n \\ \text{filled in exactly one box,} T^{(0)} \in \mathcal{P}_0(\alpha,\beta)\end{matrix}\right\}$$

For future references we will use both notions interchangeably. For a path T, denote by $c_T(i) = c(T^{(i)}/c(T^{(i-1)})$ the content of the distinct box in $T^{(i)}$ that is not in $T^{(i-1)}$.

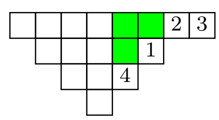
In the next section we will need operators s_i on the set of paths. To be precise we define operators on an enlarged set $\Gamma_+^\lambda = \Gamma^\lambda \cup \{\star\}$ set, where $\star$ serves as a symbol. For $T \in \Gamma^\lambda$, $1 \le i \le d-1$, define $s_i.T$ to be the tableau by interchanging the entries i and $i+1$ in T, if the resulting tableau is still semistandard. Otherwise, define $s_i.T = \star$. In addition, define $s_i.\star = \star$ for all $1 \le i \le d-1$.

Example. We use green color to highlight boxes in $T^{(0)}$ that are not in α. When T and L are the following tableaux, we have $s_2.T = L$:

$T =$ 

$L =$ 

When T is the following tableau, we have $s_2.T = \star$:

$T =$ 

Notice $s_i.T = \star$ if and only if the entries $i, i+1$ in T are in the same row or column. Interpreting $T \in \Gamma^\lambda$ as a path in the Bratteli graph, the above operation is equivalent of reversing the order of adding the i-th and $(i+1)$-th box, and taking a different pair of edges from the partition $T^{(i-1)}$ to partition $T^{(i+1)}$. The resulting path is the unique other path which shares all vertices with T except at level i, if such a path exists.

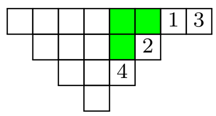
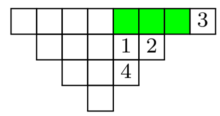
To define the operator s_0, we have the following combinatorial fact.

Lemma 5.3. *For any $T \in \Gamma^\lambda$, there is at most one other $T' \in \Gamma^\lambda$ such that $(T')^{(1)} = T^{(1)}, (T')^{(2)} = T^{(2)}, \ldots, (T')^{(d)} = T^{(d)}$.*

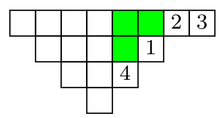
Proof. There exists such a path T' if and only if the partitions $(T')^{(0)}$ and $T^{(0)}$ are both contained in $T^{(1)}$. By Lemma 2.3, such situation exists only when $(T')^{(0)}$ has exactly one more box than $T^{(0)}$ in the first row or vice versa. $\square$

Define the action of s_0 to be $s_0.T = T'$ if there exists another path T' as in the above lemma, and $s_0.T = \star$ otherwise. In addition, define $s_0.\star = \star$.

Example. For the following tableaux, $s_0.T = L$:

$T =$  $L =$ 

For the following tableau, $s_0.T = \star$:

$T =$ 

Recall that in Lemmas 4.7 and 4.8, for a summand $L(T^{(i)})$ in $L(T^{(i-1)}) \otimes V (1 \leq i \leq d)$, or $L(T^{(0)})$ as a summand of $L(\alpha) \otimes L(\beta)$, Ω^2 acts on $L(T^{(i)})$ via a scalar $0 \leq i \leq d$. Denote this scalar by $\kappa_T(i)^2$ where $\kappa_T(i) > 0$. The following is a direct consequence of these two lemmas.

Lemma 5.4. *Let $T^{(0)} = \alpha + \gamma$, where γ is of the form $(s, 1, 1, \ldots, 1)$, and $m = s + n$ is the number of boxes in the first row of $T^{(0)}$.*

$$\kappa_T(0) = \sqrt{mp(m-p)}$$
$$\kappa_T(i) = \sqrt{c_T(i)(c_T(i)+1)} \qquad 1 \leq i \leq d$$

Lemma 5.5. *The following is true for all $T \in \Gamma^\lambda$.*

$$\kappa_T(0)^2 + \kappa_T(1)^2 = \kappa_{s_0.T}(0)^2 + \kappa_{s_0.T}(1)^2$$

Furthermore, if $s_0.T \neq \star$,

$$\kappa_0^4 + p^2\kappa_1^4 = p^3(p+1)\kappa_1^2 + p(p+1)\kappa_0^2 + 2p\kappa_0^2\kappa_1^2$$

Proof. Without a loss of generality, assume $T^{(0)} = \alpha + (s, 1, \ldots, 1)$ and $(s_0.T)^{(0)} = \alpha + (s+1, 1, \ldots, 1)$. Therefore the box $T^{(1)}/T^{(0)}$ is added to the first row with content $n + s = m$, and the box $(s_0.T)^{(1)}/(s_0.T)^{(0)}$ is added to the $(p - s + 1)$-th row with content $n - (p - s) = m - p$.

$$\kappa_T(0)^2 = mp(m-p)$$
$$\kappa_T(1)^2 = m(m+1)$$
$$\kappa_{s_0.T}(0)^2 = (m+1)p(m-p-1)$$
$$\kappa_{s_0.T}(1)^2 = (m-p)(m-p+1)$$

Therefore,

$$\begin{aligned}
&\kappa_T(0)^2 + \kappa_T(1)^2 - (\kappa_{s_0.T}(0)^2 + \kappa_{s_0.T}(1)^2)\\
&= mp(m-p) + m(m+1) - (m+1)p(m-p-1) - (m-p)(m-p+1)\\
&= m(mp - p^2 + m + 1) - (m-p+1)(mp + p + m - p)\\
&= m(m-p+1)(p+1) - (m-p+1)m(p+1) = 0
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\kappa_0^2 - p\kappa_1^2 &= -mp^2 - mp = -mp(p+1) \\
m &= -\frac{\kappa_0^2 - p\kappa_1^2}{p(p+1)} \\
\kappa_1^2 = m(m+1) &= -\frac{\kappa_0^2 - p\kappa_1^2}{p(p+1)}\left(-\frac{\kappa_0^2 - p\kappa_1^2}{p(p+1)} + 1\right) \\
p^2(p+1)^2\kappa_1^2 &= \kappa_0^4 + p^2\kappa_1^4 - 2p\kappa_0^2\kappa_1^2 - p(p+1)\kappa_0^2 + p^2(p+1)\kappa_1^2 \\
\kappa_0^4 + p^2\kappa_1^4 &= p^3(p+1)\kappa_1^2 + p(p+1)\kappa_0^2 + 2p\kappa_0^2\kappa_1^2
\end{aligned}$$

□

Define $\kappa_T = \sqrt{\kappa_T(0)^2 + \kappa_T(1)^2}$, then the above lemma implies that κ_T only depends on $T^{(1)}$. We show that the other direction is also true.

Corollary 5.6. *Let $T \in \Gamma^\lambda$ and $s_0.T \neq \star$, then $T^{(1)}$ can be uniquely determined by κ_T.*

Proof. In the above calculation, $\kappa_T^2 = m(m-p+1)(p+1)$ is an increasing positive function in m, therefore is distinct for every $m+1$, the number of boxes in the first row of $T^{(1)}$. □

Also, a path T is uniquely determined by the sequence $\kappa_T(1), \ldots, \kappa_T(d)$.

Lemma 5.7. *Given $T, S \in \Gamma^\lambda$, if $\kappa_T(i) = \kappa_S(i)$ for $1 \leq i \leq d$, then $T = S$.*

Proof. First, $T^{(d)} = S^{(d)} = \lambda$. Given $\mu \in \mathcal{P}^{d-1}(\alpha, \beta)$, $c(\lambda/\mu)$ is distance for each μ, since a removed box is uniquely determined by the diagonal, or equivalently the content. By successively removing boxes, one obtains the unique path associated to the sequence $\kappa_T(1), \ldots, \kappa_T(d)$. □

Lemma 5.8. *Denote* $\kappa_0 = \kappa_T(0)$, $\kappa_1 = \kappa_T(1)$, $\kappa_0' = \kappa_{s_0.T}(0)$, $\kappa_1' = \kappa_{s_0.T}(1)$. *Then*

$$\kappa_0' = p\kappa_1\sqrt{\frac{\kappa_0^2+\kappa_1^2}{\kappa_0^2+p^2\kappa_1^2}} \qquad \kappa_0 = p\kappa_1'\sqrt{\frac{(\kappa_0')^2+(\kappa_1')^2}{(\kappa_0')^2+p^2(\kappa_1')^2}}$$

$$\kappa_1' = \kappa_0\sqrt{\frac{\kappa_0^2+\kappa_1^2}{\kappa_0^2+p^2\kappa_1^2}} \qquad \kappa_1 = \kappa_0'\sqrt{\frac{(\kappa_0')^2+(\kappa_1')^2}{(\kappa_0')^2+p^2(\kappa_1')^2}}$$

Proof. Based on the calculation in Lemma 5.5, observe

$$\kappa_0\kappa_0' = p\sqrt{m(m+1)(m-p)(m-p+1)}$$
$$\kappa_1\kappa_1' = \sqrt{m(m+1)(m-p)(m-p+1)}$$

Then $\kappa_0\kappa_0' = p\kappa_1\kappa_1'$, or $\kappa_0 = \frac{p\kappa_1'}{\kappa_0'}\kappa_1$. Since $\kappa_0^2+\kappa_1^2 = (\kappa_0')^2+(\kappa_1')^2$,

$$\frac{p^2(\kappa_1')^2}{(\kappa_0')^2}\kappa_1^2+\kappa_1^2 = (\kappa_0')^2+(\kappa_1')^2$$
$$\kappa_1^2\Big(1+\frac{p^2(\kappa_1')^2}{(\kappa_0')^2}\Big) = (\kappa_0')^2+(\kappa_1')^2$$
$$\kappa_1^2 = \kappa_0'\sqrt{\frac{(\kappa_0')^2+(\kappa_1')^2}{(\kappa_0')^2+p^2(\kappa_1')^2}}$$

The other identities can be checked similarly. □

5.2. A Combinatorial Construction

Hill-Kujawa-Sussan [10] constructed modules for the degenerate affine Hecke-Clifford algebra $\mathcal{H}_d^{\mathrm{aff}}$ using the following combinatorial method. For a partition

$\mu \in \mathcal{P}_0(\alpha, \beta)$, let

$$\Gamma_\mu^\lambda = \{\text{paths from } \mu \text{ to } \lambda \text{ in the Bratteli graph}\}$$

or equivalently, Γ_μ^λ is the set of semistandard tableaux of skew shape λ/μ. In [10, Section 5], Hill-Kujawa-Sussan constructed a module for $\mathcal{H}_d^{\text{aff}}$ associated to each skew shape.

Proposition 5.9 (Proposition 5.1.1 [10])**.** *There is a well defined action of* $\mathcal{H}_d^{\text{aff}}$ *on* $\hat{\mathcal{H}}^{\lambda/\mu} = \bigoplus_{T \in \Gamma_\mu^\lambda} Cl_d v_T$ *via*

$$\begin{aligned} z_i.v_T &= \kappa_T(i) v_T \qquad\qquad 1 \le i \le d \\ s_i.v_T &= \left(-\frac{1}{\kappa_T(i) - \kappa_T(i+1)} + \frac{1}{\kappa_T(i) + \kappa_T(i+1)} c_i c_{i+1}\right) v_T \\ &\quad + \sqrt{1 - \frac{1}{(\kappa_T(i) + \kappa_T(i+1))^2} - \frac{1}{(\kappa_T(i) - \kappa_T(i+1))^2}}\, v_{s_i.T} \end{aligned}$$

and the Clifford generators c_i *act by multiplication in* Cl_d*.*

In [10], Hill-Kujawa-Sussan defined modules to be calibrated if $z_1, \ldots, z_d$ act semisimply. In the same spirit, we define calibrated modules for $\mathcal{H}_d^p$ as follows: a finite dimensional $\mathcal{H}_d^p$-module W is *calibrated* if it admits a basis on which $z_0, \ldots, z_d$ act by scalars.

Let λ be a fixed strict partition at level d in the Bratteli graph associated to n, p. Notice once λ is given, $n = \ell(\lambda)$, and $p = |\lambda| - (1 + 2 + \cdots + n)$ can be deduced from λ. We will construct a family of strongly calibrated $\mathcal{H}_d^p$-modules with respect to λ. Recall $\Gamma^\lambda = \{\text{all paths } T \text{ from } \alpha \text{ to } \lambda \text{ in } \Gamma_{n,p}\}$ and we defined the action of $s_0, \ldots, s_{d-1}$ on a slightly enlarged set $\Gamma^\lambda \cup \{\star\}$ in the previous section. Based on the definition of $\kappa_T(i)$ in Lemma 5.4, we define

$$\kappa_0 = \kappa_T(0), \qquad \kappa_1 = \kappa_T(1) \qquad \kappa_0' = \kappa_{s_0.T}(0), \qquad \kappa_1' = \kappa_{s_0.T}(1)$$

For our construction we also define a slighly smaller set

$$\Gamma_-^\lambda = \{T \in \Gamma^\lambda | s_0.T \neq \star\}$$

Further let $N_0 = n(n+1)$ and $f : \Gamma_-^\lambda \to \mathbb{C}$ be a set function which satisfies the following equation whenever $s_0.T \neq \star$:

$$f(T)f(s_0.T)((\kappa_0 - \kappa_0')^2 + (\kappa_1 + \kappa_1')^2) = -\frac{\kappa_0^2 + p^2\kappa_1^2}{(\kappa_0^2 + p\kappa_1^2)^2}(N_0 - \kappa_1^2)(N_0 - (\kappa_1')^2) \tag{5.2.1}$$

In particular, since $\kappa_0, \kappa_0', \kappa_1, \kappa_1'$ are real numbers, and either κ_1 or κ_1' is nonzero based on the formula in Lemma 5.4, f can be taken as

$$f(T) = f(s_0.T) = \sqrt{-\frac{(\kappa_0^2 + p^2\kappa_1^2)(N_0 - \kappa_1^2)(N_0 - (\kappa_1')^2)}{(\kappa_0^2 + p\kappa_1^2)^2((\kappa_0 - \kappa_0')^2 + (\kappa_1 + \kappa_1')^2)}}.$$

Definition 5.10. *As a superspace, let $\mathcal{D}_f^\lambda$ be the free module over the Clifford superalgebra by cases:*

1) When n is even, recall $\mathcal{H}_d^p$ is a quotient of $\mathcal{H}_d^N$. Let $\mathcal{D}_f^\lambda = \bigoplus_{T \in \Gamma^\lambda} Cl_{d+1} v_T$, where Cl_{d+1} is the subalgebra of $\mathcal{H}_d$ generated by $c_0, c_1, \ldots, c_d$.

2) When n is odd, recall $\mathcal{H}_d^p$ is a quotient of $\mathcal{H}_d^{MN}$. Let $\mathcal{D}_f^\lambda = \bigoplus_{T \in \Gamma^\lambda} Cl_{d+2} v_T$, where Cl_{d+2} is the subalgebra of $\mathcal{H}_d$ generated by $c_M, c_0, c_1, \ldots, c_d$.

In both cases, let m be the number of boxes in the first row of $T^{(0)}$, and declare v_T to be even if m is even, and v_T to be odd if m is odd. Define the degree *of* $c_{i_1}\cdots c_{i_r}v_T$ *to be* $\overline{r}+\overline{v_T}$. *The action of* Cl_{d+1} *or* Cl_{d+2} *in* $\mathcal{H}_d^p$ *is multiplication in the Clifford algebra. Since the generators* $z_0,\ldots,z_d,x_1,s_1,\ldots,s_{d-1}$ *can be moved past the Clifford generators using the commuting or anticommuting relations in Proposition 3.9, it is enough to define the action of these remaining generators on* v_T, *for each path* $T\in\Gamma^\lambda$. *They act as follows:*

$$\begin{aligned}
z_0.v_T &= \kappa_T(0)v_T\\
z_i.v_T &= \kappa_T(i)v_T \qquad\quad 1\le i\le d\\
s_i.v_T &= \Big(-\frac{1}{\kappa_T(i)-\kappa_T(i+1)}+\frac{1}{\kappa_T(i)+\kappa_T(i+1)}c_ic_{i+1}\Big)v_T\\
&\quad+\sqrt{1-\frac{1}{(\kappa_T(i)+\kappa_T(i+1))^2}-\frac{1}{(\kappa_T(i)-\kappa_T(i+1))^2}}\,v_{s_i.T}
\end{aligned}$$

In the above formula, we declare $v_\star$ *is the zero vector.*

To define the action of x_1, *we introduce the following* 2×2 *matrices:*

$$Z=\begin{bmatrix}\kappa_0-\kappa_0' & \kappa_1+\kappa_1'\\ \kappa_1+\kappa_1' & \kappa_0'-\kappa_0\end{bmatrix}. \tag{5.2.2}$$

Let $\kappa=\sqrt{\kappa_0^2+\kappa_1^2}$. *When* $s_0.T\neq\star$, *the action of* x_1 *on* v_T *is given by a* 4×4 *matrix on the subspace spanned by vectors* v_T, $c_0c_1v_T$,$c_0v_{s_0.T}$ *and* $c_1v_{s_0.T}$, *where*

$$x_1(T)=\begin{bmatrix}\frac{N_0}{\kappa^2}\begin{bmatrix}p\kappa_1 & -\kappa_0\\ -\kappa_0 & -p\kappa_1\end{bmatrix}+\begin{bmatrix}\kappa_0 & \kappa_1\\ \kappa_1 & -\kappa_0\end{bmatrix} & f(s_0.T)Z\\ f(T)Z & \frac{N_0}{\kappa^2}\begin{bmatrix}p\kappa_1' & \kappa_0'\\ \kappa_0' & -p\kappa_1'\end{bmatrix}+\begin{bmatrix}\kappa_0 & -\kappa_1'\\ -\kappa_1' & -\kappa_0\end{bmatrix}\end{bmatrix} \tag{5.2.3}$$

Recall from above we declare $v_\star=0$. *As a consequence, whenever* $s_0.T=\star$, *the* *action of x_1 on $\{v_T, c_0c_1v_T\}$ is given by the upper left 2×2 block of the above matrix.*

In order to prove two main results, Theorem 5.12 of this section and Theorem 6.1 in Section 6.1, we will need a reformulation of the matrix $x_1(T)$ in the following lemma.

Lemma 5.11. *In Definition 5.10, the matrix $x_1(T)$ can be defined alternatively via the following. Let*

$$c = \frac{2\frac{N_0}{\kappa^2}(\kappa^2 - \kappa_1^2 - (\kappa_1')^2) + (\kappa_1^2 + (\kappa_1')^2 - p(p+1))}{2(\kappa_0\kappa_1 + \kappa_0'\kappa_1')}.$$

Then c is equal to

$$\begin{aligned} c &= \frac{\kappa_0\kappa_1}{\kappa_0^2 + p\kappa_1^2}(\frac{N_0}{\kappa^2}(p-1)+1) \\ &= \frac{\kappa_0'\kappa_1'}{(\kappa_0')^2 + p(\kappa_1')^2}(\frac{N_0}{\kappa^2}(p-1)+1) \end{aligned}$$

Further let

$$X = \begin{bmatrix} \kappa_0 & \kappa_1 \\ \kappa_1 & -\kappa_0 \end{bmatrix}, \quad Y = \begin{bmatrix} \kappa_0' & -\kappa_1' \\ -\kappa_1' & -\kappa_0' \end{bmatrix} \quad Q = \begin{bmatrix} \kappa_1 & -\kappa_0 \\ -\kappa_0 & -\kappa_1 \end{bmatrix}, \quad R = \begin{bmatrix} \kappa_1' & \kappa_0' \\ \kappa_0' & -\kappa_1' \end{bmatrix}$$

Then the matrix $x_1(T)$ in Equation (5.2.3) can be written as

$$x_1(T) = \begin{bmatrix} \frac{N_0}{\kappa^2}Q + cX & f(s_0.T)Z \\ f(T)Z & \frac{N_0}{\kappa^2}R + cY \end{bmatrix} \tag{5.2.4}$$

where Z is defined in Equation (5.2.2). and the condition (5.2.1) on f is *equivalent to*

$$\frac{N_0^2}{\kappa^2} + c^2\kappa^2 + f(T)f(s_0.T)((\kappa_0 - \kappa_0')^2 + (\kappa_1 + \kappa_1')^2) = N_0 \tag{5.2.5}$$

We also have $f(T)f(s_0.T) \neq 0$ *whenever* $s_0.T \neq \star$.

Proof. We use the formulas given in Lemma 5.8 to rewrite the constant c:

$$\begin{aligned} c &= \frac{2\frac{N_0}{\kappa^2}(\kappa_0^2 - \kappa_0^2\frac{\kappa_0^2+\kappa_1^2}{\kappa_0^2+p^2\kappa_1^2}) + (\kappa_1^2 + \kappa_0^2\frac{\kappa_0^2+\kappa_1^2}{\kappa_0^2+p^2\kappa_1^2} - p(p+1))}{2(\kappa_0\kappa_1 + p\kappa_0\kappa_1\frac{\kappa_0^2+\kappa_1^2}{\kappa_0^2+p^2\kappa_1^2})} \\ &= \frac{2\frac{N_0}{\kappa^2}\kappa_0^2(p^2-1)\kappa_1^2 + (\kappa_1^2(\kappa_0^2+p^2\kappa_1^2) + \kappa_0^2(\kappa_0^2+\kappa_1^2) - p(p+1)(\kappa_0^2+p^2\kappa_1^2))}{2\kappa_0\kappa_1(p+1)(\kappa_0^2+p\kappa_1^2)} \end{aligned}$$

where according to Lemma 5.5,

$$\begin{aligned} &\kappa_1^2(\kappa_0^2+p^2\kappa_1^2) + \kappa_0^2(\kappa_0^2+\kappa_1^2) - p(p+1)(\kappa_0^2+p^2\kappa_1^2) \\ &= p^2\kappa_1^4 + \kappa_0^4 + 2\kappa_0^2\kappa_1^2 - p(p+1)(\kappa_0^2+p^2\kappa_1^2) \\ &= 2(p+1)\kappa_0^2\kappa_1^2 \end{aligned}$$

Therefore

$$\begin{aligned} c &= \frac{2\frac{N_0}{\kappa^2}\kappa_0^2(p^2-1)\kappa_1^2 + 2(p+1)\kappa_0^2\kappa_1^2}{2\kappa_0\kappa_1(p+1)(\kappa_0^2+p\kappa_1^2)} \\ &= \frac{\kappa_0\kappa_1}{\kappa_0^2+p\kappa_1^2}(\frac{N_0}{\kappa^2}(p-1)+1) \end{aligned}$$

To see that the above expression is equal to a similar version with κ_0' and κ_1', it is enough to show $\frac{\kappa_0\kappa_1}{\kappa_0^2+p\kappa_1^2} = \frac{\kappa_0'\kappa_1'}{(\kappa_0')^2+p(\kappa_1')^2}$. Indeed, this is true based on the formulas given in the proof of Lemma 5.5.

$$\frac{\kappa_0\kappa_1}{\kappa_0^2+p\kappa_1^2} = \frac{m\sqrt{(m+1)(m-p)p}}{mp(2m-p+1)},$$

while

$$\frac{\kappa_0'\kappa_1'}{(\kappa_0')^2+p(\kappa_1')^2} = \frac{(m+1-p)\sqrt{(m+1)(m-p)p}}{(m+1-p)p(2m+1-p)},$$

and the two expressions are equal. Now let us simplify the entries in the matrix of x_1 using the above formula.

$$\begin{aligned}
&\frac{N_0}{\kappa^2}Q+cX\\
&= \frac{N_0}{\kappa^2}\begin{bmatrix}\kappa_1 & -\kappa_0\\ -\kappa_0 & -\kappa_1\end{bmatrix} + \frac{\kappa_0\kappa_1}{\kappa_0^2+p\kappa_1^2}\left(\frac{N_0}{\kappa^2}(p-1)+1\right)\begin{bmatrix}\kappa_0 & \kappa_1\\ \kappa_1 & -\kappa_0\end{bmatrix}\\
&= \frac{N_0}{\kappa^2(\kappa_0^2+p\kappa_1^2)}\begin{bmatrix}\kappa_1(\kappa_0^2+p\kappa_1^2)+(p-1)\kappa_0^2\kappa_1 & -\kappa_0(\kappa_0^2+p\kappa_1^2)+(p-1)\kappa_0\kappa_1^2\\ -\kappa_0(\kappa_0^2+p\kappa_1^2)+(p-1)\kappa_0\kappa_1^2 & -\kappa_1(\kappa_0^2+p\kappa_1^2)-(p-1)\kappa_0^2\kappa_1\end{bmatrix}\\
&\quad + \frac{\kappa_0\kappa_1}{\kappa_0^2+p\kappa_1^2}\begin{bmatrix}\kappa_0 & \kappa_1\\ \kappa_1 & -\kappa_0\end{bmatrix}\\
&= \frac{N_0}{\kappa^2(\kappa_0^2+p\kappa_1^2)}\begin{bmatrix}p\kappa_1(\kappa_0^2+\kappa_1^2) & -\kappa_0(\kappa_0^2+\kappa_1^2)\\ -\kappa_0(\kappa_0^2+\kappa_1^2) & -p\kappa_1(\kappa_0^2+\kappa_1^2)\end{bmatrix} + \frac{\kappa_0\kappa_1}{\kappa_0^2+p\kappa_1^2}\begin{bmatrix}\kappa_0 & \kappa_1\\ \kappa_1 & -\kappa_0\end{bmatrix}\\
&= \frac{N_0}{\kappa_0^2+p\kappa_1^2}\begin{bmatrix}p\kappa_1 & -\kappa_0\\ -\kappa_0 & -p\kappa_1\end{bmatrix} + \frac{\kappa_0\kappa_1}{\kappa_0^2+p\kappa_1^2}\begin{bmatrix}\kappa_0 & \kappa_1\\ \kappa_1 & -\kappa_0\end{bmatrix}
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{N_0}{\kappa^2}\begin{bmatrix}\kappa_1' & \kappa_0' \\ \kappa_0' & -\kappa_1'\end{bmatrix} + cY \\
=& \frac{N_0}{(\kappa_0')^2 + p(\kappa_1')^2}\begin{bmatrix}p\kappa_1' & \kappa_0' \\ \kappa_0' & -p\kappa_1'\end{bmatrix} + \frac{\kappa_0'\kappa_1'}{(\kappa_0')^2 + p(\kappa_1')^2}\begin{bmatrix}\kappa_0' & -\kappa_1 \\ -\kappa_1 & -\kappa_0'\end{bmatrix}
\end{aligned}$$

Lastly,

$$\begin{aligned}
&N_0 - \frac{N_0^2}{\kappa^2} - c^2\kappa^2 \\
=& \frac{1}{(\kappa_0^2 + p\kappa_1^2)^2}(-\frac{N_0^2}{\kappa^2}((\kappa_0^2 + p\kappa_1^2)^2 - (p-1)^2\kappa_0^2\kappa_1^2) \\
&+ N_0((\kappa_0^2 + p\kappa_1^2)^2 - 2(p-1)\kappa_0^2\kappa_1^2) - \kappa_0^2\kappa_1^2\kappa^2)
\end{aligned}$$

where

$$\begin{aligned}
&\frac{(\kappa_0^2 + p\kappa_1^2)^2 + (p-1)^2\kappa_0^2\kappa_1^2}{\kappa^2} \\
=& \frac{\kappa_0^4 + p^2\kappa_1^4 + (p^2+1)\kappa_0^2\kappa_1^2}{\kappa_0^2 + \kappa_1^2} \\
=& \kappa_0^2 + p^2\kappa_1^2
\end{aligned}$$

Hence

$$
\begin{aligned}
& N_0 - \frac{N_0^2}{\kappa^2} - c^2\kappa^2 \\
&= \frac{1}{(\kappa_0^2 + p\kappa_1^2)^2}(-N_0^2(\kappa_0^2 + p^2\kappa_1^2) + N_0(\kappa_0^4 + p^2\kappa_1^4 + 2\kappa_0^2\kappa_1^2) - \kappa_0^2\kappa_1^2\kappa^2) \\
&= -\frac{1}{(\kappa_0^2 + p\kappa_1^2)^2}(N_0 - \kappa_1^2)(N_0(\kappa_0^2 + p^2\kappa_1^2) - \kappa_0^2(\kappa_0^2 + \kappa_1^2)) \\
&= -\frac{\kappa_0^2 + p^2\kappa_1^2}{(\kappa_0^2 + p\kappa_1^2)^2}(N_0 - \kappa_1^2)(N_0 - \kappa_0^2\frac{\kappa_0^2 + \kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2}) \\
&= -\frac{\kappa_0^2 + p^2\kappa_1^2}{(\kappa_0^2 + p\kappa_1^2)^2}(N_0 - \kappa_1^2)(N_0 - (\kappa_1')^2)
\end{aligned}
$$

where the last equality holds because of Lemma 5.8.

