ESTIMATION OF VARIANCE COMPONENTS
IN INCOMPLETE BLOCK DESIGNS

By

DAVID ELLIOTT BEE

Bachelor of Science
Oklahoma State University
Stillwater, Oklahoma
1961

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1962

Submitted to the Faculty of the Graduate School of
the Oklahoma State University
in partial fulfillment of the requirements
for the degree of
DOCTOR OF PHILOSOPHY
August, 1965

STATE UNIVERSITY
LIBRARY
NOV 24 1965

ESTIMATION OF VARIANCE COMPONENTS IN INCOMPLETE BLOCK DESIGNS

Thesis Approved:

David L. Weeks
Thesis Adviser

Robert Hultquist

Leo B. Oakley

Jeanne Agnew

Robert D Morrison

J. M. Boggs
Dean of the Graduate School

592691

ACKNOWLEDGMENTS

I wish to express my sincere gratitude to Dr. David L. Weeks for the suggestion of this thesis topic and for serving as chairman of my advisory committee. His comments, guidance, and especially, long patience were deeply appreciated. I also wish to express my appreciation to Dr. Robert D. Morrison, Dr. Robert A. Hultquist, Dr. Jeanne L. Agnew, and Dr. Leo V. Blakley for serving as members of my committee and to Dr. Carl E. Marshall whose advice and understanding has meant so much.

Acknowledgment is particularly due to those responsible for the National Defense Education Act Fellowships. The award of this fellowship gave the badly needed financial support for the continuance of my education.

Special thanks go to Mrs. Beverly Richardson for her diligent work in the typing and preparation of this thesis.

Last, but not least, I wish to express the deepest appreciation to my wife, Sherry, for all those things which cannot be said with words.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER I

INTRODUCTION

Experimental investigations and research problems often require a statistical analysis of the data from an incomplete block design. In these cases, the general model required is $y_{ijm} = \mu + \beta_i + \tau_j + \epsilon_{ijm}$, where $i = 1, \ldots, b$; $j = 1, \ldots, t$; and $m = 0, 1, \ldots, n_{ij}$. Due to the nature of the experiment, β_i, τ_j, and ϵ_{ijm} might be considered as random variables with zero means and with variances σ_1^2, σ_2^2, and σ^2 respectively. By assuming the random model, the researcher is interested in making inferences about the variance components. Estimators of the variance components are generally considered which possess the properties of being unbiased and having minimum variance.

Previously, several authors have undertaken variance component estimation in incomplete block designs based on the "method of analysis of variance." This procedure consists of equating the quadratic forms from the analysis of variance to their respective expected values and solving for the unknown parameters. This method has merit in that normality of ϵ_{ijm}, β_i, and τ_j need not be assumed and in that it is relatively simple compared to maximum likelihood procedures when distributional assumptions are made. However, there may exist estimators which are not obtained by the analysis of variance method.

From the above considerations, the problem of variance component estimation in this thesis is approached by considering an intermediate step involving the theory of minimal sufficient statistics. This "means to an end" is useful in itself due to the properties of a minimal sufficient set. That is, having found a set of minimal sufficient statistics, the experimenter has all the relevant information (based on the sample values) necessary to estimate the unknown parameters of the densities involved.

An equally important aspect of a minimal sufficient set is that if there is a minimum variance unbiased estimator, it must be a function of the statistics in the minimal set. This result has been presented in a theorem proved by Rao and Blackwell.

Thus, knowledge of a set of minimal sufficient statistics does not imply that minimum variance unbiased estimators can be found. In fact, if a parameter has more than one unbiased estimator which are functions of the statistics in a minimal set, an estimator with minimum variance may not exist. In this case the joint distribution of the set of minimal sufficient statistics is not complete.

Sets of minimal sufficient statistics for the balanced incomplete block design and for a general class of designs have been given by Weeks [3] and Weeks and Graybill [4] respectively. In both cases, the joint distribution of the minimal sufficient set is not complete since more than one unbiased estimator of each variance component exist. From these considerations, any thorough search for minimum variance unbiased estimators from these two sets will require the true variances of the estimators.

Since the statistics in each of the two minimal sets discussed above, are not all independent, one purpose of this thesis is to find both joint distributions in order to consider variances of estimators which might be functions of the dependent statistics of the minimal sets.

The statistics of the minimal set for the balanced incomplete block design are considered in detail. For this case certain estimators of the individual variance components were chosen because of their simplicity and their variances compared. There are certain conditions which are dictated by the choice of the particular design for which estimators may possess the property of minimum variance with respect to other chosen estimators. In general, however, not so clear a choice is available. In these instances the magnitudes of the variances of estimators are a function of the ratios of the true parameters (variance components) being estimated and the particular choice of design.

In view of the preceding discussion, the utility of this thesis could depend strictly upon an experimenter's knowledge of the ratios of the true variance components. This is, in fact, the case for some situations in genetics.

For those who may have some "a priori" information on the ratios of population variances, tables and graphs are given which indicate the estimator of smaller variance of the estimators considered. These tables are based on thirty balanced incomplete block designs for which the number of blocks is greater than the number of treatments. This restriction insures the existence of six statistics in the minimal sufficient set. The thirty designs were chosen for consideration because of the likelihood of their use in practical situations.

CHAPTER II

NOTATION, BASIC ASSUMPTIONS, AND LEMMAS

For the most part the notation used in this thesis will be the same as used by Weeks [3] and Weeks and Graybill [4] in their presentations of the ensuing subject matter. However, there will be some deviations from this policy in order to facilitate the differentiation between seemingly analogous situations in the balanced incomplete block and the general two-way classification.

Scalar values used in the general two-way classification model will now be defined.

1. b is the number of blocks in a design.
2. t is the number of treatments in a design.
3. r is the number of replications of a treatment.
4. k is the number of experimental units per block.
5. λ is the number of times any two treatments appear together in all blocks in a balanced incomplete block design.
6. M is the total number of observations in an experiment.
7. $\bar{y}_{\cdot\cdot\cdot}$ denotes the overall mean of the observations in a two-way classification model.
8. d_i denotes a distinct positive characteristic root of a matrix.
9. m_i denotes the multiplicity of the characteristic root d_i.

10. s denotes the number of distinct positive characteristic roots of a matrix.

11. BIB is an abbreviation for balanced incomplete block.

12. MVN is an abbreviation for multivariate normal.

13. μ is a scalar constant denoting a population mean.

The basic matrices used in this thesis with their respective dimensions will now be given.

1. $Y(M \times 1)$ is a vector of observations.

2. $X(M \times b + t + 1)$ is a design matrix for the two-way classification model.

3. J_p^q is a $(q \times p)$ matrix of all ones.

4. Σ denotes a covariance matrix.

5. ϕ will denote the null matrix. A subscript will be attached denoting the dimension of ϕ if the dimension is of importance.

6. I_p denotes a $(p \times p)$ identity matrix.

7. $X = [J_1^M, X_1, X_2]$ is a partition of X where $X_1(M \times b)$ corresponds to blocks and $X_2(M \times t)$ corresponds to treatments. The elements of X_1 and X_2 are either 0 of 1 since in this thesis the experimental design model is assumed.

8. $\bar{\mu} = \mu J_1^M$ denotes $E(Y)$ where E denotes mathematical expectation.

9. $N(t \times b)$ is the incidence matrix of the two-way cross classification design where $N = X_2' X_1$.

10. $A = X_2 - k^{-1} X_1 N'$ and has dimension $M \times t$.

11. D_B denotes a diagonal matrix with the positive characteristic roots of B on the diagonal.

12. $\mathbf{P}(M \times M)$ will denote the orthogonal matrix used in the

12. (continued) orthogonal transformation of Y in the BIB design.

13. $U(M \times M)$ will denote the orthogonal matrix used in the orthogonal transformation of Y in the general class of designs.

14. $P = [M^{-1/2} J_1^M, k^{-1/2} X_1 P_{21}, k^{-1/2} X_1 P_{22}, (k/\lambda t)^{1/2} A P_{31}, P_4]$ is a partition of P where $P_{21}(b \times b - t)$, $P_{22}(b \times t-1)$, $P_{31}(t \times t - 1)$, and $P_4(M \times M - b - t + 1)$ are partitions of other orthogonal matrices as defined by Weeks [3]. Weeks denotes the matrix P_{31} as P_3.

15. $U = [M^{-1/2} J_1^M, k^{-1/2} X_1 P_{21}^*, k^{-1/2} X_1 P_2, A P_3 D^{-1/2}_{A'A}, P_4]$ is a partition of U where $P_{21}^*(b \times m_1 + b - t)$, $P_2(b \times t - 1 - m_1)$, $P_3(t \times t - 1)$, and $P_4(M \times M - b - t + 1)$ are partitions of other orthogonal matrices. m_1 denotes the multiplicity of the characteristic root $d_1 = r$ of $A'A$. P_{21}^*, P_2, and P_3 will be considered in more detail.

16. $P_2 = [P_{22}, P_{23}, \ldots, P_{2s}]$ is a partition of P_2 where the dimension of P_{2i} is $b \times m_i$ for $i = 2, \ldots, s$.

17. $P_2' N'NP_2 = D_{NN'}(t-1-m_1 \times t-1 - m_1)$.

18. $P_{21}^{*\prime} N'NP_{21}^* = \phi_{m_1+b-t}$.

19. $P_3 = [P_{31}^*, \tilde{P}_3]$ is a partition of P_3 where P_{31}^* is $t \times m_1$ and $\tilde{P}_3$ is $t \times t - 1 - m_1$.

20. $P_3' A'A P_3 = D_{A'A}$.

21. $\tilde{P}_3' NN' \tilde{P}_3 = D_{NN'}$.

22. $\tilde{P}_3 = [P_{32}, P_{33}, \ldots, P_{3s}]$ is a partition of $\tilde{P}_3$ where P_{3i} is $t \times m_i$ for $i = 2, \ldots, s$.

23. $P_{3i}' NN'P_{3i} = k(r-d_i) I_{m_i}$ for $i = 2, \ldots, s$.

24. $P_{3i}' A'A P_{3i} = d_i I_{m_i}$ for $i = 2, \ldots, s$.

25. $P_{31}^{*}{}' A'A P_{31}^{*} = d_1 I_{m_1} = r I_{m_1}$.

26. $P_{31}^{*} NN' P_{31}^{*} = \phi_{m_1}$.

Distributional Properties, Assumptions, and Other Relationships

The two-way classification model $Y = X\gamma + \epsilon$ is assumed where X is as previously defined and where $\gamma' = [\mu, \beta', \tau']$ is a $1 \times b+t+1$ row vector. β' is $1 \times b$; τ' is $1 \times t$, and μ is a scalar constant. It will be assumed that $b > t$ for the BIB designs under consideration. Under an Eisenhart Model II the following distributional properties will be made:

1. $\epsilon \sim MVN(\phi, \sigma^2 I_M)$,
2. $Y \sim MVN(\bar{\mu}. \not{\Sigma})$,
3. $\beta \sim MVN(\phi, \sigma_1^2 I_b)$,
4. $\tau \sim MVN(\phi, \sigma_2^2 I_t)$,
5. $cov(\epsilon, \beta) = \phi$, $cov(\epsilon, \tau) = \phi$, $cov(\beta, \tau) = \phi$,
6. $P'Y \sim MVN(P'\bar{\mu}, P'\not{\Sigma}P)$ where P is as previously defined,
7. $U'Y \sim MVN(U'\bar{\mu}, U'\not{\Sigma}U)$ where U is as previously defined.

Certain relationships will prove useful in the following chapters. These are given as follows:

1. $X_1'X_1 = k I_b$ and $X_2'X_2 = r I_t$,
2. $J_M^q X_1 = k J_b^q$ and $J_M^q X_2 = r J_t^q$,
3. $J_b^1 X_1' = J_M^1$ and $J_t^1 X_2' = J_M^1$.

4. $X_2'A = A'A$,

5. $A'X_1 = \phi; J_t^1 A' = \phi$, and $J_M^1 A = \phi$,

6. $P_4'X_1 = \phi$ and $P_4'X_2 = \phi$,

7. $E(Y - \overline{\mu})(Y - \overline{\mu})' = \Sigma = [\sigma_1^2 X_1 X_1' + \sigma_2^2 X_2 X_2' + \sigma^2 I_M]$,

8. $M = kb = rt$,

9. $u = M - b - t + 1$,

10. $w = b - t$,

11. $rk - \lambda t = r - \lambda$ and $\lambda = \frac{r(k-1)}{t-1}$ for the BIB,

12. $b > t > k > 1$.

Lemmas

The derivations presented in Chapter IV require the use of certain lemmas. It should be noted that the seven lemmas presented in this chapter are all concerned with the BIB design with b, r, k, t, and λ as previously defined. These lemmas with their proofs will now be presented.

Lemma 1. For a BIB design, $M - b = \frac{\lambda t}{k}(t-1)$.

Proof: $M - b = kb - b = b(k-1)$. But $\lambda = \frac{r(k-1)}{t-1}$. Hence, $k-1 = \frac{\lambda(t-1)}{r}$ which, upon substitution, gives $M - b = \frac{b\lambda}{r}(t-1)$. But $\frac{b}{r} = \frac{t}{k}$ since $kb = rt$. Thus, $M - b = \frac{\lambda t}{k}(t-1)$ which was to be proved.

Lemma 2. For a BIB design, $M - b - t + 1 = \frac{t-1}{k}(\lambda t - k)$.

Proof: By Lemma 1, $M - b - (t-1) = \frac{\lambda t}{k}(t-1) - (t-1)$. Factoring $\frac{(t-1)}{k}$ we have $u = \frac{t-1}{k}[\lambda t - k]$ where u denotes $M - b - t + 1$ and the proof is complete.

<u>Lemma 3.</u> For a BIB design, $\lambda t > r - \lambda$.

<u>Proof:</u> Assume $\lambda t \leq r - \lambda$. Then $t \leq \frac{r}{\lambda} - 1$. From the relation $\lambda = \frac{r(k-1)}{t-1}$ we have $\frac{r}{\lambda} = \frac{t-1}{k-1}$. Hence, $t \leq \frac{t-1}{k-1} - 1$ which implies $t < \frac{t-1}{k-1}$ or $t(k-1) < t-1$. Therefore, $t(k-1) < t$. But if $t(k-1) < t$ then $k - 1 < 1$ or, $k < 2$. Since k is a positive integer, $k < 2$ implies $k = 1$. Hence, we have a contradiction since $k = 1$ implies $\lambda = \frac{r(k-1)}{t-1} = 0$. This completes the proof.

<u>Lemma 4.</u> For a BIB design in which $b > t$, $u = M - b - t + 1 \geq t - 1$.

<u>Proof:</u> Show $u < t - 1$ leads to a contradiction. By Lemma 1, $u = \frac{t-1}{k}(\lambda t - k)$. Hence, assuming $u < t - 1$, we have that $\frac{\lambda t - k}{k} < 1$ or, $\lambda t < 2k$. Now, if $\lambda \geq 2$ we have a contradiction since $t > k$. Therefore, assume $\lambda < 2$. This implies $\lambda = 1$ since $\lambda \neq 0$. Thus, if $\lambda = 1$, $t < k + k$. Subtracting 1 from both sides of the inequality gives $(t-1) < k + (k - 1)$. Since $k > 1$, multiply both sides by $\frac{1}{k-1}$ to obtain $\frac{t-1}{k-1} < \frac{k}{k-1} + 1$. But $\frac{t-1}{k-1} = r$ since $\lambda = 1$. Hence, $r < \frac{k}{k-1} + 1$. Since k is a positive integer and $k \geq 2$, the function $f(k) = \frac{k}{k-1} + 1$ attains its maximum at $k = 2$. Hence, $r < f(2) = 3$ which implies that $r \leq 2$. If $r = 1$, then $t = kb$. But, $b > t$ implies $k < 1$ which is a contradiction. If $r = 2$, then $2t = kb$. This implies $k < 2$ since $b > t$. But $k \geq 2$. Hence, we have a contradiction and the proof is complete.

<u>Lemma 5.</u> For a BIB design $b \neq 2t - 1$.

<u>Proof:</u> Assume $b = 2t - 1$. But $b = \frac{rt}{k}$. Hence, $\frac{rt}{k} = 2t - 1$ or, $rt = 2kt - k$. Rearranging, we have $k = 2kt - rt = t(2k-r)$. But

$t > k$. Hence, the integer $2k - r$ is less than 1 which implies that either (i) $2k-r = 0$, or, (ii) $2k-r < 0$. If condition (i) is true, then $k = t \cdot 0 = 0$ and this is a contradiction. If condition (ii) is true, then k is negative which again is a contradiction. Hence, the lemma is proved.

Lemma 6. If $b > t$ in a BIB design, then $b \neq \frac{r}{\lambda}\left[\frac{u+w}{w} - \frac{u(r-\lambda)}{2\lambda t(t-1)}\right]$ where $u = M - b - t + 1$ and $w = b - t$.