Finally, if $f(T)f(s_0.T) = 0$, one of the expressions $N_0 - \kappa_1^2$, $N_0 - (\kappa_1')^2$ or $\kappa_0^2 + p^2\kappa_1^2$ is zero, therefore $\kappa_1 = \sqrt{N_0}$ or $\kappa_1' = \sqrt{N_0}$. This is only the case when $T^{(1)}$ is of the form (4.3.1), or equivalently when $s_0.T = \star$. □

Theorem 5.12. *The vector space $\mathcal{D}_f^\lambda$ admits a well-defined action of $\mathcal{H}_d^p$.*

Proof. In the proof of Proposition 5.9, Hill-Kujawa-Sussan checked that the Sergeev relations and the relations $s_i z_i = z_{i+1}s_i - 1 + c_i c_{i+1}$ hold. Since the action of $s_1, \ldots, s_{d-1}$ and $z_1, \ldots, z_d$ coincide with our action, these relations hold by the same calculations. It is enough to check the remaining relations regarding x_1 and z_0. We list all the relations here from Proposition 3.9 and

Corollary 4.2.

$$
\begin{aligned}
z_0 s_i &= s_i z_0 \qquad (2 \leq i \leq d)\\
z_0 z_1 &= z_1 z_0\\
z_2 x_1 &= x_1 z_2\\
x_1 s_i &= s_i x_1 \qquad (2 \leq i \leq n)\\
x_1(s_1 x_1 s_1 + (1 - c_1 c_2)s_1) &= (s_1 x_1 s_1 + (1 - c_1 c_2)s_1)x_1\\
(z_0 c_0 c_1 + z_1 - x_1)x_1 &= -x_1(z_0 c_0 c_1 + z_1 - x_1)\\
x_1^2 &= n(n+1)\\
(x_1 - z_1)^4 &= p(p+1)(x_1 - z_1)^2\\
x_1 \Phi^2 &= \Phi^2 x_1
\end{aligned}
$$

1) The first three relations are automatically satisfied since $s_2, \ldots, s_{d-1}, z_1$ preserves the eigenspaces of z_0, and x_1 preserves the eigenspaces of z_2.

2) The relation $x_1 s_i = s_i x_1$ $(2 \leq i \leq d-1)$ holds. For $\forall i, 1 \leq i \leq d-1$, notice if $S = s_i.T$, then $\kappa_0(T) = \kappa_0(S)$ and $\kappa_1(T) = \kappa_1(S)$, therefore all the data for the action of x_1 is the same for T and S in the above definition. Also notice that the action of s_i $(2 \leq i \leq d)$ commutes with the action of s_0 on Γ^λ, therefore $s_0.S = s_i.(s_0.T)$. Let $x_1.v_T = (a_1 + a_2 c_0 c_1)v_T + (a_3 c_0 + a_4 c_1)v_{s_0.T}$, where a_i is given by the above definition, then $x_1.v_S = (a_1 + a_2 c_0 c_1)v_S + (a_3 c_0 + a_4 c_1)v_{s_0.S}$. Also notice $\kappa_i(T) = \kappa_i(s_0.T)$ and $\kappa_{i+1}(T) = \kappa_{i+1}(s_0.T)$, therefore if $s_i.v_T = (b_1 + b_2 c_i c_{i+1})v_T + b_3 v_S$, where b_i is given by the above definition, then

$s_i.v_{s_0.T} = (b_1 + b_2c_ic_{i+1})v_{s_0.T} + b_3v_{s_0.S}.$

$$\begin{aligned}
&x_1.(s_iv_T) \\
&= x_1((b_1 + b_2c_ic_{i+1})v_T + b_3v_S) \\
&= (b_1 + b_2c_ic_{i+1})((a_1 + a_2c_0c_1)v_T + (a_3c_0 + a_4c_1)v_{s_0.T}) \\
&+ b_3((a_1 + a_2c_0c_1)v_S + (a_3c_0 + a_4c_1)v_{s_0.S})
\end{aligned}$$

on the other hand,

$$\begin{aligned}
&s_i.(x_1v_T) \\
&= s_i((a_1 + a_2c_0c_1)v_T + (a_3c_0 + a_4c_1)v_{s_0.T}) \\
&= (a_1 + a_2c_0c_1)((b_1 + b_2c_ic_{i+1})v_T + b_3v_S) \\
&\quad + (a_3c_0 + a_4c_1)((b_1 + b_2c_ic_{i+1})v_{s_0.T} + b_3v_{s_0.S})
\end{aligned}$$

The two results are the same by comparison.

The last five relations are included in the following two lemmas for an easier read. □

Lemma 5.13. *The relations*

$$\begin{aligned}
(z_0c_0c_1 + z_1 - x_1)x_1 &= -x_1(z_0c_0c_1 + z_1 - x_1) \\
x_1^2 &= n(n+1) \\
(x_1 - z_1)^4 &= p(p+1)(x_1 - z_1)^2 \\
x_1\Phi^2 &= \Phi^2x_1
\end{aligned}$$

hold in the above construction.

Proof. Observe the following identities are true

$$Q^2 = R^2 = X^2 = Y^2 = \kappa^2$$

$$Z^2 = (\kappa_0 - \kappa_0')^2 + (\kappa_1 + \kappa_1')^2$$

$$QX + XQ = 0, \qquad YR + RY = 0$$

$$XZ + ZY = 0, \qquad ZX + YZ = 0$$

$$QZ + ZR = 0, \qquad RZ + ZQ = 0$$

The first two identities are straightforward. Let us check the remaining three results on the left and the others can be checked similarly.

$$\begin{aligned} &QX + XQ \\ &= \begin{bmatrix} \kappa_1 & -\kappa_0 \\ -\kappa_0 & -\kappa_1 \end{bmatrix} \begin{bmatrix} \kappa_0 & \kappa_1 \\ \kappa_1 & -\kappa_0 \end{bmatrix} + \begin{bmatrix} \kappa_0 & \kappa_1 \\ \kappa_1 & -\kappa_0 \end{bmatrix} \begin{bmatrix} \kappa_1 & -\kappa_0 \\ -\kappa_0 & -\kappa_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \kappa^2 \\ \kappa^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\kappa^2 \\ -\kappa^2 & 0 \end{bmatrix} = 0 \end{aligned}$$

$$\begin{aligned} &XZ + ZY \\ &= \begin{bmatrix} \kappa_0 & \kappa_1 \\ \kappa_1 & -\kappa_0 \end{bmatrix} \begin{bmatrix} \kappa_0 - \kappa_0' & \kappa_1 + \kappa_1' \\ \kappa_1 + \kappa_1' & \kappa_0' - \kappa_0 \end{bmatrix} + \begin{bmatrix} \kappa_0 - \kappa_0' & \kappa_1 + \kappa_1' \\ \kappa_1 + \kappa_1' & \kappa_0' - \kappa_0 \end{bmatrix} \begin{bmatrix} \kappa_0' & -\kappa_1' \\ -\kappa_1' & -\kappa_0' \end{bmatrix} \\ &= \begin{bmatrix} \kappa^2 - \kappa_0\kappa_0' + \kappa_1\kappa_1' & \kappa_0\kappa_1' + \kappa_1\kappa_0' \\ -(\kappa_0\kappa_1' + \kappa_1\kappa_0') & \kappa^2 - \kappa_0\kappa_0' + \kappa_1\kappa_1' \end{bmatrix} - \begin{bmatrix} \kappa^2 - \kappa_0\kappa_0' + \kappa_1\kappa_1' & \kappa_0\kappa_1' + \kappa_1\kappa_0' \\ -(\kappa_0\kappa_1' + \kappa_1\kappa_0') & \kappa^2 - \kappa_0\kappa_0' + \kappa_1\kappa_1' \end{bmatrix} \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
&QZ + ZR \\
&= \begin{bmatrix} \kappa_1 & -\kappa_0 \\ -\kappa_0 & -\kappa_1 \end{bmatrix} \begin{bmatrix} \kappa_0 - \kappa_0' & \kappa_1 + \kappa_1' \\ \kappa_1 + \kappa_1' & \kappa_0' - \kappa_0 \end{bmatrix} + \begin{bmatrix} \kappa_0 - \kappa_0' & \kappa_1 + \kappa_1' \\ \kappa_1 + \kappa_1' & \kappa_0' - \kappa_0 \end{bmatrix} \begin{bmatrix} \kappa_1' & \kappa_0' \\ \kappa_0' & -\kappa_1' \end{bmatrix} \\
&= \begin{bmatrix} -\kappa_0\kappa_1' - \kappa_1\kappa_0' & \kappa^2 + \kappa_1\kappa_1' - \kappa_0\kappa_0' \\ -(\kappa^2 + \kappa_1\kappa_1' - \kappa_0\kappa_0') & -\kappa_0\kappa_1' - \kappa_1\kappa_0' \end{bmatrix} \\
&\quad - \begin{bmatrix} -\kappa_0\kappa_1' - \kappa_1\kappa_0' & \kappa^2 + \kappa_1\kappa_1' - \kappa_0\kappa_0' \\ -(\kappa^2 + \kappa_1\kappa_1' - \kappa_0\kappa_0') & -\kappa_0\kappa_1' - \kappa_1\kappa_0' \end{bmatrix} \\
&= 0
\end{aligned}
$$

1) The relation $x_1^2 = n(n+1)$ is satisfied. On the subspace spanned by $\{v_T, c_0c_1v_T, c_0v_{s_0.T}, c_1v_{s_0.T}\}$,

$$
x_1^2 = \begin{bmatrix} \frac{N_0}{\kappa^2}Q + cX & f(s_0.T)Z \\ f(T)Z & \frac{N_0}{\kappa^2}R + cY \end{bmatrix}^2
$$

Let us calculate the 2×2 blocks individually,

$$
\begin{aligned}
(x_1^2)_{11} &= \frac{N_0^2}{\kappa^4}Q^2 + c^2X^2 + \frac{cN_0}{\kappa^2}(QX + XQ) + f(T)f(s_0.T)Z^2 \\
&= \frac{N_0^2\kappa^2}{\kappa^4} + c^2\kappa^2 + f(T)f(s_0.T)((\kappa_0 - \kappa_0') + (\kappa_1 + \kappa_1')^2) \\
&= N_0
\end{aligned}
$$

by the condition imposed on f. Similarly,

$$\begin{aligned}(x_1^2)_{22} &= \frac{N_0^2}{\kappa^4}R^2 + c^2X^2 + \frac{cN_0}{\kappa^2}(RX + XR) + f(T)f(s_0.T)Z^2 \\ &= \frac{N_0^2\kappa^2}{\kappa^4} + c^2\kappa^2 + f(T)f(s_0.T)((\kappa_0 - \kappa_0') + (\kappa_1 + \kappa_1')^2) \\ &= N_0\end{aligned}$$

Moreover,

$$\begin{aligned}(x_1^2)_{12} &= f(s_0.T)(\frac{N_0}{\kappa^2}(QZ + ZR) + f(s_0.T)c^2(XZ + ZY)) = 0 \\ (x_1^2)_{12} &= f(T)(\frac{N_0}{\kappa^2}(ZQ + RZ) + f(T)c^2(ZX + YZ)) = 0\end{aligned}$$

Therefore $x_1^2 = N_0 = n(n+1)$.

2) The relation $(z_0c_0c_1 + z_1 - x_1)x_1 = -x_1(z_0c_0c_1 + z_1 - x_1)$ is satisfied. Notice the matrices for z_0, c_0c_1, z_1 under the given basis are

$$z_0 = \begin{bmatrix} \kappa_0 & 0 & 0 & 0 \\ 0 & -\kappa_0 & 0 & 0 \\ 0 & 0 & -\kappa_0' & 0 \\ 0 & 0 & 0 & \kappa_0' \end{bmatrix}, c_0c_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, z_1 = \begin{bmatrix} \kappa_1 & 0 & 0 & 0 \\ 0 & -\kappa_1 & 0 & 0 \\ 0 & 0 & \kappa_1' & 0 \\ 0 & 0 & 0 & -\kappa_1' \end{bmatrix}$$

and $z_0c_0c_1 + z_1$ acts as

$$z_0c_0c_1 + z_1 = \begin{bmatrix} \kappa_1 & -\kappa_0 & 0 & 0 \\ -\kappa & -\kappa_1 & 0 & 0 \\ 0 & 0 & \kappa_1' & \kappa_0' \\ 0 & 0 & \kappa_0' & -\kappa_1' \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$$

Therefore,

$$\begin{aligned}
&(z_0c_0c_1 + z_1)x_1 + x_1(z_0c_0c_1 + z_1) \\
&= \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} \frac{N_0}{\kappa^2}Q + cX & f(s_0.T)Z \\ f(T)Z & \frac{N_0}{\kappa^2}R + cY \end{bmatrix} + \begin{bmatrix} \frac{N_0}{\kappa^2}Q + cX & f(s_0.T)Z \\ f(T)Z & \frac{N_0}{\kappa^2}R + cY \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \\
&= \begin{bmatrix} \frac{2N_0}{\kappa^2}Q^2 + c(QX + XQ) & f(s_0.T)(QZ + ZR) \\ f(T)(ZQ + RZ) & \frac{2N_0}{\kappa^2}Q^2 + c(RY + YR) \end{bmatrix} = 2N_0 = 2x_1^2
\end{aligned}$$

3) The relation $x_1\Phi^2 = \Phi^2 x_1$ is satisfied.

This relation is equivalent to $(x_1\Phi - \Phi x_1)\Phi + \Phi(x_1\Phi - \Phi x_1) = 0$. Using the matrices for z_0, z_1 in 2),

$$\begin{aligned}
\Phi &= z_0^2 - pz_1^2 + \frac{p(p+1)(p-1)}{2} \\
&= \begin{bmatrix} (\kappa_0^2 - p\kappa_1^2 + \frac{p(p+1)(p-1)}{2})I_2 & 0 \\ 0 & ((\kappa_0')^2 - p(\kappa_1')^2 + \frac{p(p+1)(p-1)}{2})I_2 \end{bmatrix}
\end{aligned}$$

where I_2 is the 2×2 identity matrix. As a consequence of the proof of Lemma 4.10,

$$\Phi = (\kappa_0^2 - p\kappa_1^2 + \frac{p(p+1)(p-1)}{2}) \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}$$

Therefore, when

$$x_1 = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

$$x_1\Phi - \Phi x_1 = 2(\kappa_0^2 - p\kappa_1^2 + \frac{p(p+1)(p-1)}{2}) \begin{bmatrix} 0 & -x_{12} \\ x_{21} & 0 \end{bmatrix}$$

and any skew-diagonal matrix anticommutes with Φ.

4) To check the last relation, first claim that

$$p(p+1) = \frac{(\kappa_1 + \kappa_1')^2((\kappa_0 - \kappa_0')^2 + (\kappa_1 - \kappa_1')^2)}{(\kappa_0 - \kappa_0')^2 + (\kappa_1 + \kappa_1')^2}$$

We will use the formulas for κ_0 and κ_1 we developed in the proof of Lemma 5.5. In particular, the following quantities will be useful.

$$\begin{aligned} \kappa_1^2 + (\kappa_1)^2 &= m(m+1) + (m-p)(m-p+1) \\ \kappa_0\kappa_0' &= p\sqrt{m(m+1)(m-p)(m-p+1)} = p\kappa_1\kappa_1' \\ \kappa_1\kappa_1' &= \sqrt{m(m+1)(m-p)(m-p+1)} \\ \kappa^2 &= m(p+1)(m-p+1) \end{aligned}$$

$$\begin{aligned} &\frac{(\kappa_1 + \kappa_1')^2((\kappa_0 - \kappa_0')^2 + (\kappa_1 - \kappa_1')^2)}{(\kappa_0 - \kappa_0')^2 + (\kappa_1 + \kappa_1')^2} \\ &= \frac{(\kappa_1^2 + (\kappa_1')^2 + 2\kappa_1\kappa_1')(2\kappa^2 - 2\kappa_0\kappa_0' - 2\kappa_1\kappa_1')}{2\kappa^2 - 2\kappa_0\kappa_0' + 2\kappa_1\kappa_1'} \\ &= \frac{(\kappa_1^2 + (\kappa_1')^2 + 2\kappa_1\kappa_1')(\kappa^2 - (p+1)\kappa_1\kappa_1')}{\kappa^2 - (p-1)\kappa_1\kappa_1'} \\ &= \frac{\kappa^2(\kappa_1^2 + (\kappa_1')^2) - 2(p+1)(\kappa_1\kappa_1')^2 + (2\kappa^2 - (p+1)(\kappa_1^2 + (\kappa_1')^2))\kappa_1\kappa_1'}{\kappa^2 - (p-1)\kappa_1\kappa_1'} \end{aligned}$$

where

$$\begin{aligned}
&\frac{\kappa^2(\kappa_1^2+(\kappa_1')^2)-2(p+1)(\kappa_1\kappa_1')^2}{\kappa^2}\\
&=\kappa_1^2+(\kappa_1')^2-\frac{2(p+1)(\kappa_1\kappa_1')^2}{\kappa^2}\\
&=m(m+1)+(m-p)(m-p+1)-2(m+1)(m-p)=p^2+p
\end{aligned}$$

and

$$\begin{aligned}
&\frac{2\kappa^2-(p+1)(\kappa_1^2+(\kappa_1')^2)}{-(p-1)}\\
=&\frac{2m(p+1)(m-p+1)-(p+1)(m(m+1)+(m-p)(m-p+1))}{-(p-1)}\\
=&\frac{2m(p+1)(-p+1)-2(p+1)m(1-p)+(p+1)p(-p+1)}{-p+1}=(p+1)p
\end{aligned}$$

4) The relation $(x_1-z_1)^4=p(p+1)(x_1-z_1)^2$ holds. Let $J=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$, notice that on the four dimensional subspace,

$$x_1-z_1=\begin{bmatrix}\frac{N_0}{\kappa^2}Q+cX-\kappa_1 J & f(s_0.T)Z\\ f(T)Z & \frac{N_0}{\kappa^2}R+cY-\kappa_1'J\end{bmatrix}$$

Let us compute each 2×2 block in $(x_1-z_1)^2=(A_{ij})$ individually:

$$\begin{aligned}
A_{11}=&\left(\frac{N_0^2}{\kappa^4}Q^2+c^2X^2+f(T)f(s_0.T)Z^2\right)+\kappa_1^2-\frac{N_0c}{\kappa^2}(QX+XQ)\\
&-\frac{N_0\kappa_1}{\kappa^2}(JQ+QJ)-c\kappa_1(JX+XJ)\\
=&N_0+\kappa_1^2-\frac{2N_0\kappa_1^2}{\kappa^2}-2c\kappa_0\kappa_1
\end{aligned}$$

similarly

$$
\begin{aligned}
A_{22} &= (\frac{N_0^2}{\kappa^4}R^2 + c^2Y^2 + f(T)f(s_0.T)Z^2) + (\kappa_1')^2 - \frac{N_0 c}{\kappa^2}(RX + XR) \\
&\quad - \frac{N_0\kappa_1}{\kappa^2}(JR + RJ) - c\kappa_1(JY + YJ) \\
&= N_0 + \kappa_1^2 - \frac{2N_0(\kappa_1')^2}{\kappa^2} - 2c\kappa_0'\kappa_1'
\end{aligned}
$$

Also we have

$$
\begin{aligned}
A_{12} &= f(s_0.T)(\frac{N_0^2}{\kappa^4}(QZ + ZR) + c(XZ + ZY) - (\kappa_1 JZ + \kappa_1' ZJ)) \\
&= -f(s_0.T)(\kappa_1 JZ + \kappa_1' ZJ) \\
A_{21} &= f(T)(\frac{N_0^2}{\kappa^4}(ZQ + RZ) + c(ZX + YZ) - (\kappa_1' JZ + \kappa_1 ZJ)) \\
&= -f(T)(\kappa_1' JZ + \kappa_1 ZJ)
\end{aligned}
$$

Since A_11 and A_22 are scalar matrices, the condition $A^2 = p(p+1)A$ is equivalent to the following

$$
\begin{bmatrix} A_{11}^2 + A_{12}A_{21} & (A_{11} + A_{22})A_{12} \\ (A_{11} + A_{22})A_{21} & A_{22}^2 + A_{21}A_{12} \end{bmatrix} = p(p+1)\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
$$

which is equivalent to

$$
\begin{aligned}
A_{11}^2 + A_{12}A_{21} - p(p+1)A_{11} &= 0 \\
A_{22}^2 + A_{21}A_{12} - p(p+1)A_{22} &= 0 \\
(A_{11} + A_{22}) &= p(p+1)
\end{aligned}
$$

The last condition is automatically satisifed due to our choice of c. In particular,

$$\begin{aligned}
&A_{11}+A_{22}\\
&=(N_0+\kappa_1^2-\frac{2N_0\kappa_1^2}{\kappa^2}-2c\kappa_0\kappa_1)+(N_0+(\kappa_1')^2-\frac{2N_0(\kappa_1')^2}{\kappa^2}-2c\kappa_0'\kappa_1')\\
&=\frac{2N_0}{\kappa^2}(\kappa^2-\kappa_1^2-(\kappa_1')^2)+(\kappa_1^2+(\kappa_1')^2)-2c(\kappa_0\kappa_1+\kappa_0'\kappa_1')\\
&=p(p+1)
\end{aligned}$$

Let us check $A_{11}^2+A_{12}A_{21}-p(p+1)A_{11}=0$ and the other relation can be checked via symmetry. To calculate $A_{12}A_{21}$, observe

$$\begin{aligned}
JZJZ&=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}\kappa_0-\kappa_0'&\kappa_1+\kappa_1'\\\kappa_1+\kappa_1'&-(\kappa_0-\kappa_0')\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}\kappa_0-\kappa_0'&\kappa_1+\kappa_1'\\\kappa_1+\kappa_1'&-(\kappa_0-\kappa_0')\end{bmatrix}\\
&=\begin{bmatrix}\kappa_0-\kappa_0'&\kappa_1+\kappa_1'\\-(\kappa_1+\kappa_1')&\kappa_0-\kappa_0'\end{bmatrix}^2=(\kappa_0-\kappa_0')^2-(\kappa_1+\kappa_1')^2
\end{aligned}$$

$$\begin{aligned}
JZZJ&=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}\kappa_0-\kappa_0'&\kappa_1+\kappa_1'\\\kappa_1+\kappa_1'&-(\kappa_0-\kappa_0')\end{bmatrix}^2\begin{bmatrix}1&0\\0&-1\end{bmatrix}\\
&=(\kappa_0-\kappa_0')^2+(\kappa_1+\kappa_1')^2
\end{aligned}$$

$$ZJJZ=Z^2=(\kappa_0-\kappa_0')^2+(\kappa_1+\kappa_1')^2$$

and

$$ZJZJ = \begin{bmatrix} \kappa_0 - \kappa_0' & \kappa_1 + \kappa_1' \\ \kappa_1 + \kappa_1' & -(\kappa_0 - \kappa_0') \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \kappa_0 - \kappa_0' & \kappa_1 + \kappa_1' \\ \kappa_1 + \kappa_1' & -(\kappa_0 - \kappa_0') \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \kappa_0 - \kappa_0' & -(\kappa_1 + \kappa_1') \\ \kappa_1 + \kappa_1' & \kappa_0 - \kappa_0' \end{bmatrix}^2 = (\kappa_0 - \kappa_0')^2 - (\kappa_1 + \kappa_1')^2$$

Therefore

$$\begin{aligned} A_{12}A_{21} &= f(T)f(s_0.T)(\kappa_1 JZ + \kappa_1' ZJ)(\kappa_1' JZ + \kappa_1 ZJ) \\ &= f(T)f(s_0.T)(2\kappa_1\kappa_1'((\kappa_0 - \kappa_0')^2 - (\kappa_1 + \kappa_1')^2) \\ &\quad + (\kappa_1^2 + (\kappa_1')^2)((\kappa_0 - \kappa_0')^2 + (\kappa_1 + \kappa_1')^2)) \\ &= f(T)f(s_0.T)(\kappa_1^2 + \kappa_1')^2((\kappa_0 - \kappa_0')^2 + (\kappa_1 - \kappa_1')^2) \end{aligned}$$

by the claim in 6), and the condition in f imposed in the construction, this quantity is equal to

$$\begin{aligned} A_{12}A_{21} &= f(T)f(s_0.T)((\kappa_0 - \kappa_0')^2 + (\kappa_1 + \kappa_1')^2)p(p+1) \\ &= p(p+1)(N_0 - \frac{N_0^2}{\kappa^2} - c^2\kappa^2) \end{aligned}$$

Therefore it remains to check the following quantity is zero.

$$
\begin{aligned}
&A_{11}^2 + A_{12}A_{21} - p(p+1)A_{11} \\
&= (N_0 + \kappa_1^2 - \frac{2N_0\kappa_1^2}{\kappa^2} - 2c\kappa_0\kappa_1)^2 + p(p+1)(N_0 - \frac{N_0^2}{\kappa^2} - c^2\kappa^2) \\
&\quad - p(p+1)(N_0 + \kappa_1^2 - \frac{2N_0\kappa_1^2}{\kappa^2} - 2c\kappa_0\kappa_1) \\
&= (\frac{N_0}{\kappa^2}(\kappa^2 - 2\kappa_1^2) + \kappa_1^2 - 2c\kappa_0\kappa_1)^2 + p(p+1)(\kappa^2(\frac{N_0}{\kappa^2} - (\frac{N_0}{\kappa^2})^2) - (c\kappa_0\kappa_1)^2\frac{\kappa^2}{\kappa_0^2\kappa_1^2}) \\
&\quad - p(p+1)(\frac{N_0}{\kappa^2}(\kappa^2 - 2\kappa_1^2) + \kappa_1^2 - 2c\kappa_0\kappa_1)
\end{aligned}
$$

Let us substitute $\frac{N_0}{\kappa^2}$ with x, and organize the above equation according to the power of $(c\kappa_0\kappa_1)$:

$$
\begin{aligned}
&A_{11}^2 + A_{12}A_{21} - p(p+1)A_{11} \\
&= (4 - \frac{p(p+1)\kappa^2}{\kappa_0^2\kappa_1^2})(c\kappa_0\kappa_1)^2 + (2p(p+1) - 4(x(\kappa^2 - 2\kappa_1^2) + \kappa_1^2))c\kappa_0\kappa_1 \\
&\quad + (x(\kappa^2 - 2\kappa_1^2) + \kappa_1^2)^2 + p(p+1)\kappa^2(x - x^2) - p(p+1)(x(\kappa^2 - 2\kappa_1^2) + \kappa_1^2))
\end{aligned}
$$

Now we plug in all formulas we developed for $\kappa, \kappa_0, \kappa_1$ which are mentioned in claim 6). In particular, the coefficients for the variable $(c\kappa_0\kappa_1)$ are as follows

$$
\begin{aligned}
&4 - \frac{p(p+1)\kappa^2}{\kappa_0^2\kappa_1^2} \\
&= 4 - \frac{p(p+1)^2 m(m-p+1)}{m^2 p(m-p)(m+1)} \\
&= 4 - \frac{(m-p+1)(p+1)^2}{m(m-p)(m+1)}
\end{aligned}
$$

$$
\begin{aligned}
&2p(p+1) - 4(x(\kappa^2 - 2\kappa_1^2) + \kappa_1^2) \\
&= 2p(p+1) - 4m(m+1) - 4xm(pm - m - p^2 - 1)
\end{aligned}
$$

and lastly,

$$
\begin{aligned}
&(x(\kappa^2 - 2\kappa_1^2) + \kappa_1^2)^2 + p(p+1)\kappa^2(x - x^2) - p(p+1)(x(\kappa^2 - 2\kappa_1^2) + \kappa_1^2)) \\
&= x^2((\kappa^2 - 2\kappa_1^2)^2 - p(p+1)\kappa^2) + x(2\kappa_1^2(\kappa^2 - 2\kappa_1^2) + p(p+1)\kappa^2 \\
&\quad - p(p+1)(\kappa^2 - 2\kappa_1^2)) + \kappa_1^4 - p(p+1)\kappa_1^2 \\
&= x^2(m^2(mp - p^2 - m - 1)^2 - p(p+1)^2 m(m - p + 1)) \\
&\quad + x(2m(m+1)m(mp - p^2 - m - 1) + 2p(p+1)m(m+1)) \\
&\quad + m(m+1)(m(m+1) - p(p+1)) \\
&= m(m-p)(m+1)(p-1)(m(p-1) - (p+1)^2)x^2 \\
&+ 2m(m+1)(m-p)(mp - m - p - 1)x + m(m+1)(m-p)(m+p+1)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&A_{11}^2 + A_{12}A_{21} - p(p+1)A_{11} \\
&= (4 - \frac{(m-p+1)(p+1)^2}{m(m-p)(m+1)})(c\kappa_0\kappa_1)^2 \\
&\quad + (2p(p+1) - 4m(m+1) - 4xm(pm - m - p^2 - 1))(c\kappa_0\kappa_1) \\
&+ m(m-p)(m+1)(p-1)(m(p-1) - (p+1)^2)x^2 \\
&+ 2m(m+1)(m-p)(mp - m - p - 1)x + m(m+1)(m-p)(m+p+1)
\end{aligned}
\tag{**}
$$

To calculate $c\kappa_0\kappa_1$, observe

$$\begin{aligned}
\kappa_0\kappa_1 &= m\sqrt{(m+1)(m-p)p} \\
\kappa_0'\kappa_1' &= (m+1-p)\sqrt{(m+1)(m-p)p} \\
\kappa_1^2 + (\kappa_1')^2 - p(p+1) &= m(m+1) + (m-p)(m-p+1) - p(p+1) \\
&= (m-p)((m-p+1) + (m+p+1)) \\
&= 2(m-p)(m+1)
\end{aligned}$$

and

$$\begin{aligned}
&\kappa^2 - (\kappa_1^2 + (\kappa_1')^2) \\
&= m(p+1)(m-p+1) - m(m+1) - (m-p)(m-p+1) \\
&= (m-p)(m(p+1) - m - (m-p+1)) \\
&= (m-p)(m+1)(p-1)
\end{aligned}$$

Therefore

$$\begin{aligned}
c\kappa_0\kappa_1 &= \frac{2\frac{N_0}{\kappa^2}(\kappa^2 - \kappa_1^2 - (\kappa_1')^2) + (\kappa_1^2 + (\kappa_1')^2 - p(p+1))}{2(1 + \frac{\kappa_0'\kappa_1'}{\kappa_0\kappa_1})} \\
&= \frac{m(m-p)(m+1)(p-1)x}{2m-p+1} + \frac{m(m-p)(m+1)}{2m-p+1}
\end{aligned}$$

We claim that all the coefficients of x in Equation (**) are zero. The coefficient of x^2 is:

$$
\begin{aligned}
&(4-\frac{(m-p+1)(p+1)^2}{m(m-p)(m+1)})(\frac{m(m-p)(m+1)(p-1)}{2m-p+1})^2 \\
&-4m(pm-m-p^2-1)\frac{m(m-p)(m+1)(p-1)}{2m-p+1} \\
&+m(m-p)(m+1)(p-1)(m(p-1)-(p+1)^2) \\
=&\frac{m(m-p)(m+1)(p-1)}{(2m-p+1)^2}\cdot B
\end{aligned}
$$

where

$$
\begin{aligned}
B =& 4m(m-p)(m+1)(p-1)-(m-p+1)(p+1)^2(p-1) \\
&-4m(pm-m-p^2-1)(2m-p+1) \\
&+(m(p-1)-(p+1)^2)(2m-p+1)^2 \\
=&(2m-p+1)(-4m(pm-m-p^2-1) \\
&+(m(p-1)-(p+1)^2)(2m-p+1) \\
&+(2m^2-(p-1)m-(p+1)^2)(p-1)) \\
=&(2m-p+1)^2(m(p-1)-(p+1)^2-m(p-1)+(p+1)^2)=0
\end{aligned}
$$

The coefficient of x is:

$$
\begin{aligned}
&2(4-\frac{(m-p+1)(p+1)^2}{m(m-p)(m+1)})\frac{m^2(m-p)^2(m+1)^2(p-1)}{(2m-p+1)^2}\\
&+(2p(p+1)-4m(m+1))\frac{m(m-p)(m+1)(p-1)}{2m-p+1}\\
&-4m(pm-m-p^2-1)\frac{m(m-p)(m+1)}{2m-p+1}\\
&+2m(m+1)(m-p)(mp-m-p-1)\\
=&\frac{m(m-p)(m+1)}{(2m-p+1)^2}C
\end{aligned}
$$

where

$$
\begin{aligned}
C=&\,8m(m-p)(m+1)(p-1)-2(m-p+1)(p+1)^2(p-1)\\
&+(2m-p+1)((2p(p+1)-4m(m+1))(p-1)\\
&-4m(pm-m-p^2-1))+2(2m-p+1)^2(mp-m-p-1)\\
=&\,2(2m-p+1)((p(p+1)-2m(m+1))(p-1)-2m(pm-m-p^2-1)\\
&+(2m-p+1)(mp-m-p-1)+(p-1)(m(2m-p+1)-(p+1)^2))\\
=&\,2(2m-p+1)^2((mp-m-p-1)+m(p-1)-2m(p-1)+(p+1))=0
\end{aligned}
$$

Finally, the constant term in Equation (**) with respect to x is:

$$
\begin{aligned}
&(4-\frac{(m-p+1)(p+1)^2}{m(m-p)(m+1)})(\frac{m(m-p)(m+1)}{2m-p+1})^2\\
&+(2p(p+1)-4m(m+1))\frac{m(m-p)(m+1)}{2m-p+1}\\
&+m(m+1)(m-p)(m+p+1)\\
=&\frac{m(m-p)(m+1)}{(2m-p+1)^2}D
\end{aligned}
$$

where

$$\begin{aligned}D &= 4m(m-p)(m+1) - (m-p+1)(p+1)^2 \\ &\quad + (2p(p+1) - 4m(m+1))(2m-p+1) \\ &\quad + (m+p+1)(2m-p+1)^2 \\ &= (2m-p+1)((m+p+1)(2m-p+1) + (2p(p+1) - 4m(m+1)) \\ &\quad - (p+1)^2 + m(2m-p+1)) \\ &= (2m-p+1)^2((m+p+1) + m - 2m - (p+1)) = 0\end{aligned}$$

Therefore we have verified the last relation. □

Lemma 5.14. *The relation*

$$x_1(s_1x_1s_1 + (1-c_1c_2)s_1) = (s_1x_1s_1 + (1-c_1c_2)s_1)x_1$$

is satisfied.

Proof. Let C_3 be the Clifford algebra generated by c_0, c_1, c_2 and ϕ_0, ϕ_1 be the $\mathbb{C}$-linear map on C_3 such that $x_1a = \phi_0(a)x_1$, $s_1a = \phi_1(a)s_1$ for any $a \in C_3$. Notice $\phi_0(c_1) = -c_1$, $\phi_1(c_1) = c_2$ and $\phi_1(c_2) = c_1$. For any path $T \in \Gamma^\lambda$, let $x_1.v_T = a_Tv_T + b_Tv_{s_0.T}$, $s_1.v_T = d_Tv_T + e_{s_1.T}v_T$, where $a_T, b_T, d_T \in C_3$, and $e_T \in \mathbb{C}$ as given in the construction, and $\phi_0(e_T) = \phi_1(e_T)$ for any $T \in \Gamma^\lambda$.

$$\begin{aligned}x_1s_1.v_T &= x_1(d_Tv_T + e_Tv_{s_1.T}) \\ &= \phi_0(d_T)(a_Tv_T + b_Tv_{s_0.T}) + e_T(a_{s_1.T}v_{s_1.T} + b_{s_1.T}v_{s_0s_1.T})\end{aligned}$$

$$
\begin{aligned}
&x_1 s_1 x_1 s_1 . v_T \\
&= x_1 s_1 . (\phi_0(d_T)(a_T v_T + b_T v_{s_0.T}) + e_T(a_{s_1.T} v_{s_1.T} + b_{s_1.T} v_{s_0 s_1.T})) \\
&= x_1 . (\phi_1\phi_0(d_T)\phi_1(a_T)(d_T v_T + e_T v_{s_1.T}) \\
&\quad + \phi_1\phi_0(d_T)\phi_1(b_T)(d_{s_0.T} v_{s_0.T} + e_{s_0.T} v_{s_1 s_0.T}) \\
&\quad + e_T \phi_1(a_{s_1.T})(d_{s_1.T} v_{s_1.T} + e_{s_1.T} v_T) \\
&\quad + e_T \phi_1(b_{s_1.T})(d_{s_0 s_1.T} v_{s_0 s_1.T} + e_{s_0 s_1.T} v_{s_1 s_0 s_1.T})) \\
&= \phi_0\phi_1\phi_1(d_T)\phi_0\phi_1(a_T)\phi_0(d_T)(a_T v_T + b_T v_{s_0.T}) \\
&\quad + \phi_0\phi_1\phi_0(d_T)\phi_0\phi_1(a_T) e_T (a_{s_1.T} v_{s_1.T} + b_{s_1.T} v_{s_0 s_1.T}) \\
&\quad + \phi_0\phi_1\phi_0(d_T)\phi_0\phi_1(b_T)\phi_0(d_{s_0.T})(a_{s_0.T} v_{s_0.T} + b_{s_0.T} v_T) \\
&\quad + \phi_0\phi_1\phi_0(d_T)\phi_0\phi_1(b_T) e_{s_0.T}(a_{s_1 s_0.T} v_{s_1 s_0.T} + b_{s_1 s_0.T} v_{s_0 s_1 s_0.T}) \\
&\quad + e_T \phi_0\phi_1(a_{s_1.T})\phi_0(d_{s_1.T})(a_{s_1.T} v_{s_1.T} + b_{s_1.T} v_{s_0 s_1.T}) \\
&\quad + e_T \phi_0\phi_1(a_{s_1.T}) e_{s_1.T}(a_T v_T + b_T v_{s_0.T}) \\
&\quad + e_T \phi_0\phi_1(b_{s_1.T})\phi_0(d_{s_0 s_1.T})(a_{s_0 s_1.T} v_{s_0 s_1.T} + b_{s_0 s_1.T} v_{s_1 T}) \\
&\quad + e_T \phi_0\phi_1(b_{s_1.T}) e_{s_0 s_1.T}(a_{s_1 s_0 s_1.T} v_{s_1 s_0 s_1.T} + b_{s_1 s_0 s_1.T} v_{s_0 s_1 s_0 s_1.T})
\end{aligned}
$$

Similarly,

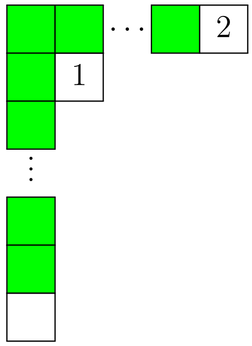
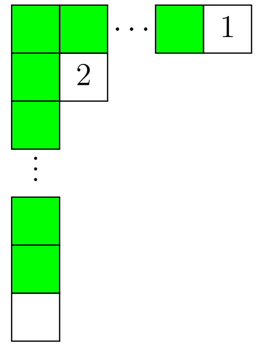
$$
\begin{aligned}
s_1 x_1 . v_T &= s_1(a_T v_T + b_T v_{s_0.T}) \\
&= \phi_1(a_T)(d_T v_T + e_T v_{s_1.T}) + \phi_1(b_T)(d_{s_0.T} v_{s_0.T} + e_{s_0.T} v_{s_1 s_0.T})
\end{aligned}
$$

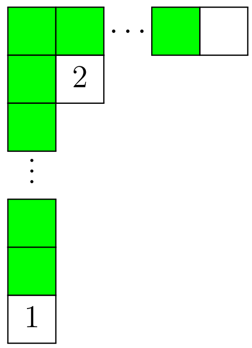
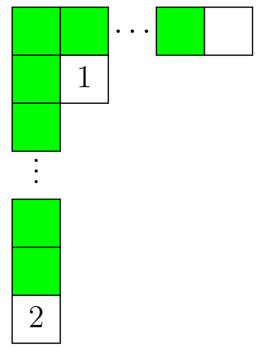
$$
\begin{aligned}
&s_1 x_1 s_1 x_1 . v_T \\
&= s_1(\phi_0\phi_1(a_T)\phi_0(d_T)(a_T v_T + b_T v_{s_0.T}) + \phi_0\phi_1(a_T)e_T(a_{s_1.T}v_{s_1.T} + b_{s_1.T}v_{s_0 s_1.T}) \\
&\quad + \phi_0\phi_1(b_T)\phi_0(d_{s_0.T})(a_{s_0.T}v_{s_0.T} + b_{s_0.T}v_T) \\
&\quad + \phi_0\phi_1(b_T)e_{s_0.T}(a_{s_1 s_0.T}v_{s_1 s_0.T} + b_{s_1 s_0.T}v_{s_0 s_1 s_0.T})) \\
&= \phi_1\phi_0\phi_1(a_T)\phi_1\phi_0(d_T)\phi_1(a_T)(d_T v_T + e_T v_{s_1.T}) \\
&\quad + \phi_1\phi_0\phi_1(a_T)\phi_1\phi_0(d_T)\phi_1(b_T)(d_{s_0.T}v_{s_0.T} + e_{s_0.T}v_{s_1 s_0.T}) \\
&\quad + \phi_1\phi_0\phi_1(a_T)e_T\phi_1(a_{s_1.T})(d_{s_1.T}v_{s_1.T} + e_{s_1.T}v_T) \\
&\quad + \phi_1\phi_0\phi_1(a_T)e_T\phi_1(b_{s_1.T})(d_{s_0 s_1.T}v_{s_0 s_1.T} + e_{s_0 s_1.T}v_{s_1 s_0 s_1.T}) \\
&\quad + \phi_1\phi_0\phi_1(b_T)\phi_1\phi_0(d_{s_0.T})\phi_1(a_{s_0.T})(d_{s_0.T}v_{s_0.T} + e_{s_0.T}v_{s_1 s_0.T}) \\
&\quad + \phi_1\phi_0\phi_1(b_T)\phi_1\phi_0(d_{s_0.T})\phi_1(b_{s_0.T})(d_T v_T + e_T v_{s_1.T}) \\
&\quad + \phi_1\phi_0\phi_1(b_T)e_{s_0.T}\phi_1(a_{s_1 s_0.T})(d_{s_1 s_0.T}v_{s_1 s_0.T} + e_{s_1 s_0.T}v_{s_0.T}) \\
&\quad + \phi_1\phi_0\phi_1(b_T)e_{s_0.T}\phi_1(b_{s_1 s_0.T})(d_{s_0 s_1 s_0.T}v_{s_0 s_1 s_0.T} + e_{s_0 s_1 s_0.T}v_{s_1 s_0 s_1 s_0.T})
\end{aligned}
$$

To check

$$
\begin{aligned}
&x_1(s_1 x_1 s_1 + (1 - c_1 c_2)s_1) - (s_1 x_1 s_1 + (1 - c_1 c_2)s_1)x_1 \\
&= x_1 s_1 x_1 s_1 - s_1 x_1 s_1 x_1 + (1 + c_1 c_2)x_1 s_1 - (1 - c_1 c_2)s_1 x_1
\end{aligned}
$$

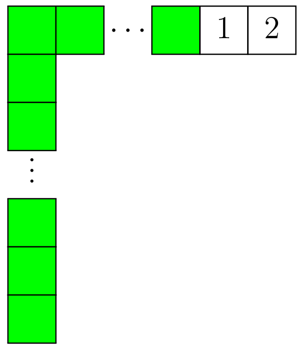
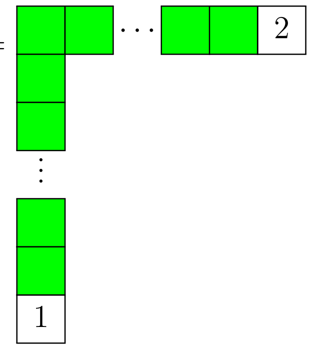
is equal to zero, let us check the coefficients for each v_T of the above expression. We will let T be one of the following tableaux in two cases. Again we omit the staircase portion of the tableaux, as well as boxes filled with integers 3 and above, and green boxes have no entries.

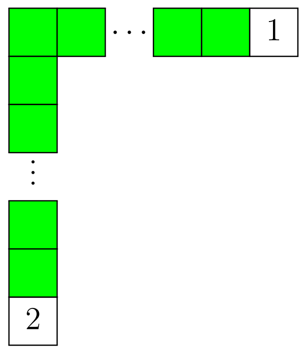
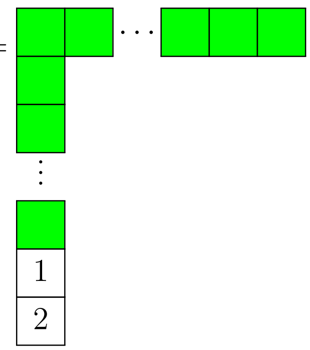
Case 1) L_1=  (via $s_1 \leftrightarrow$) L_2= 

(via $s_0 \leftrightarrow$) L_3=  (via $s_1 \leftrightarrow$) L_4= 

Notice $s_0.L_1 = L_1$, $s_0.L_4 = L_4$, because given the partitions $L_1^{(1)}$ and $L_4^{(1)}$, the position of the first added box is uniquely determined. Therefore $v_{s_0.L_1} = v_{s_0.L_4} = 0$ and $b_T = 0$ for $T = L_1, s_0.L_1, L_4, s_0.L_4$. Also let m be the number of boxes in the first row of $L_2^{(0)}$, consistent with the notation in the proof of Lemma 5.5, and recall $N_0 = n(n+1)$. The eigenvalues for z_0, z_1, z_2 of the above tableaux are given as follows

	$\kappa(0)$	$\kappa(1)$	$\kappa(2)$
L_1	$\sqrt{mp(m-p)}$	$\sqrt{N_0}$	$\sqrt{m(m+1)}$
L_2	$\sqrt{mp(m-p)}$	$\sqrt{m(m+1)}$	$\sqrt{N_0}$
L_3	$\sqrt{(m+1)p(m+1-p)}$	$\sqrt{(m-p)(m-p+1)}$	$\sqrt{N_0}$
L_4	$\sqrt{(m+1)p(m+1-p)}$	$\sqrt{N_0}$	$\sqrt{(m-p)(m-p+1)}$

Case 2) T_1=  (via $s_0 \leftrightarrow$) T_2= 

(via $s_1 \leftrightarrow$) T_3=  (via $s_0 \leftrightarrow$) T_4= 

Let m be the number of boxes in the first row of $T_3^{(0)}$, then the eigenvalues for z_0, z_1, z_2 are as follows

	$\kappa(0)$	$\kappa(1)$	$\kappa(2)$
T_1	$\sqrt{(m-1)p(m-1-p)}$	$\sqrt{m(m-1)}$	$\sqrt{m(m+1)}$
T_2	$\sqrt{mp(m-p)}$	$\sqrt{(m-p-1)(m-p)}$	$\sqrt{m(m+1)}$
T_3	$\sqrt{mp(m-p)}$	$\sqrt{m(m+1)}$	$\sqrt{(m-p-1)(m-p)}$
T_4	$\sqrt{(m+1)p(m+1-p)}$	$\sqrt{(m-p)(m-p+1)}$	$\sqrt{(m-p-1)(m-p)}$

For the rest of the discussion, for any one of the paths T above, Let κ_0, κ_1, κ_2 be the eigenvalues associated to T, then the eigenvalues associated to the other three paths in the same orbit of T are as follows:

	$s_0s_1.T$	$s_1.T$	T	$s_0.T$	$s_1s_0.T$	$s_0s_1s_0.T$
z_0	κ_0''	κ_0	κ_0	κ_0'	κ_0'	κ_0'''
z_1	κ_2''	κ_2	κ_1	κ_1'	κ_2	κ_2'''
z_2	κ_1	κ_1	κ_2	κ_2	κ_1'	κ_1'

We also list the following quantities based on the above eigenvalues: The quantities a_T:

$$\begin{aligned}
a_T &= \frac{(N_0 p + \kappa_0^2)\kappa_1}{\kappa_0^2 + p\kappa_1^2} + \frac{(-N_0 + \kappa_1^2)\kappa_0}{\kappa_0^2 + p\kappa_1^2} c_0 c_1 = \gamma + \delta c_0 c_1 \\
a_{s_0.T} &= \frac{(N_0 p + (\kappa_0')^2)\kappa_1'}{(\kappa_0')^2 + p(\kappa_1')^2} + \frac{(-N_0 + (\kappa_1')^2)\kappa_0'}{(\kappa_0')^2 + p(\kappa_1')^2} c_0 c_1 = \gamma' + \delta' c_0 c_1 \\
a_{s_1.T} &= \frac{(N_0 p + \kappa_0^2)\kappa_2}{\kappa_0^2 + p\kappa_2^2} + \frac{(-N_0 + \kappa_2^2)\kappa_0}{\kappa_0^2 + p(\kappa_2)^2} c_0 c_1 = \gamma_1 + \delta_1 c_0 c_1 = \gamma'' + \delta'' c_0 c_1 \\
a_{s_1 s_0.T} &= \frac{(N_0 p + (\kappa_0')^2)\kappa_2}{(\kappa_0')^2 + p\kappa_2^2} + \frac{(-N_0 + \kappa_2^2)\kappa_0'}{(\kappa_0')^2 + p\kappa_2^2} c_0 c_1 = \gamma''' + \delta''' c_0 c_1 \\
a_{s_0 s_1.T} &= \frac{(N_0 p + (\kappa_0'')^2)\kappa_2}{(\kappa_0'')^2 + p(\kappa_2'')^2} + \frac{(-N_0 + (\kappa_2'')^2)\kappa_0''}{(\kappa_0'')^2 + p(\kappa_2'')^2} c_0 c_1 = \gamma_2 + \delta_2 c_0 c_1
\end{aligned}$$

The quantities b_T:

$$\begin{aligned}
b_T &= f(T)((\kappa_0 - \kappa_0')c_0 + (\kappa_1 + \kappa_1')c_1) = Ac_0 + Bc_1 \\
&= f(T)(A'c_0 + B'c_1) \\
b_{s_0.T} &= f(s_0.T)(-(\kappa_0 - \kappa_0')c_0 + (\kappa_1 + \kappa_1')c_1) = Dc_0 + Ec_1 \\
&= f(s_0.T)(-A'c_0 + B'c_1) \\
b_{s_1.T} &= f(s_1.T)((\kappa_0 - \kappa_0'')c_1 + (\kappa_2 + \kappa_2'')c_1) \\
&= f(s_1.T)(A_1 c_0 + B_1 c_1) = f(s_1.T)(Fc_0 + Gc_1) \\
b_{s_0 s_1.T} &= f(s_0 s_1.T)(-Fc_0 + Gc_1) \\
b_{s_1 s_0.T} &= f(s_1 s_0.T)((\kappa_0' - \kappa_0''')c_0 + (\kappa_2 + \kappa_2''')c_1) = f(s_1 s_0.T)(Hc_0 + Ic_1)
\end{aligned}$$

Note that $AE = -DB$. The quantities d_T:

$$\begin{aligned}
d_T &= -\frac{1}{\kappa_1 - \kappa_2} + \frac{1}{\kappa_1 + \kappa_2} c_1 c_2 = \alpha + \beta c_1 c_2 \\
d_{s_0.T} &= -\frac{1}{\kappa_1' - \kappa_2} + \frac{1}{\kappa_1' + \kappa_2} c_1 c_2 = \alpha' + \beta' c_1 c_2 \\
d_{s_1.T} &= -\frac{1}{\kappa_2 - \kappa_1} + \frac{1}{\kappa_2 + \kappa_1} c_1 c_2 = \alpha_1 + \beta_1 c_1 c_2 \\
d_{s_0 s_1.T} &= -\frac{1}{\kappa_2'' - \kappa_1} + \frac{1}{\kappa_2'' + \kappa_1} c_1 c_2 = \alpha_2 + \beta_2 c_1 c_2 \\
d_{s_0 s_1 s_0.T} &= -\frac{1}{\kappa_2''' - \kappa_1'} + \frac{1}{\kappa_2''' + \kappa_1'} c_1 c_2 = \alpha_3 + \beta_3 c_1 c_2
\end{aligned}$$

The quantities e_T:

$$\begin{aligned}
e_T = e_{s_1.T} &= \sqrt{1 - \frac{1}{(\kappa_1 - \kappa_2)^2} - \frac{1}{(\kappa_1 + \kappa_2)^2}} \\
e_{s_0.T} = e_{s_1 s_0.T} &= \sqrt{1 - \frac{1}{(\kappa_1' - \kappa_2)^2} - \frac{1}{(\kappa_1' + \kappa_2)^2}}
\end{aligned}$$

We now check the coefficients individually: the coefficient for v_T will be checked manually. The other coefficients will be checked using MAGMA, by computing rational functions and check that after simplification, the numerator is a polynomial in the variables $\sqrt{m}$, $\sqrt{m-1}$, $\sqrt{m+1}$, $\sqrt{p}$, $\sqrt{m-p}$, $\sqrt{m-p-1}$, $\sqrt{m-p+1}$, $\sqrt{n}$, $\sqrt{n+1}$ which is identically zero.

1) The coefficient for v_T.