Proof: Assume $b = \frac{r}{\lambda}\left[\frac{u+w}{w} - \frac{u(r-\lambda)}{2\lambda t(t-1)}\right]$. Solving for the last term in the brackets, we have $\frac{u(r-\lambda)}{2\lambda t(t-1)} = \frac{u+w}{w} - \frac{\lambda b}{r} = \frac{u}{w} + 1 - \frac{\lambda t}{k}$. By Lemma 2,

$$u = \frac{t-1}{k}(\lambda t - k)$$

or,

$$\frac{u}{t-1} = \left(\frac{\lambda t}{k} - 1\right).$$

Hence,

$$\frac{u(r-\lambda)}{2\lambda t(t-1)} = \frac{u}{w} - \frac{u}{t-1}.$$

Multiplying both sides by $\frac{t-1}{u}$, we have

$$\frac{r-\lambda}{2\lambda t} = \frac{t-1}{w} - 1 = \frac{t-1-w}{w} = \frac{2t-b-1}{b-t}.$$

Expanding

$$\frac{r-\lambda}{2\lambda t} = \frac{2t-b-1}{b-t},$$

the equation

$$b[k + \lambda - 2\lambda t - r] = \lambda t[3 - 4t]$$

is obtained.

But, $b > t$. Hence, $k + \lambda - 2\lambda t - r < \lambda(3-4t)$ and this inequality reduces to $2\lambda(t-1) < r - k$. Since $\lambda(t-1) = r(k-1)$ for the BIB design, we have that $2r(k-1) < r - k$ or, $r(2k - 3) + k < 0$. The only case when $r(2k - 3) + k < 0$ could hold is for $k = 1$. But $k \neq 1$. Hence, we have a contradiction and the lemma is proved.

Lemma 7. If $b > t$ in a BIB design, then $b > \frac{r(r-1)}{3\lambda} + 1$.

Proof: Assume

$$b \leq \frac{r(r-1)}{3\lambda} + 1.$$

Then

$$3\lambda b \leq r(r-1) + 3\lambda$$

or,

$$3\lambda(b-1) \leq r(r-1).$$

But,

$$\lambda = \frac{r(k-1)}{t-1} .$$

Having substituted for λ, algebraic manipulation gives

$$3kb - rt + r + t \leq 3k + 3b - 2.$$

Replacing kb by rt, we have

$$2\,rt + (r + t) \leq 3(k + b) - 2.$$

Now,

$$r + t = rt\left(\frac{1}{t} + \frac{1}{r}\right)$$

and

$$k + b = \frac{rt}{b} + \frac{rt}{k} = rt\left(\frac{1}{b} + \frac{1}{k}\right).$$

Substitution for $(r + t)$ and $(k + b)$ in the last inequality gives

$$2 + \left(\frac{1}{t} + \frac{1}{r}\right) \leq 3\left(\frac{1}{b} + \frac{1}{k}\right) - \frac{2}{rt} .$$

In a general BIB design for which $b > t$ the minimum block size is six. This restriction will give the desired contradiction. To see that the restriction on block size is true, we have in the general case that $k \neq 1$ since $k = 1$ implies that $\lambda = 0$. Hence, $k \geq 2$. Also, under the assumption that $b > t$ we have $r > k$. Thus, $r \geq 3$ and $t \geq 3$ since $t > k$. This in turn implies $b \geq 4$. However, the minimum values of these constants do not conform to the condition $kb = rt$ of the BIB.

If b is increased to 5 and r and k are held fixed, the relation $kb = rt$ is still not satisfied for $t = 3$ or $t = 4$. However, if $b = 6$, $t = 4$, $k = 2$, and $r = 3$, a BIB is defined. Thus the minimum block size for a general BIB when $b > t$ is $b = 6$.

From these considerations, the maximum value of $(\frac{1}{b} + \frac{1}{k})$ is $(\frac{1}{6} + \frac{1}{2}) = \frac{2}{3}$. Thus, $3(\frac{1}{b} + \frac{1}{k}) - \frac{2}{rt} \leq 3 \cdot \frac{2}{3} - \frac{2}{rt} = 2 - \frac{2}{rt}$. Hence, $2 + (\frac{1}{t} + \frac{1}{r}) \leq 2 - \frac{2}{rt}$ which is a contradiction. This completes the proof of the lemma.

CHAPTER III

THE JOINT DISTRIBUTION OF A SET OF MINIMAL SUFFICIENT STATISTICS FOR THE BIB DESIGN

General Discussion

The purpose of this chapter is to derive the joint distribution of a given set of minimal sufficient statistics for the BIB design. Since minimum variance unbiased estimators of the variance components σ^2, σ_1^2, and σ_2^2 must be based on functions of the statistics from the minimal set, the joint distribution of the statistics of the set should be found. The marginal distribution of each statistic in the set has previously been found, but certain dependencies exist among the statistics.

The derivation of the set of minimal sufficient statistics is based upon the matrix model $Y = X\gamma + \epsilon$ (defined in Chapter II) under the assumption of an Eisenhart Model II. By invoking the restriction that the number of blocks is greater than the number of treatments, the minimal set will contain six statistics. Throughout this thesis, only BIB designs in which this restriction holds will be considered.

A Set of Minimal Sufficient Statistics for the BIB Design

The six statistics of the minimal set and their individual distributions as given by Weeks [3] are as follows:

1. $s_1 = M^{-1} J_M^1 Y = \bar{y} \ldots$.

$s_1 \sim N[\mu, M^{-1}(\sigma^2 + k\sigma_1^2 + r\sigma_2^2)]$.

2. $s_2 = k^{-1} Y' X_1 P_{21} P_{21}' X_1' Y$.

$s_2 \sim (\sigma^2 + k\sigma_1^2) \chi^2(w)$, where $w = b - t$.

3. $s_3 = [k(r-\lambda)]^{-1} Y' X_1 N' P_3 P_3' N X_1' Y$.

$s_3 \sim (\sigma^2 + k\sigma_1^2 + k^{-1}(r-\lambda)\sigma_2^2) \chi^2(t-1)$.

4. $s_4 = k^{-1}(\lambda t)^{1/2} Y' X_1 N' P_3 P_3' A' Y$.

$s_4 \sim \Sigma p_i \chi^2(1)$ where the p_i are the non-zero characteristic roots of $2^{-1}(A_4 + A_4') \not{\Sigma}$ and where $A_4 = k^{-1} X_1 N' P_3 P_3' A'$.

5. $s_5 = k(\lambda t)^{-1} Y' A P_3 P_3' A' Y$.

$s_5 \sim (\sigma^2 + k^{-1}\lambda t \sigma_2^2) \chi^2(t-1)$.

6. $s_6 = Y' P_4 P_4' Y$.

$s_6 \sim \sigma^2 \chi^2(u)$, where $u = M - b - t + 1$.

Weeks [3] has also shown that the statistics s_i ($i = 1, 2, \ldots, 6$) are pairwise independent except for the pairs (s_3, s_4), (s_3, s_5), and (s_4, s_5). Hence, due to independence properties of the minimal set, the problem of finding the joint distribution of the s_i ($i = 1, \ldots, 6$) reduces to that of finding the joint distribution of s_3, s_4, and s_5.

Before proceeding to the derivation of the joint distribution of the statistics s_3, s_4, and s_5 the matrix notation used in defining the six statistics will be simplified by expressing them in terms of a Z matrix which will be defined.

As was noted in Chapter II, the $M \times 1$ vector $P'Y$ has a multivariate normal distribution with mean $P'\bar{\mu}$ and covariance matrix $P' \not{\Sigma} P$.

That is,

$$P'Y \sim MVN\,[P'\bar{\mu}\,,\; P'\Sigma\; P].$$

To simplify notation let $P'Y = Z$ and partition Z as

$$Z' = [\, Z_1',\, Z_2',\, Z_3', Z_4', Z_5'\,]$$

to correspond to the partitioning of P as explained in Chapter II. Now, expressing $P'Y$ in terms of Z we have

$$P'Y = \begin{bmatrix} M^{-1/2}\, J_M^1 Y \\ k^{-1/2} P_{21}'\, X_1' Y \\ [\,k(r-\lambda)\,]^{-1/2} P_3' N X_1' Y \\ (k/\lambda\, t)^{1/2}\, P_3' A' Y \\ P_4'\, Y \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \end{bmatrix} = Z\,,$$

where the diminsions of Z_1, Z_2, Z_3, Z_4, and Z_5 are 1×1, $w \times 1$, $t-1 \times 1$, $t-1 \times 1$, and $u \times 1$ respectively.

From the way the six statistics are defined, we can now express them in terms of the Z vector as follows:

1. $Z_1'Z_1 = M\, s_1^2.$
2. $Z_2'Z_2 = s_2.$
3. $Z_3'\, Z_3 = s_3.$
4. $Z_4'\, Z_4 = s_5.$
5. $Z_5'\, Z_5 = s_6.$
6. $Z_3' Z_4 = k_o\, s_4,$ where $k_o = (k/\lambda\, t)(r-\lambda)^{-1/2}.$

The occurence of the constant k_o in $Z_3'\, Z_4$ is due to the definition of s_4. To see this relationship, we have

$$Z_3' Z_4 = [k(r-\lambda)]^{-1/2}(k/\lambda t)^{1/2} Y'X_1N'P_3P_3'A'Y$$

$$= (\frac{k}{\lambda t}) (r - \lambda)^{-1/2}[\frac{(\lambda t)}{k}^{1/2} Y'X_1N'P_3P_3'A'Y]$$

$$= k_o\, s_4.$$

The Joint Distribution of s_3, s_4, and s_5

The derivation of the joint distribution of the statistics s_3, s_4, and s_5 requires that the form of $E(Z)$ and $cov(Z)$ be known. The $E(Z)$ will be considered first.

Now, $E(Z) = E(P'Y) = P'\overline{\mu}$. But the orthogonal matrix P has been so constructed as to make every element of the first column equal. [3] This first column(by the partitioning indicated in Chapter II) has the form $M^{-1/2} J_1^M$. Since P is orthogonal, the form of the first column insures that the elements of any other column add to zero. Also, $\overline{\mu} = \mu J_1^M$. Therefore, by partitioning P as $P = [M^{-1/2} J_1^M, P^*]$, where P^* has dimension $M \times M-1$, we have that

$$E(Z') = [EY'][M^{-1/2}J_1^M , P^*]$$

$$= [\mu J_M^1][M^{-1/2}J_1^M , P^*]$$

$$= [\mu M^{1/2}, \phi], \text{ where } \phi \text{ has dimension } 1 \times M-1.$$

Hence, $E(Z_1) = \mu M^{1/2}$ and $E(Z_i) = \phi$ for $i = 2, \ldots, 5$.

The covariance matrix of Z as given by Weeks [3] is

$$P' \not\Sigma P = \begin{bmatrix} B_1 & \phi & \phi & \phi & \phi \\ \phi & B_2 & \phi & \phi & \phi \\ \phi & \phi & B_3 & B_{34} & \phi \\ \phi & \phi & B_{34} & B_4 & \phi \\ \phi & \phi & \phi & \phi & B_5 \end{bmatrix},$$

where

$$B_1 = (\sigma^2 + k\sigma_1^2 + r\sigma_2^2),$$

$$B_2 = (\sigma^2 + k\sigma_1^2)\, I_w,$$

$$B_3 = [\sigma^2 + k\sigma_1^2 + k^{-1}(r - \lambda)\,\sigma_2^2]\, I_{t-1},$$

$$B_4 = (\sigma^2 + (\lambda\, t/k)\,\sigma_2^2)\, I_{t-1},$$

$$B_5 = \sigma^2 I_u,$$

and

$$B_{34} = [k^{-2}\lambda t(r-\lambda)]^{1/2}\,\sigma_2^2\, I_{t-1}\,.$$

From the form of $P' \not\Sigma P$, we see that the only dependency which exists among the Z_i is between Z_3 and Z_4. Hence, from multivariate normal theory, we may state that

$$\begin{bmatrix} Z_3 \\ Z_4 \end{bmatrix} \sim MVN\left(\begin{bmatrix} \phi \\ \phi \end{bmatrix} , \begin{bmatrix} B_3 & B_{34} \\ B_{34} & B_4 \end{bmatrix} \right).$$

Let the covariance matrix of the vector $\begin{bmatrix} Z_3 \\ Z_4 \end{bmatrix}$ be denoted by $\not\Sigma_{34}$. Then, rewriting B_3, B_4, and B_{34} in terms of constants times identity matrices (as previously defined), we have

$$\not\Sigma_{34} = \begin{bmatrix} cI_{t-1} & eI_{t-1} \\ e\,I_{t-1} & dI_{t-1} \end{bmatrix},$$

where

$$c = [\sigma^2 + k\sigma_1^2 + k^{-1}(r-\lambda)\,\sigma_2^2],$$

$$d = \sigma^2 + (\lambda t/k)\,\sigma_2^2,$$

and

$$e = [k^{-2}\lambda\, t(r-\lambda)\,]^{1/2}\,\sigma_2^2\,.$$

Now denote the $t - 1$ elements of Z_3 and Z_4 by $z_{3\alpha}$ and $z_{4\alpha}$ respectively, ($\alpha = 1, 2, \ldots, t-1$), and denote the sub-subvector $\begin{bmatrix} z_{3\alpha} \\ z_{4\alpha} \end{bmatrix}$ by Q_α. From the form of Σ_{34} we see that

$$\text{cov}\,(Q_\alpha, Q_{\alpha'}) = \phi \quad \text{for } \alpha \neq \alpha'$$

and

$$\text{cov}\,(Q_\alpha, Q_{\alpha'}) = \begin{bmatrix} c & e \\ e & d \end{bmatrix} \text{for } \alpha = \alpha'.$$

Therefore,

$$Q_\alpha \sim \text{BVN}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} c & e \\ e & d \end{bmatrix} \right) \text{for } \alpha = 1, \ldots, t-1.$$

Denote the covariance matrix of Q_α by Σ^*.

We can now quote a theorem as given by Anderson [1].

Suppose the p-component vectors $Z_1, \ldots, Z_n (n \geq p)$ are independent, each distributed according to $N(\phi, \Sigma)$. Then the density of $A = \sum_{\alpha=1}^{n} Z_\alpha Z_{\alpha'}$ is

$$\frac{|A|^{\frac{1}{2}(n-p-1)} \; e^{-\frac{1}{2} \text{tr} A \Sigma^{-1}}}{2^{\frac{1}{2}np} \; \pi^{p(p-1)/4} \; |\Sigma|^{\frac{1}{2}n} \prod_{i=1}^{p} \Gamma[\frac{1}{2}(n+1-i)]}$$

for A positive definite and 0 otherwise.

In other terminology, we say that A is distributed as the Wishart with parameters n and Σ. Applying this theorem to the Q_α vectors we have that $\sum_{\alpha=1}^{t-1} Q_\alpha Q_{\alpha'} = A_1$ (say) has the Wishart distribution with parameters $t - 1$ and Σ^*.