It is helpful to observe that

$$\alpha+\beta = \frac{-2\kappa_2}{\kappa_1^2-\kappa_2^2}, \qquad \alpha'+\beta' = \frac{-2\kappa_2}{(\kappa_1')^2-\kappa_2^2}$$

$$\alpha^2-\beta^2 = \frac{4\kappa_1\kappa_2}{(\kappa_1^2-\kappa_2^2)^2}, \qquad (\alpha')^2-(\beta')^2 = \frac{4\kappa_1'\kappa_2}{((\kappa_1')^2-\kappa_2^2)^2}$$

$$\alpha\alpha'-\beta\beta' = \frac{2(\kappa_1+\kappa_1')\kappa_2}{((\kappa_1')^2-\kappa_2^2)(\kappa_1^2-\kappa_2^2)}, \qquad \beta\alpha'-\alpha\beta' = \frac{2(\kappa_1'-\kappa_1)\kappa_2}{(\kappa_1^2-\kappa_2^2)((\kappa_1')^2-\kappa_2^2)}$$

$$\alpha'\beta+\alpha\beta' = -\frac{2(\kappa_1\kappa_1'-\kappa_2^2)}{((\kappa_1')^2-\kappa_2^2)(\kappa_1^2-\kappa_2^2)}, \qquad \alpha\alpha'+\beta\beta = \frac{2(\kappa_1\kappa_1'+\kappa_2^2)}{((\kappa_1')^2-\kappa_2^2)(\kappa_1^2-\kappa_2^2)}$$

It is a straightforward calculation to see that

$$\phi_0\phi_1\phi_0(d_T)\phi_0\phi_1(a_T)\phi_0(d_T)a_T - \phi_1\phi_0\phi_1(a_T)\phi_1\phi_0(d_T)\phi_1(a_T)d_T$$

$$= -2(\alpha^2+\beta^2)\delta^2 c_1c_2 - 4\gamma^2\alpha\beta c_1c_2$$

$$(1+c_1c_2)\phi_0(d_T)a_T - (1-c_1c_2)\phi_1(a_T)d_T = -2\gamma\beta c_1c_2 + 2\alpha\gamma c_1c_2$$

$$\phi_0\phi_1\phi_0(d_T)\phi_0\phi_1(b_T)\phi_0(d_{s_0.T})b_{s_0.T} - \phi_1\phi_0\phi_1(b_T)\phi_1\phi_0(d_{s_0.T})\phi_1(b_{s_0.T})d_T$$

$$= 2AD(\alpha'\beta+\alpha\beta')c_1c_2 - 2BE(\alpha\alpha'+\beta\beta')c_1c_2$$

$$e_T\phi_0\phi_1(a_{s_1.T})e_{s_1.T}a_T - \phi_1\phi_0\phi_1(a_T)e_T\phi_1(a_{s_1.T})e_{s_1.T}$$

$$= 2\delta\delta' c_1c_2$$

Therefore it is enough to check the following quantity is zero for all $T \in \Gamma^\lambda$:

$$-4\alpha\beta\gamma^2 - 2\delta^2(\alpha^2+\beta^2) - 2e_Te_{s_1.T}\delta\delta' - 2\beta\gamma + 2\alpha\gamma$$
$$+2AD(\alpha\beta'+\alpha'\beta) - 2BE(\alpha\alpha'+\beta\beta') \qquad (*)$$

Therefore

$$
\begin{aligned}
&AD(\alpha'\beta+\alpha\beta')-BE(\alpha\alpha'+\beta\beta')\\
=&\frac{2f(T)f(s_0.T)}{((\kappa_1')^2-\kappa_2^2)(\kappa_1^2-\kappa_2^2)}((\kappa_0-\kappa_0')^2(\kappa_1\kappa_1'-\kappa_2^2)-(\kappa_1+\kappa_1')^2(\kappa_1\kappa_1'+\kappa_2^2))
\end{aligned}
$$

Notice this expression is the same when for T and $s_0.T$. Based on the condition on f, notice that

$$
\begin{aligned}
&\frac{(\kappa_0-\kappa_0')^2(\kappa_1\kappa_1'-\kappa_2^2)-(\kappa_1+\kappa_1')^2(\kappa_1\kappa_1'+\kappa_2^2)}{(\kappa_0-\kappa_0')^2+(\kappa_1+\kappa_1')^2} \qquad (\dagger)\\
=&\frac{\kappa_1\kappa_1'((\kappa_0-\kappa_0')^2-(\kappa_1+\kappa_1')^2)}{(\kappa_0-\kappa_0')^2+(\kappa_1+\kappa_1')^2}-\kappa_2^2\\
=&\kappa_1\kappa_1'\frac{\kappa_0^2-\kappa_1^2+(\kappa_0')^2-(\kappa_1')^2-2\kappa_0\kappa_0'-2\kappa_1\kappa_1'}{2\kappa^2-2\kappa_0\kappa_0'+2\kappa_1\kappa_1'}-\kappa_2^2\\
=&\frac{\kappa_0^2-\kappa_1^2+(\kappa_0')^2-(\kappa_1')^2-2\kappa_0\kappa_0'-2\kappa_1\kappa_1'}{2\frac{\kappa^2}{\kappa_1\kappa_1'}-2(p-1)}-\kappa_2^2
\end{aligned}
$$

And the last equality holds since $\kappa_0\kappa_0'=p\kappa_1\kappa_1'$ in the calculation in Lemma 5.5. Moreover, when m is the number of boxes in $T^{(1)}$,

$$
\begin{aligned}
&\kappa_0^2-\kappa_1^2+(\kappa_0')^2-(\kappa_1')^2\\
=&mp(m-p)-m(m+1)+(m+1)p(m+1-p)-(m-p)(m-p+1)\\
=&2(p-1)(m-p)(m+1)
\end{aligned}
$$

and

$$
\frac{-2\kappa_0\kappa_0'-2\kappa_1\kappa_1'}{2\frac{\kappa^2}{\kappa_1\kappa_1'}}=-(p+1)\frac{(\kappa_1\kappa_1')^2}{\kappa^2}=-(m+1)(m-p)
$$

Therefore the quantity in (†) is equal to $-(m+1)(m-p)-\kappa_2^2$ and

$$\begin{aligned}
&\Sigma_1 = 2AD(\alpha'\beta+\alpha\beta') - 2BE(\alpha\alpha'+\beta\beta')\\
&= -\frac{4f(T)f(s_0.T)}{((\kappa_1')^2-\kappa_2^2)(\kappa_1^2-\kappa_2^2)}((\kappa_0-\kappa_0')^2+(\kappa_1+\kappa_1')^2)(\kappa_2^2+(m-p)(m+1))\\
&= 4\frac{\kappa_2^2+(m-p)(m+1)}{((\kappa_1')^2-\kappa_2^2)(\kappa_1^2-\kappa_2^2)}\frac{\kappa_0^2+p^2\kappa_1^2}{(\kappa_0^2+p\kappa_1^2)^2}(N_0-\kappa_1^2)(N_0-(\kappa_1')^2)
\end{aligned}$$

where the last equality holds because of the condition on f shown in Lemma 5.11, and the quantity is still the same for T and $s_0.T$ as argued earlier.

On the other hand,

$$\begin{aligned}
&\Sigma_2 = -4\alpha\beta\gamma^2 - 2\delta^2(\alpha^2+\beta^2) - 2\beta\gamma + 2\alpha\gamma\\
&= \frac{4}{\kappa_1^2-\kappa_2^2}\frac{\kappa_1^2}{(\kappa_0^2+p\kappa_1^2)^2}(N_0p+\kappa_0)^2 - 2\frac{\kappa_0^2}{(\kappa_0^2+p\kappa_1^2)^2}(-N_0+\kappa_1)^2\frac{2(\kappa_1^2+\kappa_2^2)}{(\kappa_1^2-\kappa_2^2)^2}\\
&\quad - \frac{\kappa_1}{(\kappa_0^2+p\kappa_1^2)}(N_0p+\kappa_0)\frac{4\kappa_1}{\kappa_1^2-\kappa_2^2}\\
&= \frac{4(N_0-\kappa_1^2)}{(\kappa_0^2+p\kappa_1^2)^2(\kappa_1^2-\kappa_2^2)^2}((N_0p+\kappa_0^2)\kappa_1^2(\kappa_1^2-\kappa_2^2)p-(\kappa_1^2+\kappa_2^2)(N_0-\kappa_1^2)\kappa_0^2)
\end{aligned}$$

Also observe in Lemma 5.8, $\kappa_0\kappa_0' = p\kappa_1\kappa_1'$ and

$$(\kappa_0')^2+p(\kappa_1')^2 = (\kappa_0')^2 + p\frac{\kappa_0^2(\kappa_0')^2}{p^2\kappa_1^2} = (\kappa_0')^2\frac{\kappa_0^2+p\kappa_1^2}{p\kappa_1^2}$$

therefore

$$\frac{1}{(\kappa_0')^2+p(\kappa_1')^2} = \frac{1}{\kappa_0^2+p\kappa_1^2}\frac{p\kappa_1^2}{(\kappa_0')^2} = \frac{1}{\kappa_0^2+p\kappa_1^2}\frac{\kappa_0^2}{p(\kappa_1')^2} \tag{‡}$$

We now discuss by cases: When $T = L_1$ or L_4, $\kappa_1 = \sqrt{N_0}$, $\delta = 0$, $\gamma = \sqrt{N_0}$,

$A = B = 0$, $\Sigma_1 = 0$, $\alpha = -\frac{1}{\kappa_1 - \kappa_2}$ and $\beta = \frac{1}{\kappa_1 + \kappa_2}$. The above quantity becomes

$$
\begin{aligned}
& -4\alpha\beta\gamma^2 - 2\beta\gamma + 2\alpha\gamma \\
= & \frac{4}{\kappa_1^2 - \kappa_2^2} N_0 - \frac{4\kappa_1}{\kappa_1^2 - \kappa_2^2}\sqrt{N_0} = 0
\end{aligned}
$$

When $T = L_2$, notice $\delta' = 0$. Also notice $\kappa_2 = \sqrt{N_0}$ in either case, and

$$
\begin{aligned}
\Sigma_1 &= 4(N_0 + (m-p)(m+1)) \frac{\kappa_0^2 + p^2\kappa_1^2}{(\kappa_0^2 + p\kappa_1^2)^2} \\
\Sigma_2 &= \frac{-4}{(\kappa_0^2 + p\kappa_1^2)^2}((N_0 p + \kappa_0^2)\kappa_1^2 p + (\kappa_1^2 + \kappa_2^2)\kappa_0^2)
\end{aligned}
$$

where $\kappa_0^2 + p^2\kappa_1^2 = m^2 p(p+1)$,

$$
\begin{aligned}
& (N_0 p + \kappa_0^2)\kappa_1^2 p + (\kappa_1^2 + \kappa_2^2)\kappa_0^2 \\
&= (N_0 p + mp(m-p))m(m+1)p + (N_0 + m(m+1))mp(m-p) \\
&= N_0 m^2 p(p+1) + m^2(m+1)p(m-p)(p+1) \\
&= m^2 p(p+1)(N_0 + (m+1)(m-p))
\end{aligned}
$$

therefore $\Sigma_1 + \Sigma_2 = 0$ and the coefficient (*) is zero.

When $T = L_3$, since Σ_1 remains the same, it is enough to show Σ_2 also remains the same. Here

$$\begin{aligned}
&(N_0p+(\kappa_0')^2)\kappa_1^2p+(\kappa_1^2+\kappa_2^2)\kappa_0^2\\
=&\,(N_0p+(m+1)p(m+1-p))p(m-p)(m-p+1)\\
&+((m-p)(m-p+1)+N_0)(m+1)p(m+1-p)\\
=&\,N_0p(p+1)(m-p+1)^2+(m-p)(m+1)p(p+1)(m-p+1)^2\\
=&\,p(p+1)(m-p+1)^2(N_0+(m-p)(m+1))
\end{aligned}$$

Moreover, $\kappa_0^2+p\kappa_1^2=mp(2m-p+1)$, $(\kappa_0')^2+p(\kappa_1')^2=p(m+1-p)(2m-p+1)$, therefore Σ_2 remains invariant after changing the path from L_2 to L_3.

When $T=T_1$ or T_2,

$$\begin{aligned}
&N_0p+\kappa_0^2=p(N_0+m(m-p)), && \kappa_0^2+p\kappa_2^2=mp(2m-p+1)\\
&\kappa_0^2+p\kappa_1^2=(m-p)p(2m-p-1), && \kappa_1^2+\kappa_2^2=2m(m-p)+p(p+1)\\
&\kappa_1^2-\kappa_2^2=-(2m-p)(p+1), && \kappa_2^2+(m-p-1)m=m(2m-p)\\
&\kappa_0^2+p^2\kappa_1^2=p(m-p)^2(p+1), && (\kappa_1')^2-\kappa_2^2=-2m
\end{aligned}$$

Since the number of boxes in the first row of T_2 is $m-1$, the formula for Σ_1 is similar to the earlier, by changing m to $m-1$,

$$\Sigma_1=\frac{N_0-\kappa_1^2}{(\kappa_1^2-\kappa_2^2)(\kappa_0^2+p\kappa_1^2)^2}\frac{1}{-2m}4(N_0-m(m-1))mp(m-p)^2(2m-p)(p+1)$$

When $T=T_2$, let $\kappa_0,\kappa_1,\kappa_2$ be the eigenvalues associated to T_2. Based on the expression in $(*)$, let

$$\Sigma_2'=\Sigma_2-2\delta\delta' e_T e_{s_1.T}$$

where $e_T = e_{s_1.T} = \sqrt{1 - \frac{1}{(\kappa_1-\kappa_2)^2} - \frac{1}{(\kappa_1+\kappa_2)^2}}$. Then

$$\delta\delta' e_T e_{s_1.T} = \frac{(\kappa_1^2-\kappa_2^2)^2 - 2(\kappa_1^2+\kappa_2^2)}{(\kappa_1^2-\kappa_2^2)^2} \cdot \frac{\kappa_0(N_0-\kappa_1^2)(N_0-\kappa_2^2)}{(\kappa_0^2+p\kappa_1^2)(\kappa_0^2+p\kappa_2^2)}$$

$$\begin{aligned}\Sigma_2' &= \frac{N_0-\kappa_1^2}{(\kappa_0^2+p\kappa_1^2)^2(\kappa_0^2+p\kappa_2^2)(\kappa_1^2-\kappa_2^2)}(4(N_0p+\kappa_0^2)\kappa_1^2p(\kappa_0^2+p\kappa_2^2) \\ &\quad + 4\kappa_0^2(\kappa_1^2+\kappa_2^2)(\kappa_0^2+pN_0) - 2(\kappa_1^2-\kappa_2^2)\kappa_0^2(N_0-\kappa_0^2)(\kappa_2^2+p\kappa_1^2)) \\ &= \frac{N_0-\kappa_1^2}{(\kappa_0^2+p\kappa_1^2)^2(\kappa_0^2+p\kappa_2^2)(\kappa_1^2-\kappa_2^2)}2mp^2(m-p)^2 \cdot \\ &\quad (N_0-m(m-1))(p+1)(2m-p)(2m-p+1)\end{aligned}$$

Since $\kappa_0^2 + p\kappa_2^2 = mp(2m-p+1)$, it follows that $\Sigma_1 + \Sigma_2' = 0$, therefore the coefficient in $(*)$ is zero for $T = T_2$.

When $T = T_1$, since $s_1.T_1 = T_1$, $e_{T_1} = 0$. Again let $\kappa_0, \kappa_1, \kappa_2$ be the eigenvalue associated to the path T_1,

$$\begin{aligned}\kappa_2^2 + m(m-1-p) &= m(2m-p) \\ (\kappa_1') - \kappa_2^2 &= (p+1)(2m-p) \\ \kappa_0^2 + p^2\kappa_1^2 &= (m-1)^2p(p+1) \\ \Sigma_1 &= \frac{N_0-\kappa_1^2}{(\kappa_1^2-\kappa_2^2)(\kappa_0+p\kappa_1^2)^2}4(N_0-(m-p-1)(m-p)) \\ &\quad \cdot \frac{(m-1)^2p(p+1)m(2m-p)}{(p+1)(2m-p)} \\ \Sigma_2 &= \frac{N_0-\kappa_1^2}{(\kappa_1^2-\kappa_2^2)(\kappa_0+p\kappa_1^2)^2}\frac{4}{2m}(-2)m^2p(m-1)^2 \\ &\quad \cdot (N_0(m-1)+(m-1)(p-m)(m-1-p))\end{aligned}$$

Therefore $\Sigma_1 + \Sigma_2 = 0$.

When $T = T_4$, notice $s_1.T = T$ and therefore $e_T = 0$. let $\kappa_0, \kappa_1, \kappa_2$ be the eigenvalues associated to T_4,

$$\kappa_1^2 - \kappa_2^2 = 2(m-p), \qquad \kappa_1^2 + \kappa_2^2 = 2(m-p)^2$$
$$(\kappa_1')^2 - \kappa_2^2 = (2m-p)(p+1), \qquad \kappa_0^2 + p^2\kappa_1^2 = p(p+1)(m-p+1)^2$$

Therefore

$$\Sigma_1 = -\Sigma_2 = \frac{4(N_0 - \kappa_1^2)}{(\kappa_1^2 - \kappa_2^2)(\kappa_0^2 + p\kappa_1^2)^2} p(m-p)(m-p+1)^2(N_0 - (m+1))$$

When $T = T_3$, let let $\kappa_0, \kappa_1, \kappa_2$ be the eigenvalues associated to T_3, and let $\Sigma_2' = \Sigma_2 - 2\delta\delta' e_{T_4} e_{s_1.T_4}$ as in the case $T = T_2$. We have the following

$$N_0 p + \kappa_0^2 = p(N_0 + m(m-p)), \qquad \kappa_0^2 + p\kappa_2^2 = p(m-p)(2m-p-1)$$
$$\kappa_1^2 + \kappa_2^2 = 2m(m-p) + p(p+1), \qquad \kappa_1^2 - \kappa_2^2 = (p+1)(2m-p)$$
$$\kappa_0^2 + p\kappa_1^2 = mp(2m-p+1), \qquad \kappa_0^2 + p\kappa_2^2 = m(m-p)(2m-p-1)$$
$$(\kappa_1')^2 - \kappa_2^2 = 2(m-p)$$

Therefore,

$$\Sigma_1 = -\Sigma_2' = \frac{N_0 - \kappa_1^2}{(\kappa_1^2 - \kappa_2^2)(\kappa_0^2 + p\kappa_1^2)^2} 2(2m-p)m^2 p(p+1)(N_0 - (m-p)(m-p+1))$$

2) The coefficient for $v_{s_0.T}$.

Notice $v_{s_0.T} = 0$ for $T = L_1$ and $T = L4$, therefore it is enough to check the following quantity is zero for all other six paths.

$$\begin{aligned}
&\phi_0\phi_1\phi_0(d_T)\phi_0\phi_1(a_T)\phi_0(d_T)b_T - \phi_1\phi_0\phi_1(a_T)\phi_1\phi_0(d_T)\phi_1(b_T)d_{s_0.T} \\
&\quad + \phi_0\phi_1\phi_0(d_T)\phi_0\phi_1(b_T)\phi_0(d_{s_0.T})a_{s_0.T} - \phi_1\phi_0\phi_1(b_T)\phi_1\phi_0(d_{s_0.T})\phi_1(a_{s_0.T})d_{s_0.T} \\
&\quad + e_T\phi_0\phi_1(a_{s_1.T})e_{s_1.T}b_T - \phi_1\phi_0\phi_1(b_T)e_{s_0.T}\phi_1(a_{s_1s_0.T})e_{s_1s_0.T} \\
&\quad + (1+c_1c_2)\phi_0(d_T)b_T - (1-c_1c_2)\phi_1(b_T)d_{s_0.T} \\
&= (\alpha-\beta c_1c_2)(\gamma+\delta c_0c_2)(\alpha-\beta c_1c_2)(Ac_0+Bc_1) \\
&\quad - (\gamma+\delta c_0c_1)(\alpha+\beta c_1c_2)(Ac_0+Bc_2)(\alpha'+\beta' c_1c_2) \\
&\quad + (\alpha-\beta c_1c_2)(Ac_0+Bc_2)(\alpha'-\beta' c_1c_2)(\gamma'+\delta' c_0c_1) \\
&\quad - (Ac_0+Bc_1)(\alpha'+\beta' c_1c_2)(\gamma'+\delta' c_0c_2)(\alpha'+\beta' c_1c_2) \\
&\quad + e_T^2(\gamma''+\delta'' c_0c_2)(Ac_0+Bc_1) - e_{s_0.T}^2(Ac_0+Bc_1)(\gamma'''+\delta''' c_0c_2) \\
&\quad + (1+c_1c_2)(\alpha-\beta c_1c_2)(Ac_0+Bc_1) - (1-c_1c_2)(Ac_0-Bc_2)(\alpha'+\beta' c_1c_2) \\
&= M_0c_0 + M_1c_1 + M_2c_2 + M_4c_0c_1c_2
\end{aligned}$$

By substuting in the quantities at the beginning and expanding, we obtain the expression for each M_i.

$$\begin{aligned}
M_0 &= -A(\alpha'+\beta) + A\gamma'(\alpha+\beta) - A((\alpha')^2-(\beta')^2) + A(\gamma'-\gamma)(\alpha\alpha'-\beta\beta') \\
&\qquad + B(\delta'-\delta)(\beta\alpha'-\alpha\beta') + A\gamma(\alpha^2-\beta^2) + A(e_T^2\gamma''-e_{s_0.T}^2\gamma''') \\
&= \kappa_2(\kappa_0-\kappa_0')(M_{01}+M_{02})
\end{aligned}$$

where the following is true after substitution in (‡),

$$
\begin{aligned}
M_{01} = {} & \frac{-2}{\kappa_1^2-\kappa_2^2} + \frac{2}{(\kappa_1')^2-\kappa_2^2} - \frac{4\kappa_0^2(N_0+\frac{(\kappa_0')^2}{p})}{((\kappa_1')^2-\kappa_2^2)(\kappa_0^2+p\kappa_1^2)} \\
& + \frac{2(p\kappa_1^2(-N_0+(\kappa_1')^2)-\kappa_0^2(-N_0+\kappa_1^2))}{((\kappa_1')^2-\kappa_2^2)(\kappa_1^2-\kappa_2^2)(\kappa_0^2+p\kappa_1^2)} \\
& + \frac{2(p\kappa_1^2(-N_0+(\kappa_1')^2)-\kappa_0^2(-N_0+\kappa_1^2))}{((\kappa_1')^2-\kappa_1^2)(\kappa_1^2-\kappa_2^2)(\kappa_0^2+p\kappa_1^2)(\kappa_0^2-(\kappa_0')^2)} + \frac{4\kappa_1^2(N_0p+\kappa_0^2)}{(\kappa_1^2-\kappa_2^2)^2(\kappa_0^2+p\kappa_1^2)} \\
& + \frac{(\kappa_1^2-\kappa_2^2)^2-2((\kappa_1^2+\kappa_2^2))}{(\kappa_1^2-\kappa_2^2)^2}\frac{N_0p+\kappa_0^2}{\kappa_0^2+p\kappa_2^2} \\
& + \frac{((\kappa_1')^2-\kappa_2^2)^2-2((\kappa_1')^2+\kappa_2^2)}{((\kappa_1')^2-\kappa_2^2)^2}\frac{N_0p+(\kappa_0')^2}{(\kappa_0')^2+p\kappa_2^2}
\end{aligned}
$$

After clearing the denominators, the above quantity is a polynomial in κ_i^2 and $(\kappa_i')^2$ $(i=0,1,2,)$ and therefore is a polynomial in m, N_0, p for all tableaux T, and can be checked as zero using MAGMA (the code is included in the appendix.)

$$
\begin{aligned}
M_{02} &= \frac{2\kappa_2(\kappa_0-\kappa_0')}{(\kappa_1^2-\kappa_2^2)((\kappa_1')^2-\kappa_2^2)}M_{03} \\
M_{03} &= -\frac{\kappa_1\kappa_1'(N_0p+\kappa_0)}{\kappa_0^2+p\kappa_1^2} + \frac{\kappa_1\kappa_1'(N_0p+(\kappa_0')^2)}{(\kappa_0')^2+p(\kappa_1')^2} \\
&\quad + \frac{(\kappa_1')^2-\kappa_1^2}{\kappa_0^2-(\kappa_0')^2}\left(\frac{\kappa_0\kappa_0'(-N_0+(\kappa_1')^2)}{(\kappa_0')^2+p(\kappa_1')^2} - \frac{\kappa_0\kappa_0'(-N_0+\kappa_1^2)}{\kappa_0^2+p\kappa_1^2}\right) \\
&= \frac{\kappa_1\kappa_1'}{\kappa_0^2+p\kappa_1^2}M_{04} \\
M_{04} &= -(N_0p+\kappa_0^2)+(N_0p+(\kappa_0')^2)\frac{\kappa_1^2p}{(\kappa_0')^2} \\
&\quad + p(-N_0+(\kappa_1')^2)\frac{\kappa_0^2}{p(\kappa_1')^2} - p(-N_0+\kappa_1^2) = 0
\end{aligned}
$$

$$
\begin{aligned}
M_1 &= B(\alpha+\beta-\alpha'-\beta') - B\gamma'((\alpha')^2-(\beta')^2) + B(\gamma+\gamma')(\alpha'\beta-\alpha\beta') \\
&\quad - A(\delta+\delta')(\alpha\alpha'-\beta\beta') + B\gamma(\alpha^2-\beta^2) + (e_T^2\gamma''-e_{s_0.T}^2\gamma''')B \\
&\quad = f(T)\kappa_2(\kappa_1+\kappa_1')(M_{11}+M_{12})
\end{aligned}
$$

where

$$
\begin{aligned}
M_{11} = & \frac{-2}{\kappa_1^2-\kappa_2^2} + \frac{2}{(\kappa_1')^2-\kappa_2^2} \\
& - \frac{N_0p+(\kappa_0')^2}{(\kappa_0')^2+p(\kappa_1')^2}\left(\frac{4(\kappa_1')^2}{((\kappa_1')^2-\kappa_2^2)^2} - \frac{2(\kappa_1')^2}{(\kappa_1^2-\kappa_2^2)((\kappa_1')^2-\kappa_2^2)}\right) \\
& + \frac{2}{(\kappa_1^2-\kappa_2^2)((\kappa_1')^2-\kappa_2^2)}\left(\frac{\kappa_0^2(-N_0+\kappa_1^2)}{\kappa_0^2+p\kappa_1^2} - \frac{(\kappa_0')^2(-N_0+(\kappa_1'^2))}{((\kappa_0')^2+p(\kappa_1')^2)}\right) \\
& + \frac{N_0p+\kappa_0^2}{\kappa_0^2+p\kappa_1^2}\left(\frac{4\kappa_1^2}{(\kappa_1^2-\kappa_2^2)} + \frac{-2\kappa_1^2}{((\kappa_1')^2-\kappa_2^2)(\kappa_1^2-\kappa_2^2)}\right) \\
& + \frac{(\kappa_1^2-\kappa_2^2)^2-2(\kappa_1^2+\kappa_2^2)}{(\kappa_1^2-\kappa_2^2)^2}\frac{N_0p+\kappa_0^2}{\kappa_0^2+p\kappa_2^2} \\
& - \frac{((\kappa_1')^2-\kappa_2^2)^2-2((\kappa_1')^2+\kappa_2^2)}{((\kappa_1')^2-\kappa_2^2)^2}\frac{N_0p+(\kappa_0')^2}{(\kappa_0')^2+p\kappa_2^2}
\end{aligned}
$$

This is zero by a MAGMA computation (see Appendix.)

$$
\begin{aligned}
M_{12} &= \frac{2}{(\kappa_1^2-\kappa_2^2)((\kappa_1')^2-\kappa_2^2)}M_{13} \\
M_{13} &= -\frac{\kappa_1\kappa_1'(N_0p+(\kappa_0')^2)}{(\kappa_0')^2+p(\kappa_1')^2} + \frac{\kappa_1\kappa_1'(N_0p+\kappa_0^2)}{\kappa_0^2+p\kappa_1^2} \\
&\quad + \frac{\kappa_0\kappa_0'(-N_0+\kappa_1^2)}{\kappa_0^2+p\kappa_1^2} - \frac{\kappa_0\kappa_0'(-N_0+(\kappa_1')^2)}{(\kappa_0')^2+p(\kappa_1')^2} \\
&= \frac{\kappa_1\kappa_1'}{\kappa_0^2+p\kappa_1^2}(-(N_0p+(\kappa_0')^2)\frac{p\kappa_1^2}{(\kappa_0')^2}) + (N_0p+\kappa_0^2) + p(-N_0+\kappa_1^2) \\
&\quad - p(-N_0+(\kappa_1')^2)\frac{\kappa_0^2}{p(\kappa_1')^2}) = 0
\end{aligned}
$$

$$\begin{aligned} M_2 = B(\gamma' - \gamma)(\alpha\alpha' + \beta\beta') + A(\delta + \delta')(\alpha\beta' + \alpha'\beta) \\ + A(\delta' - \delta''')((\alpha')^2 + (\beta')^2) + A(\delta - \delta'')(\alpha^2 + \beta^2) \\ + A(\delta'' + \delta''') + B(\alpha - \beta - 2\alpha\beta\gamma) - B(\alpha' - \beta' - 2\alpha'\beta'\gamma') \end{aligned}$$

where

$$\begin{aligned} \gamma - \gamma' &= -\frac{1}{\kappa_0^2 + p\kappa_1^2}(N_0 - \kappa_1\kappa_1')(\kappa_0 - \kappa_0')\frac{\kappa_0}{\kappa_1'} \\ \delta + \delta' &= -\frac{1}{\kappa_0^2 + p\kappa_1^2}(N_0 - \kappa_1\kappa_1')(\kappa_1 + \kappa_1')\frac{p\kappa_1}{\kappa_0'} \\ &B(\gamma' - \gamma)(\alpha\alpha' + \beta\beta') + A(\delta + \delta')(\alpha\beta' + \alpha'\beta) \\ &= \frac{4\kappa_0\kappa_1(N_0 - \kappa_1\kappa_1')(\kappa_1 + \kappa_1')(\kappa_0 - \kappa_0')}{(\kappa_0^2 + p\kappa_1^2)(\kappa_1^2 - \kappa_2^2)((\kappa_1')^2 - \kappa_2^2)} \\ \delta - \delta'' &= \frac{\kappa_1^2 - \kappa_2^2}{(\kappa_0^2 + p\kappa_1^2)(\kappa_0^2 + p\kappa_2^2)}\kappa_0(N_0 p + \kappa_0^2) \\ \delta' - \delta''' &= \frac{(\kappa_1')^2 - \kappa_2^2}{((\kappa_0')^2 + p(\kappa_1')^2)((\kappa_0')^2 + p\kappa_2^2)}\kappa_0'(N_0 p + (\kappa_0')^2) \\ \delta'' + \delta''' &= -\frac{\kappa_0 + \kappa_0'}{(\kappa_0^2 + p\kappa_2^2)((\kappa_0')^2 + p(\kappa_2')^2)}(\kappa_0\kappa_0' + p\kappa_2^2)(N_0 - \kappa_2^2) \end{aligned}$$