Expressing A_1 in terms of the Q_α vectors, we have

$$A_1 = \sum_{\alpha=1}^{t-1} Q_\alpha Q_\alpha' = \sum_{\alpha=1}^{t-1} \begin{bmatrix} z_{3\alpha} \\ z_{4\alpha} \end{bmatrix} \begin{bmatrix} z_{3\alpha}, & Z_{4\alpha} \end{bmatrix}$$

$$= \sum_{\alpha=1}^{t-1} \begin{bmatrix} z_{3\alpha}^2 & z_{3\alpha} z_{4\alpha} \\ z_{4\alpha} z_{3\alpha} & z_{4\alpha}^2 \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma\, z_{3\alpha}^2 & \Sigma z_{3\alpha} z_{4\alpha} \\ \Sigma\, z_{4\alpha} z_{3\alpha} & \Sigma z_{4\alpha}^2 \end{bmatrix}$$

$$= \begin{bmatrix} Z_3' Z_3 & Z_3' Z_4 \\ Z_4' Z_3 & Z_4' Z_4 \end{bmatrix}$$

$$= \begin{bmatrix} s_3 & k_o s_4 \\ k_o s_4 & s_5 \end{bmatrix} .$$

Hence, the elements of A_1 are functions of the statistics whose joint distribution is desired. We then have the result that

$$A_1 = \begin{bmatrix} s_3 & k_o s_4 \\ k_o s_4 & s_5 \end{bmatrix} \sim W(\Sigma^*, \ t-1)$$

or equivalently,

$$h_1(s_3, \ s_4, \ s_5) = \frac{|A_1|^{\frac{t-4}{2}} \ e^{-\frac{1}{2} \operatorname{tr} A_1 \Sigma^{*-1}}}{2^{t-1} \ \pi^{1/2} \ |\Sigma^*|^{(t-1)/2} \Gamma(\frac{t-1}{2}) \Gamma(\frac{t-2}{2})} .$$

This function is easily simplified by finding $|A_1|$, $|\Sigma^*|$, and $\operatorname{tr} A_1 \Sigma^{*-1}$. After some algebraic manipulation $\operatorname{tr} A_1 \Sigma^{*-1}$ was found to be

$$\text{tr } A_1 \Sigma^{*-1} = a^{-1}[s_3(\sigma^2 + (\lambda t/k)\sigma_2^{\,2}) + s_5(\sigma^2 + k\sigma_1^{\,2} + \frac{r-\lambda}{k}\sigma_2^{\,2}) - 2\, s_4(\lambda t)^{-1/2}\, \sigma_2^{\,2}] ,$$

where

$$a = |\Sigma^*| = \sigma^4 + r\,\sigma^2\sigma_2^{\,2} + k\,\sigma^2\sigma_1^{\,2} + \lambda\, t\sigma_1^{\,2}\sigma_2^{\,2} .$$

The joint distribution of s_3, $k_o s_4$, and s_5 is then

$$h_1(s_3, s_4, s_5) = \frac{(s_3 s_5 - k_o^2 s_4^2)^{\frac{t-4}{2}} \exp -\frac{1}{2} a^{-1}[s_3(\sigma^2 + \frac{\lambda t}{k}\sigma_2^{\,2}) + s_5(\sigma^2 + k\sigma_1^2 + \frac{r-\lambda}{k}\sigma_2^2) - 2 s_4(\lambda t)^{-\frac{1}{2}}\sigma_2^{\,2}]}{2^{t-1}\ \pi^{\frac{1}{2}}\ a^{\frac{t-1}{2}}\ \Gamma(\frac{t-1}{2})\ \Gamma(\frac{t-2}{2})}$$

The Joint Distribution of s_1, s_2, and s_6

As was stated previously, the statistics s_1, s_2, and s_6 are mutually independent. [3] Therefore, the joint distribution (denoted by h_2) of these three statistics can be expressed as

$$h_2(s_1,\ s_2,\ s_6) = g_1(s_1)g_2(s_2)g_6(s_6)$$

where g_1, g_2, and g_6 denote the functional forms of the three distributions. The density functions of s_1, s_2, and s_6 are as follows:

1. $s_1 \sim N[\mu,\ M^{-1}(\sigma^2 + k\sigma_1^{\,2} + r\sigma_2^{\,2})]$.

$$g_1(s_1) = [M^{\frac{1}{2}}/(2\pi)^{\frac{1}{2}}(\sigma^2 + k\sigma_1^{\,2} + r\sigma_2^{\,2})] \exp - \frac{M(s_1 - \mu)^2}{2(\sigma^2 + k\sigma_1^2 + r\sigma_2^2)} .$$

2. $s_6 \sim \sigma^2\chi^2(u)$ or, $\sigma^{-2}s_6 \sim \chi^2(u)$.

$$g_6(s_6) = [(\sigma^{-2}s_6)^{\frac{u}{2}-1} \exp -\frac{1}{2}\,\sigma^{-2}s_6]/\Gamma(\tfrac{u}{2})\ 2^{\frac{u}{2}} ,$$

where $u = M - b - t + 1$.

3. $s_2 \sim (\sigma^2+k\sigma_1^2)\chi^2(w)$ or, $(\sigma^2+k\sigma_1^2)^{-1}s_2 \sim \chi^2(w)$.

$$g_2(s_2)=\left\{[(\sigma^2+k\sigma_1^2)^{-1}s_2]^{\frac{w}{2}-1}\exp -\tfrac{1}{2}(\sigma^2+k\sigma_1^2)^{-1}s_2\right\}/\Gamma(\tfrac{w}{2})\, 2^{\frac{w}{2}},$$

where $w = b - t$.

Now letting $\xi_1 = \sigma^2 + k\sigma_1^2$ and $\xi_2 = \sigma^2 + k\sigma_1^2 + r\sigma_2^2$ we have that

$$h_2(s_1, s_2, s_6) = g_1(s_1)g_2(s_2)g_6(s_6)$$

$$= \frac{M^{\frac{1}{2}}(\sigma^{-2}s_6)^{\frac{u}{2}-1}(\xi_1^{-1}s_2)^{\frac{w}{2}-1}\exp -\frac{1}{2}[\xi_2^{-1}M(s_1-\mu)^2+\sigma^{-2}s_6+\xi_1^{-1}s_2]}{(\pi\,\xi_2)^{\frac{1}{2}}\,2^{\frac{u+w+1}{2}}\,\Gamma(\frac{u}{2})\,\Gamma(\frac{w}{2})}.$$

The Joint Distribution of s_1, s_2, s_3, s_4, s_5, and s_6

Due to the independence of the sets (s_1, s_2, s_6) and (s_3, s_4, s_5), the joint distribution of the set of minimal sufficient statistics (denoted by h) is simply the product of the densities h_1 and h_2 which are as previously defined.

Therefore,

$$h(s_1, s_2, s_3, s_4, s_5, s_6) = h_1(s_1, s_2, s_6)h_2(s_3, s_4, s_5).$$

$$= g_1(s_1)g_2(s_2)g_6(s_6)h_2(s_3, s_4 s_5)$$

and the joint distribution of the set of minimal sufficient statistics is the product of a normal, two independent chi-squares, and a Wishart.

Actually the importance if this joint distribution is in the form of $h_2(s_3, s_4, s_5)$. From the result that h_2 is a Wishart, the variances and covariances of s_3, $k_o s_4$, and s_5 may be easily obtained since the moments of the elements of A_1 (i.e., s_3, $k_o s_4$, s_5) have been given in multivariate analysis theory. [1]

The expected values, variances, and covariances of the statistics in the minimal sufficient set will be investigated in detail in the following chapter.

CHAPTER IV

ESTIMATION OF σ^2, σ_1^2, AND σ_2^2 IN THE BIB DESIGN

General Discussion

The derivations of this chapter result from considering a special case (the BIB design) of the two-way classification model $y_{ijm} = \mu + \beta_i + \tau_j + \epsilon_{ijm}$ where $i = 1, \ldots, b$; $j = 1, \ldots, t$; and $m = 0, 1, \ldots, n_{ij}$. It is assumed here that β_i, τ_j, and ϵ_{ijm} are independent normal random variables with zero means and variances σ_1^2, σ_2^2, and σ^2 respectively. The complete distributional properties are discussed in Chapter II.

Under certain conditions the above model represents the BIB design. These conditions are as follows:

1. There are b blocks of k experimental units each.
2. The number of treatments t is greater than k.
3. Each treatment appears exactly r times.
4. Every pair of treatments must appear together in the same number (λ) of blocks.

For this special case, the subscript m used in the general model will take on the value 0 or 1. Expressed mathematically, we have $m = n_{ij}$ where $n_{ij} = 0$ if treatment j does not appear in block i, and $n_{ij} = 1$ if treatment j appears in block i.

Under the assumption of the BIB design, the purpose of this chapter will be to find unbiased estimators of the variance components σ^2, σ_1^2, and σ_2^2 based on the set of minimal sufficient statistics and to compare the variances of the estimators. To accomplish this, the expected values, variances, and covariances of the minimal sufficient set must be shown. As previously noted, only those BIB designs for which $b > t$ will be considered.

Expected Values of the s_i $(i = 1, 2, \ldots, 6)$

In Chapter III a minimal sufficient set containing the six statistics s_i $(i = 1, \ldots, 6)$ was given, and the 2×2 matrix $A_1 = (A_{1_{ij}})$, where

$$A_1 = \sum_{\alpha=1}^{t-1} Q_\alpha Q_\alpha' = \begin{bmatrix} s_3 & k_o s_4 \\ k_o s_4 & s_5 \end{bmatrix} ,$$

was shown to have a Wishart distribution with parameters $t - 1$ and

$$\not\Sigma^* = \begin{bmatrix} c & e \\ e & d \end{bmatrix} .$$

The values of c, d, and e are as follows:

$$c = [\sigma^2 + k\sigma_1^2 + k^{-1}(r-\lambda)\sigma_2^2] ,$$

$$d = [\sigma^2 + k^{-1}\lambda t \sigma_2^2],$$

and

$$e = [\lambda t (r-\lambda)/k^2]^{1/2} \sigma_2^2 .$$

Knowing this, the expected values of the elements of A_1 may be found using the fact that $E(A_{1_{ij}}) = (t-1)\sigma_{ij}$ where σ_{ij} is the i, j-th element of $\not\Sigma^*$. [1]

Thus we have

$$E(s_3) = E(A_{1_{11}}) = (t-1)c = (t-1)[\sigma^2 + k\sigma_1^2 + k^{-1}(r-\lambda)\sigma_2^2],$$

$$E(s_5) = E(A_{1_{22}}) = (t-1)d = (t-1)[\sigma^2 + k^{-1}\lambda t\sigma_2^2],$$

and

$$E(k_o s_4) = E(A_{1_{12}}) = (t-1)[\lambda t(r-\lambda)/k^2]^{\frac{1}{2}}\sigma_2^2.$$

From the last equation

$$E(s_4) = k_o^{-1}(t-1)[\lambda t(r-\lambda)/k^2]^{\frac{1}{2}}\sigma_2^2$$

or,

$$E(s_4) = (t-1)(\lambda t)^{\frac{3}{2}}(r-\lambda)\, k^{-2}\sigma_2^2$$

since

$$k_o = (k/\lambda t)(r-\lambda)^{-\frac{1}{2}}.$$

For simplicity denote the coefficient of σ_2^2 in the expression for $E(s_4)$ by f_1. Then, $E(s_4) = f_1\sigma_2^2$. It should be noted that these three expected values are identical to those obtained by Weeks [3]. He has also shown the expected values of s_1, s_2, and s_3 to be μ, $w(\sigma^2+k\sigma_1^2)$, and $u\,\sigma^2$ respectively where $w = b - t$ and $u = M - b - t + 1$.

In summary, the six expected values are presented as part of a table given after the derivation of the covariance matrix.

Covariance Matrix of the s_i $(i = 1, 2, \ldots, 6)$

Since s_1, s_2, and s_6 are mutually independent and the distribution of each is known, their variances are as follows:

1. $\text{var } s_1 = M^{-1}(\sigma^2 + k\sigma_1^2 + r\sigma_2^2)$.

2. $\text{var}\,[(\sigma^2 + k\sigma_1^2)^{-1} s_2] = 2(b-t) = 2w$

or,

$$\text{var } s_2 = 2w(\sigma^2 + k\sigma_1^{\,2})^2$$

$$= 2w[\sigma^4 + 2k\sigma^2\sigma_1^{\,2} + k^2\sigma_1^{\,4}] .$$

3. $$\text{var } (\sigma^{-2}s_6) = 2(M - b - t + 1) = 2u$$

or,

$$\text{var } s_6 = 2u\,\sigma^4 .$$

The variances of s_3, s_4, and s_5 are obtained by using the fact that Anderson [1] has given the general expression for cov (A_{ij}, A_{kl}) when $A \sim W(\Sigma, n)$. Denoting Σ by (σ_{ij}), this general expression is

$$\text{cov } (A_{ij}, A_{kl}) = E\,(A_{ij} - n\sigma_{ij})(A_{kl} - n\sigma_{kl})$$

$$= n(\sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}).$$

For our case $n = t - 1$. Therefore, the variances and covariances of the statistics s_3, s_4, and s_5 are as follows:

1. $$\text{var } s_3 = \text{var } A_{1_{11}}$$

$$= (t-1)(\sigma_{11}\sigma_{11} + \sigma_{11}\sigma_{11})$$

$$= 2(t-1)\sigma_{11}^{\,2}$$

$$= 2(t-1)c^2$$

$$= 2(t-1)[\sigma^2 + k\sigma_1^{\,2} + k^{-1}(r-\lambda)\sigma_2^{\,2}]^2.$$

2. $$\text{var } (k_o s_4) = \text{var } A_{1_{12}}$$

$$= (t-1)[\sigma_{12}^{\,2} + \sigma_{11}\sigma_{22}]$$

$$= (t-1)[e^2 + cd]$$

$$\text{var}\ (k_o s_4) = (t-1)[k^{-2}\lambda t(r-\lambda)\sigma_2^{\ 4}+(\sigma^2+k\sigma_1^{\ 2}+k^{-1}(r-\lambda)\sigma_2^{\ 2})(\sigma^2 + k^{-1}\lambda t\, \sigma_2^{\ 2})]$$

$$=(t-1)[2\lambda t(r-\lambda)\sigma_2^{\ 4}/k^2+r\sigma^2\sigma_2^{\ 2}+k\sigma^2\sigma_1^{\ 2}+\lambda t\sigma_1\sigma_2^{\ 2} + \sigma^4] .$$

But, $\text{var}\ (k_o s_4) = k_o^{\ 2}\ \text{var}\ s_4$. Therefore,

$$\text{var}\ s_4=(\lambda t)^{1/2} f_1[2\lambda t(r-\lambda)\sigma_2^4/k^2+r\sigma^2\sigma_2^2+k\sigma^2\sigma_1^{\ 2}+\lambda t\sigma_1^2\sigma_2^2+\sigma^4]$$

where

$$f_1 = k^{-2}(\lambda t)^{3/2}(r-\lambda)(t-1) .$$

3. $$\text{var}\ (s_5) = \text{var}\ A_{1_{22}}$$
$$= 2(t-1)\ \sigma_{22}^{\ 2}$$
$$= 2(t-1)d^2$$
$$= 2(t-1)[\sigma^2 + k^{-1}\lambda t\sigma_2^{\ 2}]^2$$

or,

$$\text{var}\ (s_5) = 2(t-1)(\sigma^4+2k^{-1}\lambda t\sigma^2\sigma_2^{\ 2}+ (k^{-1}\lambda t)^2\sigma_2^{\ 4}).$$

4. $$\text{cov}\ (s_3, s_5) = \text{cov}\ (A_{1_{11}}, A_{1_{22}})$$
$$= (t-1)(\sigma_{12}\,\sigma_{12} + \sigma_{12}\,\sigma_{12})$$
$$= 2(t-1)k^{-2}\lambda t\ (r-\lambda)\ \sigma_2^{\ 4} .$$

5. $$\text{cov}\ (s_3, k_o s_4) = \text{cov}\ (A_{1_{11}}, A_{1_{12}})$$
$$= (t-1)(\sigma_{11}\sigma_{12} + \sigma_{12}\,\sigma_{11})$$
$$= 2(t-1)[\sigma^2+k\sigma_1^{\ 2}+k^{-1}(r-\lambda)\sigma_2^{\ 2}][(k^{-2}\lambda t(r-\lambda))^{\frac{1}{2}}\sigma_2^{\ 2}].$$

But $\text{cov}\ (s_3, k_o s_4) = k_o\ \text{cov}\ (s_3, s_4)$. Therefore, after some algebraic manipulation, we obtain

$$\text{cov}(s_3, s_4) = 2f_1[\sigma^2\sigma_2^2 + k\sigma_1^2\sigma_2^2 + k^{-1}(r-\lambda)\sigma_2^4] .$$

6. $$\text{cov}(k_o s_4, s_5) = \text{cov}(A_{1_{12}}, A_{1_{22}})$$
$$= (t-1)(\sigma_{12}\sigma_{22} + \sigma_{12}\sigma_{22})$$
$$= 2(t-1)[k^{-2}\lambda t(r-\lambda)]^{1/2}\sigma_2^2(\sigma^2 + k^{-1}\lambda t\sigma_2^2)$$
$$= 2(t-1)k^{-1}[\lambda t(r-\lambda)]^{1/2}[\sigma^2\sigma_2^2 + k^{-1}\lambda t\sigma_2^4].$$

Since $\text{cov}(k_o s_4, s_5) = k_o \text{cov}(s_4, s_5)$, we have that

$$\text{cov}(s_4, s_5) = 2f_1[\sigma^2\sigma_2^2 + k^{-1}\lambda t\sigma_2^4].$$

To summarize the preceding derivations, the covariance matrix of the s_i $(i = 1, 2, \ldots, 6)$ is shown in Table I with the corresponding expected values.

Unbiased Estimators of σ^2, σ_1^2, and σ_2^2 and Their Variances

In looking at the expected values of the statistics in the minimal sufficient set as given in Table I, we see that an obvious unbiased estimator of σ^2 is $u^{-1}s_6$ since $E(u^{-1}s_6) = \sigma^2$. However, further investigation reveals many other unbiased estimators of the same variance component σ^2. In fact, as will be shown later, there exist an infinite number of unbiased estimators of σ^2. The same may be said of the other components σ_1^2 and σ_2^2.

Since it is the purpose of this chapter to choose unbiased estimators of the variance components and compare their variances, a systematic approach to their selection should be taken. Such a procedure will now be discussed.

TABLE I

EXPECTED VALUES AND COVARIANCE MATRIX
OF THE s_i $(i = 1, \ldots, 6)$

Statistic	Expected Value	Covariance Matrix					
s_1	μ	S_1	ϕ	ϕ	ϕ	ϕ	ϕ
s_2	$w(\sigma^2+k\sigma_1^2)$	ϕ	S_2	ϕ	ϕ	ϕ	ϕ
s_3	$(t-1)[\sigma^2+k\sigma_1^2+\frac{r-\lambda}{k}\sigma_2^2]$	ϕ	ϕ	S_3	S_{34}	S_{35}	ϕ
s_4	$f_1\sigma_2^2$	ϕ	ϕ	S_{34}	S_4	S_{45}	ϕ
s_5	$(t-1)[\sigma^2+\frac{\lambda t}{k}\sigma_2^2]$	ϕ	ϕ	S_{35}	S_{45}	S_5	ϕ
s_6	$u\sigma^2$	ϕ	ϕ	ϕ	ϕ	ϕ	S_6

$f_1 = k^{-2}(\lambda t)^{3/2}(r-\lambda)(t-1)$

$u = M - b - t + 1$

$w = b - t$

$S_1 = M^{-1}[\sigma^2 + k\sigma_1^2 + r\sigma_2^2]$

$S_2 = 2w[\sigma^4 + 2k\sigma^2\sigma_1^2 + k^2\sigma_1^4]$

$S_3 = 2(t-1)[\sigma^4+k^2\sigma_1^4+k^{-2}(r-\lambda)^2\sigma_2^4+2k\sigma^2\sigma_1^2+2k^{-1}(r-\lambda)\sigma^2\sigma_2^2+2(r-\lambda)\sigma_1^2\sigma_2^2]$

$S_{34} = 2f_1[k^{-1}(r-\lambda)\sigma_2^4 + \sigma^2\sigma_2^2 + k\sigma_1^2\sigma_2^2]$

$S_{35} = 2(t-1)k^{-2}\lambda t\,(r-\lambda)\,\sigma_2^4$

$S_4 = (\lambda t)^{1/2} f_1[\sigma^4 + 2\lambda t(r-\lambda)k^{-2}\sigma_2^4+k\sigma^2\sigma_1^2+r\sigma^2\sigma_2^2 + \lambda t\sigma_1^2\sigma_2^2]$

$S_{45} = 2f_1[(\lambda t/k)\,\sigma_2^4 + \sigma^2\sigma_2^2]$

$S_5 = 2(t-1)[\sigma^4 + (\lambda t/k)^2\,\sigma_2^4 + 2(\lambda t/k)\sigma^2\sigma_2^2]$

$S_6 = 2u\,\sigma^4$

Let

$$F = g_1 s_1 + g_2\left(\frac{s_2}{w}\right) + g_3\left(\frac{s_3}{t-1}\right) + g_4\left(\frac{s_4}{f_1}\right) + g_5\left(\frac{s_5}{t-1}\right) + g_6\left(\frac{s_6}{u}\right).$$

be a linear function of all the statistics of the minimal sufficient set where the g_i are arbitrary real numbers. Then,

$$E(F) = g_1\mu + g_2(\sigma^2 + k\sigma_1^2) + g_3(\sigma^2 + k\sigma_1^2 + \frac{r-\lambda}{k}\sigma_2^2) + g_4\sigma_2^2$$
$$+ g_5(\sigma^2 + \frac{\lambda t}{k}\sigma_2^2) + g_6\sigma^2.$$

Collecting coefficients of the variance components, we have

$$E(F) = g_1\mu + (g_2+g_3+g_5+g_6)\sigma^2 + (kg_2+kg_3)\sigma_1^2$$
$$+ [(\frac{r-\lambda}{k})g_3 + g_4 + \frac{\lambda t}{k} g_5]\sigma_2^2 .$$

We now want to find the values of g_i which will reduce $E(F)$ to that variance component which is to be estimated. This is done by setting $E(F)$ equal to the component under consideration and equating coefficients. The method described will yield a system of equations with the g_i's as unknowns. Once the system is solved, the determined g values may be substituted into the F function to obtain a linear combination of the minimal sufficient statistics whose expected value is the variance component under consideration.

Unbiased Estimators of σ^2, σ_1^2, and σ_2^2

Unbiased estimators of the variance components will now be found using the procedure described above. A separate system will be required for each of the three variance components.

Case 1: Estimation of σ^2.

Letting $E(F) = \sigma^2$ we have

$$g_1 = 0$$

$$g_2 + g_3 + g_5 + g_6 = 1$$

$$kg_2 + kg_3 = 0$$

$$\frac{r-\lambda}{k} g_3 + g_4 + \frac{\lambda t}{k} g_5 = 0$$

as a system of three equations in five unknowns. g_1 may be disregarded for it will always yield a value of zero when estimating variance components. Solution of the system in terms of g_3 and g_6 (chosen for convenience) yields

$$g_2 = -g_3,$$

$$g_4 = \frac{\lambda t}{k}(g_6 - 1) - \left(\frac{r-\lambda}{k}\right) g_3,$$

and

$$g_5 = 1 - g_6.$$

Thus, for any value of g_3 and g_6 an unbiased estimator of σ^2 can be defined. Two simple solutions are as follows:

1. Let $g_3 = 0$ and $g_6 = 1$. Then, $g_2 = g_4 = g_5 = 0$.

Hence,

$$F = \frac{s_6}{u} \quad \text{and} \quad E(F) = \sigma^2 .$$

2. Let $g_3 = 0$ and $g_6 = 0$. Then, $g_2 = 0$, $g_4 = -\frac{\lambda t}{k}$, and $g_5 = 1$. Hence,

$$F = \left(-\frac{\lambda t}{k} \cdot \frac{s_4}{f_1} + \frac{s_5}{t-1}\right) \quad \text{and} \quad E(F) = \sigma^2.$$

Let $u^{-1}s_6 = \theta_1$ and $(t-1)^{-1}s_5 - (\lambda t/k)f_1^{-1} s_4 = \theta_2$.

Certainly, other estimators (depending upon the values of g_3 and g_6) could have been chosen. These two were selected because of their simplicity.

<u>Case 2:</u> Estimation of σ_1^2.

Letting $E(F) = \sigma_1^2$ we have

$$g_1 = 0$$

$$g_2 + g_3 + g_5 + g_6 = 0$$

$$kg_2 + kg_3 = 1$$

$$\frac{r-\lambda}{k} g_3 + g_4 + \frac{\lambda t}{k} g_5 = 0$$

as a system of three equations in five unknowns. Solution of this system in terms of g_3 and g_5 gives

$$g_2 = \frac{1}{k} - g_3,$$

$$g_4 = -\left(\frac{r-\lambda}{k} g_3 + \frac{\lambda t}{k} g_5\right),$$

and

$$g_6 = -\left(g_5 + \frac{1}{k}\right).$$

Two simple solutions are as follows:

1. Let $g_3 = g_5 = 0$. Then, $g_2 = \frac{1}{k}$, $g_4 = 0$, and $g_6 = -\frac{1}{k}$.

Hence,

$$F = \frac{1}{k}\left(\frac{s_2}{w}\right) - \frac{1}{k}\left(\frac{s_6}{u}\right) \text{ and } E(F) = \sigma_1^2.$$

2. Let $g_3 = \frac{1}{k}$ and $g_5 = 0$. Then, $g_2 = 0$, $g_4 = -\frac{r-\lambda}{k^2}$, and $g_6 = -\frac{1}{k}$. Hence,

$$F = [k(t-1)]^{-1} s_3 - k^{-2}(r-\lambda) f_1^{-1} s_4 - (ku)^{-1} s_6 \text{ and } E(F) = \sigma_1^2.$$

Let

$$(kw)^{-1} s_2 - (ku)^{-1} s_6 = \theta_3$$

and

$$[k(t-1)]^{-1}s_3 - k^{-2}(r-\lambda) f_1^{-1} s_4 - (ku)^{-1}s_6 = \theta_4 .$$

<u>Case 3:</u> Estimation of σ_2^2.

Letting $E(F) = \sigma_2^2$, we have

$$g_1 = 0$$

$$g_2 + g_3 + g_5 + g_6 = 0$$

$$kg_2 + kg_3 = 0$$

$$\frac{r-\lambda}{k} g_3 + g_4 + \frac{\lambda t}{k} g_5 = 1$$

as a system of three equations in five unknowns. Solving the system in terms of g_3 and g_5 gives

$$g_2 = -g_3,$$

$$g_4 = 1 - (\frac{r-\lambda}{k} g_3 + \frac{\lambda t}{k} g_5),$$

and

$$g_6 = -g_5.$$

Three simple solutions are as follows:

1. Let $g_3 = g_5 = 0$. Then, $g_2 = g_6 = 0$ and $g_4 = 1$. Hence,

$$F = f_1^{-1} s_4 \text{ and } E(F) = \sigma_2^2.$$

2. Let $g_3 = 0$ and $g_5 = \frac{k}{\lambda t}$. Then, $g_2 = g_4 = 0$, and $g_6 = -\frac{k}{\lambda t}$. Hence,

$$F = k[\lambda t (t-1)]^{-1}s_5 - k(\lambda tu)^{-1}s_6 \text{ and } E(F) = \sigma_2^2.$$

3. Let $g_3 = \frac{k}{r-\lambda}$ and $g_5 = 0$. Then, $g_2 = -\frac{k}{r-\lambda}$ and $g_4 = g_6 = 0$. Hence,

$$F = -k[(r-\lambda)w]^{-1}s_2 + k[(r-\lambda)(t-1)]^{-1}s_3 \text{ and } E(F) = \sigma_2^2 .$$

Let $(f_1^{-1}s_4) = \theta_5$, $k[\lambda t\,(t-1)]^{-1}s_5 - k(\lambda tu)^{-1}s_6 = \theta_6$, and

$$k[(r-\lambda)(t-1)]^{-1}s_3 - k[(r-\lambda)w]^{-1}s_2 = \theta_7.$$

In summary of the preceding derivations, we have

$$\theta_1 = u^{-1}s_6,$$

$$\theta_2 = (t-1)^{-1}s_5 - k^{-1}\lambda t f_1^{-1}s_4,$$

$$\theta_3 = k^{-1}[w^{-1}s_2 - u^{-1}s_6],$$

$$\theta_4 = k^{-1}[(t-1)^{-1}s_3 - k^{-1}(r-\lambda) f_1^{-1}s_4 - u^{-1} s_6] ,$$

$$\theta_5 = f_1^{-1} s_4 ,$$

$$\theta_6 = k(\lambda t)^{-1} [(t-1)^{-1}s_5 - u^{-1}s_6],$$

and

$$\theta_7 = k(r-\lambda)^{-1}[(t-1)^{-1}s_3 - w^{-1}s_2],$$

where

$$E(\theta_1) = E(\theta_2) = \sigma^2,$$

$$E(\theta_3) = E(\theta_4) = \sigma_1^2,$$

and

$$E(\theta_5) = E(\theta_6) = E(\theta_7) = \sigma_2^2.$$

In order to make a decision as to which one of the estimators to use for any particular variance component, the variance of each should be found. The criterion for the "best" estimator will be that of minimum variance. This does not mean to imply that an estimator has been found for each variance component which has the smallest variance of the entire class of estimators of that component. We are interested here in comparing variances of only the above particular estimators. For example, in comparing the variances of θ_3 and θ_4, it will be shown that the sign of $\mathrm{var}(\theta_3) - \mathrm{var}(\theta_4)$ will change according to the BIB design used and the true magnitude of the ratios of the variance components. Hence, we cannot say θ_3 is uniformly better than θ_4 for all BIB designs.

Having obtained the covariance matrix (Table I) of the set of minimal sufficient statistics $s_i (i = 1, 2, \ldots, 6)$, we can now find the variances of the seven estimators.

<u>Variance of $\theta_1 = u^{-1}s_6$</u>

$$\mathrm{var}\ \theta_1 = u^{-2} \quad \mathrm{var}\ s_6 = u^{-2}[2u\sigma^4] = 2u^{-1}\sigma^4.$$

<u>Variance of $\theta_2 = (t-1)^{-1}s_5 - k^{-1}\lambda t f_1^{-1}s_4$</u>

$$\begin{aligned}
\mathrm{var}\ \theta_2 &= \mathrm{var}\ [(t-1)^{-1} s_5 - k^{-1} \lambda t f_1^{-1} s_4] \\
&= (t-1)^{-2}\mathrm{var}\ s_5 + (k^{-1}\lambda t)^2 f_1^{-2}\mathrm{var}\ s_4 - 2(t-1)^{-1}k^{-1}\lambda t f_1^{-1}\mathrm{cov}(s_5, s_4) \\
&= 2(t-1)^{-1}[\sigma^4 + 2k^{-1}\lambda t\, \sigma^2\sigma_2^2 + (\lambda t/k)^2\sigma_2^4] \\
&\quad + (\lambda t/k)^2(\lambda t)^{1/2} f_1^{-1}[2\lambda t(r-\lambda)k^{-2}\sigma_2^4 + r\sigma^2\sigma_2^2 + k\sigma^2\sigma_1^2 + \lambda t\sigma_1^2\sigma_2^2 + \sigma^4] \\
&\quad - 4[(t-1)kf_1]^{-1}\lambda t f_1[\sigma^2\sigma_2^2 + (\lambda t/k)\sigma_2^4].
\end{aligned}$$

By collecting terms and simplifying, the above equation becomes,

$$\mathrm{var}\ \theta_2 = \left(\frac{2}{(t-1)} + p\right)\sigma^4 + kp\sigma^2\sigma_1^2 + rp\sigma^2\sigma_2^2 + \lambda tp\,\sigma_1^2\sigma_2^2,$$

where $p = \lambda t/[(r-\lambda)(t-1)]$.

Now factoring p, we have

$$\mathrm{var}\ \theta_2 = p\left[\left(1 + \frac{2}{p(t-1)}\right)\sigma^4 + k\sigma^2\sigma_1^2 + r\sigma^2\sigma_2^2 + \lambda t\sigma_1^2\sigma_2^2\right].$$

<u>Variance of $\theta_3 = k^{-1}[w^{-1}s_2 - u^{-1}s_6]$</u>

$$\begin{aligned} \operatorname{var}\theta_3 &= \operatorname{var}[(kw)^{-1}s_2 - (ku)^{-1}s_6] \\ &= (kw)^{-2}\operatorname{var}s_2 + (ku)^{-2}\operatorname{var}s_6. \end{aligned}$$

Now consulting Table I, we see that

$$\operatorname{var}\theta_3 = 2(kw)^{-2}w[\sigma^4 + 2k\sigma^2\sigma_1^2 + k^2\sigma_1^4] + 2(ku)^{-2}u\sigma^4$$

which reduces to

$$\operatorname{var}\theta_3 = 2w^{-1}k^{-2}[u^{-1}(u+w)\sigma^4 + k^2\sigma_1^4 + 2k\sigma^2\sigma_1^2] .$$

<u>Variance of $\theta_4 = k^{-1}[(t-1)^{-1}s_3 - k^{-1}(r-\lambda)f_1^{-1}s_4 - u^{-1}s_6]$</u>

$$\begin{aligned} \operatorname{var}\theta_4 &= \operatorname{var}[k^{-1}(t-1)^{-1}s_3 - (ku)^{-1}s_6 - k^{-2}f_1^{-1}(r-\lambda)s_4] \\ &= [k(t-1)]^{-2}\operatorname{var}s_3 + (ku)^{-2}\operatorname{var}s_6 + k^{-4}f_1^{-2}(r-\lambda)^2\operatorname{var}s_4 \\ &\quad -2[k(t-1)]^{-1}k^{-2}f_1^{-1}(r-\lambda)\operatorname{cov}(s_3, s_4). \end{aligned}$$

Substitution for the variance and covariance terms gives

$$\begin{aligned} \operatorname{var}\theta_4 = {} & 2k^{-2}(t-1)^{-1}[\sigma^4 + k^2\sigma_1^4 + k^{-2}(r-\lambda)^2\sigma_2^4 + 2k\sigma^2\sigma_1^2 \\ & +2k^{-1}(r-\lambda)\sigma^2\sigma_2^2 + 2(r-\lambda)\sigma_1^2\sigma_2^2] + 2k^{-2}u^{-1}\sigma^4 \\ & +k^{-4}f_1^{-1}(r-\lambda)^2(\lambda t)^{1/2}[2\lambda t(r-\lambda)k^{-2}\sigma_2^4 + r\sigma^2\sigma_2^2 + k\sigma^2\sigma_1^2 \\ & +\lambda t\sigma_1^2\sigma_2^2 + \sigma^4] - 4(t-1)^{-1}k^{-3}(r-\lambda)[\sigma^2\sigma_2^2 + k\sigma_1^2\sigma_2^2 + k^{-1}(r-\lambda)\sigma_2^4]. \end{aligned}$$

Collecting the coefficientsof the variance components and their combinations we have that

$$\begin{aligned} \operatorname{var}\theta_4 = {} & \frac{2}{k^2}\left[\frac{u+(t-1)}{u(t-1)} + \frac{r-\lambda}{2\lambda(t-1)}\right]\sigma^4 + \frac{2}{(t-1)}\sigma_1^4 + \frac{1}{k(t-1)}\left(4+\frac{r-\lambda}{\lambda t}\right)\sigma^2\sigma_1^2 \\ & + \frac{r(r-\lambda)}{k^2\lambda t(t-1)}\sigma^2\sigma_2^2 + \frac{r-\lambda}{k^2(t-1)}\sigma_1^2\sigma_2^2 . \end{aligned}$$