By using $\kappa_0\kappa_0' = p\kappa_1\kappa_1'$ and arranging the terms as rational expressions of κ_i^2, $(\kappa_i')^2$ and $\kappa_1\kappa_1'$, we obtain

$$M_2 = \kappa_1\kappa_1' M_{21} + M_{22}$$

where

$$M_{21} = \frac{4(-\kappa_1^2\kappa_0^2 + N_0\kappa_0^2 - pN_0\kappa_1^2 + p\kappa_1^2(\kappa_1')^2)}{(\kappa_0^2 + p\kappa_1^2)(\kappa_1^2 - \kappa_2^2)((\kappa_1')^2 - \kappa_2^2)}$$
$$+ \frac{2p(N_0p + (\kappa_0')^2)((\kappa_1')^2 + \kappa_2^2)}{((\kappa_0')^2 + p(\kappa_1')^2)((\kappa_0')^2 + p\kappa_2^2)((\kappa_1')^2 - \kappa_2^2)}$$
$$- \frac{2p(N_0p + \kappa_0^2)(\kappa_1^2 + \kappa_2^2)}{(\kappa_1^2 - \kappa_2^2)(\kappa_0^2 + p\kappa_1^2)(\kappa_0^2 + p\kappa_2^2)} - \frac{p(\kappa_0^2 - (\kappa_0')^2)(N_0 - \kappa_2^2)}{(\kappa_0^2 + p\kappa_2^2)((\kappa_0')^2 - \kappa_2^2)}$$
$$+ \frac{2p(N_0 - \kappa_1^2)}{(\kappa_1^2 - \kappa_2^2)(\kappa_0^2 + p\kappa_1^2)} - \frac{2p(N_0 - (\kappa_1')^2)}{((\kappa_1')^2 - \kappa_2^2)((\kappa_0')^2 + p(\kappa_1')^2)}$$

$$M_{22} = \frac{4(N_0\kappa_0^2\kappa_1^2 - N_0p\kappa_1^2(\kappa_1')^2 + p\kappa_1^4(\kappa_1')^2 - \kappa_0^2\kappa_1^2(\kappa_1')^2)}{(\kappa_0^2 + p\kappa_1^2)(\kappa_1^2 - \kappa_2^2)((\kappa_1')1^2 - \kappa_2^2)}$$
$$- \frac{2(\kappa_0')^2(N_0p + (\kappa_0')^2)((\kappa_1')^2 + \kappa_2^2)}{((\kappa_0')^2 + p(\kappa_1')^2)T((\kappa_1')1^2 - \kappa_2^2)} + \frac{2\kappa_0^2(N_0p + \kappa_0^2)(\kappa_1^2 + \kappa_2^2)}{(\kappa_1^2 - \kappa_2^2)(\kappa_0^2 + p\kappa_2^2)(\kappa_0^2 + p\kappa_1^2)}$$
$$- \frac{p(\kappa_0^2 - (\kappa_0')^2)\kappa_2^2(N_0 - \kappa_2^2)}{((\kappa_0')^2 + p\kappa_2^2)(\kappa_0^2 + p\kappa_2^2)} + \frac{2p\kappa_1^2(N_0 - \kappa_1^2)}{(\kappa_1^2 - \kappa_2^2)(\kappa_0^2 + p\kappa_1^2)}$$
$$- \frac{2p(\kappa_1')^2(N_0 - (\kappa_1')^2)}{((\kappa_1')1^2 - \kappa_2^2)((\kappa_0')^2 + p(\kappa_1')^2)}$$

Both can be checked to be zero by MAGMA for all relevant tableaux T in the two cases listed in the beginning of this proof (see Appendix.)

$$M_3 = -A(\gamma + \gamma')(\alpha\beta' + \alpha'\beta) - B(\delta - \delta')(\alpha\alpha' + \beta\beta') + A(\alpha - \beta - 2\alpha\beta\gamma)$$
$$+ A(\alpha' - \beta' - 2\alpha'\beta'\gamma') - B(\alpha^2 + \beta^2)(\delta - \delta'')$$
$$+ B((\alpha')^2 + (\beta')^2)(\delta' - \delta''') + (\delta''' - \delta'')B$$

where

$$
\begin{aligned}
\gamma + \gamma' &= \frac{1}{\kappa_0^2 + p\kappa_1^2}(\kappa_0 + \kappa_0')(N_0 + \kappa_1\kappa_1')\frac{\kappa_0}{\kappa_1'} \\
\delta - \delta' &= -\frac{1}{\kappa_0^2 + p\kappa_1^2}(\kappa_1' - \kappa_1)(N_0 + \kappa_1\kappa_1')\frac{p\kappa_1}{\kappa_0'} \\
&\quad - A(\gamma + \gamma')(\alpha\beta' + \alpha'\beta) - B(\delta - \delta')(\alpha\alpha' + \beta\beta') \\
&= \frac{4\kappa_0\kappa_1(N_0 + \kappa_1\kappa_1')(\kappa_0^2 - (\kappa_0')^2)}{(\kappa_1^2 - \kappa_2^2)((\kappa_1')^2 - \kappa_2^2)(\kappa_0^2 + p\kappa_1^2)}
\end{aligned}
$$

and the rest of the calculation is similar to that in M_2. Hence

$$
M_3(\kappa_0 + \kappa_0')(\kappa_1 - \kappa_1') = \kappa_1\kappa_1' M_{31} + M_{32}
$$

where

$$
\begin{aligned}
M_{31} = &\frac{4(\kappa_0^2\kappa_1^2 + pN_0\kappa_1^2 - N_0\kappa_0^2 - p\kappa_1^2(\kappa_1')^2)X}{LMK} \\
&- \frac{2p(N_0 - \kappa_1^2)X}{LK} + \frac{2p(N_0 - (\kappa_1')^2)X}{MJ} - \frac{pXY(N_0 - \kappa_2^2)}{QR} \\
&- \frac{2pY(\kappa_1^2 + \kappa_2^2)(N_0 p + \kappa_0^2)}{LKQ} + \frac{2pY((\kappa_1')^2 + \kappa_2^2)(N_0 p + (\kappa_0')^2)}{MJR} \\
M_{32} = &\frac{4(N_0\kappa_0^2\kappa_1^2 - N_0 p\kappa_1^2(\kappa_1')^2 - \kappa_0^2\kappa_1^2(\kappa_1')^2 + p\kappa_1^4(\kappa_1')^2)X}{LMK} + \\
&\frac{2\kappa_1^2 p(N_0 - \kappa_1^2)X}{LK} - \frac{2p(\kappa_1')^2(N_0 - (\kappa_1')^2)X}{MJ} + \frac{p\kappa_2^2(N_0 - \kappa_2^2)XY}{QR} \\
&- \frac{2(\kappa_1^2 + \kappa_2^2)\kappa_0^2(N_0 p + \kappa_0^2)Y}{LKQ} + \frac{2((\kappa_1')^2 + \kappa_2^2)(\kappa_0')^2(N_0 p + (\kappa_0')^2)Y}{MJR}
\end{aligned}
$$

and

$$
L = \kappa_1^2 - \kappa_2^2, \quad M = (\kappa_1')^2 - \kappa_2^2, \quad K = \kappa_0^2 + p\kappa_1^2, \quad J = (\kappa_0')^2 + p(\kappa_1')^2
$$

$$
Q = \kappa_0^2 + p\kappa_2^2, \quad R = (\kappa_0')^2 + p\kappa_2^2, \quad X = \kappa_0^2 - (\kappa_0')^2, \quad Y = \kappa_1^2 - (\kappa_1')^2
$$

both can be checked to be zero using MAGMA (see Appendix.) Since $\kappa_0 \neq \kappa_0'$, $\kappa_1 \neq \kappa_1'$ for $T = L_2, L_3, T_1, T_2, T_3, T_4$, it follows that M_3 is zero for all the above paths.

3) The coefficient for $v_{s_1.T}$. It is enough to show the quantity $e_i W$ is zero, where

$$\begin{aligned}
W &= (1 + c_1c_2)a_{s_1.T} - (1 - c_1c_2)\phi_1(a_T) + \phi_0\phi_1\phi_0(d_T)\phi_0\phi_1(a_T)a_{s_1.T} \\
&\quad + \phi_0\phi_1(a_{s_1.T})\phi_0(d_{s_1.T})a_{s_1.T} - \phi_1\phi_0\phi_1(a_T)\phi_1\phi_0(d_T)\phi_1(a_T) \\
&\quad - \phi_1\phi_0\phi_1(a_T)\phi_1(a_{s_1.T})d_{s_1.T} + \phi_0\phi_1(b_{s_1.T})\phi_0(d_{s_0s_1.T})b_{s_0s_1.T} \\
&\quad - \phi_1\phi_0\phi_1(b_T)\phi_1\phi_0(d_{s_0.T})\phi_1(b_{s_0.T}) \\
&= W_0c_1c_2 + W_1c_0c_2 + W_2c_0c_1 + W_3
\end{aligned}$$

and

$$\begin{aligned}
W_0 &= M(-F^2\beta_2 - G^2\alpha_2) - K((A')^2\beta' + (B')^2\alpha') - \gamma\beta\gamma_1 - \alpha\delta\delta_1 - \gamma_1^2\beta_1 \\
&\quad - \delta_1^2\alpha_1 - \gamma^2\beta - \delta^2\alpha - \delta\delta_1\alpha_1 - \gamma\gamma_1\beta_1 + \gamma_1 + \gamma \\
W_1 &= M(GF\alpha_2 - FG\beta_2) - K(-A'B'\beta' + A'B'\alpha') + \alpha\delta\gamma_1 - \gamma\beta\delta_1 + \delta\alpha\gamma_1 \\
&\quad - \gamma_1\beta_1\delta_1 + \delta\beta\gamma - \gamma\alpha\delta - \gamma\delta_1\alpha_1 + \delta\gamma_1\beta_1 + \delta_1 - \delta \\
W_2 &= M(-GF\beta_2 + GF\alpha_2) - K(A'B'\alpha' - A'B'\beta') + \beta\delta\gamma_1 + \alpha\gamma\delta_1 - \delta_1\beta_1\gamma_1 \\
&\quad + \gamma_1\alpha_1\delta_1 - \delta\alpha\gamma + \gamma\beta\delta - \delta\gamma_1\alpha_1 - \gamma\delta_1\beta_1 + \delta_1 - \delta \\
W_3 &= M(F^2\alpha_2 + G^2\beta_2) - K((A')^2\alpha' + (B')^2\beta') + \alpha\gamma\gamma_1 - \beta\delta\delta_1 + \gamma_1^2\alpha_1 \\
&\quad + \delta_1^2\beta_1 - \gamma^2\alpha - \delta^2\beta - \gamma\gamma_1\alpha_1 + \delta\delta_1\beta_1 + \gamma_1 - \gamma
\end{aligned}$$

where according to Lemma 5.11,

$$K = f(T)f(s_0.T) = -\frac{\kappa_0^2 + p^2\kappa_1^2}{(\kappa_0^2 + p\kappa_1^2)^2}\frac{(N_0 - \kappa_1^2)(N_0 - (\kappa_1')^2)}{(\kappa_0 - \kappa_0')^2 + (\kappa_1 + \kappa_1')^2}$$
$$M = f(T)f(s_0.T) = -\frac{\kappa_0^2 + p^2\kappa_2^2}{(\kappa_0^2 + p\kappa_2^2)^2}\frac{(N_0 - \kappa_2^2)(N_0 - (\kappa_2'')^2)}{(\kappa_0 - \kappa_0'')^2 + (\kappa_2 + \kappa_2'')^2}$$

These expressions are zero for all relevant tableaux T_i and L_i in the two cases listed in the beginning of this proof, by a MAGMA computation (see Appendix.)

4) The coefficient for $v_{s_0 s_1.T}$.

It is enough to show the quantity $e_T D$ is zero, where

$$\begin{aligned} D &= \phi_0\phi_1\phi_0(d_T)\phi_0\phi_1(a_T)b_{s_1.T} + \phi_0\phi_1(a_{s_1.T})\phi_0(d_{s_1.T})b_{s_1.T} \\ &+ \phi_1\phi_0(b_{s_1.T})\phi_0(d_{s_0s_1.T})a_{s_0s_1.T} + (1 + c_1c_2)b_{s_1.T} - \phi_1\phi_0\phi_1(a_T)\phi_1(b_{s_1.T})d_{s_0s_1.T} \\ &= f(s_1.T)(D_0c_0 + D_1c_1 + D_2c_2 + D_3c_0c_1c_2) \end{aligned}$$

and

$$D_0 = \alpha\gamma A_1 - \beta\gamma B_1 + \gamma_1\alpha_1 A_1 + \delta_1\beta_1 B_1 + A_1\alpha_2\gamma_2 - B_1\beta_2\delta_2 - \gamma A_1\alpha_2 + B_1\delta\beta_2 + A_1$$
$$D_1 = \beta\delta A_1 + \alpha\gamma B_1 - \delta_1\beta_1 A_1 + \gamma_1\alpha_1 B_1 - B_1\beta_2\gamma_2 - A_1\alpha_2\delta_2 - \delta A_1\alpha_2 - B_1\gamma\beta_2 + B_1$$
$$D_2 = \alpha\delta A_1 - \beta\delta B_1 + \delta_1\alpha_1 A_1 - \gamma_1\beta_1 B_1 + B_1\alpha_2\gamma_2 + A_1\beta_2\delta_2 - B_1\gamma\alpha_2 + \delta A_1\beta_2 + B_1$$
$$D_3 = -\beta\gamma A_1 - \alpha\delta B_1 - \gamma_1\beta_1 A_1 - \delta_1\alpha_1 B_1 - A_1\beta_2\gamma_2 + B_1\alpha_2\delta_2 - B_1\delta\alpha_2 - \gamma A_1\beta_2 + A_1$$

These expressions are zero for all relevant tableaux T_i and L_i in the two cases listed in the beginning of this proof, by a MAGMA computation (see Appendix.)

5) The coefficient for $v_{s_1 s_0.T}$.

It is enough to check that the quantity $e_{s_0.T}E$ is zero, where:

$$E = (1 - c_1c_2)\phi_1(b_T) + \phi_1\phi_0\phi_1(a_T)\phi_1\phi_0(d_T)\phi_1(b_T) + \phi_1\phi_0\phi_1(b_T)\phi_1\phi_0(d_{s_0.T})\phi_1(a_{s_0.T})$$
$$+ \phi_1\phi_0\phi_1(b_T)\phi_1(a_{s_1s_0.T})d_{s_1s_0.T} - \phi_0\phi_1\phi_0(d_T)\phi_0\phi_1(b_T)a_{s_1s_0.T}$$
$$= f(T)(E_0c_0 + E_1c_1 + E_2c_2 + E_3c_0c_1c_2)$$

and

$$E_0 = A'\alpha'\gamma' - B'\beta'\delta' + A'\gamma''\alpha'' + B'\delta''\beta'' - \alpha A'\gamma'' - \beta B'\delta'' + \delta\beta B' + \gamma\alpha A' + A'$$
$$E_1 = B'\alpha'\gamma' + A'\beta'\delta' + B'\gamma''\alpha'' - A'\delta''\beta'' - \beta B'\gamma'' + \alpha A'\delta'' + \delta\alpha A' - \gamma\delta B' + B'$$
$$E_2 = -B'\beta'\gamma' - A'\alpha'\delta' - A'\delta''\alpha'' - B'\gamma''\beta'' - \alpha B'\gamma'' - \beta A'\delta'' - \delta\beta A' + \gamma\alpha B' + B'$$
$$E_3 = A'\beta'\gamma' - B'\alpha'\delta' - B'\delta''\alpha'' + A'\gamma''\beta'' + \beta A'\gamma'' - \alpha B'\delta'' + \gamma\beta A' + \delta\alpha B' - A'$$

These expressions are zero for all relevant tableaux T_i and L_i in the two cases listed in the beginning of this proof, by a MAGMA computation (see Appendix.)

6) The coefficient for $v_{s_0s_1s_0.T}$.

It is enough to check $e_{s_0.T}E = 0$, where

$$E = \phi_0\phi_1\phi_0(d_T)\phi_0\phi_1(b_T)b_{s_1s_0.T} - \phi_1\phi_0\phi_1(b_T)\phi_1(b_{s_1s_0.T})d_{s_0s_1s_0.T}$$
$$= f(T)f(s_1s_0.T)(E_0c_1c_2 + E_1c_0c_2 + E_2c_0c_1 + E_3)$$

and

$$E_0 = \beta A'H - \alpha B'I - B'I\alpha_3 + A'H\beta_3$$
$$E_1 = -\alpha B'H - \beta A'I - A'I\alpha_3 - B'H\beta_3$$
$$E_2 = -\beta B'H + \alpha A'I + B'H\alpha_3 - A'I\beta_3$$
$$E_3 = -\alpha A'H - \beta B'I + A'H\alpha_3 + B'I\beta_3$$

These expressions are zero for all relevant tableaux T_i and L_i in the two cases listed in the beginning of this proof, by a MAGMA computation (see Appendix.)

7) The coefficient for $v_{s_1 s_0 s_1.T}$

Notice when $T \neq L_1$ or L_4, at least one of $v_{s_1.T}$, $v_{s_0 s_1.T}$, and $v_{s_1 s_0 s_1.T} = 0$ is zero, the term $v_{s_1 s_0 s_1.T}$ vanishes automatically. When $T \neq L_1$ or L_4, $a_T = a_{s_1 s_0 s_1.T} = \sqrt{N_0}$, and it is enough to show that $e_T e_{s_0 s_1.T} M = 0$, where

$$\begin{aligned} M &= \phi_0\phi_1(b_{s_1.T})a_{s_1 s_0 s_1.T} - \phi_1\phi_0\phi_1(a_T)\phi_1(b_{s_1.T}) \\ &= \sqrt{N_0}(\phi_0\phi_1(b_{s_1.T}) - \phi_1(b_{s_1.T})) = 0 \end{aligned}$$

because $b_{s_1.T} = f(s_1.T)(Fc_0 + Gc_1)$ and $\phi_0\phi_1(b_{s_1.T}) = \phi_1(b_{s_1.T})f(s_1.T)(Fc_0 + Gc_2)$.

So far we have checked all coefficients are zero, hence the relation is satisfied.

□

Based on the surjection $\mathcal{H}_d \to \mathcal{H}_d^p$, we also have the following result.

Corollary 5.15. *The superspace $\mathcal{D}_f^\lambda$ in Definition 5.10 admits a well-defined action of $\mathcal{H}_d$.*

5.3. Irreducibility

First we need some some combinatorial results. Given a semistandard tableau T of a certain skew shape, denote by T^{row} the tableau of the same skew shape as T, filled with $1, 2, \ldots$ successively along each row from left to right. Recall the action of $s_0, \ldots, s_{d-1}$ on semistandard tableaux in Γ^λ explained after Definition 5.2. Given a word w in $s_0, \ldots, s_{d-1}$, we denote by $w.T$ the resulting tableau after applying the letters in w successively. The following result is well known.

Lemma 5.16. *Let T be a semistandard tableau of any skew shape. There exists a word w in $s_1, \ldots, s_{d-1}$ such that $w.T = T^{\mathrm{row}}$.*

We first give the criteria for the two modules $\mathcal{D}_f^\lambda$ and $\mathcal{D}_g^\lambda$ to be isomorphic. Recall the set $\mathcal{P}_0(\alpha, \beta)$ of strict partitions at level 0 as defined in Definition 5.1, and the partition $T^{(0)}$ at level 0 associated to a path T as defined in Definition 5.2.

Theorem 5.17. *For $f, g : \Gamma^\lambda \to \mathbb{C}$ which satisfy Equation (5.2.1), $D_f^\lambda \simeq D_g^\lambda$ if and only if there exists a function $H : \Gamma^\lambda \to \mathbb{C}^\times$, $H(T)$ is constant on all T's which go through the same partition $T^{(0)}$ at level 0, such that for any $T \in \Gamma^\lambda$, $s_0.T \neq \star$,*

$$\frac{f(T)}{g(T)} = \frac{H(T)}{H(s_0.T)}$$

Proof. ($\Rightarrow$) Let $\mathcal{D}_f^\lambda = \bigoplus_{T \in \Gamma^\lambda} Cl_{d+1} v_T$ and $\mathcal{D}_g^\lambda = \bigoplus_{T \in \Gamma^\lambda} Cl_{d+1} w_T$ according to the construction in Definition 5.10. If there exists an isomorphism $\phi : D_f^\lambda \to D_g^\lambda$, then by the grading convention in the construction, ϕ is even, and

$$z_i(\phi(v_T)) = \phi(z_i v_T) = \kappa_T(i)\phi(v_T), \quad 0 \le i \le d$$

According to Lemma 5.7, the list of eigenvalues for each basis vector is distinct for each T, therefore $\phi(v_T) = H(T)w_T$ for some $H(T) \in \mathbb{C}^\times$.

Denote $s_i.v_T = d_T v_T + e_T v_{s_i.T}$, where $d_T, e_T \in Cl_2$, the Clifford algebra generated by c_i and c_{i+1}, and they are given via explicit formulas which only depends on T in Definition 5.10. Similarly $s_i.w_T = d_T w_T + e_T w_{s_i.T}$ and we have

$$\begin{aligned}
\phi(s_i.v_T) &= d_T\phi(v_T) + e_T\phi(v_{s_i.T}) \\
&= d_T h(T)w_T + e_T H(s_i.T)w_{s_i.T}, \\
s_i.\phi(v_T) &= s_i(h(T)w_T) \\
&= H(T)(d_T w_T + e_T w_{s_i.T}).
\end{aligned}$$

Whenever $s_i.T \neq \star$, we have $e_T \neq 0$ and $w_{s_i.T} \neq 0$, hence $H(s_i.T) = H(T)$, for any $1 \le i \le d-1$. By the previous lemma, any two semistandard tableaux of the same skew shape can be obtained from each other via a word in $s_1, \dots, s_{d-1}$, hence H is constant on any semistandard tableaux of the same skew shape, and only depends on $T^{(0)} \in \mathcal{P}_0(\alpha)$.

Denote $x_1.v_T = a_T v_T + f(T)b'_T v_{s_0.T}$, where $a_T, b'_T \in Cl_2$, the Clifford algebra generated by c_0 and c_1, and a_T, b'_T have explicit formula in Definition 5.10 which only depends on T. In other words, $x_1.w_T = a_T w_T + g(T)b'_T w_{s_0.T}$ and we have

$$\begin{aligned}\phi(x_1.v_T) &= a_T\phi(v_T) + f(T)b'_T\phi(v_{s_0.T})\\ &= a_T h(T)w_T + f(T)b'_T H(s_0.T)w_{s_0.T},\\ x_1.\phi(v_T) &= x_1.(h(T)w_T)\\ &= H(T)(a_T w_T + b'_T g(T)w_{s_0.T}).\end{aligned}$$

When $s_0.T \neq \star$, we have $w_{s_0.T} \neq 0$ and $b'_T \neq 0$. Also $g(T) \neq 0$ by Lemma 5.11, therefore $f(T)H(s_0.T) = g(T)H(T)$ and

$$\frac{f(T)}{g(T)} = \frac{H(T)}{H(s_0.T)}.$$

($\Leftarrow$) On the other hand, if there exists such a function H , then the map $\phi : \mathcal{D}_f^\lambda \to \mathcal{D}_g^\lambda$ defined by $v_T \mapsto H(T)w_T$ defines an $\mathcal{H}_d^p$-module isomorphism, based on the calculations given above.

$\square$

Corollary 5.18. *The modules* $\mathcal{D}_f^\lambda \simeq \mathcal{D}_g^\lambda$ *if and only if*

$$\frac{f(T)}{g(T)} = \frac{f(s_i.T)}{g(s_i.T)}$$

for all $2 \leq i \leq d$ *and* $s_0.T \neq \star$*.*

Proof. ($\Longrightarrow$) For $i \geq 2$, $s_0 s_i.T = s_i s_0.T$ for all $T \in \Gamma^\lambda$, and the condition in the previous Lemma implies

$$\frac{f(s_i.T)}{g(s_i.T)} = \frac{H(s_i.T)}{H(s_0 s_i.T)} = \frac{H(s_i.T)}{H(s_i s_0.T)} = \frac{H(T)}{H(s_0.T)} = \frac{f(T)}{g(T)}$$

($\Longleftarrow$) Notice there is a bijection

$$\begin{aligned}\phi : \mathcal{P}_0 &\leftrightarrow \{n+p, n+p-1, \dots, \max\{n+1, p+1\}\}\\ \mu &\mapsto \#\mu_1\end{aligned}$$

Let $s = \max\{n+1, p+1\}$, and define the function $H : \mathcal{P}_0 \to \mathbb{C}$ to be the unique function (up to rescaling) such that

$$\frac{H(i)}{H(i+1)} = \frac{f(T)}{g(T)}$$

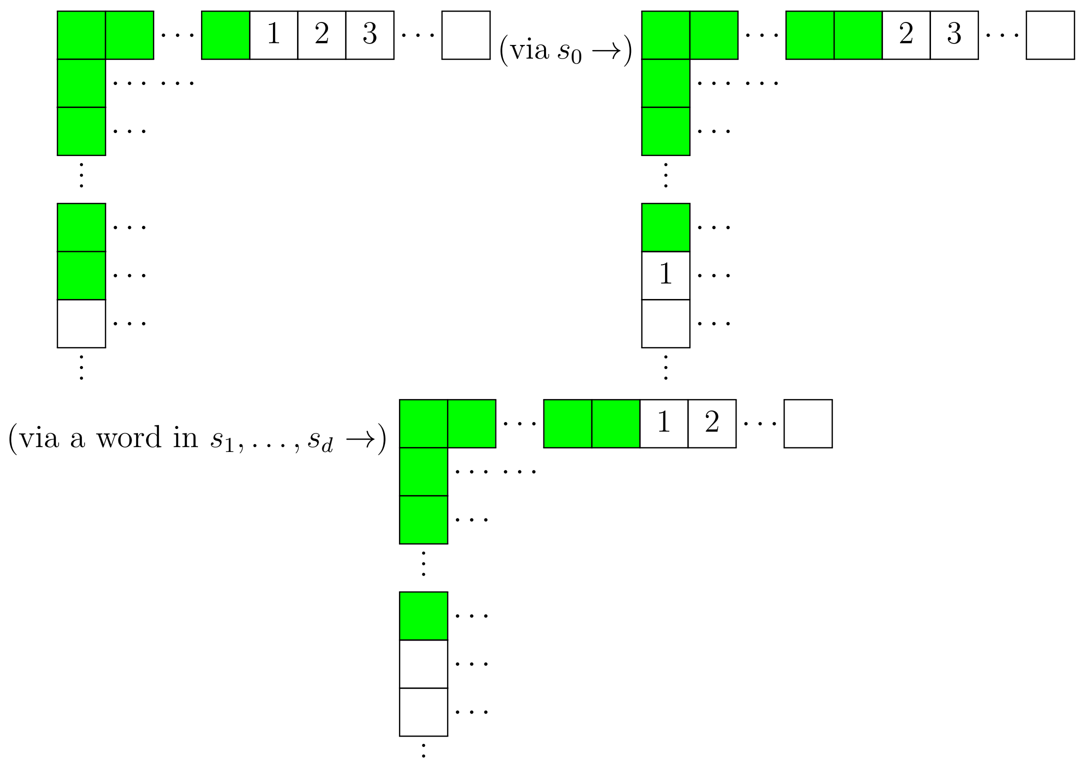
for some T such that $\phi(T^{(0)}) = i$ and $\phi((s_0.T)^{(0)}) = i+1$. We check that the function is well defined: if T is a path that satisfies the condition, then both $T^{(0)}$ and $(s_0.T)^{(0)}$ are uniquely determined. Since $T^{(1)}$ is the unique partition containing both $T^{(0)}$ and $(s_0.T)^{(0)}$, it is also uniquely determined. The first two edges in T are fixed, therefore T is well defined up to operators s_i for $i \geq 2$, and the value on the right hand side is invariant under these operators s_i.

When H is defined as above, the condition in the previous lemma is automatically satisfied. □

Remark 5.19. *It is straighforward to construct functions f and g that fail this requirement, and hence the modules constructed in Definition 5.10 often belong to different isomorphism classes.*

Lemma 5.20. *Let $T_1, T_2 \in \Gamma^\lambda$, then there is a word w in $s_0, s_1, \dots, s_d$ such that $w.T_1 = T_2$.*

Proof. Let $L_1 = T_1^{\text{row}}$ and $L_2 = T_2^{\text{row}}$. Because of the previous lemma, it is enough to show there exists a word w in $s_0, \dots, s_{d-1}$ such that $w.L_1 = L_2$. We first assume μ_1 and μ_2 differ by a box in the first row. They can be obtained from each other via the following moves. (For brevity we will denote tableaux by omitting the staircase α, and empty boxes are emphasized with color. White boxes are filled with certain integer entries.)



By repeatedly applying the above procedure, one can successively add green boxes to the first row and obtain L_2 from L_1 via a word w in $s_0, \ldots, s_d$.