Now, if we let $p_1 = \frac{(r-\lambda)}{k^2\lambda t(t-1)}$ and factor p_1 out of the expression for var θ_4, we then have

$$\text{var } \theta_4 = p_1 \Big\{ [2p(t-1)u^{-1}(u+t-1)+1]\sigma^4 + 2p(t-1)k^2\sigma_1^{\ 4} + [4p(t-1)+1]k\sigma^2\sigma_1^{\ 2} + r\sigma^2\sigma_2^{\ 2} + \lambda t\sigma_1^{\ 2}\sigma_2^{\ 2} \Big\}$$

where

$$p = \frac{\lambda t}{(r-\lambda)(t-1)} \quad \text{and} \quad p_1 = \frac{(r-\lambda)}{k^2\lambda t(t-1)} .$$

<u>Variance of $\theta_5 = f_1^{-1}s_4$</u>

$$\text{var } \theta_5 = \text{var } f_1^{-1} s_4 = f_1^{-2} \text{ var } s_4$$

$$= (\lambda t)^{1/2} f_1^{-1}[\sigma^4 + 2\lambda t(r-\lambda)k^{-2}\sigma_2^{\ 4} + k\sigma^2\sigma_1^{\ 2} + r\sigma^2\sigma_2^{\ 2} + \lambda t\sigma_1^{\ 2}\sigma_2^{\ 2}] .$$

<u>Variance of $\theta_6 = k(\lambda t)^{-1}[(t-1)^{-1}s_5 - u^{-1}s_6]$</u>

$$\text{var } \theta_6 = \text{var } \Big\{ k[\lambda t(t-1)]^{-1}s_5 - k(\lambda t\, u)^{-1} s_6 \Big\}$$

$$= k^2[\lambda t(t-1)]^{-2} \text{ var } s_5 + k^2 (\lambda t u)^{-2} \text{ var } s_6$$

$$= 2k^2(\lambda t)^{-2}(t-1)[\sigma^4 + 2(\lambda t/k)\sigma^2\sigma_2^{\ 2} + (\lambda t/k)^2\sigma_2^{\ 4}] + 2(k/\lambda t)^2 u^{-1}\sigma^4$$

$$= [2(k/\lambda t)^2 (t-1)^{-1} + 2(k/\lambda t)^2 u^{-1}]\sigma^4 + 2(k/\lambda t)^2 (t-1)^{-1}[2(\lambda t/k)\sigma^2\sigma_2^{\ 2} + (\lambda t/k)^2\sigma_2^{\ 4}].$$

This equation simplifies to

$$\text{var } \theta_6 = 2(t-1)^{-1}\Big\{(k/\lambda t)^2[1+u^{-1}(t-1)]\sigma^4 + \sigma_2^4 + 2(k/\lambda t)\sigma^2\sigma_2^{\ 2}\Big\} .$$

Variance of $\theta_7 = k(r-\lambda)^{-1}[(t-1)^{-1}s_3 - w^{-1}s_2]$

$$\begin{aligned}
\text{var } \theta_7 &= \text{var}\left\{k[(r-\lambda)(t-1)]^{-1}s_3 - k[w(r-\lambda)]^{-1}s_2\right\} \\
&= k^2[(r-\lambda)(t-1)]^{-2}\text{ var } s_3 + k^2[w(r-\lambda)]^{-2}\text{ var } s_2 \\
&= 2k^2(r-\lambda)^{-2}(t-1)^{-1}[\sigma^4 + k^2\sigma_1^4 + k^{-2}(r-\lambda)^2\sigma_2^4 + 2k\sigma^2\sigma_1^2 \\
&\quad + 2k^{-1}(r-\lambda)\sigma^2\sigma_2^2 + 2(r-\lambda)\sigma_1^2\sigma_2^2] \\
&\quad + 2k^2(r-\lambda)^{-2}w^{-1}[\sigma^4 + 2k\sigma^2\sigma_1^2 + k^2\sigma_1^4] \\
&= 2k^2(r-\lambda)^{-2}[(t-1)^{-1} + w^{-1}]\sigma^4 + 2k^4(r-\lambda)^{-2}[(t-1)^{-1} + w^{-1}]\sigma_1^4 \\
&\quad + 2(t-1)^{-1}\sigma_2^4 + 4k^3(r-\lambda)^{-2}[(t-1)^{-1} + w^{-1}]\sigma^2\sigma_1^2 \\
&\quad + 4k(r-\lambda)^{-1}(t-1)\sigma^2\sigma_2^2 + 4k^2[(r-\lambda)(t-1)]^{-1}\sigma_1^2\sigma_2^2.
\end{aligned}$$

Let

$$g = 2k^2(r-\lambda)^{-2}[(t-1)^{-1} + w^{-1}]$$

and

$$g_2 = 4k[(r-\lambda)(t-1)]^{-1}$$

to obtain

$$\begin{aligned}
\text{var } \theta_7 &= g\sigma^4 + gk^2\sigma_1^4 + 2(t-1)^{-1}\sigma_2^4 + 2kg\sigma^2\sigma_1^2 + g_2\sigma^2\sigma_2^2 \\
&\quad + kg_2\sigma_1^2\sigma_2^2 \\
&= g[\sigma^4 + k^2\sigma_1^4 + 2g^{-1}(t-1)\sigma_2^4 + 2k\sigma^2\sigma_1^2 + g^{-1}g_2\sigma^2\sigma_2^2 \\
&\quad + kg^{-1}g_2\sigma_1^2\sigma_2^2].
\end{aligned}$$

Now,

$$g_2/g = \frac{2(r-\lambda)}{k[(t-1)^{-1} + w^{-1}]} = \frac{2w\,(r-\lambda)}{k(b-1)} = g_1 \text{ (say)}.$$

Expressing the coefficient of σ_2^4 in terms of g_1, we have

$$\frac{2}{g(t-1)} = \frac{w(r-\lambda)^2}{k^2(b-1)} = \frac{g_1(r-\lambda)}{2k} .$$

The variance of θ_7 can now be rewritten as

$$\text{var } \theta_7 = g[\sigma^4 + k^2\sigma_1^{\ 4} + g_1(r-\lambda)(2k)^{-1}\sigma_2^{\ 4} + 2k\sigma^2\sigma_1^{\ 2}$$
$$+ g^{-1}g_2\sigma^2\sigma_2^{\ 2} + kg^{-1}g_2\sigma_1^{\ 2}\sigma_2^{\ 2}]$$

where

$$g = 2k^2(r-\lambda)^{-2}[(t-1)^{-1} + w^{-1}],$$

$$g_1 = 2w(r-\lambda)[k(b-1)]^{-1},$$

and

$$g_2 = 4k[(r-\lambda)(t-1)]^{-1} .$$

The results are summarized in Table II which gives the parameter being estimated, the unbiased estimators of the parameter, and the variance of the estimator. In Table II, the variance of θ_j $(j = 1, 2, \ldots, 7)$ is denoted by V_j.

Comparison of V_j $(j = 1, 2, \ldots, 7)$

In looking at the different variances, it seems evident that comparisons between the V_j will depend upon the variance components themselves as well as the particular BIB design. Under certain conditions, however, an estimator can have minimum variance with respect to other chosen estimators regardless of the values of σ^2, $\sigma_1^{\ 2}$, or $\sigma_2^{\ 2}$. In this section such conditions on the variance components and type of design will be investigated. Cochran and Cox have given a list of BIB designs most likely to be used in practical situations. Thirty of those designs for which $b > t$ have been chosen for consideration. Without

TABLE II

UNBIASED ESTIMATORS OF THE PARAMETERS
FOR THE BIB AND THEIR VARIANCES

Parameter	Unbiased Estimators	$V_j (V_j = \text{var}\ \theta_j)$
σ^2	$\theta_1 = u^{-1} s_6$	$2u^{-1}\sigma^4$
	$\theta_2 = (t-1)^{-1} s_5 - (\lambda t)(kf_1)^{-1} s_4$	$p[(1+2p^{-1}(t-1)^{-1})\sigma^4 + k\sigma^2\sigma_1^2 + r\sigma^2\sigma_2^2 + \lambda t\sigma_1^2\sigma_2^2]$
σ_1^2	$\theta_3 = (kw)^{-1} s_2 - (ku)^{-1} s_6$	$2w^{-1}k^{-2}[u^{-1}(u+w)\sigma^4 + k^2\sigma_1^4 + 2k\sigma^2\sigma_1^2]$
	$\theta_4 = [k(t-1)]^{-1} s_3 - (ku)^{-1} s_6 - k^{-2}(r-\lambda) f_1^{-1} s_4$	$p_1\left\{[p_3 u^{-1}(u+t-1)+1]\sigma^4 + p_3 k^2\sigma_1^4 + (2p_3+1)k\sigma^2\sigma_1^2 + r\sigma^2\sigma_2^2 + \lambda t\sigma_1^2\sigma_2^2\right\}$
σ_2^2	$\theta_5 = f_1^{-1} s_4$	$(\lambda t)^{1/2} f_1^{-1}[\sigma^4 + 2\lambda t(r-\lambda)k^{-2}\sigma_2^4 + k\sigma^2\sigma_1^2 + r\sigma^2\sigma_2^2 + \lambda t\sigma_1^2\sigma_2^2]$
	$\theta_6 = k[\lambda t(t-1)]^{-1} s_5 - k(u\lambda t)^{-1} s_6$	$2(t-1)^{-1}\left\{k^2(\lambda t)^{-2}[1+u^{-1}(t-1)]\sigma^4 + \sigma_2^4 + 2k(\lambda t)^{-1}\sigma^2\sigma_2^2\right\}$
	$\theta_7 = k[(r-\lambda)(t-1)]^{-1} s_3 - k[w(r-\lambda)]^{-1} s_2$	$g[\sigma^4 + k^2\sigma_1^4 + g_1(r-\lambda)(2k)^{-1}\sigma_2^4 + 2k\sigma^2\sigma_1^2 + g_1\sigma^2\sigma_2^2 + kg_1\sigma_1^2\sigma_2^2]$

$u = M - b - t + 1$

$w = b - t$

$f_1 = k^{-2}(\lambda t)^{3/2}(r-\lambda)(t-1)$

$p = (\lambda t)/(r-\lambda)(t-1)$

$p_1 = (r-\lambda)/k^2\lambda t\,(t-1)$

$p_3 = 2p\,(t-1)$

$g = 2k^2(b-1)/w(r-\lambda)^2\,(t-1)$

$g_1 = 2w(r-\lambda)/k(b-1)$

the restriction $b > t$, the statistic s_2, of which inter-error is a function, (as given in the A.O.V., Weeks [3]) would not be defined.

Comparison of V_1 and V_2

First to be considered are the variances V_1 and V_2 of θ_1 and θ_2 respectively. The results of the comparison are given in Theorem IV-1.

Theorem IV-1. Let V_1 and V_2 be as given in Table II.

a. If $\sigma^2 = 0$ and if one or both of σ_1^2, σ_2^2 is zero, then $V_2 - V_1 = 0$.

b. If $\sigma^2 = 0$ and neither σ_1^2 nor σ_2^2 is zero, then $V_2 - V_1 > 0$.

c. If $\sigma^2 > 0$, then $V_2 - V_1 > 0$.

Proof: a. If $\sigma^2 = 0$ then $V_1 = 0$ and $V_2 = p\lambda t \sigma_1^2 \sigma_2^2$. Hence, if σ_1^2 or σ_2^2 or both are zero, we have that $V_2 = 0$. Therefore, $V_2 - V_1 = 0$.

b. $\sigma^2 = 0$ implies $V_1 = 0$ and $V_2 = p\lambda t\sigma_1^2\sigma_2^2$. But $p = \dfrac{\lambda t}{(r - \lambda)(t-1)} > 0$ since $r > \lambda$, $t > 1$, and λt is a positive integer. Hence, $V_2 - V_1 > 0$.

c. If $\sigma^2 > 0$, we may write V_2 as

$$V_2 = p\sigma^4[1 + 2p^{-1}(t-1)^{-1} + k\gamma_1^2 + r\gamma_2^2 + \lambda t\gamma_1^2\gamma_2^2]$$

where

$$\gamma_1^2 = \sigma_1^2/\sigma^2,$$

and

$$\gamma_2^{\,2} = \sigma_2^{\,2}/\sigma^2.$$

Then,

$$V_2 - V_1 = \sigma^4[p+2(t-1)^{-1}+pk\,\gamma_1^{\,2}+pr\gamma_2^{\,2}$$
$$+p\lambda t\gamma_1^{\,2}\gamma_2^{\,2}-2u^{-1}].$$

Since

$$pk\gamma_1^{\,2}+pr\gamma_2^{\,2}+p\,\lambda t\gamma_1^{\,2}\gamma_2^{\,2} \geq 0,$$

we need only to show that $p + 2(t-1)^{-1} - 2u^{-1} > 0$. To accomplish this, assume $p + 2(t-1)^{-1} - 2u^{-1} \leq 0$. This implies $(p/2) + (t-1)^{-1} - u^{-1} \leq 0$, or, $[(\lambda t)/2(r-\lambda)(t-1)] + \frac{1}{t-1} - \frac{1}{u} \leq 0$. But by Lemma 4, $u \geq t-1$. Hence, $1/u \leq 1/(t-1)$ and we have a contradiction. This proves that $V_2 - V_1 > 0$.

The above theorem shows that regardless of the BIB design and the true value of the variance components the variance of θ_1 is smaller than the variance of θ_2 and hence, that of the two estimators, θ_1 is uniformly better than θ_2.

Comparison of V_3 and V_4

From Table II we have that

$$V_3 = 2w^{-1}k^{-2}[u^{-1}(u+w)\sigma^4 + 2k\sigma^2\sigma_1^{\,2}+k^2\sigma_1^{\,4}]$$

and

$$V_4 = p_1\Big\{[2p(t-1)u^{-1}(u+t-1)+1]\sigma^4+2p(t-1)k^2\sigma_1^{\,4}$$
$$+[4p(t-1)+1]k\sigma^2\sigma_1^{\,2}+r\sigma^2\sigma_2^{\,2} + \lambda t\sigma_1^{\,2}\sigma_2^{\,2}\Big\}.$$

Subtracting V_3 from V_4 and collecting coefficients, the following equation is obtained.

$$V_4 - V_3 = \sigma^4[2p_1p(t-1)u^{-1}(u+t-1)+p_1-2(wu)^{-1}k^{-2}(u+w)]$$
$$+\sigma_1^4[2p_1p(t-1)k^2-2w^{-1}]+\sigma^2\sigma_1^2[4p_1pk(t-1)$$
$$+p_1k-4(kw)^{-1}]+p_1[r\sigma^2\sigma_2^2+\lambda t\sigma_1^2\sigma_2^2].$$

Simplifying the coefficients we have

$$V_4-V_3=\sigma^4\left[\frac{2\lambda t(M-b)+u(r-\lambda)}{uk^2\lambda t(t-1)} - \frac{2(u+w)}{k^2uw}\right]+\sigma_1^4\left[\frac{2}{t-1} - \frac{2}{w}\right]$$
$$+\sigma^2\sigma_1^2\left[\frac{4\lambda t+(r-\lambda)}{k\lambda t(t-1)} - \frac{4}{kw}\right]+p_1[r\sigma^2\sigma_2^2+\lambda t\sigma_1^2\sigma_2^2].$$

Let the coefficients of σ^4, σ_1^4, and $\sigma^2\sigma_1^2$ be denoted by c_3, c_1, and c_2 respectively. Then,

$$V_4-V_3 = c_3\sigma^4+c_1\sigma_1^4+c_2\sigma^2\sigma_1^2+p_1r\sigma^2\sigma_2^2+p_1\lambda t\sigma_1^2\sigma_2^2. \quad (1)$$

The examination of the equation involves two caes, namely $\sigma^2 = 0$ and $\sigma^2 > 0$.

<u>Case 1:</u> $\sigma^2 = 0$.

If $\sigma^2 = 0$ in (1), then

$$V_4 - V_3 = c_1\sigma_1^4 + p_1\lambda t\sigma_1^2\sigma_2^2 \quad (2)$$

Setting $\sigma_1^2 = 0$ in (2), we have $V_4 - V_3 = 0$. On the other hand, if $\sigma_2^2 = 0$ in (2), then

$$V_4 - V_3 = c_1\sigma_1^4 > 0 \text{ if } c_1 > 0.$$

$c_1 > 0$ is equivalent to $b > 2t - 1$.

For the next situation assume neither σ_1^2 nor σ_2^2 is zero. This restriction implies $V_4 - V_3 > 0$ if $c_1\sigma_1^2+p_1\lambda t\sigma_2^2 > 0$. Solving for σ_2^2 we obtain $\sigma_2^2 > -(c_1/p_1\lambda t)\sigma_1^2$. $p_1\lambda t$ being positive insures that $V_4 - V_3 > 0$ if $c_1 > 0$. As noted above, $c_1 > 0$ when $b>2t-1$.

It should be pointed out that $b \neq 2t-1$ by Lemma 5. Thus, if $b < 2t-1$ the sign of $V_4 - V_3$ depends upon the true values of σ_1^2 and σ_2^2 when assuming $\sigma^2 = 0$.

Case 2: $\sigma^2 > 0$.

If $\sigma^2 > 0$, we can rewrite (1) as

$$V_4 - V_3 = \sigma^4[c_3+c_1\gamma_1^4+c_2\gamma_1^2+p_1r\gamma_2^2+p_1\lambda t\,\gamma_1^2\gamma_2^2],$$

where $\gamma_1^2 = \sigma_1^2/\sigma^2$ and $\gamma_2^2 = \sigma_2^2/\sigma^2$.

Setting $V_4 - V_3 = 0$ and solving for γ_2^2 we have

$$\gamma_2^2 = -[c_1\gamma_1^4+c_2\gamma_1^2+c_3]/(p_1r+p_1\lambda t\,\gamma_1^2) \tag{3}$$

which is the ratio of a quadratic in γ_1^2 to a linear function of γ_1^2. It is evident that $V_4 - V_3 > 0$ when γ_2^2 is greater than the right hand side of (3).

Clearly, the better estimator might depend upon the true values of the variance components, and for one who might have an "a priori" knowledge of the ratios of σ^2, σ_1^2, and σ_2^2, the constants involved in (3), as well as the roots of the equation, have been tabulated by the use of the IBM 1410 for the thirty different BIB designs previously mentioned. The calculations for only ten of the thirty designs are given in Table III. The remaining twenty designs satisfy certain conditions which insure that $V_4 - V_3 > 0$. Thus, for these particular designs, θ_3 is preferred to θ_4 in the estimation of σ_1^2.

Without considering the true values of σ^2, σ_1^2, or σ_2^2, the conditions which guarantee that $V_4 - V_3$ is positive are $c_1 > 0$, $c_2 > 0$, and $c_3 > 0$ in (3). These are sufficient conditions since p_1r and $p_1\lambda t$ are

TABLE III

EQUATION AND CONSTANT VALUES FOR COMPARING V_3 AND V_4

$$V_4 - V_3 = 0 \text{ implies } (p_1 r + p_1 \lambda t \gamma_1^2)\gamma_2^2 = -(c_1\gamma_1^4 + c_2\gamma_1^2 + c_3)$$

Design No.	r	k	t	$p_1 r$	$p_1 \lambda t$	c_1	c_2	c_3	Root 1	Root 2	γ_2^2 intercept
1	3	2	4	.1250	.1667	-.3333	-.2500	-.0417	-.5000	-.2500	.3333
2	5	3	6	.0278	.0667	-.1000	-.0500	-.0056	-.3333	-.1667	.2000
3	7	4	8	.0104	.0357	-.0476	-.0179	-.0015	-.2500	-.1250	.1429
4	8	6	9	.0019	.0104	-.4167	-.1375	-.0113	-.1667	-.1633	6.1250
5	6	4	10	.0083	.0278	-.1778	-.0833	-.0097	-.2500	-.2187	1.1667
6	9	5	10	.0050	.0222	-.0278	-.0083	-.0006	-.2000	-.1000	.1111
7	9	6	10	.0022	.0123	-.1778	-.0578	-.0047	-.1667	-.1583	2.1111
8	9	6	16	.0021	.0111	-.1167	-.0375	-.0030	-.1667	-.1548	1.4444
9	10	7	21	.0011	.0071	-.1222	-.0341	-.0024	-.1429	-.1364	2.1000
10	9	7	28	.0009	.0053	-.1759	-.0496	-.0035	-.1429	-.1391	4.1111

Other BIB Designs Investigated

Design No.	r	k	t	Design No.	r	k	t
11	10	5	9	21	8	4	9
12	6	3	5	22	9	2	10
13	10	3	6	23	9	3	10
14	10	4	6	24	10	2	11
15	10	5	41	25	6	3	13
16	4	2	5	26	7	3	15
17	5	2	6	27	9	3	19
18	6	2	7	28	10	3	21
19	7	2	8	29	8	4	25
20	8	2	9	30	9	4	28

positive. Now, $c_1 = \frac{2}{t-1} - \frac{2}{w} > 0$ whenever $w > t - 1$. But $w = b - t$. Hence, $c_1 > 0$ when $b > 2t - 1 = \delta_2$ (say). The condition for c_3 to be positive is more complicated. To see this, set

$$c_3 = \frac{2\lambda t(M-b) + u(r-\lambda)}{uk^2\lambda\, t(t-1)} - \frac{2(u + w)}{k^2 u\; w} > 0.$$

This reduces to

$$\frac{M-b}{t-1} + \frac{u(r-\lambda)}{2\lambda t(t-1)} - (\frac{u+w}{w}) > 0.$$

But, as a consequence of Lemma 1, we have that

$$\frac{M-b}{t-1} = b\frac{\lambda}{r} .$$

Thus, if c_3 is to be greater than zero, then

$$b > \frac{r}{\lambda}\left[\frac{u+w}{w} - \frac{u(r-\lambda)}{2\lambda t(t-1)}\right] = \delta_1 \text{ (say)}.$$

It should be noted that $b \neq \delta_1$ by Lemma 6. Next,

$$c_2 = \left[\frac{4\lambda t + r - \lambda}{k\lambda t(t-1)} - \frac{4}{kw}\right] > 0$$

implies that

$$\frac{4\lambda t + r - \lambda}{\lambda t(t-1)} > \frac{4}{w} \qquad \text{or,} \qquad 4 + \frac{r-\lambda}{\lambda t} > 4(\frac{t-1}{w}) .$$

From the condition on c_1 (i.e., $w > t - 1$), it is clear that $c_2 > 0$ when $c_1 > 0$. Hence, a sufficient conditon for $V_4 - V_3 > 0$ is $b > \max(\delta_1, \delta_2)$. Of the thirty designs investigated only 10 fail to meet this sufficient condition. As was previously stated, these are the first ten designs in Table III. It is easily shown that c_1, c_2, and c_3 are positive for the remaining twenty designs. As an example, the roots of (3) for design 11 are -.2 and -.3 while the γ_2^2 intercept is -.3. Hence, the graph of (3) does not fall in the first quadrant to which γ_1^2 and γ_2^2 are restricted. The graphs of (3), for designs 12 through 20 possess this same property.

The results of the preceding section can be summarized in the following theorem.

Theorem IV-2. Let V_3 and V_4 be as given in Table II. Let

$$\delta_1 = \left(\frac{r}{\lambda}\right)\left[\frac{u+w}{w} - \frac{u(r-\lambda)}{2\lambda t(t-1)}\right]$$

and

$$\delta_2 = 2t - 1.$$

Case 1: Assume $\sigma^2 = 0$.

a. If $\sigma_1^2 = 0$ and $\sigma_2^2 > 0$, then $V_4 - V_3 = 0$

b. If $\sigma_1^2 > 0$ and $\sigma_2^2 = 0$, then $V_4 - V_3 > 0$ whenever $b > \delta_2$. If $b < \delta_2$, $V_4 - V_3 < 0$.

c. If $\sigma_1^2 > 0$ and $\sigma_2^2 > 0$, then $V_4 - V_3 > 0$ whenever $b > \delta_2$. For $b < \delta_2$, the sign of $V_4 - V_3$ depends upon σ_1^2 and σ_2^2.

Case 2: Assume $\sigma^2 > 0$.

d. If $\sigma_1^2 = \sigma_2^2 = 0$, then $V_4 - V_3 > 0$ whenever $b > \delta_1$. If $b < \delta_1$, $V_4 - V_3 < 0$.

e. If $\sigma_1^2 = 0$ and $\sigma_2^2 > 0$, then $V_4 - V_3 > 0$ whenever $b > \delta_1$. If $b < \delta_1$, the sign of $V_4 - V_3$ depends upon γ_2^2.

f. If $\sigma_1^2 > 0$, then $V_4 - V_3 > 0$ whenever $b > \max(\delta_1, \delta_2)$.

g. If $b < \min(\delta_1, \delta_2)$, then

(i) $V_4 - V_3 < 0$ for $\sigma_2^2 = 0$.

(ii) Sign of $V_4 - V_3$ depends on γ_1^2 and γ_2^2 for $\sigma_2^2 > 0$.

The results of parts d and e of Case 2 are obvious when considering (3). However, the restriction $b < \min(\delta_1, \delta_2)$ in part g could lead to some confusion. From the discussion of Case 2 we have

$$V_4 - V_3 = \sigma^4 [c_3 + c_1\gamma_1^4 + c_2\gamma_1^2] \text{ if } \gamma_2^2 = 0.$$

Theoretically, if $b < \max(\delta_1, \delta_2)$ we could have $\delta_1 < b < \delta_2$ or $\delta_2 < b < \delta_1$ depending on the maximum. Hence, from the manner in which δ_1 and δ_2 were obtained it would be possible to have $c_3 > 0$ and $c_1 < 0$ or $c_1 > 0$ and $c_3 < 0$. For either of these cases the sign of $V_4 - V_3$ would depend upon the magnitude of γ_1^2. However, for the ten designs which fail to meet the sufficient condition $b > \max(\delta_1, \delta_2)$ all the c_j $(j = 1, 2, 3)$ are negative. Thus, for these ten designs in particular the sign of $V_4 - V_3$ is negative regardless of γ_1^2 when $\gamma_2^2 = 0$ and $\sigma^2 > 0$.

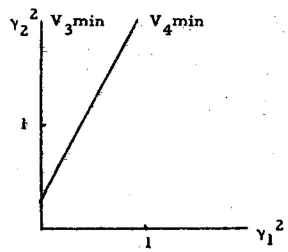
Disregarding the trivial cases of zero variance components (ii) of part g in Case 2 remains the most important. For this nontrivial case, the graph of (3) for designs 1 through 10 is shown in Table IV.

<u>Comparison of V_5 and V_6</u>

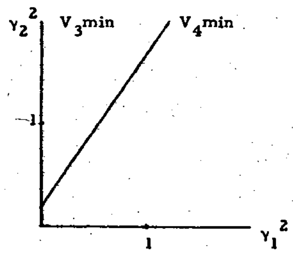
As in the preceding section an expression for $V_5 - V_6 = 0$ will be derived. From Table II we have that

TABLE IV

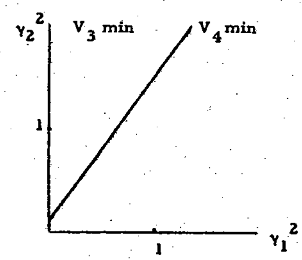
GRAPHICAL COMPARISONS OF V_3 AND V_4 FOR DESIGNS IN TABLE III



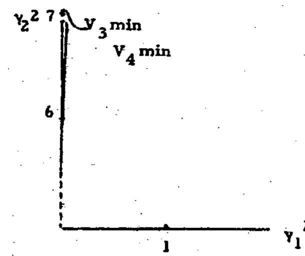
Design 1



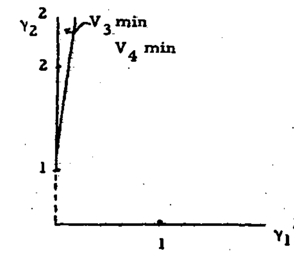
Design 2



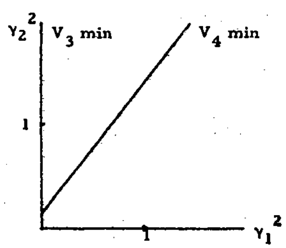
Design 3



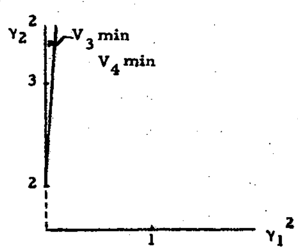
Design 4



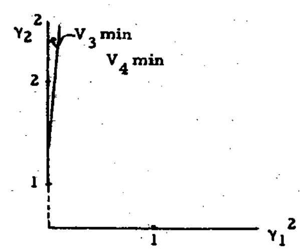
Design 5



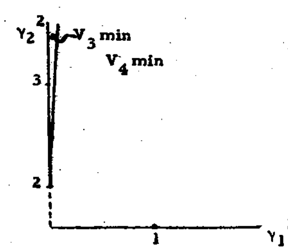
Design 6



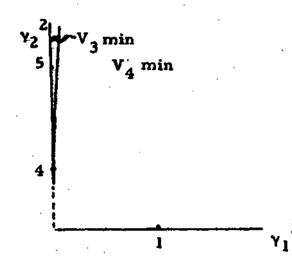
Design 7



Design 8



Design 9



Design 10

$$V_5 = (\lambda t)^{1/2} f_1^{-1}[2\lambda t(r-\lambda)k^{-2}\sigma_2^4 + r\sigma^2\sigma_2^2 + k\sigma^2\sigma_1^2$$
$$+ \lambda t\sigma_1^2\sigma_2^2 + \sigma^4]$$

and

$$V_6 = 2(t-1)^{-1}\left\{k^2(\lambda t)^{-2}[1+u^{-1}(t-1)]\sigma^4 + 2k(\lambda t)^{-1}\sigma^2\sigma_2^2 + \sigma_2^4\right\} .$$

Taking the difference and collecting coefficients of the variance components, we have that

$$V_5 - V_6 = \sigma^4\left\{(\lambda t)^{1/2} f_1^{-1} - 2(t-1)^{-1}k^2(\lambda t)^{-2}[1+u^{-1}(t-1)]\right\}$$
$$+\sigma_2^4[2(\lambda t)^{3/2} f_1^{-1}(r-\lambda)k^{-2} - 2(t-1)^{-1}]$$
$$+\sigma^2\sigma_2^2[r(\lambda t)^{1/2} f_1^{-1} - 4k(t-1)^{-1}(\lambda t)^{-1}]$$
$$+(\lambda t)^{1/2} f_1^{-1}[k\sigma^2\sigma_1^2 + \lambda t\sigma_1^2\sigma_2^2].$$

Simplifying the coefficients, the above equation becomes

$$V_5 - V_6 = \sigma^4 \frac{k^2}{\lambda t\ (t-1)}\left[\frac{1}{r-\lambda} - \frac{2(M-b)}{u\lambda t}\right]$$
$$+\sigma^2\sigma_2^2 \frac{k}{\lambda t(t-1)}\left[\frac{rk}{r-\lambda} - 4\right]$$
$$+ \frac{k^2}{\lambda t(r-\lambda)(t-1)}[k\sigma^2\sigma_1^2 + \lambda t\ \sigma_1^2\sigma_2^2] .$$

To simplify even further let

$$c_5 = \left[\frac{1}{r-\lambda} - \frac{2(M-b)}{u\lambda\ t}\right] ,$$

$$c_4 = \frac{1}{k}\left[\frac{rk}{r-\lambda} - 4\right] ,$$

and

$$c_o = \frac{k^2}{\lambda t(t-1)} .$$

Then,

$$V_5 - V_6 = c_o c_5\sigma^4 + c_o c_4\sigma^2\sigma_2^2 + c_o(r-\lambda)^{-1}[k\sigma^2\sigma_1^2 + \lambda t\sigma_1^2\sigma_2^2].$$

The remaining investigation of this equation will be considered in two cases.

<u>Case 1:</u> $\sigma^2 = 0$.

If $\sigma^2 = 0$ and either σ_1^2 or σ_2^2 is zero, then $V_5 - V_6 = 0$. If $\sigma^2 = 0$ and neither σ_1^2 nor σ_2^2 is zero, then,

$$V_5 - V_6 = \frac{k^2}{(r-\lambda)(t-1)} \sigma_1^2 \sigma_2^2 > 0.$$

<u>Case 2:</u> $\sigma^2 > 0$.

If $\sigma^2 > 0$, then $V_5 - V_6$ can be written as

$$V_5 - V_6 = c_o \sigma^4 [c_5 + c_4 \gamma_2^2 + (r-\lambda)^{-1} k \gamma_1^2 + (r-\lambda)^{-1} \lambda t \gamma_1^2 \gamma_2^2], \tag{4}$$

where γ_1^2 and γ_2^2 are as previously defined. Now, set $V_5 - V_6$ equal to zero to obtain

$$[c_4 + (r-\lambda)^{-1} \lambda t \gamma_1^2]\, \gamma_2^2 = -[c_5 + k(r-\lambda)^{-1} \gamma_1^2]. \tag{5}$$

Hence, $V_5 - V_6 > 0$ when

$$\gamma_2^2 > \frac{-[c_5 + k(r-\lambda)^{-1} \gamma_1^2]}{c_4 + \lambda t (r-\lambda)^{-1} \gamma_1^2} \quad \text{for } \gamma_1^2 > \frac{-c_4}{\lambda t (r-\lambda)^{-1}} \tag{5.1}$$

or,

$$\gamma_2^2 < \frac{-[c_5 + k(r-\lambda)^{-1} \gamma_1^2]}{c_4 + \lambda t (r-\lambda)^{-1} \gamma_1^2} \quad \text{for } \gamma_1^2 < \frac{-c_4}{\lambda t (r-\lambda)^{-1}}. \tag{5.2}$$

If $\gamma_1^2 = \dfrac{-c_4}{\lambda t (r-\lambda)^{-1}}$, (5.1) and (5.2) are not defined.