$\square$

Lemma 5.21. *Given a path $T \in \Gamma^\lambda$, if $s_i.T \neq \star$, then the scalar*

$$\mathcal{Y}_T(i) = (\kappa_T(i)^2 - \kappa_T(i+1)^2)^2 - (\kappa_T(i) - \kappa_T(i+1))^2 - (\kappa_T(i) + \kappa_T(i+1))^2$$

is nonzero.

Proof. Let $a = \kappa_T(i)$ and $b = \kappa_T(i+1)$, we have

$$\begin{aligned}
\mathcal{Y}_T(i) &= (a^2 - b^2)^2 - (a-b)^2 - (a+b)^2 \\
&= (a^2 + b^2)^2 - 4a^2b^2 - 2(a^2 + b^2) \\
&= (a^2 + b^2 - 1)^2 - 1 - 4a^2b^2 \\
&= (a^2 + b^2 - 1 + 2ab)(a^2 + b^2 - 1 - 2ab) - 1 \\
&= ((a+b)^2 - 1)((a-b)^2 - 1) - 1
\end{aligned}$$

if $s_i.T \neq \star$, then the boxes containing i and $i+1$ are not adjacent to each other, therefore $|a-b| \geq 2$. Moreover, all boxes in a shifted tableau have contents at least 0, hence one of a, b is at least 0 and $|a+b| \geq 2$. Therefore $\mathcal{Y}_T(i) \geq 3 \cdot 3 - 1 = 8$.

□

Theorem 5.22. *For any $\lambda \in \mathcal{P}_d(\alpha, \beta)$ and f satisfying the condition in* (5.2.1)*, the module $\mathcal{D}_f^\lambda$ is simple.*

Proof. Denote $E = \{0,1\}^{d+1}$. For any sequence $\epsilon = (\epsilon_0, \ldots, \epsilon_d) \in E$ and $T \in \Gamma^\lambda$ define

$$\begin{aligned}
P_{T,\epsilon} = &\prod_{\sigma \in E, S \in \Gamma^\lambda, S \neq T} \frac{(z_0 - \sigma_0\kappa_0(S))^2 + \cdots + (z_d - \sigma_d\kappa_d(S))^2}{(\epsilon_0\kappa_0(T) - \sigma_0\kappa_0(S))^2 + \cdots + (\epsilon_d\kappa_d(T) - \sigma_d\kappa_d(S))^2} \\
&\times \prod_{\sigma \in E, \sigma \neq \epsilon} \frac{(z_0 - \sigma_0\kappa_0(S))^2 + \cdots + (z_d - \sigma_d\kappa_d(S))^2}{(\epsilon_0\kappa_0(T) - \sigma_0\kappa_0(S))^2 + \cdots + (\epsilon_d\kappa_d(T) - \sigma_d\kappa_d(S))^2}
\end{aligned} \tag{5.3.1}$$

Denote $c^\epsilon = c_1^{\epsilon_1} \cdots c_d^{\epsilon_d}$. Since $z_i.(c^\epsilon v_T) = (-1)^{\epsilon_i} c^\epsilon v_T$, it is straighforward that $P_{T,\epsilon}.(c^\sigma v_S) = 0$ for all $S \neq T$, $P_{T,\epsilon}.(c^\sigma v_T) = 0$ for all $\sigma \neq \epsilon$, and therefore

$P_{T,\epsilon}.(c^\epsilon v_T) = c^\epsilon v_T$.

Specifically, suppose W is a nonzero submodule of $\mathcal{D}^\lambda$. Since $\mathcal{D}^\lambda$ admits a basis $\{c^\sigma v_T\}$ where $\sigma \in E$, given $w \in W$, there exists $T \in \Gamma$ and $\sigma \in E$ such that $P_{T,\sigma} w = ac^\sigma v_T$ for some $a \neq 0$, and $v_T \in W$. Note

$$\begin{aligned}&(s_i + \frac{1}{\kappa_T(i) - \kappa_T(i+1)} - \frac{1}{\kappa_T(i) + \kappa_T(i+1)} c_i c_{i+1}) v_T \\ = &\sqrt{1 - \frac{1}{(\kappa_T(i) - \kappa_T(i+1))^2} - \frac{1}{(\kappa_T(i) + \kappa_T(i+1))^2}} v_{s_i.T}\end{aligned}$$

Moreover, from the previous lemma the scalar

$$\sqrt{1 - \frac{1}{(\kappa_T(i) - \kappa_T(i+1))^2} - \frac{1}{(\kappa_T(i) + \kappa_T(i+1))^2}} = \sqrt{\frac{\mathcal{Y}_T(i)}{(\kappa_T(i)^2 - \kappa_T(i+1)^2)^2}}$$

is nonzero, therefore $v_{s_i.T} \in W$ if $v_T \in W$.

On the other hand, notice

$$\begin{aligned}&((\kappa_T(0) - \kappa'_T(0))c_0 + (\kappa_T(1) + \kappa'_T(1))c_1)\cdot \\ &(x_1 - \frac{N_0}{\kappa^2}(\kappa_T(1) - \kappa_T(0)c_0c_1) - c(\kappa_T(0) + \kappa_T(1)c_0c_1))v_T \\ = &((\kappa_T(0) - \kappa'_T(0))^2 + (\kappa_T(1) + \kappa'_T(1))^2)v_{s_0.T}\end{aligned}$$

The constant $(\kappa_T(0) - \kappa'_T(0))^2 + (\kappa_T(1) + \kappa'_T(1))^2$ is nonzero since

$$\kappa_T(0), \kappa'_T(0), \kappa_T(1), \kappa'_T(1) \in \mathbb{R}$$

and $\kappa_T(1) + \kappa'_T(1) > 0$. Therefore $v_{s_0.T} \in W$ if $v_T \in W$.

Therefore, for any word w in $s_0, \ldots, s_d$, $v_{w.T} \in W$. By Lemma 5.20, $v_T \in W$ for all $T \in \Gamma^\lambda$, hence $c^\sigma v_T \in W$ for all $\sigma \in E$, $T \in \Gamma$ and $W = \mathcal{D}^\lambda$ which proves the claim.

□

Chapter 6

$\mathcal{H}_d^p$-Modules via Restrictions

6.1. A Classification Result

Recall that the defining relations in $\mathcal{H}_d^p$ are dependent on the fixed choice of integers $n = \ell(\alpha)$ and $p = |\beta|$. In addition, the Bratteli diagram $\Gamma_{\alpha,\beta}$ in Definition 5.2 also depends on n and p, and so does the modules $\mathcal{D}_f^\lambda$. In this section we focus on the case when n is even, and we will give a classification result about a certain family of calibrated modules defined in Section 5.2. The module structure is largely determined by the action of the generators z_i. Recall that for $\sigma \in E = \{0,1\}^{d+1}$, $c^\sigma = c_0^{\sigma_0} \cdots c_d^{\sigma_d}$.

Proposition 6.1. *When n is even, let $\mathcal{W}^\lambda$ be a module satisfying the following properties:*

1) It admits a homogenous basis $\{c^\sigma v_T\}_{\sigma \in E, T \in \Gamma^\lambda}$.

2) For each $0 \le i \le d$, $z_i.v_T = \kappa_T(i) v_T$.

Then the following is true:

1) The action of x_1 admits the formula in (5.2.4)*, or equivalently* (5.2.3)*.*

2) The grading on $\mathcal{W}^\lambda$ is one of the two cases:

a) $w_T \in \mathcal{W}^\lambda$ is even if and only if $v_T \in \mathcal{D}^\lambda$ is even, or

b) $w_T \in \mathcal{W}^\lambda$ is even if and only if $v_T \in \mathcal{D}^\lambda$ is odd.

Proof. Since $x_1 z_j = z_j x_1$ for $j \ge 2$, x_1 preserves the eigenspace with eigenvalues $\kappa_T(2), \ldots, \kappa_T(d)$ for $z_2, \ldots, z_d$. This subspace is spanned by vectors $\mathcal{B} =$

$\{v_T, c_0c_1v_T, c_0v_S, v_1v_S, v_S, c_0c_1v_S, c_0v_T, c_1v_T\}$, where $S = s_0.T$.

The linear maps will be given as matrices in the basis $\mathcal{B}$. Specifically, the matrices for c_0c_1, z_0, z_1 are as follows

$$c_0c_1 = \begin{bmatrix} J & 0 & 0 & 0 \\ 0 & J & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & J \end{bmatrix}, z_0 = \begin{bmatrix} \kappa_0 H & 0 & 0 & 0 \\ 0 & -\kappa_0' H & 0 & 0 \\ 0 & 0 & \kappa_0' H & 0 \\ 0 & 0 & 0 & -\kappa_0 H \end{bmatrix},$$

$$z_1 = \begin{bmatrix} \kappa_1 H & 0 & 0 & 0 \\ 0 & \kappa_1' H & 0 & 0 \\ 0 & 0 & \kappa_1' H & 0 \\ 0 & 0 & 0 & \kappa_1 H \end{bmatrix}$$

Where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Therefore

$$z_0c_0c_1 + z_1 = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

where

$$P_1 = \begin{bmatrix} \kappa_0 HJ + \kappa_1 H & 0 \\ 0 & -\kappa_0' HJ + \kappa_1' H \end{bmatrix}, P_2 = \begin{bmatrix} \kappa_0' HJ + \kappa_1' H & 0 \\ 0 & -\kappa_0 HJ + \kappa_1 H \end{bmatrix}$$

Now we will determine the action of x_1. Assume

$$x_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with each A_{ij} a 2×2 matrix, and the notation is similar for other blocks.

1) Claim that for each 2×2 block is in the form of

$$A_{ij} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

where $a, b \in \mathbb{C}$ are potentially different constants for various blocks, and similar is true for B_{ij}, C_{ij}, D_{ij}.

This is because $x_1(c_0c_1) + (c_0c_1)x_1 = 0$, therefore

$$JA_{ij} + A_{ij}J = 0$$

and similarly for other blocks B_{ij}, C_{ij}, D_{ij}. Let

$$A_{ij} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then the above condition becomes

$$\begin{bmatrix} -c & -d \\ a & b \end{bmatrix} + \begin{bmatrix} b & -a \\ d & -c \end{bmatrix} = 0$$

therefore $a = -d$ and $b = c$, hence proving the claim.

2) Given the matrices Q, R, X, Y, Z in Lemma 5.11, where $\kappa_0 = \kappa_T(0), \kappa_1 =$

$\kappa_T(1), \kappa_0' = \kappa_S(0), \kappa_1' = \kappa_S(1), \kappa^2 = \kappa_0^2 + \kappa_1^2$, claim the following is true for some constants a, b, c, d:

$$A_{11} = \frac{N_0}{\kappa^2} Q + cX, \qquad A_{12} = eZ$$
$$A_{21} = fZ, \qquad A_{22} = \frac{N_0}{\kappa^2} R + dY$$

Similarly, let Q', R', X', Y', Z' be the analogue of Q, R, X, Y, Z by switching the role of κ_0 with that of κ_0', and the role of κ_1 with that of κ_1', then

$$D_{11} = \frac{N_0}{\kappa^2} Q' + c'X', \qquad D_{12} = e'Z'$$
$$D_{21} = f'Z, \qquad D_{22} = \frac{N_0}{\kappa^2} R' + d'Y'$$

On the other hand,

$$B_{11} = b_{11}E, \qquad B_{12} = b_{12} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$B_{21} = b_{21} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B_{22} = b_{22}F$$
$$C_{11} = c_{11}E, \qquad C_{12} = c_{12} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$C_{21} = c_{21} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad C_{22} = c_{22}F$$

where

$$E = \begin{bmatrix} \kappa_0 + \kappa_0' & \kappa_1 + \kappa_1' \\ \kappa_1 + \kappa_1' & -(\kappa_0 + \kappa_0') \end{bmatrix}, F = \begin{bmatrix} -(\kappa_0 + \kappa_0') & \kappa_1 + \kappa_1' \\ \kappa_1 + \kappa_1' & \kappa_0 + \kappa_0' \end{bmatrix}$$

Indeed, these are all consequences of the relation $x_1(z_0c_0c_1 + z_1) + (z_0c_0c_1 + z_1)x_1 = 2n(n+1)$. For example,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} + \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = 2n(n+1)$$

implies that

$$P_1A + AP_1 = 2n(n+1)$$
$$P_2D + DP_2 = 2n(n+1)$$
$$P_1B + BP_2 = 0$$
$$P_2C + CP_1 = 0$$

Similar to the previous convention write $N_0 = n(n+1)$. The first equation implies that

$$\begin{bmatrix} \kappa_0 HJ + \kappa_1 H & 0 \\ 0 & -\kappa_0' HJ + \kappa_1' H \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \kappa_0 HJ + \kappa_1 H & 0 \\ 0 & -\kappa_0' HJ + \kappa_1' H \end{bmatrix} = 2N_0$$

$$(\kappa_0 HJ + \kappa_1 H)A_{11} + A_{11}(\kappa_0 HJ + \kappa_1 H) = 2N_0$$

$$(\kappa_0 HJ + \kappa_1 H)A_{12} + A_{12}(-\kappa_0' HJ + \kappa_1' H) = 0$$

$$(-\kappa_0' HJ + \kappa_1' H)A_{21} + A_{21}(\kappa_0 HJ + \kappa_1 H) = 0$$

$$(-\kappa_0' HJ + \kappa_1' H)A_{22} + A_{22}(-\kappa_0' HJ + \kappa_1' H) = 2N_0$$

Let us work out the first two equations in details and the other two equations follow a similar calculation. Specifically, let

$$A_{11} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

then

$$\kappa_0 \begin{bmatrix} -b & a \\ -a & -b \end{bmatrix} + \kappa_1 \begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \kappa_0 \begin{bmatrix} -b & -a \\ a & -b \end{bmatrix} + \kappa_1 \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = 2N_0$$

$$a\kappa_1 - b\kappa_0 = N_0$$

therefore

$$a = \frac{N_0}{\kappa^2}\kappa_1 + c\kappa_0$$
$$b = -\frac{N_0}{\kappa^2}\kappa_0 + c\kappa_1$$

for some constant c, and A_{11} coincides with the claim. For the second equation, Assume

$$A_{12} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

where a, b are potentially different constants than those used in A_{11}. Then the equation regarding A_{12} becomes

$$\kappa_0 \begin{bmatrix} -b & a \\ -a & -b \end{bmatrix} + \kappa_1 \begin{bmatrix} a & b \\ -b & a \end{bmatrix} - \kappa_0' \begin{bmatrix} -b & -a \\ a & -b \end{bmatrix} + \kappa_1' \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = 0$$
$$a(\kappa_1 + \kappa_1') - b(\kappa_0 - \kappa_0') = 0$$

Therefore A_{12} is in the form given in the claim. Similarly, the blocks A_{21}, A_{22} can be deduced from the same calculation, replacing κ_0 and κ_1 with the appropriate version of κ_0' and κ_1'. We have concluded the claim regarding the 4×4 matrix A.

For the matrix B, we use the equation $P_1 B + B P_2 = 0$, or equivalently

$$\begin{bmatrix} \kappa_0 HJ + \kappa_1 H & 0 \\ 0 & -\kappa_0' HJ + \kappa_1' H \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \kappa_0' HJ + \kappa_1' H & 0 \\ 0 & -\kappa_0 HJ + \kappa_1 H \end{bmatrix} = 0$$

Similarly to the earlier argument, assume

$$B_{11} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

then

$$\begin{aligned} &(\kappa_0 HJ + \kappa_1 H)B_{11} + B_{11}\kappa_0' HJ + \kappa_1' H \\ = \kappa_0 &\begin{bmatrix} -b & a \\ -a & -b \end{bmatrix} + \kappa_1 \begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \kappa_0' \begin{bmatrix} -b & -a \\ a & -b \end{bmatrix} + \kappa_1' \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = 0 \\ &a(\kappa_1 + \kappa_1') - b(\kappa_0 + \kappa_0') = 0 \end{aligned}$$

Hence B_{11} is of the form in the claim. Lastly,

$$\begin{aligned} &(\kappa_0 HJ + \kappa_1 H)B_{12} + B_{12}(-\kappa_0 HJ + \kappa_1 H) \\ = \kappa_0 &\begin{bmatrix} -b & a \\ -a & -b \end{bmatrix} + \kappa_1 \begin{bmatrix} a & b \\ -b & a \end{bmatrix} - \kappa_0 \begin{bmatrix} -b & -a \\ a & -b \end{bmatrix} + \kappa_1 \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = 0 \\ a &= 0 \end{aligned}$$

Therefore B_{12} is in the form claimed. The calculations for B_{21}, B_{22} and the

4×4 matrix C are similar.

3) Claim v_T and v_S have the opposite parities. Assume on the contrary that their parities agree, then x_1 acts on the basis $v_T, c_0c_1v_T, v_S, c_0c_1v_S$ via the matrix

$$\begin{bmatrix} A_{11} & B_{11} \\ C_{11} & D_{11} \end{bmatrix} = \begin{bmatrix} \frac{N_0}{\kappa^2}Q + cX & b_{11}E \\ c_{11}E & \frac{N_0}{\kappa^2}Q' + c'X' \end{bmatrix}$$

It is straighforward to check $QE + EQ' = 0$ and $XE = EX'$, and recall in the proof of Theorem 5.12 we showed $Q^2 = X^2 = (Q')^2 = (X')^2 = \kappa^2$ and $XQ + QX = X'Q' + Q'X' = 0$, using the relation $x_1^2 = N_0$ we obtain

$$\begin{bmatrix} \frac{N_0^2}{\kappa^2} + c^2\kappa^2 + b_{11}c_{11}E^2 & b_{11}(c - c')XE \\ c_{11}(c - c')EX & \frac{N_0^2}{\kappa^2} + (c')^2\kappa^2 b_{11}c_{11}E^2 \end{bmatrix} = 0$$

where $E^2 = (\kappa_0 + \kappa_0')^2 + (\kappa_1 + \kappa_1')^2$, therefore $c = c'$, and

$$\frac{N_0^2}{\kappa^2} + c^2\kappa^2 + b_{11}c_{11}((\kappa_0 + \kappa_0')^2 + (\kappa_1 + \kappa_1')^2) = N_0 \qquad (**)$$

Lastly let us use the relation $(x_1 - z_1)^4 = p(p+1)(x_1 - z_1)^2$. Observe

$$(x_1 - z_1)^2 = x_1^2 + z_1^2 - G = N_0 + \begin{bmatrix} \kappa_1^2 I_2 & 0 \\ 0 & (\kappa_1')^2 I_2 \end{bmatrix} - G \qquad (\dagger)$$

$$G = x_1z_1 + z_1x_1 = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

where

$$z_1 = \begin{bmatrix} \kappa_1 K & 0 \\ 0 & \kappa_1' K \end{bmatrix}, K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned}
G_{11} &= \kappa_1(\frac{N_0}{\kappa^2}(KQ + QK) + c(KX + KX)) \\
&= 2\kappa_1(\frac{N_0}{\kappa^2}\kappa_1 + c\kappa_0) \\
G_{22} &= \kappa_1'(\frac{N_0}{\kappa^2}(KQ' + Q'K) + c'(KX' + X'K)) \\
&= 2\kappa_1'(\frac{N_0}{\kappa^2}\kappa_1' - c\kappa_0') \\
G_{12} &= b_{11}(\kappa_1 KE + \kappa_1' EK) \\
&= b_{11}(\kappa_1 \begin{bmatrix} \kappa_0 + \kappa_0' & \kappa_1 + \kappa_1' \\ -(\kappa_1 + \kappa_1') & \kappa_0 + \kappa_0' \end{bmatrix} + \kappa_1' \begin{bmatrix} \kappa_0 + \kappa_0' & -(\kappa_1 + \kappa_1') \\ \kappa_1 + \kappa_1' & \kappa_0 + \kappa_0' \end{bmatrix}) \\
&= b_{11} \begin{bmatrix} (\kappa_0 + \kappa_0')(\kappa_1 + \kappa_1') & \kappa_1^2 - (\kappa_1')^2 \\ (\kappa_1')^2 - \kappa_1^2 & (\kappa_0 + \kappa_0')(\kappa_1 + \kappa_1') \end{bmatrix} \\
G_{21} &= c_{11}(\kappa_1' KE + \kappa_1 EK) \\
&= c_{11} \begin{bmatrix} (\kappa_0 + \kappa_0')(\kappa_1 + \kappa_1') & (\kappa_1')^2 - \kappa_1^2 \\ \kappa_1^2 - (\kappa_1')^2 & (\kappa_0 + \kappa_0')(\kappa_1 + \kappa_1') \end{bmatrix}
\end{aligned}$$

For future use let us compute

$$\begin{aligned}
G_{12}G_{21} &= b_{11}c_{11}((\kappa_0 + \kappa_0')^2(\kappa_1 + \kappa_1')^2 + (\kappa_1^2 - (\kappa_1')^2)^2) \\
&= b_{11}c_{11}(\kappa_1 + \kappa_1')^2((\kappa_0 + \kappa_0')^2 + (\kappa_1 - \kappa_1')^2) \qquad (***)
\end{aligned}$$

Also, claim that

$$\frac{(\kappa_1 + \kappa_1')^2((\kappa_0 + \kappa_0')^2 + (\kappa_1 - \kappa_1')^2)}{(\kappa_0 + \kappa_0')^2 + (\kappa_1 + \kappa_1')^2} = \frac{(p+1)p^3\kappa_1^2 + (p+1)p\kappa_0^2 + 4p\kappa_0^2\kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2}$$

where the definition of m follows that in the construction in Section 5.2, that $T^{(0)} = \alpha + (s, 1, \ldots, 1)$ and $m = n + s$.

Following the previous lemma, let us compute the pieces separately,

$$
\begin{aligned}
&(\kappa_1 + \kappa_1')^2 \\
&= (\kappa_1 + \kappa_0\sqrt{\frac{\kappa_0^2 + \kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2}})^2 \\
&= \kappa_1^2 + \kappa_0^2\frac{\kappa_0^2 + \kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2} + 2\kappa_0\kappa_1\sqrt{\frac{\kappa_0^2 + \kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2}} \\
&= \frac{\kappa_0^4 + p^2\kappa_1^4 + (p^2+1)\kappa_0^2\kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2} + 2\kappa_0\kappa_1\sqrt{\frac{\kappa_0^2 + \kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2}} \\
&= \frac{p^3(p+1)\kappa_1^2 + p(p+1)\kappa_0^2 + 2p\kappa_0^2\kappa_1^2 + 2\kappa_0^2\kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2} + 2\kappa_0\kappa_1\sqrt{\frac{\kappa_0^2 + \kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2}} \\
&= (p+1)\frac{p^3\kappa_1^2 + p\kappa_0^2 + 2\kappa_0^2\kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2} + 2\kappa_0\kappa_1\sqrt{\frac{\kappa_0^2 + \kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2}}
\end{aligned}
$$

$$
\begin{aligned}
&(\kappa_0 + \kappa_0')^2 + (\kappa_1 - \kappa_1')^2 \\
&= (\kappa_0 + p\kappa_1\sqrt{\frac{\kappa_0^2 + \kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2}})^2 + (\kappa_1 - \kappa_0\sqrt{\frac{\kappa_0^2 + \kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2}})^2 \\
&= \kappa_0^2(1 + \frac{\kappa_0^2 + \kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2}) + \kappa_1^2(1 + p^2\frac{\kappa_0^2 + \kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2}) + 2(p-1)\kappa_0\kappa_1\sqrt{\frac{\kappa_0^2 + \kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2}} \\
&= \kappa_0^2\frac{2\kappa_0^2 + (p^2+1)\kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2} + \kappa_1^2\frac{(p^2+1)\kappa_0^2 + 2p^2\kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2} + 2(p-1)\kappa_0\kappa_1\sqrt{\frac{\kappa_0^2 + \kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2}} \\
&= \frac{2\kappa_0^4 + 2p^2\kappa_1^4 + 2(p^2+1)\kappa_0^2\kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2} + 2(p-1)\kappa_0\kappa_1\sqrt{\frac{\kappa_0^2 + \kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2}} \\
&= 2(p+1)\frac{p^3\kappa_1^2 + p\kappa_0^2 + (p+1)\kappa_0^2\kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2} + 2(p-1)\kappa_0\kappa_1\sqrt{\frac{\kappa_0^2 + \kappa_1^2}{\kappa_0^2 + p^2\kappa_1^2}}
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&(\kappa_0+\kappa_0')^2+(\kappa_1+\kappa_1')^2\\
&=2(p+1)\frac{p^3\kappa_1^2+p\kappa_0^2+(p+1)\kappa_0^2\kappa_1^2}{\kappa_0^2+p^2\kappa_1^2}+2(p+1)\kappa_0\kappa_1\sqrt{\frac{\kappa_0^2+\kappa_1^2}{\kappa_0^2+p^2\kappa_1^2}}
\end{aligned}
$$

Denote the original quotient in the claim as $\frac{x+y\sqrt{t}}{z+w\sqrt{t}}$, where $t=\frac{\kappa_0^2+\kappa_1^2}{\kappa_0^2+p^2\kappa_1^2}$, then it can be checked that

$$
\frac{x}{z}=\frac{y}{w}=\frac{(p+1)p^3\kappa_1^2+(p+1)p\kappa_0^2+4p\kappa_0^2\kappa_1^2}{\kappa_0^2+p^2\kappa_1^2}
$$

Hence proving the claim. We include the calculation here

$$
\begin{aligned}
\frac{x}{z}&=(p+1)\frac{p^3\kappa_1^2+p\kappa_0^2+(p+1)\kappa_0^2\kappa_1^2}{\kappa_0^2+p^2\kappa_1^2}+\frac{2(p-1)\kappa_0^2\kappa_1^2(\kappa_0^2+\kappa_1^2)}{(p+1)(p^3\kappa_1^2+p\kappa_0^2+(p+1)\kappa_0^2\kappa_1^2)}\\
&=(p+1)\frac{p^3\kappa_1^2+p\kappa_0^2+2\kappa_0^2\kappa_1^2}{\kappa_0^2+p^2\kappa_1^2}+\frac{2(p-1)\kappa_0^2\kappa_1^2(\kappa_0^2+\kappa_1^2)}{(\kappa_0^2+\kappa_1^2)(\kappa_0^2+p^2\kappa_1^2)}\\
\frac{y}{w}&=(p-1)\frac{p^3\kappa_1^2+p\kappa_0^2+2\kappa_0^2\kappa_1^2}{\kappa_0^2+p^2\kappa_1^2}+2\frac{p^3\kappa_1^2+p\kappa_0^2+(p+1)\kappa_0^2\kappa_1^2}{\kappa_0^2+p^2\kappa_1^2}
\end{aligned}
$$

We write this ratio in m (see proof of Lemma 5.5):

$$
\begin{aligned}
&\frac{(\kappa_1+\kappa_1')^2((\kappa_0+\kappa_0')^2+(\kappa_1-\kappa_1')^2)}{(\kappa_0+\kappa_0')^2+(\kappa_1+\kappa_1')^2}\\
&=\frac{(p+1)p^3\kappa_1^2+(p+1)p\kappa_0^2+4p\kappa_0^2\kappa_1^2}{\kappa_0^2+p^2\kappa_1^2}\\
&=\frac{(p+1)p^3m(m+1)+(p+1)mp^2(m-p)+4p^2m^2(m+1)(m-p)}{m^2p(1+p)}\\
&=p(p+1)+\frac{4(m+1)(m-p)p}{1+p}
\end{aligned}
$$

Combined with Equations (**) and (***), this implies

$$b_{11}c_{11}G_{12}G_{21} = (N_0 - \frac{N_0^2}{\kappa^2} - c^2\kappa^2)(p(p+1) + \frac{4(m+1)(m-p)p}{1+p}) \qquad (\dagger\dagger)$$

Now we conclude by contradition that the matrix for x_1 fails to satisfy the relation $(x_1 - z_1)^4 - p(p+1)(x_1 - z_1) = 0$. Using equation (†),

$$\left(N_0 + \begin{bmatrix} \kappa_1^2 I_2 & 0 \\ 0 & (\kappa_1')^2 I_2 \end{bmatrix} - G \right)^2 - p(p+1) \left(N_0 + \begin{bmatrix} \kappa_1^2 I_2 & 0 \\ 0 & (\kappa_1')^2 I_2 \end{bmatrix} - G \right) = 0$$