Since $(r-\lambda)$, k, λ, and t are positive integers, a sufficient condition for $V_5 - V_6$ to be positive is that c_4 and c_5 are greater than zero. For $c_4 > 0$ and $c_5 > 0$, (5.2) is redundant since γ_1^2 and γ_2^2 cannot be negative.

Now,

$$c_4 = \frac{1}{k}\left[\frac{rk}{r-\lambda} - 4\right] > 0$$

when $rk > 4(r-\lambda)$. But, $r-\lambda = rk - \lambda t$. Hence, the condition on c_4 reduces to $4\lambda t > 3rk$. Expressing t as $t = \frac{kb}{r}$ we have

$$\frac{4\lambda\, kb}{r} > 3rk \qquad \text{or,} \qquad b > \frac{3}{4}\,\frac{r^2}{\lambda} = \delta_3 \text{ (say).}$$

Thus, $c_4 > 0$ when $b > \delta_3$.

Next, set

$$c_5 = \left[\frac{1}{r-\lambda} - \frac{2(M-b)}{u\lambda t}\right] > 0.$$

By Lemma 1,

$$M - b = \frac{\lambda t}{k}(t-1)$$

which, upon substitution, reduces the above condition to

$$\frac{1}{r-\lambda} > \frac{2(t-1)}{uk} .$$

But, by Lemma 2,

$$u = \frac{(t-1)}{k}(\lambda t - k).$$

Hence, the restriction on c_5 is equivalent to

$$\frac{1}{r-\lambda} > \frac{2}{\lambda t - k} \qquad \text{or,} \qquad \lambda t > 2(r-\lambda) + k.$$

This last inequality may be expressed as a restriction on block size by making the substitutions $r - \lambda = rk - \lambda t$ and $k = rt/b$. After some algebraic manipulation the inequality

$$b > \frac{r}{3\lambda}(2r + 1)$$

is obtained. Let

$$\frac{r}{3\lambda}(2r + 1) = \delta_4 .$$

We can now state that $V_5 - V_6 > 0$ if $b > \max(\delta_3, \delta_4)$. There are other conditions, however, which dictate the maximum of δ_3 and δ_4. Setting $\delta_3 \geq \delta_4$, we have $3r \geq (4/3)(2r+1)$ or, $r \geq 4$. Hence, for $r > 4$, $\delta_3 > \delta_4$; for $r = 4$, $\delta_3 = \delta_4$; and for $r < 4$, $\delta_3 < \delta_4$.

Of the thirty designs under consideration only two are such that $r \leq 4$. Hence, except for these two designs, one needs only to note that $b > 3/4\,(r^2/\lambda) = \delta_3$ to conclude that $V_5 > V_6$.

The trivial cases under Case 2 are easily determined from (4). If $\sigma_2^2 = 0$, then $V_5 - V_6 > 0$ whenever $c_5 > 0$ or equivalently, when $b > \delta_4$. If $b < \delta_4$ the sign of $V_5 - V_6$ depends on γ_1^2. If $\sigma_1^2 = 0$, then $V_5 - V_6 > 0$ if $b > \max(\delta_3, \delta_4)$ and $V_5 - V_6 < 0$, if $b < \min(\delta_3, \delta_4)$.

For the non-trivial case, the sign of $V_5 - V_6$ will depend upon the values of γ_1^2 and γ_2^2 for those designs which fail to meet the sufficient condition $b > \max(\delta_3, \delta_4)$. Thirteen designs of those considered fall into this category. For these, the coefficients of the true ratios of variances in (5), as well as the root of the equation, have been calculated and are presented in Table V. Also, that portion of the graph of (5) which falls in the first quadrant is shown for these thirteen designs in Table VI.

The preceding discussion is summarized in Theorem IV-3 and the following corollaries.

<u>Theorem IV-3.</u> Let V_5 and V_6 be as given in Table II. Let $\delta_3 = 3/4\,(r^2/\lambda)$ and $\delta_4 = (r/3\lambda)(2r+1)$.

<u>Case 1:</u> Assume $\sigma^2 = 0$.

a. If either σ_1^2 or σ_2^2 is zero, then $V_5 - V_6 = 0$.

TABLE V

EQUATION AND CONSTANT VALUES FOR COMPARING V_5 AND V_6

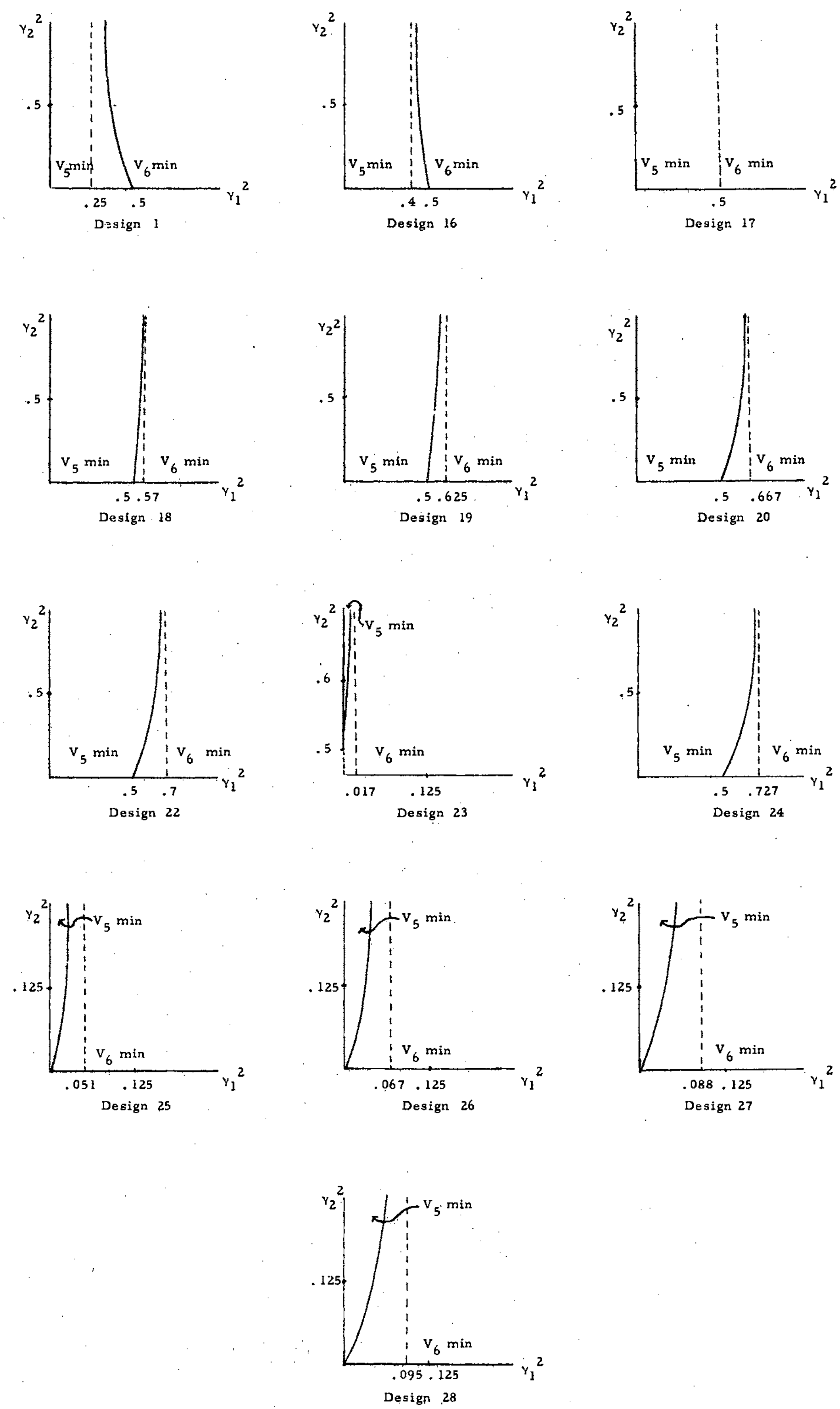
$$V_6 - V_6 = 0 \text{ implies } (c_4 + q_4\gamma_1^2)\gamma_2^2 = -(c_5 + q_5\ \gamma_1^2)$$

Design No.	r	k	t	b	δ_3	δ_4	c_4	q_4	c_5	q_5	Root	γ_2^2 intercept	Undefined For $\gamma_1^2 =$
1	3	2	4	6	6.750	7.00	-.500	2.000	-.500	1.000	.500	-1.000	.250
16	4	2	5	10	12.000	12.00	-.667	1.667	-.333	.667	.500	- .500	.400
17	5	2	6	15	18.750	18.33	-.750	1.500	-.250	.500	.500	- .333	.500
18	6	2	7	21	27.000	26.00	-.800	1.400	-.200	.400	.500	- .250	.571
19	7	2	8	28	36.750	35.00	-.833	1.333	-.167	.333	.500	- .200	.625
20	8	2	9	36	48.000	45.33	-.857	1.286	-.143	.286	.500	- .167	.667
22	9	2	10	45	60.750	57.00	-.875	1.250	-.125	.250	.500	- .143	.700
23	9	3	10	30	30.375	28.50	-.048	2.857	.025	.429	-.059	.529	.017
24	10	2	11	55	75.000	70.00	-.889	1.222	-.111	.222	.500	- .125	.727
25	6	3	13	26	27.000	26.00	-.133	2.600	.000	.600	.000	.000	.051
26	7	3	15	35	36.750	35.00	-.167	2.500	.000	.500	.000	.000	.067
27	9	3	19	57	60.750	57.00	-.208	2.375	.000	.375	.000	.000	.088
28	10	3	21	70	75.000	70.00	-.222	2.333	.000	.333	.000	.000	.095

$\delta_3 = \frac{3}{4}\frac{r^2}{\lambda}$ $\delta_4 = (r/3\lambda)(2r+1)$ $q_4 = \lambda t/(r-\lambda)$ $q_5 = k/(r-\lambda)$

TABLE VI

GRAPHICAL COMPARISONS OF V_5 AND V_6 FOR DESIGNS IN TABLE V



b. If neither σ_1^2 nor σ_2^2 is zero, then $V_5 - V_6 > 0$.

Case 2: Assume $\sigma^2 > 0$.

c. If $\sigma_i^2 \geq 0$ $(i = 1, 2)$, $V_5 - V_6 > 0$ whenever $b > \max(\delta_3, \delta_4)$.

d. If $\sigma_i^2 \geq 0$ $(i = 1, 2)$, the sign of $V_5 - V_6$ depends on γ_2^2 whenever $b < \min(\delta_3, \delta_4)$.

Corollary IV-1: If $\sigma^2 > 0$ and $\sigma_2^2 = 0$, then

a. $V_5 - V_6 > 0$ whenever $b > \delta_4$.

b. The sign of $V_5 - V_6$ depends on γ_1^2 whenever $b < \delta_4$.

Proof:

a. $b > \delta_3$ implies $c_4 > 0$. But, $\sigma_2^2 = 0$ implies $c_4\gamma_2^2 = 0$ and the proof is complete by (c) of Theorem IV-3.

b. The proof is analogous to (a) using (d) of Theorem IV-3.

Corollary IV-2: Assume $\sigma^2 > 0$ and $\sigma_i^2 > 0$ $(i = 1, 2)$.
If $r > 4$ and $b > \frac{3}{4}(r^2/\lambda)$, then $V_5 - V_6 > 0$.

Proof: $r > 4$ implies $\frac{3}{4}\frac{r^2}{\lambda} = \delta_3 > \delta_4$. Hence, $b > \frac{3}{4}\frac{r^2}{\lambda}$ implies $b > \max(\delta_3, \delta_4)$ and the proof is complete by (c) of Theorem IV-3.

Corollary IV-3: Assume $\sigma^2 > 0$ and $\sigma_i^2 > 0$ $(i = 1, 2)$. If $r > 4$ and $k \geq 4$, then $V_5 - V_6 > 0$.

Proof: $b > \frac{3}{4}\frac{r^2}{\lambda}$ implies $k > 4\left(\frac{r-\lambda}{r}\right)$. But $\frac{r-\lambda}{r} < 1$. Since k is an integer we then have $k \geq 4$. Hence, by Corollary IV-2,

$V_5 - V_6 > 0.$

Comparison of V_5 and V_7

From Table II

$$V_5 = (\lambda t)^{1/2} f_1^{-1}[2\lambda t(r-\lambda)k^{-2}\sigma_2^4 + r\sigma^2\sigma_2^2 + k\sigma^2\sigma_1^2 + \lambda t\sigma_1^2\sigma_2^2 + \sigma^4]$$

and

$$V_7 = g[g_1(r-\lambda)(2k)^{-1}\sigma_2^4 + g_1\sigma^2\sigma_2^2 + 2k\sigma^2\sigma_1^2 + kg_1\sigma_1^2\sigma_2^2 + k^2\sigma_1^4 + \sigma^4],$$

where

$$f_1 = k^{-2}(\lambda t)^{3/2}(r-\lambda)(t-1),$$

$$g = 2k^2(b-1)/w(r-\lambda)^2(t-1),$$

and

$$g_1 = 2w(r-\lambda)/k(b-1).$$

Collecting coefficients of like terms, the difference $V_7 - V_5$ can be written as

$$V_7 - V_5 = \sigma^4[g-(\lambda t)^{1/2}f_1^{-1}] + gk^2\sigma_1^4 + \sigma_2^4[gg_1(r-\lambda)(2k)^{-1} - 2(\lambda t)^{3/2}f_1^{-1}(r-\lambda)k^{-2}] + \sigma^2\sigma_1^2[2gk - k(\lambda t)^{1/2}f_1^{-1}] + \sigma^2\sigma_2^2[gg_1 - r(\lambda t)^{1/2}f_1^{-1}] + \sigma_1^2\sigma_2^2[kgg_1 - (\lambda t)^{3/2}f_1^{-1}].$$

By simplifying the coefficients and letting $g_2 = k/(r-\lambda)(t-1)$, and $g_3 = 2(b-1)/w(r-\lambda)$, we have

$$V_7 - V_5 = kg_2[g_3 - (\lambda t)^{-1}]\sigma^4 + g_2g_3k^3\sigma_1^4 + k^2g_2[2g_3 - (\lambda t)^{-1}]\sigma^2\sigma_1^2 + g_2[4 - rk(\lambda t)^{-1}]\sigma^2\sigma_2^2 + 3kg_2\sigma_1^2\sigma_2^2.$$

To simplify even further, let

$$c_6 = k[g_3 - (\lambda t)^{-1}], \quad c_7 = k^2[2g_3 - (\lambda t)^{-1}],$$

and

$$c_8 = 4 - rk(\lambda t)^{-1}.$$

Then,

$$V_7 - V_5 = g_2[c_6\sigma^4 + c_7\sigma^2\sigma_1^2 + c_8\sigma^2\sigma_2^2 + g_3k^3\sigma_1^4 + 3k\sigma_1^2\sigma_2^2].$$

The examination of this equation consists of two cases.

Case 1: Assume $\sigma^2 = 0$.

Under the assumption that $\sigma^2 = 0$, we have that

$$V_7 - V_5 = g_2g_3k^3\sigma_1^4 + 3g_2k\sigma_1^2\sigma_2^2.$$

Now, $\sigma_1^2 = 0$ implies $V_7 - V_5 = 0$ and $\sigma_2^2 = 0$ implies that $V_7 - V_5 > 0$ since g_2g_3 is positive. If neither σ_1^2 nor σ_2^2 is zero, then $V_7 - V_5 > 0$.

Case 2: Assume $\sigma^2 > 0$.

Letting $\sigma_1^2/\sigma^2 = \gamma_1^2$ and $\sigma_2^2/\sigma^2 = \gamma_2^2$ as before, we have

$$V_7 - V_5 = g_2\sigma^4[c_6 + c_7\gamma_1^2 + c_8\gamma_2^2 + g_3k^3\gamma_1^4 + 3k\gamma_1^2\gamma_2^2]. \qquad (6)$$

Now, if $\sigma_1^2 = 0$ then $V_7 - V_5 = g_2\sigma^4[c_6 + c_8\gamma_2^2]$ which is greater than zero whenever c_6 and c_8 are positive. If c_6 and c_8 are both negative then $V_7 - V_5 < 0$. If c_6 and c_8 do not have the same sign, then the sign of $V_7 - V_5$ depends upon γ_2^2.