Or equivalently,

$$\begin{aligned} N_0^2 - p(p+1)N_0 + \begin{bmatrix} \kappa_1^4 I_2 & 0 \\ 0 & (\kappa_1')^4 I_2 \end{bmatrix} + G^2 + (2N_0 - p(p+1)) \begin{bmatrix} \kappa_1^2 I_2 & 0 \\ 0 & (\kappa_1')^2 I_2 \end{bmatrix} \\ - (2N_0 + p(p+1))G - \left(\begin{bmatrix} \kappa_1^2 I_2 & 0 \\ 0 & (\kappa_1')^2 I_2 \end{bmatrix} G + G \begin{bmatrix} \kappa_1^2 I_2 & 0 \\ 0 & (\kappa_1')^2 I_2 \end{bmatrix} \right) = 0 \end{aligned} \qquad (\ddagger)$$

Since the top right 2×2 block is zero,

$$G_{11}G_{12} + G_{12}G_{22} - (2N_0 + p(p+1))G_{12} - (\kappa_1 + \kappa_1')G_{12} = 0$$

Using the scalar values obtained for G_{11}, G_{22} earlier,

$$2\kappa_1(\frac{N_0}{\kappa^2}\kappa_1 + c\kappa_0) + 2\kappa_1'(\frac{N_0}{\kappa^2}\kappa_1' - c\kappa_0') - (2N_0 + p(p+1)) - (\kappa_1^2 + (\kappa_1')^2) = 0$$

Therefore

$$c = \left(\frac{N_0}{\kappa^2}\right)\frac{\kappa^2 - \kappa_1^2 - (\kappa_1')^2}{\kappa_0\kappa_1 - \kappa_0'\kappa_1'} + \frac{\kappa_1^2 + (\kappa_1')^2 - p(p+1)}{2(\kappa_0\kappa_1 - \kappa_0'\kappa_1')}$$

Using the formulas developed in Lemma 5.5, it is a straighforward calculation to see that

$$c\kappa_0\kappa_1 = \frac{N_0}{\kappa^2}m(m+1)(m-p) + \frac{m(m-p)(m+1)}{p-1}$$

We now claim that equation ‡ is never satisifed, for m associated to any path T. The top left block becomes

$$\begin{aligned}
&N_0^2 - p(p+1)N_0 + \kappa_1^4 + G_{11}^2 + b_{11}c_{11}G_{12}G_{21}\\
&+ (2N_0 - p(p+1))\kappa_1^2 - (2N_0 + p(p+1) + 2\kappa_1^2)G_{11}\\
= &N_0^2 - p(p+1)N_0 + \kappa_1^4 + (\frac{2N_0\kappa_1^2}{\kappa^2} + 2c\kappa_0\kappa_1)^2\\
&+ (N_0 - \frac{N_0^2}{\kappa^2} - c^2\kappa^2)(p(p+1) + \frac{4(m+1)(m-p)p}{1+p})\\
&+ (2N_0 - p(p+1))\kappa_1^2 - (2N_0 + p(p+1) + 2\kappa_1^2)(\frac{2N_0\kappa_1^2}{\kappa^2} + 2c\kappa_0\kappa_1) \quad (\#)
\end{aligned}$$

Substituting the expression for $c\kappa_0\kappa_1$, G_{11}, G_{22} and $b_{11}c_{11}G_{12}G_{21}$ as computed in Equation (††) earlier. Notice N_0 and p are fixed constants, and m is only dependent on the path T. We organize the above expression as a polynomial in the variable $\frac{N_0}{\kappa^2}$, with coefficients in the field $\mathbb{C}(m, p)$. We obtain the coefficients for $\frac{N_0^2}{\kappa^4}$ is

$$
\begin{aligned}
&\kappa^4 + 4\kappa_1^4 + 8\kappa_1^2 m(m+1)(m-p) + 4m^2(m+1)^2(m-p)^2 \\
&-(\kappa^2 + m^2(m+1)^2(m-p)^2\frac{\kappa^2}{\kappa_0^2\kappa_1^2})(p(p+1) + \frac{4(m+1)(m-p)p}{1+p}) \\
&-4\kappa^2\kappa_1^2 - 4\kappa^2 m(m+1)(m-p) \\
=&\ \kappa^2(\kappa^2 + \frac{4\kappa^2(m+1)^2}{(p+1)^2}) - (1 + \frac{m^2(m+1)^2(m-p)^2}{\kappa_0^2\kappa_1^2}) \\
&\cdot(\frac{4(m+1)(m-p)p}{1+p} + p(p+1)) - 4\kappa_1^2 - 4m(m+1)(m-p) \\
=&\ \kappa^2(m-p+1)(m(m+1)(1 + \frac{4(m+1)^2}{(p+1)^2}) - m\frac{4(m+1)(m-p) + (p+1)^2}{p+1} \\
&-4m(m+1)) \\
=&\ \frac{4(m+1)(p+1-m)}{p+1}\kappa^4
\end{aligned}
$$

The coefficient for $\frac{N_0}{\kappa^2}$ is

$$
\begin{aligned}
&-p(p+1)\kappa^2 + 2\kappa_1^2\kappa^2 + 8(\kappa_1^2 + m(m+1)(m-p))\frac{m(m-p)(m+1)}{p-1} \\
&+(p(p+1) + \frac{4(m+1)(m-p)p}{1+p}) \\
&\cdot(\kappa^2 - \frac{\kappa^2}{\kappa_0^2\kappa_1^2}2m(m+1)(m-p)\frac{m(m-p)(m+1)}{p-1}) \\
&-4\kappa^2\frac{m(m-p)(m+1)}{p-1} + 2p(p+1)(\kappa_1^2 + m(m+1)(m-p)) \\
&-4\kappa_1^4 - 4\kappa_1^2 m(m+1)(m-p) \\
=&\ (m-p+1)m(-p(p+1)^2 + 2m(m+2)(p+1) + \frac{8m(m+1)^2(m-p)}{p-1} + \\
&(p(p+1) + \frac{4(m+1)(m-p)p}{1+p})((p+1) - \frac{2(p+1)(m-p)(m+1)}{p(p-1)}) \\
&-4(p+1)\frac{m(m-p)(m+1)}{p-1} + 2p(p+1)(m+1) - 4m(m+1)^2)
\end{aligned}
$$

(continued from last page)

$$= (m-p+1)m(m+1)(2m(p+1) + \frac{8m(m+1)(m-p)}{p-1} + 4(m-p)p$$
$$- \frac{2(p+1)^2(m-p)}{p-1} - \frac{8(m-p)^2(m+1)}{p-1} - \frac{4m(m-p)(p+1)}{p-1}$$
$$+ 2p(p+1) - 4m(m+1))$$
$$= \frac{(m-p+1)m(m+1)(m-p)}{p-1}(8m(m+1) - 2(p+1)^2 - 8(m-p)(m+1)$$
$$- 4m(p+1) + (-4m+2p-2)(p-1))$$
$$= \frac{(m-p+1)m(m+1)^2(m-p)}{p-1}(8m - 8(m-p) - 8p) = 0$$

The constant term is

$$\kappa_1^4 - p(p+1)\kappa_1^2 + \frac{4m^2(m+1)^2(m-p)^2}{(p-1)^2} - \frac{\kappa^2}{\kappa_0^2\kappa_1^2}\frac{m^2(m+1)^2)(m-p)^2}{(p-1)^2}$$
$$\cdot (p(p+1) + \frac{4(m+1)(m-p)p}{1+p}) + \frac{2p(p+1)m(m-p)(m+1)}{p-1}$$
$$- \frac{4m^2(m+1)^2(m-p^2)}{(p-1^2)}$$
$$= m(m+1)(m(m+1) - p(p+1) + \frac{4m(m+1)(m-p)^2}{(p-1)^2}$$
$$- \frac{(p+1)(m-p)(m-p+1)}{(p-1)^2}(p(p+1) + \frac{4(m+1)(m-p)p}{1+p})$$
$$+ \frac{2p(p+1)(m-p)}{p-1} - \frac{4m(m+1)(m-p)}{p-1})$$
$$= \frac{m(m+1)(m-p)}{(p+1)^2}((m+p+1)(p-1)^2 + 4m(m+1)(m-p)$$
$$- \frac{(p+1)(m-p+1)}{p} \cdot (p(p+1) + \frac{4(m+1)(m-p)p}{1+p})$$
$$+ 2p(p+1)(p-1) - 4m(m+1)(p-1))$$
$$= \frac{-4p^2m(m+1)(m-p)(m-p+1)}{(p-1)^2}$$

Therefore, Equation (#) becomes

$$N_0^2 \frac{4(m+1)(p+1-m)}{p+1} - \frac{4p^2m(m+1)(m-p)(m-p+1)}{(p-1)^2} = 0$$

Or

$$N_0^2 = \frac{p^2(p+1)m(m-p)(m-p+1)}{(p-1)^2(p+1-m)}$$

We claim that the right hand is always negative. Recall p is the number of boxes in $(s, 1, \dots, 1)$ where $\alpha + (s, 1, \dots, 1)$ is the partition $T^{(0)}$. Let l be the length of the partition $(s, 1, \dots, 1)$ and we have $l + s - 1 = p$. On the other hand, $l \leq n$ hence $p \leq n + s - 1 = m - 1$, $m - p \geq 1$. On the other hand, $(m - p + 1) + (p + 1 - m) = 2$ hence they are both positive if and only if $(m - p + 1) = (p + 1 - m) = 1$, or $m = p$, which violates the result above. Therefore, one of $(m - p + 1)$ and $(p + 1 - m)$ is zero or negative, and the quantity on the right hand of the equation is negative.

The assumption in the beginning of the claim 3) is false, therefore v_T and v_S have the opposite parities.

4) The action of x_1 on the subspace spanned by $\{v_T, c_0c_1v_T, c_0v_S, c_1v_S\}$ is given by A. Using the relation $x_1^2 = N_0$,

$$\begin{bmatrix} (\frac{N_0}{\kappa^2}Q + cX)^2 + efZ^2 & e((\frac{N_0}{\kappa^2}Q + cX)Z + Z(\frac{N_0}{\kappa^2}R + dY)) \\ f(Z(\frac{N_0}{\kappa^2}Q + cX) + (\frac{N_0}{\kappa^2}R + dY)Z) & (\frac{N_0}{\kappa^2}R + dY)^2 + efZ^2 \end{bmatrix}^2 = N_0$$

It was shown in the proof of Theorem 5.12 that $QX + XQ = RY + YR =$

$XZ + ZY = YZ + ZX = 0$, $Q^2 = R^2 = X^2 = Y^2 = \kappa^2$ therefore

$$\begin{bmatrix} \frac{N_0^2}{\kappa^2} + c\kappa^2 + efZ^2 & e(c-d)XZ \\ f(c-d)ZX & \frac{N_0^2}{\kappa^2}R + d\kappa^2 + efZ^2 \end{bmatrix} = N_0 \qquad (\#\#)$$

On the other hand, we follow a similar calculation to use the relation $(x_1 - z_1)^4 - p(p+1)(x_1 - z_1) = 0$.

$$(x_1 - z_1)^2 = x_1^2 + z_1^2 - L = N_0 + \begin{bmatrix} \kappa_1^2 I_2 & 0 \\ 0 & (\kappa_1')^2 I_2 \end{bmatrix} - L \qquad (\dagger)$$

$$L = x_1 z_1 + z_1 x_1 = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

and a similar version of Equation (‡) is true,

$$L_{11}L_{12} + L_{12}L_{22} - (2N_0 + p(p+1))L_{22} - (\kappa_1 + \kappa_1')L_{22} = 0$$

where

$$z_1 = \begin{bmatrix} \kappa_1 K & 0 \\ 0 & \kappa_1' K \end{bmatrix}, K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned} L_{11} &= \kappa_1(\frac{N_0}{\kappa^2}(KQ + QK) + c(KX + KX)) \\ &= 2\kappa_1(\frac{N_0}{\kappa^2}\kappa_1 + c\kappa_0) \\ L_{22} &= \kappa_1'(\frac{N_0}{\kappa^2}(KR + RK) + c(KY + YK)) \\ &= 2\kappa_1'(\frac{N_0}{\kappa^2}\kappa_1' + c\kappa_0') \end{aligned}$$

therefore c is given as in the combinatorial construction in Section 5.2 and

Theorem 5.12. The top left block in Equation (##) implies the condition on the product ef as specificed in the construction. $\square$

Regarding the action of s_i, we have the following result:

Theorem 6.2. *Let $\mathcal{W}^\lambda$ be a module satisfying the conditions in the previous theorem, then there exists a homogeneous module isomorphism $\phi : \mathcal{W}^\lambda \to \mathcal{D}^\lambda_f$ for some $f : \Gamma^\lambda_- \to \mathbb{C}$.*

We need a few lemmas to prove the result. In particular, to understand the action of s_i, it is helpful to study elements Φ_i associated to s_i. In [16], Wan mentioned the intertwiners

$$\Phi_i = s_i(z_i^2 - z_{i+1}^2) + (z_i + z_{i+1}) - c_0c_1(z_i - z_{i+1})$$

and it can be checked that

$$\begin{aligned} \Phi_i z_i &= z_{i+1}\Phi_i \\ \Phi_i z_{i+1} &= z_i \Phi_i \\ \Phi_i z_j &= z_j \Phi_i \qquad\qquad j \neq i, i+1 \\ \Phi_i^2 &= 2(z_i^2 + z_{i+1}^2) - (z_i^2 - z_{i+1}^2)^2 \\ &= -(z_i^2 - z_{i+1}^2)^2 + (z_i - z_{i+1})^2 + (z_i + z_{i+1})^2 \end{aligned}$$

Following Hill-Kujawa-Sussan[10], define

$$\begin{aligned} \mathcal{Y}_T(i) &= \mathcal{Y}(\kappa_T(i), \kappa_T(i+1)) \\ &= (\kappa_T(i)^2 - \kappa_T(i+1)^2)^2 - (\kappa_T(i) - \kappa_T(i+1))^2 - (\kappa_T(i) + \kappa_T(i+1))^2 \end{aligned}$$

Then $\mathcal{Y}_T(i) = \mathcal{Y}_{s_i.T}(i)$ and $\Phi_i^2 v_T = -\mathcal{Y}_T(i)v_T$.

Lemma 6.3. *The following is true*

$$\mathcal{Y}_T(i)\mathcal{Y}_{s_i.T}(i+1)\mathcal{Y}_{s_{i+1}s_i.T}(i) = \mathcal{Y}_T(i+1)\mathcal{Y}_{s_{i+1}.T}(i)\mathcal{Y}_{s_is_{i+1}.T}(i+1)$$

Proof. Let $\kappa_i = \kappa_T(i), \kappa_{i+1} = \kappa_T(i+1), \kappa_{i+2} = \kappa_T(i+2)$, then the values of κ on the following paths are as follows

	$\kappa(i)$	$\kappa(i+1)$	$\kappa(i+2)$
T	κ_i	κ_{i+1}	κ_{i+2}
$s_i.T$	κ_{i+1}	κ_i	κ_{i+2}
$s_{i+1}.T$	κ_i	κ_{i+2}	κ_{i+1}
$s_is_{i+1}.T$	κ_{i+2}	κ_i	κ_{i+1}
$s_{i+1}s_i.T$	κ_{i+1}	κ_{i+2}	κ_i

Then both sides are equal to the quantity $\mathcal{Y}(\kappa_i, \kappa_{i+1})\mathcal{Y}(\kappa_{i+1}, \kappa_{i+2})\mathcal{Y}(\kappa_i, \kappa_{i+2})$.

□

Proof of Proposition 6.2. For notation purposes let us replace v_T with w_T in Theorem 6.1 so that $\mathcal{W}^\lambda = \bigoplus_{T\in\Gamma^\lambda} Cl_{d+1}w_T$ and $z_iw_T = \kappa_T(i)w_T$, $0 \le i \le d$.

Since

$$\kappa_T(i)\Phi_iw_T = \Phi_i(z_iw_T) = z_{i+1}(\Phi_iw_T)$$
$$\kappa_T(i+1)\Phi_iw_T = \Phi_i(z_{i+1}w_T) = z_i(\Phi_iw_T)$$
$$\kappa_T(j)\Phi_iw_T = \Phi_i(z_jw_T) = z_j(\Phi_iw_T) \qquad j \neq i, i+1$$

It follows Φ_iw_T is a vector with eigenvalues $\kappa_T(0), \dots, \kappa_T(i+1), \kappa_T(i), \dots, \kappa_T(d)$ for $z_0, \dots, z_d$. By since $w_{s_i.T}$ is such a vector, by Lemma 5.7 Φ_iw_T is a scalar multiple of $w_{s_i.T}$. By a similar argument, $\Phi_i w_{s_i.T}$ is also a scalar multiple of w_T. Let $a_T(i)$, $a_{s_i.T}(i)$ be constants such that $\Phi_i w_T = \frac{\sqrt{-\mathcal{Y}_T(i)}}{a_T(i)} w_{s_i.T}$ and $\Phi_i w_{s_i.T} = \frac{\sqrt{-\mathcal{Y}_{s_i.T}(i)}}{a_{s_i.T}} w_T$. Since Φ_i^2 acts on v_T by the scalar $-\mathcal{Y}_T(i)$, it follows that $a_T(i)a_{s_i.T}(i) = 1$.

Define the Cl_{d+1}-superlinear map $\phi : \mathcal{W}^\lambda \to \mathcal{D}^\lambda$ as follows: let L_0 be the standard tableau of the skew shape λ/μ, where μ is the partition with $p-n+1$ boxes attached to the first row of α, and one box attached to each row from the second row down. As a result of the proof for Lemma 5.20, for any two tableaux standard tableax T and S, if $T^{(1)}$ has exactly one more box than $S^{(1)}$ in the first row, there is a word w in $s_1, \ldots, s_{d-1}$ such that $ws_0.S = T$. Similarly, for any semistandard tableau $T = (T^{(0)}, \ldots, T^{(d)} = \lambda)$, let T_{st} be the standard tableau of the skew shape $\lambda/T^{(0)}$, then there are words $w_1, \ldots, w_t$ in $s_1, \ldots, s_{d-1}$, such that $w_1 s_0 w_2 s_0 \cdots w_t s_0 L_0 = T_{st}$, where t is the difference between the number of boxes in the first row of T_{st} and L_0. Therefore there is another word w_1' such that $w_1' s_0 w_2 s_0 \cdots w_t s_0 L_0 = T$.

Define the map ϕ recursively via $\phi(w_{L_0}) = w_{L_0}$, $\phi(w_{s_i.T}) = a_T(i) v_{s_i.T}$ and $\phi(w_{s_0.T}) = w_T$. For example, when $w = s_{k_1} \cdots s_{k_t}$, and $1 \le k_1, \ldots, k_t \le d$, such that $wT_{st} = T$. Define

$$\phi(w_{s_{k_1} \cdots s_{k_t}.T}) = a_{s_{k_2} s_{k_3} \cdots s_{k_t} T}(k_1) a_{s_{k_3} \cdots s_{k_t} T}(k_2) \cdots a_{s_{k_t}.T}(k_{t-1}) a_T(k_t) v_{s_{k_1} \cdots s_{k_t}.T}$$

To see the well-definedness of the map, we only need to show it is independent of the choice of the words $w_1', w_2, \ldots, w_t$ under the symmetric group relations.

For $|i-j|>1$,

$$\Phi_i\Phi_j w_T = \Phi_i(\frac{\sqrt{-\mathcal{Y}_T(j)}}{a_T(j)} w_{s_j.T}) = \frac{\sqrt{(-\mathcal{Y}_T(j))(-\mathcal{Y}_{s_j.T}(i))}}{a_T(j)a_{s_j.T}(i)} w_{s_is_j.T}$$
$$\Phi_j\Phi_i w_T = \Phi_j(\frac{\sqrt{-\mathcal{Y}_T(i)}}{a_T(i)} w_{s_i.T}) = \frac{\sqrt{(-\mathcal{Y}_T(i))(-\mathcal{Y}_{s_i.T}(j))}}{a_T(i)a_{s_i.T}(j)} w_{s_js_i.T}$$

Since $\mathcal{Y}_{s_i.T}(j) = \mathcal{Y}_T(j)$ and $\mathcal{Y}_T(i) = \mathcal{Y}_{s_j.T}(i)$, it follows that $a_{s_j.T}(i)a_T(j) = a_{s_i.T}(j)a_T(i)$ for any tableau T, therefore

$$\phi(w_{s_is_j.T}) = a_{s_j.T}(i)a_T(j)w_T = a_{s_i.T}(j)a_T(i) = \phi(w_{s_js_i.T})$$

Similarly,

$$\Phi_{i+1}\Phi_i\Phi_{i+1}w_T = \frac{\sqrt{(-\mathcal{Y}_{s_is_{i+1}.T}(i+1))(-\mathcal{Y}_{s_{i+1}.T}(i))(-\mathcal{Y}_T(i+1))}}{a_{s_is_{i+1}.T}(i+1)a_{s_{i+1}.T}(i)a_T(i+1)} w_T$$
$$\Phi_i\Phi_{i+1}\Phi_i w_T = \frac{\sqrt{(-\mathcal{Y}_{s_{i+1}s_i.T}(i))(-\mathcal{Y}_{s_i.T}(i+1))(-\mathcal{Y}_T(i))}}{a_{s_{i+1}s_i.T}(i)a_{s_i.T}(i+1)a_T(i)} w_T$$

and by the previous lemma $a_{s_is_{i+1}.T}(i+1)a_{s_{i+1}.T}(i)a_T(i+1) = a_{s_{i+1}s_i.T}(i)a_{s_i.T}(i+1)a_T(i)$. Therefore

$$\phi(w_{s_is_{i+1}s_i.T}) = a_{s_{i+1}s_i.T}(i)a_{s_i.T}(i+1)a_T(i)w_T$$
$$= a_{s_is_{i+1}.T}(i+1)a_{s_{i+1}.T}(i)a_T(i+1)w_T = \phi(w_{s_{i+1}s_is_{i+1}.T})$$

The map ϕ intertwines the action of $z_0,\dots,z_d$ since both v_T and w_T are eigenvectors for z_i with the same eigenvalue. It also intertwines the action of $s_1,\dots,s_{d-1}$ by design. We only show the work for $T=T_{st}$ and the other cases can be proved inductively via the recursive definition of ϕ. In particular, since

$\Phi_i w_{T_{st}} = \frac{\sqrt{-\mathcal{Y}_T(i)}}{a_T(i)} w_{T_{st}}$, it follows that

$$\begin{aligned}(s_i(z_i^2 - z_{i+1}^2) + (z_i + z_{i+1}) - c_0c_1(z_i - z_{i+1}))w_{T_{st}} &= \frac{\sqrt{-\mathcal{Y}_T(i)}}{a_T(i)} w_{s_i.T_{st}} \\ s_i w_{T_{st}} &= (-\frac{1}{\kappa_T(i) - \kappa_T(i+1)} + \frac{1}{\kappa_T(i) + \kappa_T(i+1)} c_0c_1)w_{T_{st}} \\ &+ \frac{\sqrt{1 - \frac{1}{(\kappa_T(i) - \kappa_T(i+1)^2)} - \frac{1}{(\kappa_T(i) + \kappa_T(i+1))^2}}}{a_T(i)} w_{s_i.T_{st}}\end{aligned}$$

Therefore

$$\begin{aligned}\phi(s_i.w_{T_{st}}) = (&-\frac{1}{\kappa_T(i) - \kappa_T(i+1)} + \frac{1}{\kappa_T(i) + \kappa_T(i+1)} c_0c_1)v_{T_{st}} \\ &+ \sqrt{1 - \frac{1}{(\kappa_T(i) - \kappa_T(i+1)^2)} - \frac{1}{(\kappa_T(i) + \kappa_T(i+1))^2}}\, v_{s_i.T} = s_i.\phi(w_{T_{st}})\end{aligned}$$

We now show that ϕ intertwines the action x_1. Since $x_1 z_i = z_i x_1$ for all $i \geq 2$, x_1 preserves the $\kappa_T(i)$ eigenspace of z_i for $2 \leq i \leq d$, therefore $x_1.w_T = h_T w_T + b_T w_{s_0.T}$, where h_T, b_T is in the Clifford algebra generated by c_0 and c_1. Since w_T and $w_{s_0.T}$ have different parities, h_T is even and b_T is odd. By Propositions 6.1, $x_1.v_T = h_T w_t + b_T w_{s_0.T}$ with the same coefficient h_T, b_T in $\mathcal{D}^\lambda$, and ϕ is a homogeneous map. Let a be the constant such that $\phi(w_T) = av_T$ from the definition above. We discuss by cases:

1) When w_T has the same parity as v_T, ϕ is an even map

$$\begin{aligned}\phi(x_1.w_T) &= \phi(h_T w_T + b_T w_{s_0.T}) \\ &= ah_T v_T + ab_T v_{s_0.T} \\ x_1.\phi(w_T) &= a(x_1.v_T) = ah_T v_T + ab_T v_{s_0.T}.\end{aligned}$$

2) When w_T has the opposite parity as v_T, ϕ is an odd map, and $x_1.v_T =$

$h_T v_T - b_T v_{s_0.T}$ in $\Pi\mathcal{D}^\lambda$.

$$\begin{aligned}\phi(x_1.w_T) &= \phi(h_T w_T + b_T w_{s_0.T}) \\ &= a h_T v_T - a b_T v_{s_0.T} \\ x_1.\phi(w_T) &= a(x_1.v_T) = a h_T v_T - a b_T v_{s_0.T}.\end{aligned}$$

The other tableaux can be checked inductively. □

For the moment let us distinguish the two algebras $H^p_{d,\text{even}}$ and $H^p_{d,\text{odd}}$, in the case when n is even or odd. Recall in Section 4.1 we defined $H^p_{d,\text{even}}$ to be a quotient of $\mathcal{H}^M_d$ and $H^p_{d,\text{odd}}$ a quotient of $\mathcal{H}^{MN}_d$, under the same relations involving generators and parameters of the same name. Also in Section 3.3 we defined $\mathcal{H}^{MN}_d = \mathcal{H}^M_d \otimes \mathrm{Cl}_1$. As a consequence, $H^p_{d,\text{odd}} = H^p_{d,\text{even}} \otimes \mathrm{Cl}_1$.

In [9], Kang-Kashiwara-Tsuchioka introduced the notion of a supercategory and its Clifford twist. As pointed out in [3] by Brundan-Davidson, this implies that the supercategory of $H^p_{d,\text{odd}}$-modules is a Clifford twist of the supercategory of $H^p_{d,\text{even}}$-modules, hence the two supercategories are closely related to each other. Moreover, the relation is given by a specific functor, which is compatible with our construction of $\mathcal{D}^\lambda_f$ in the two cases. We therefore expect a similar result to be true for Proposition 6.2, in the case when n is odd.

6.2. Centralizer Algebras for $\mathfrak{q}(n)$

Recall in Section 4.3 we defined the action

$$\phi : \mathcal{H}^p_d \to \mathrm{End}_{\mathfrak{q}(n)}(L(\alpha) \otimes L(\beta) \otimes V^{\otimes d}).$$

Hence $\phi(\mathcal{H}_d^p)$ is a subalgebra of the centralizer

$$\mathcal{Z}_d := \mathrm{End}_{\mathfrak{q}(n)}(L(\alpha) \otimes L(\beta) \otimes V^{\otimes d}).$$

In this section we will study $\mathcal{H}_d^p$-modules that are restrictions of $\mathcal{Z}_d$-modules. As explained in [17, Section 3.1.3], if A, B are semisimple superalgebras, W is an A-module, U is a B-module, then $W \otimes U$ naturally admits a module structure for the superalgebra $A \otimes B$ defined in Section 3.3. Moreover, if W and U are both of Type Q, then $W \otimes U$ decomposes into two isomorphic copies of a simple module of Type M, denoted as $2^{-1}W \otimes U$. As a consequence we have $\dim(2^{-1}W \otimes U) = 2^{-1} \dim W \otimes U$. In the other cases $W \otimes U$ is a simple $A \otimes B$-module. There is a super version of the well-known double centralizer theorem.

Theorem 6.4. *Double Centralizer Theorem, [17, Proposition 3.5] Let W be a finite dimensional vector superspace, A a semisimple subalgebra of $\mathrm{End}_{\mathbb{C}}(W)$. Let B be the centralizer of A in $\mathrm{End}_{\mathbb{C}}(W)$, then A is the full centralizer of B in $\mathrm{End}_{\mathbb{C}}(W)$, and W admits a decomposition into irreducible (A, B)-bimodules*

$$W \simeq \bigoplus_{\lambda} 2^{-\delta(\lambda)} W(\lambda) \otimes W^{\lambda} \tag{6.2.1}$$

where $W(\lambda)$ is an irreducible A-module, W^{λ} an irreducible B-module, and $W(\lambda)$ is of Type M if and only if W^{λ} is of Type M. Here, $\delta(\lambda) = 0$ if $W(\lambda)$ is of Type M, and $\delta(\lambda) = 1$ if $W(\lambda)$ is of Type Q. In particular, the multiplicity of $W(\lambda)$, is equal to $2^{-\delta(\lambda)} \dim W^{\lambda}$.

In the case when n is even, we apply the above theorem to the representation $\rho : U(\mathfrak{q}(n)) \to \mathrm{End}(L(\alpha) \otimes L(\beta) \otimes V^{\otimes d})$, where $A = \rho(U(\mathfrak{q}(n)))$ and $B = \mathcal{Z}_d$.

All irreducible summands $L(\lambda)$ as $\mathfrak{q}(n)$-modules have $\ell(\lambda) = n$ which is even by assumption, therefore $L(\lambda)$ is of Type M. We obtain the decomposition

$$L(\alpha) \otimes L(\beta) \otimes V^{\otimes d} \simeq \bigoplus_{\lambda \in \mathcal{P}_d(\alpha,\beta)} L(\lambda) \otimes \mathcal{L}^\lambda \tag{6.2.2}$$

where $\mathcal{L}^\lambda$ is an irreducible $\mathcal{Z}_d$-module, $\mathcal{P}_d(\alpha, \beta)$ is the set of partitions defined in Definition 5.2.

In particular, the summand $L(\lambda) \otimes \mathcal{L}^\lambda$ is the $L(\lambda)$-isotypic component of $L(\alpha) \otimes L(\beta) \otimes V^{\otimes d}$ as a $\mathfrak{q}(n)$-module, and at the same time, the $\mathcal{L}^\lambda$-isotypic component of $L(\alpha) \otimes L(\beta) \otimes V^{\otimes d}$ as a $\mathcal{Z}_d$-module.

By the argument in Section 2.4, if $L(\gamma)$ is a direct summand of $L(\mu) \otimes V$, then it has multiplicity 2, and similarly, if $L(\gamma)$ is a direct summand of $L(\alpha) \otimes L(\beta)$, it also has multiplicity 2. Therefore each path in Γ^λ corresponds to obtaining the summand $L(\lambda)$ with the multiplicity 2^{d+1}, and the total multiplicity of $L(\lambda)$ is $2^{d+1} \# \Gamma^\lambda$. As a consequence of the double centralizer theorem, this implies that when n is even, $\dim \mathcal{L}^\lambda = 2^{d+1} \# \Gamma^\lambda$.

Theorem 6.5. *When n is even, the $\mathcal{H}_d^p$-modules $\operatorname{Res}^{\mathcal{Z}_d}_{\phi(\mathcal{H}_d^p)} \mathcal{L}^\lambda$ are isomorphic to $\mathcal{D}_f^\lambda$ for some $f : \Gamma_-^\lambda \to \mathbb{C}$ and therefore irreducible.*

Proof. Let $W = L(\lambda) \otimes \mathcal{L}^\lambda$ as in the decomposition (6.2.2). We argue that a few claims are true:

1) We claim that for each $T \in \Gamma^\lambda$, there exists $w_T \in W$ such that $z_i . w_T = \kappa_T(i) w_T$, $0 \leq i \leq d$.

Given $T \in \Gamma^\lambda$, the sequence of partitions in T corresponds to a (potentially nonunique) choice of irreducible summand at each level of the decomposition. By Lemma 4.9, the resulting summand $L(\lambda)$ at level d has the property that for any $u_T \in L(\lambda)$, $z_i.u_T = \kappa_T(i)$ or $z_i.u_T = -\kappa_T(i)u_T$, $0 \leq i \leq d$. In particular, choose u_T to be a homogeneous nonzero vector. Since $u_T \in L(\lambda) \subset W$ and W is an $\mathcal{H}_d^p$-module, one can choose a suitable $\sigma = (\sigma_0, \ldots, \sigma_d) \in \{0,1\}^{d+1}$, such that $w_T := c^\sigma u_T = c_0^{\sigma_0} \cdots c_d^{\sigma_d} u_T$ is again homogeneous and has the eigenvalues $\kappa_T(0), \ldots, \kappa_T(d)$ for $z_0, \ldots, z_d$.

2) Fix an irreducible summand $\mathcal{L}^\lambda \subset W$. We claim that for each $T \in \Gamma^\lambda$, there exists homogeneous $v_T \in \mathcal{L}^\lambda$ such that $z_i.v_T = \kappa_T(i)v_T$, $0 \leq i \leq d$.

Since W is a direct sum of $\mathcal{L}^\lambda$'s, there exists a grade preserving isomorphism $W \simeq (\mathcal{L}^\lambda)^{\oplus a} \oplus (\Pi\mathcal{L}^\lambda)^{\oplus b}$, for some $a, b \in \mathbb{Z}$. Given T, let w_T be the vector chosen in Claim 1. Let $w_T = w_1 + \cdots + w_{a+b}$ be the direct sum decomposition of the vector w_T under the previous isomorphism. We have

$$\begin{aligned} z_i.w_T &= \kappa_T(i)w_1 + \cdots + \kappa_T(i)w_{m_\lambda} \\ &= z_i w_1 + \cdots + z_i w_{a+b}. \end{aligned}$$

Since the decomposition is unique, and $z_i w_k$ is in the k-th copy of $\mathcal{L}^\lambda$ (or $\Pi\mathcal{L}^\lambda$), it follows that $z_i w_k = \kappa_T(i)w_k$ for all $1 \leq k \leq a+b$. Since $w_T \neq 0$, there exists some j such that $w_j \neq 0$, and w_j is homogeneous. Denote by $\mathcal{L}_j^\lambda$ the j-th copy of $\mathcal{L}^\lambda$ or $\Pi\mathcal{L}^\lambda$. Thus, via the isomorphism $\mathcal{L}_j^\lambda \simeq \mathcal{L}^\lambda$, there is a vector v_T as claimed.

3) We claim that $\mathcal{L}^\lambda \simeq \bigoplus_{T \in \Gamma^\lambda} Cl_{d+1} v_T$ as superspaces.

In Lemma 4.9, we showed that $\kappa_T(i)$ is never zero for any $0 \leq i \leq d$, $T \in \Gamma^\lambda$. Therefore, for all $\sigma \in \{0,1\}^{d+1}$, each $c^\sigma v_T$ has a distinct list of eigenvalues for $z_0, \ldots, z_d$. Moreover, we showed in Lemma 5.7 that the lists of eigenvalues for v_T and v_S are different for $T, S \in \Gamma^\lambda$, whenever $T \neq S$, and each $\kappa_T(i)$ or $\kappa_S(i)$ is positive. Therefore, $\{c^\sigma v_T\}_{\sigma \in \{0,1\}^{d+1}, T \in \Gamma^\lambda}$ all have distinct lists of eigenvalues, and so are linearly independent. Since $\mathcal{L}^\lambda$ is an $\mathcal{H}_d^p$-module, $\bigoplus_{T \in \Gamma^\lambda} Cl_{d+1} v_T \subset \mathcal{L}^\lambda$, and since both have dimension $2^{d+1} \# \Gamma$, our claim follows.

By showing Claims 1,2 and 3, we have shown that $\mathcal{L}^\lambda$ satisfies the conditions in Propositions 6.1 and 6.2, and the conclusion follows.

$\square$

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Appendix: Code for Lemma 5.14

We include the MAGMA computation in the proof of Lemma 5.14. We organize it according to the various coefficients in the same sequence as that in the proof. The following code is for a given tableau T, and one needs to manually change the values for various κ_i and κ_i', based on the chart given in the proof of Lemma 5.14, for all other tableaux.

1) The coefficient for v_T was checked by hand.

2) Coefficient for $v_{s_0.T}$:

The code for checking $M_{01} = 0$ for the tableau $T = L_2$ is as follows

```
C:=IntegerRing();
R<m,N,p>:=PolynomialRing(C,3);
k0:=m*p*(m-p);
k1:=m*(m+1);
k2:=N;
k0p:=(m+1)*p*(m+1-p);
k1p:=(m-p)*(m-p+1);
k0p_p:=(m+1)*(m+1-p);
L:=k1p-k2;
M:=k1-k2;
K:=k0-k0p;
```

```
Q:=k0+p*k1;
R:=k0+p*k2;
S:=k0p+p*k2;
X:=N*p+k0;
Y:=N*p+k0p;
Y_p:=k0p_p+N;
D:=-2*M*L*L*Q*R*S*K
+2*M*M*L*Q*R*S*K
-4*k0*Y_p*M*M*R*S*K
+2*M*L*R*S*K*(-X*k1+k0*Y_p)
+2*(k1p-k1)*L*M*R*S*(p*k1*(-N+k1p)-k0*(-N+k1))+4*k1*X*L*L*R*S*K
+(M*M-2*(k1+k2))*X*L*L*Q*S*K
-(L*L-2*(k1p+k2))*Y*M*M*Q*R*K;
D;
```

The code for checking $M_{11} = 0$ for the tableau $T = L_2$ is as follows.

```
C:=IntegerRing();
R<m,N,p>:=PolynomialRing(C,3);
k0:=m*p*(m-p);
k0p_p:=(m+1)*(m+1-p);
k1:=m*(m+1);
k2:=N;
k0p:=(m+1)*p*(m+1-p);
k1p:=(m-p)*(m-p+1);
M:=(k0+p*k2)*(k0p+p*k2);
K:=k0+p*k1;
```

```
L:=k1-k2;
Q:=k1p-k2;
D:=-2*L*Q*Q*M*K+2*L*L*Q*M*K
-k0*M*(N+k0p_p)*(4*L*L-2*L*Q)
-2*M*L*Q*(k0*(-N+k1)-p*k1*(-N+k1p))
+M*(N*p+k0)*(4*k1*Q*Q-2*k1*L*Q)
+K*(L*L-2*(k1+k2))*(N*p+k0)*(k0p+p*k2)*Q*Q
-K*(Q*Q-2*(k1p+k2))*(N*p+k0p)*L*L*(k0+p*k2);
D;
```

Code for checking $M_{21} = M_{22} = 0$, for $T = L_2$:

```
C:=IntegerRing();
R<m,N,p>:=PolynomialRing(C,3);
k0:=m*p*(m-p);
k1:=m*(m+1);
k2:=N;
k0p:=(m+1)*p*(m+1-p);
k1p:=(m-p)*(m-p+1);
L:=k0+p*k1;
M:=k0p+p*k1p;
Q:=k1-k2;
R:=k1p-k2;
S:=k0+p*k2;
T:=k0p+p*k2;
D:=4*(-k0*k1+N*k0-p*N*k1+p*k1*k1p)*M*S*T
```

```
+2*(N*p+k0p)*p*(k1p+k2)*L*Q*S
-2*p*(N*p+k0)*(k1+k2)*M*R*T
+2*p*(N-k1)*M*R*S*T
-2*(N*p-p*k1p)*L*Q*S*T
-p*(N-k2)*(k0-k0p)*L*M*Q*R;
D;
F:=4*(N*k0*k1-N*p*k1*k1p+k1*p*k1*k1p-k0*k1*k1p)*M*T*S
-2*k0p*(N*p+k0p)*(k1p+k2)*L*Q*S
+2*k0*(N*p+k0)*(k1+k2)*R*M*T
-(k0-k0p)*p*k2*(N-k2)*L*Q*R*M
+2*p*k1*(N-k1)*R*M*T*S
-2*k1p*(N-k1p)*p*L*Q*T*S;
F;
```

Code for checking $M_{31} = M_{32} = 0$, for $T = L_2$:

```
C:=IntegerRing();
R<m,N,p>:=PolynomialRing(C,3);
k0:=m*p*(m-p);
k1:=m*(m+1);
k2:=N;
k0p:=(m+1)*p*(m+1-p);
k1p:=(m-p)*(m-p+1);
k0p_p:=(m+1)*(m+1-p);
L:=k1-k2;
M:=k1p-k2;
K:=k0+p*k1;
```

```
J:=k0p+p*k1p;

Q:=k0+p*k2;

R:=k0p+p*k2;

X:=k0-k0p;

Y:=k1-k1p;

W:=k0*k1+p*N*k1-N*k0-p*k1*k1p;

V:=N*k0*k1-N*p*k1*k1p-k0*k1*k1p+p*k1*k1*k1p;

D:=4*W*X*J*Q*R
-2*X*(N-k1)*p*M*J*Q*R
+2*(N-k1p)*p*X*L*K*Q*R
-X*Y*p*(N-k2)*L*M*J*K
-2*(k1+k2)*p*Y*(N*p+k0)*M*J*R
+2*(k1p+k2)*Y*p*(N*p+k0p)*L*K*Q;

D;

F:=4*V*X*J*R*Q
+2*k1*p*X*(N-k1)*M*J*Q*R
-2*k1p*(N-k1p)*p*X*L*K*Q*R
+p*k2*(N-k2)*X*Y*L*M*K*J
-2*(k1+k2)*k0*(N*p+k0)*Y*M*J*R
+2*(k1p+k2)*Y*k0p*(N*p+k0p)*L*K*Q;

F;
```

3) The coefficient for $v_{s_1.T} = 0$. We check that the constant and the coefficients for c_0c_1, c_0c_2, c_1c_2 are zeros. The code for $T = L_1$ is as follows

```
C := IntegerRing();

R<m,p,mmp,mp1,n,np1,mmpp1,mmpm1,mm1>
```

```
:= PolynomialRing(C,9);
Ideal := ideal<R|m^2-1-mm1^2, m^2-1-p^2-mmpm1^2,
m^2+1-mp1^2,  n^2+1-np1^2, m^2+1-p^2-mmpp1^2,
m^2-p^2-mmp^2>;
N:=n*np1;
k0:=(m*p*(mmp));
k1:=(n*(np1));
k2:=(m*(mp1));
k0p:=k0;
k1p:=k1;
k0pp:=((mp1)*p*(mmpp1));
k2pp:=((mmp)*(mmpp1));
alpha:=-1/(k1-k2);
beta:=1/(k1+k2);
alpha1:=-1/(k2-k1);
beta1:=1/(k1+k2);
alpha2:=-1/(k2pp-k1);
beta2:=1/(k2pp+k1);
alphap:=-1/(k1p-k2);
betap:=1/(k1p+k2);
A:=k0-k0p;
B:=k1+k1p;
F:=k0-k0pp;
G:=k2+k2pp;
delta:=(-N^2+k1*k1)*k0/(k0*k0+p^2*k1*k1);
delta1:=(-N^2+k2*k2)*k0/(k0*k0+p^2*k2*k2);
```

```
gamma:=(N^2*p^2+k0*k0)*k1/(k0*k0+p^2*k1*k1);
gamma1:=(N^2*p^2+k0*k0)*k2/(k0*k0+p^2*k2*k2);
M:=-(k0*k0+p^2*p^2*k2*k2)*(N^2-k2*k2)*(N^2-k2pp*k2pp)/
((k0*k0+p^2*k2*k2)*(k0*k0+p^2*k2*k2)*((k0-k0pp)*(k0-k0pp)+
(k2+k2pp)*(k2+k2pp)));
K:=-(k0*k0+p^2*p^2*k1*k1)*(N^2-k1*k1)*(N^2-k1p*k1p)/
((k0*k0+p^2*k1*k1)*(k0*k0+p^2*k1*k1)*((k0-k0p)*(k0-k0p)+
(k1+k1p)*(k1+k1p)));
C0:=-beta*gamma*gamma1-alpha*delta*delta1-delta1*delta1*alpha1
-gamma1*gamma1*beta1-gamma*gamma*beta-delta*delta*alpha
-delta*delta1*alpha1-gamma*gamma1*beta1
+M*(-F*F*beta2-G*G*alpha2)
-K*(A*A*betap+B*B*alphap)+gamma1+gamma;
C3:=M*(-F^2*beta2-G^2*alpha2)-K*(A^2*betap+B^2*alphap)
-gamma*beta*gamma1-alpha*delta*delta1-gamma1^2*beta1
-delta1^2*alpha1-gamma^2*beta-delta^2*alpha
-delta*delta1*alpha1-gamma*gamma1*beta1+gamma1+gamma;
C2:=M*(-G*F*beta2+G*F*alpha2)-K*(B*A*alphap-A*B*betap)
+beta*delta*gamma1+alpha*gamma*delta1
-delta1*beta1*gamma1+gamma1*alpha1*delta1
-delta*alpha*gamma+gamma*beta*delta-delta*gamma1*alpha1
-gamma*delta1*beta1+delta1-delta;
C1:=M*(G*F*alpha2-G*F*beta2)-K*(-A*B*betap+B*A*alphap)
+alpha*delta*gamma1-gamma*beta*delta1
+delta1*alpha1*gamma1-gamma1*beta1*delta1
+delta*beta*gamma-gamma*alpha*delta
```

```
-gamma*delta1*alpha1+delta*gamma1*beta1+delta1-delta;
Numerator(C0) in Ideal;
Numerator(C1) in Ideal;
Numerator(C2) in Ideal;
Numerator(C3) in Ideal;
```

We change the eigenvalues for other tableaux as follows:

$T = L_2$:

```
k0:=(m*p*(mmp));
k1:=(m*(mp1));
k2:=(n*(np1));
k0pp:=k0;
k2pp:=k2;
k0p:=((mp1)*p*(mmpp1));
k1p:=((mmp)*(mmpp1));
```

$T = L_3$: switch the values for κ_0, κ_0' and κ_1, κ_1' above.

$T = L_4$:

```
k2:=((mmp)*(mmpp1));
k0:=((mp1)*p*(mmpp1));
k1:=(N);
k0pp:=(m*p*(mmp));
k2pp:=(m*(mp1));
k0p:=k0;
k1p:=k1;
```

$T = T_2$:

```
k2:=(m*(mp1));

k0:=((m)*p*(mmp));

k1:=((mmpm1)*(mmp));

k0p:=((mm1)*p*(mmpm1));

k1p:=(m*(mm1));

k0pp:=((mp1)*p*(mmpp1));

k2pp:=((mmp)*(mmpp1));
```

$T = T_3$:

```
k2:=((mmp)*(mmpm1));

k0p:=((mp1)*p*(mmpp1));

k1p:=((mmpp1)*(mmp));

k0:=(m*p*(mmp));

k1:=(m*(mp1));

k0pp:=((mm1)*p*(mmpm1));

k2pp:=((m)*(mm1));
```

4) Coefficient for $v_{s_0 s_1 . T}$.

We check that the coefficients for c_0, c_1, c_2, $c_0 c_1 c_2$ are zero.

```
C := IntegerRing();

R<m,p,mmp,mp1,n,np1,mmpp1,mmpm1,mm1>

:= PolynomialRing(C,9);

Ideal := ideal<R|m^2-1-mm1^2, m^2-1-p^2-mmpm1^2,

m^2+1-mp1^2,  n^2+1-np1^2, m^2+1-p^2-mmpp1^2,

m^2-p^2-mmp^2>;

N:=n*np1;
```

```
k0:=(m*p*(mmp));
k1:=((mmpm1)*(mmp));
k2:=(m*(mp1));
k0p:=((mm1)*p*(mmpm1));
k1p:=(m*(mm1));
k0pp:=((mp1)*p*(mmpp1));
k2pp:=((mmp)*(mmpp1));
alpha:=-1/(k1-k2);
beta:=1/(k1+k2);
alpha1:=-1/(k2-k1);
beta1:=1/(k2+k1);
alpha2:=-1/(k2pp-k1);
beta2:=1/(k2pp+k1);
A1:=k0-k0pp;
B1:=k2+k2pp;
delta:=(-N^2+k1*k1)*k0/(k0*k0+p^2*k1*k1);
delta1:=(-N^2+k2*k2)*k0/(k0*k0+p^2*k2*k2);
delta2:=(-N^2+k2pp*k2pp)*k0pp/(k0pp*k0pp+p^2*k2pp*k2pp);
gamma:=(N^2*p^2+k0*k0)*k1/(k0*k0+p^2*k1*k1);
gamma1:=(N^2*p^2+k0*k0)*k2/(k0*k0+p^2*k2*k2);
gamma2:=(N^2*p^2+k0pp*k0pp)*k2pp/(k0pp*k0pp+p^2*k2pp*k2pp);
D1:=alpha*gamma*A1-beta*delta*B1+gamma1*alpha1*A1+delta1*beta1*B1
+A1*alpha2*gamma2-B1*beta2*delta2-gamma*A1*alpha2+B1*delta*beta2+A1;
Numerator(D1) in Ideal;
D2:=beta*delta*A1+alpha*gamma*B1
-delta1*beta1*A1+gamma1*alpha1*B1-B1*beta2*gamma2
```

```
-A1*alpha2*delta2-delta*A1*alpha2
-B1*gamma*beta2+B1;
 Numerator(D2) in Ideal;
 D3:=alpha*delta*A1-beta*gamma*B1+delta1*alpha1*A1
-gamma1*beta1*B1
+B1*alpha2*gamma2+A1*beta2*delta2-B1*gamma*alpha2+delta*A1*beta2+B1;
 Numerator(D3) in Ideal;
 D4:=-beta*gamma*A1-alpha*delta*B1-gamma1*beta1*A1
-delta1*alpha1*B1
-A1*beta2*gamma2+B1*alpha2*delta2-B1*delta*alpha2
-gamma*A1*beta2+A1;
 Numerator(D4) in Ideal;
```

For other paths, one can switch out the code for defining various κ values as follows:

When $T = T3$:

```
k0:=(m*p*(mmp));
k2:=((mmpm1)*(mmp));
k1:=(m*(mp1));
k0pp:=((mm1)*p*(mmpm1));
k2pp:=(m*(mm1));
k0p:=((mp1)*p*(mmpp1));
k1p:=((mmp)*(mmpp1));
```

When $T = L_1$:

```
k0:=(m*p*(mmp));
```

```
k2:=(m*(mp1));

k1:=(N);

k0pp:=((mp1)*p*(mmpp1));

k2pp:=((mmp)*(mmpp1));

k0p:=k0;

k1p:=k1;
```

When $T = L_4$:

```
k0pp:=SquareRoot(m*p*(m-p));

k2pp:=SquareRoot(m*(m+1));

k1:=SquareRoot(N);

k0:=SquareRoot((m+1)*p*(m+1-p));

k2:=SquareRoot((m-p)*(m-p+1));

k0p:=k0;

k1p:=k1;
```

For other paths either $v_{s_1.T}$ or $v_{s_0 s_1.T} = 0$

5) The coefficient for $v_{s_1 s_0.T}$.

When $T = L_2$, the code is as follows:

```
C := IntegerRing();

R<m,p,mmp,mp1,n,np1,mmpp1,mmpm1,mm1>

:= PolynomialRing(C,9);

Ideal := ideal<R|m^2-1-mm1^2, m^2-1-p^2-mmpm1^2,

 m^2+1-mp1^2,  n^2+1-np1^2, m^2+1-p^2-mmpp1^2,

  m^2-p^2-mmp^2>;
```

```
N:=n*np1;
k0:=(m*p*(mmp));
k1:=(m*(mp1));
k2:=(N);
k0p:=((mp1)*p*(mmpp1));
k1p:=((mmp)*(mmpp1));
gamma:=(N^2*p^2+k0*k0)*k1/(k0*k0+p^2*k1*k1);
gammap:=(N^2*p^2+k0p*k0p)*k1p/(k0p*k0p+p^2*k1p*k1p);;
gammapp:=(N^2*p^2+k0p*k0p)*k2/(k0p*k0p+p^2*k2*k2);
delta:=(-N^2+k1*k1)*k0/(k0*k0+p^2*k1*k1);
deltap:=(-N^2+k1p*k1p)*k0p/(k0p*k0p+p^2*k1p*k1p);
deltapp:=(-N^2+k2*k2)*k0p/(k0p*k0p+p^2*k2*k2);
Ap:=k0-k0p;
Bp:=k1+k1p;
alpha:=-1/(k1-k2);
alphap:=-1/(k1p-k2);
alphapp:=-1/(k2-k1p);
beta:=1/(k1+k2);
betap:=1/(k1p+k2);
betapp:=1/(k2+k1p);
E0:=Ap*alphap*gammap-Bp*betap*deltap+Ap*gammapp*alphapp
+Bp*deltapp*betapp-alpha*Ap*gammapp-beta*Bp*deltapp
+delta*beta*Bp+gamma*alpha*Ap+Ap;
Numerator(E0) in Ideal;
E1:=Bp*alphap*gammap+Ap*betap*deltap+Bp*gammapp*alphapp
-Ap*deltapp*betapp-beta*Bp*gammapp+alpha*Ap*deltapp
```

```
+delta*alpha*Ap-gamma*beta*Bp+Bp;
Numerator(E1) in Ideal;
E2:=-Bp*betap*gammap-Ap*alphap*deltap-Ap*deltapp*alphapp
-Bp*gammapp*betapp-alpha*Bp*gammapp-beta*Ap*deltapp
-delta*beta*Ap+gamma*alpha*Bp+Bp;
Numerator(E2) in Ideal;
E3:=Ap*betap*gammap-Bp*alphap*deltap-Bp*deltapp*alphapp
+Ap*gammapp*betapp+beta*Ap*gammapp-alpha*Bp*deltapp
+gamma*beta*Ap+delta*alpha*Bp-Ap;
Numerator(E3) in Ideal;
```

When $T = T_3$, switch the values for κ_0, κ_0' and κ_1, κ_1'. When $T = T_1$, set the eigenvalues to be the following

```
k0p:=(m*p*(mmp));
k1p:=((mmpm1)*(mmp));
k2:=(m*(mp1));
k0:=((mm1)*p*(mmpm1));
k1:=((m)*(mm1));
```

When $T = T_4$, set the eigenvalues to be the following

```
k0p:=(m*p*(mmp));
k1p:=((m)*(mp1));
k2:=((mmpm1)*(mmp));
k0:=((mp1)*p*(mmpp1));
k1:=((mmp)*(mmpp1));
```

For all other tableaux, either $v_{s_0.T} = 0$ or $v_{s_1 s_0.T} = 0$.

6) The coefficient for $v_{s_0 s_1 s_0 . T}$.

When $T = T_1$:

```
C := IntegerRing();
R<m,p,mmp,mp1,n,np1,mp1,p,mmpp1,mm1mp,mm1> := PolynomialRing(C,11);
Ideal := ideal<R|m^2-1-mm1^2, m^2-1-p^2-mm1mp^2,
m^2+1-mp1^2, m^2-p^2-mmp^2, n^2+1-np1^2,
m^2+1-p^2-mmpp1^2, m^2-p^2-mmp^2>;
N:=n*np1;
k0:=((mm1)*p*(mm1mp));
k1:=(m*(mm1));
k2:=(m*(mp1));
k0p:=(m*p*(mmp));
k1p:=((mm1mp)*(mmp));
k0pp:=((mp1)*p*(mmpp1));
k2pp:=((mmp)*(mmpp1));
alphai:=-(k1-k2);
alpha3i:=-(k2pp-k1p);
betai:=(k1+k2);
beta3i:=(k2pp+k1p);
Ap:=k0-k0p;
Bp:=k1+k1p;
H:=k0p-k0pp;
I:=k2+k2pp;
E0:=alphai*alpha3i*beta3i*Ap*H-betai*alpha3i*beta3i*Bp*I
-Bp*I*alphai*betai*beta3i+Ap*H*alpha3i*alphai*betai;
```

```
E1:=-betai*alpha3i*beta3i*Bp*H-alphai*alpha3i*beta3i*Ap*I
-Ap*I*alphai*betai*beta3i-Bp*H*betai*alphai*alpha3i;
E2:=-alpha3i*beta3i*alphai*Bp*H+alpha3i*beta3i*betai*Ap*I
+Bp*H*alphai*alpha3i*beta3i-Ap*I*betai*alpha3i*beta3i;
E3:= -betai*alpha3i*beta3i*Ap*H-alphai*alpha3i*beta3i*Bp*I
+Ap*H*alphai*betai*beta3i+Bp*I*alpha3i*alphai*betai;
E0 in Ideal; E1 in Ideal; E2 in Ideal; E3 in Ideal;
```

When $T = T_4$, part of the code is modified as

```
k0pp:=((mm1)*p*(mmpm1));
k2pp:=(m*(mm1));
k1p:=(m*(mp1));
k0p:=(m*p*(mmp));
k2:=((mmpm1)*(mmp));
k0:=((mp1)*p*(mmpp1));
k1:=((mmp)*(mmpp1));
```

For all other tableaux, one of $v_{s_0.T}$, $v_{s_1 s_0.T}$, $v_{s_0 s_1 s_0.T}$ is zero.