If $\sigma_2^2 = 0$, (6) reduces to a quadratic in γ_1^2. For this case, $V_7 - V_5 > 0$ whenever $g_3k^3\gamma_1^4 + c_7\gamma_1^2 + c_6 > 0$. The conditions which guarantee that $V_7 - V_5 > 0$ when $\sigma_2^2 = 0$ will be considered in conjunction with the restriction of non-zero variance components.

Setting $V_7 - V_5 > 0$, we have from (6) that

$$\gamma_2^2 > \frac{-(c_6 + c_7\gamma_1^2 + g_3k^3\gamma_1^4)}{(c_8 + 3k\gamma_1^2)} \quad \text{when } \gamma_1^2 > \frac{-c_8}{3k}. \qquad (7)$$

Since any meaningful interpretation of (7) will depend upon the constants involved, conditions to insure that $V_7 - V_5$ is positive should be considered.

Now, $c_6 = k[g_3 - (\lambda t)^{-1}]$ and is positive if $g_3 > (\lambda t)^{-1}$. Algebraically, this inequality reduces to $b > \frac{r(r-1)}{3\lambda} + 1$. Since $c_7 = k^2[2g_3 - (\lambda t)^{-1}]$, it is evident that $c_6 > 0$ implies $c_7 > 0$. Also, $c_8 > 0$ if $4 > rk(\lambda t)^{-1}$ which is equivalent to $b > \frac{r^2}{4\lambda}$. Letting $\frac{r(r-1)}{3\lambda} + 1 = \delta_5$ and $(r^2/4\lambda) = \delta_6$, we can then state that $V_7 - V_5 > 0$ if $b > \max(\delta_5, \delta_6)$. In other words, the right hand side of (7) is negative.

However, the sitpulation that $b > \max(\delta_5, \delta_6)$ will reduce to $b > \delta_5$ since $\delta_5 > \delta_6$ for all permissible values of r and λ. To see this, assume $\delta_5 \leq \delta_6$. Under this assumption,

$$\frac{r(r-1)}{3\lambda} + 1 \leq \frac{r^2}{4\lambda}$$

which reduces to

$$r^2 + 12\lambda \leq 4r. \tag{8}$$

The minimum value of the left side of (8) occurs when $r = 2$ and $\lambda = 1$. Hence, $16 \leq 8$ is a contradiction and $\delta_5 > \delta_6$. We can now say that $V_7 - V_5 > 0$ whenever $b > \frac{r(r-1)}{3\lambda} + 1 = \delta_5$. But, by Lemma 7 of Chapter II, we see that $b > [r(r-1)/3\lambda] + 1$ for every permissible value of b, r, and λ. Hence, $V_7 - V_5 > 0$ for every BIB design for which $b > t$. These results are stated more precisely in the following theorem.

Theorem IV - 4. Let V_5 and V_7 be as given in Table II.

Case 1: Assume $\sigma^2 = 0$.

a. If $\sigma_1^2 = 0$, then $V_7 - V_5 = 0$.

b. If $\sigma_1^2 > 0$, then $V_7 - V_5 > 0$.

Case 2: Assume $\sigma^2 > 0$.

If $\sigma^2 \geq 0$, then $V_7 - V_5 > 0$.

Comparison of V_6 and V_7

It now remains to compare V_6 and V_7. But, by Theorem IV-4, V_7 is always larger than V_5 for the non-trivial case. Thus, in those cases for which V_7 might be smaller than V_6, one would naturally pick θ_5 as the estimator of σ_2^2. On the other hand, in those cases for which $V_7 > V_6$, θ_6 would be chosen unless $V_5 < V_6$ for the particular situation.

It is then evident that V_7 can be elininated from further consideration and that only the comparison of V_6 and V_5 is of any value in choosing an estimator of σ_2^2.

Summary

In order to summarize the various comparisons presented in this chapter, Table VII exhibits the thirty designs of interest with the suggested estimators when the true variances are irrelevant. Also shown are conditions on γ_1^2 and γ_2^2 when knowledge of the true ratios of variances is assumed.

The graphs of Table IV are of special interest in that $V_4 < V_3$ regardless of γ_1^2 when γ_2^2 is below its corresponding intercept. These cases are of interest since θ_4 is a function of the intra-error, treatment component under blocks (ignoring treatments), and the additional statistic s_4 which is not found as a sum of squares in the

TABLE VII

PARTIAL CONDITIONS FOR THE SELECTION OF UNIBASED ESTIMATORS IN THIRTY BIB DESIGNS FOR WHICH b > t *

Design No.	r	k	t	Estimator of σ^2	Estimator of σ_1^2	Estimator of σ_2^2
1	3	2	4	θ_1	θ_4 if $\gamma_2^2 \leq 0.333$	θ_5 if $\gamma_1^2 < .25$; θ_6 if $\gamma_1^2 > .500$
2	5	3	6	θ_1	θ_4 if $\gamma_2^2 \leq 0.200$	θ_6
3	7	4	8	θ_1	θ_4 if $\gamma_2^2 \leq 0.143$	θ_6
4	8	6	9	θ_1	θ_4 if $\gamma_2^2 \leq 6.125$	θ_6
5	6	4	10	θ_1	θ_4 if $\gamma_2^2 \leq 1.167$	θ_6
6	9	5	10	θ_1	θ_4 if $\gamma_2^2 \leq 0.111$	θ_6
7	9	6	10	θ_1	θ_4 if $\gamma_2^2 \leq 2.111$	θ_6
8	9	6	16	θ_1	θ_4 if $\gamma_2^2 \leq 1.444$	θ_6
9	10	7	21	θ_1	θ_4 if $\gamma_2^2 \leq 2.100$	θ_6
10	9	7	28	θ_1	θ_4 if $\gamma_2^2 \leq 4.111$	θ_6
11	10	5	9	θ_1	θ_3	θ_6
12	6	3	5	θ_1	θ_3	θ_6
13	10	3	6	θ_1	θ_3	θ_6
14	10	4	6	θ_1	θ_3	θ_6
15	10	5	41	θ_1	θ_3	θ_6
16	4	2	5	θ_1	θ_3	θ_5 if $\gamma_1^2 < .40$; θ_6 if $\gamma_1^2 > .500$
17	5	2	6	θ_1	θ_3	θ_5 if $\gamma_1^2 < .50$; θ_6 if $\gamma_1^2 > .500$
18	6	2	7	θ_1	θ_3	θ_5 if $\gamma_1^2 < .50$; θ_6 if $\gamma_1^2 > .571$
19	7	2	8	θ_1	θ_3	θ_5 if $\gamma_1^2 < .50$; θ_6 if $\gamma_1^2 > .625$
20	8	2	9	θ_1	θ_3	θ_5 if $\gamma_1^2 < .50$; θ_6 if $\gamma_1^2 > .667$

TABLE VII

(Continued)

Design No.	r	k	t	Estimator of σ^2	σ_1^2	σ_2^2
21	8	4	9	θ_1	θ_3	θ_6
22	9	2	10	θ_1	θ_3	θ_5 if $\gamma_1^2 < .50$; θ_6 if $\gamma_1^2 > .700$
23	9	3	10	θ_1	θ_3	θ_6 if $\gamma_1^2 > .017$
24	10	2	11	θ_1	θ_3	θ_5 if $\gamma_1^2 < .50$; θ_6 if $\gamma_1^2 > .727$
25	6	3	13	θ_1	θ_3	θ_6 if $\gamma_1^2 > .051$
26	7	3	15	θ_1	θ_3	θ_6 if $\gamma_1^2 > .067$
27	9	3	19	θ_1	θ_3	θ_6 if $\gamma_1^2 > .088$
28	10	3	21	θ_1	θ_3	θ_6 if $\gamma_1^2 > .095$
29	8	4	25	θ_1	θ_3	θ_6
30	9	4	28	θ_1	θ_3	θ_6

θ_i $(i = 1, \ldots, 6)$ are as defined in Table II

*A choice of estimators may be obtained for those regions of γ_1^2 and γ_2^2 not specified in the above table by consulting the equations of Tables III and V or the graphs of Tables IV and VI.

analysis of variance. [3] On the other hand, θ_3 is a function of inter-error and intra-error. Thus, the statistic not normally computed in the analysis of variance (namely, s_4) has a useful purpose in the estimation of σ_1^2 for those designs under investigation when the choice of θ_3 or θ_4 depends upon the true ratios of variance components.

It should be pointed out that some of the unbiased estimators found in this chapter could give negative estimates of the block and treatment variances. If such a result is deemed negligible on comparison with the intra-error estimate of variance, a zero estimate could be used. However, if a relatively large negative estimate occurs, a re-examination of the entire experimental procedure might prove more feasible.

Example

In conclusion of this chapter an example will be given to illustrate how to choose and compute the different unbiased estimators under consideration. Design 2 (as given in Table III) will be examined using artificial data. Table VIII gives the statistical layout of this design where $r = 5$, $k = 3$, $t = 6$, $b = 10$, and $\lambda = 2$.

TABLE VIII

STATISTICAL LAYOUT

Treatment	1	2	3	4	5	6	Block Totals
Block 1	7.0	5.4			8.2		20.6
2	5.0	3.8				3.0	11.8
3	10.6		9.0	11.3			30.9
4	8.3		7.5			6.0	21.8
5	5.9			7.3	7.2		20.4
6		6.8	7.0	8.7			22.5
7		5.2	5.6		8.4		19.2
8		7.4		9.4		6.4	23.2
9			5.9		8.7	5.0	19.6
10				8.7	9.3	5.5	23.5
Treatment Totals	36.8	28.6	35.0	45.4	41.8	25.9	213.5 Grand Total

Let B_j ($j = 1, \ldots, 6$) denote the j-th treatment total and T_j the total of all blocks containing the j-th treatment. Let

$$B_j - k^{-1} T_j = Q_j .$$

These quantities for the data of Table VIII are as follows:

j	B_j	T_j	Q_j	T_jQ_j
1	36.8	105.5	1.6	168.80
2	28.6	97.3	-3.8	-369.74
3	35.0	114.0	-3.0	-342.00
4	45.4	120.5	5.2	626.60
5	41.8	103.3	7.4	764.42
6	25.9	99.9	-7.4	-739.26
Totals	213.5	640.5	0.0	108.82

The analysis of variance may be obtained from these calculations and is given in Table IX.

TABLE IX

ANALYSIS OF VARIANCE FOR THE DATA IN TABLE VIII

Source	d.f.	S.S.	M.S.
Total	29	108.46	
Blocks (ignoring treatments)	9	67.04	
Treatment Component	5	43.48	8.696
Inter-error	4	23.56	5.890
Treatments (adjusted for blocks)	5	40.64	8.128
Intra-error	15	.78	.052

Weeks [3] has shown that Inter-error S.S. = s_2, Treatment Component S.S. = s_3, Treatments (adj.) S.S. = s_5, Intra-error S.S. = s_6, and $(\lambda t)^{-1/2} k^{-1} \Sigma T_j Q_j = s_4$. Therefore, from the way the unbiased estimators are defined in Table II, the following relations exist:

$$\theta_1 = \text{Intra-error M.S.}$$

$$\theta_3 = k^{-1}[\text{Inter-error M.S. - Intra-error M.S.}],$$

$$\theta_4 = k^{-1}[\text{Treatment Component M.S. - Intra-error M.S.} - \frac{r-\lambda}{k^2} f_1^{-1} \Sigma T_j Q_j].$$

$$\theta_5 = (kf_1)^{-1} \Sigma T_j Q_j, \text{ where } f_1^{-1} = k^2[(\lambda t)^{3/2}(r-\lambda)(t-1)]^{-1},$$

$$\theta_6 = (k/\lambda t)[\text{Treatment (adj.) M.S. - Intra-error M.S.}].$$

Using these results and the mean squares in Table IX we have

$$\theta_1 = .052,$$

$$\theta_3 = \frac{1}{3}[5.89 - .052] = 1.95,$$

$$\theta_4 = \frac{1}{3}[8.696 - .052 - \frac{9}{623.52}(10.472)] = 2.83,$$

$$\theta_5 = .15,$$

and

$$\theta_6 = \frac{1}{4}[8.128 - .052] = 2.02.$$

Consulting Table VII, we see that the estimators of σ^2 and σ_2^2, for Design 2, are θ_1 and θ_6 respectively. Hence, the estimate of σ^2 is .052 and the estimate of σ_2^2 is 2.02.

The decision of whether to pick $\theta_3 = 1.95$ or $\theta_4 = 2.83$ as the estimate of σ_1^2 must now be made. Since $\gamma_2^2 = \sigma_2^2/\sigma^2$, we can estimate the ratio by computing $\hat{\gamma}_2^2 = \frac{\theta_6}{\theta_1} = \frac{2.02}{.052} = 38.8$. Also, two estimates of γ_1^2 are available, namely, $\hat{\gamma}_{1_1}^2 = \frac{\theta_3}{\theta_1} = 37.5$ and $\hat{\gamma}_{1_2}^2 = \frac{\theta_4}{\theta_1} = 54.4$.

Now, using the values in Table III, the equation

$$(.028 + .067\ \gamma_1^2)\gamma_2^2 = .1\gamma_1^4 + .05\gamma_1^2 + .006 \tag{9}$$

is obtained. If

$$\gamma_2^2 > (.1\,\gamma_1^4 + .05\,\gamma_1^2 + .006)/(.028 + .067\,\gamma_1^2)\,,$$

then V_4 is greater than V_3 and θ_3 has smaller variance. If the inequality is reversed then $V_4 < V_3$ and θ_4 has smaller variance.

Substitution of $\hat{\gamma}_{1_1}^2$ and $\hat{\gamma}_{1_2}^2$ into (9) gives

$$\gamma_2^2 = 56.1 \quad \text{and} \quad \gamma_2^2 = 81.3$$

respectively. But $\hat{\gamma}_2^2 = 38.8 < 56.1 < 81.3$.

Thus, since both results indicate that

$$\gamma_2^2 < (.1\,\gamma_1^4 + .05\,\gamma_1^2 + .006)/(.028 + .067\,\gamma_1^2),$$

we could conclude that θ_4 has minimum variance with respect to θ_3. On the basis of this information, $\theta_1 = .052$, $\theta_4 = 2.83$, and $\theta_6 = 2.02$ would be used to estimate σ^2, σ_1^2, and σ_2^2 respectively.

It should be pointed out that the method used for finding the estimates of γ_1^2 did not involve any "a priori" information about the true ratios of variances. However, the method could be misleading since it was not rigorously defined and investigated.

CHAPTER V

THE JOINT DISTRIBUTION OF A MINIMAL SUFFICIENT STATISTIC FOR A GENERAL CLASS OF DESIGNS

General Discussion

In Chapter III the joint distribution of a set of minimal sufficient statistics for a BIB design was found. This chapter will be an extension of that derivation in the sense that a general class of designs will be considered. This general class of designs will include the BIB and the partially-balanced incomplete block design as subsets.

The principal differences between this chapter and Chapter III result from the development of the orthogonal transformation on the vector Y as given by Weeks [3].

The construction of the orthogonal matrix **P**(which is used in the special case of the BIB) makes use of the known characteristic roots of $A'A$ where $A'A$ is the matrix in the system $A'A\hat{\tau} = A'Y$. However, for the general class of designs all of the characteristic roots of $A'A$ are not known. It is assumed that there are s distinct positive characteristic roots of $A'A$ denoted by $d_1, d_2, \ldots, d_s$ of multiplicities $m_1, m_2, \ldots, m_s$ respectively. This distinction has mainly dictated the construction of the orthogonal matrix U as defined in Chapter II.

In addition to the joint distribution of the minimal set of sufficient statistics their expected values and variances will be found.

A Set of Minimal Sufficient Statistics for a General Class of Designs

A set of minimal sufficient statistics for a general class of designs has been given by Weeks and Graybill [4] assuming Eisenhart's Model II. Before giving these statistics it should be noted that the condition $b > t$ is not imposed for the general class of designs.

The $3s + 1$ statistics of the minimal set are as follows:

$$s_1 = \bar{y}\ldots$$

$$s_2 = k^{-1} Y' X_1 P_{21}^* P_{21}^{*\prime} X_1' Y$$

$$s_3 = r^{-1} Y' A P_{31}^* P_{31}^{*\prime} A' Y$$

$$s_4 = Y' P_4 P_4' Y$$

$$s_{5i} = k^{-1} Y' X_1 P_{2i} P_{2i}' X_1' Y \qquad i = 2, \ldots, s$$

$$s_{6i} = d_i^{-1} Y' A P_{3i} P_{3i}' A' Y \qquad i = 2, \ldots, s$$

$$s_{7i} = k^{-1} Y' X_1 N P_{3i} P_{3i}' A' Y \qquad i = 2, \ldots, s$$

where P_{21}^*, P_{31}^* and P_{3i} are as defined in Chapter II. The matrix notation for these statistics will now be expressed in terms of the partitions of Z which is to be defined.

From the distributional properties given in Chapter II we have that $U'Y \sim MVN\ (U'\bar{\mu}, U' \Sigma U)$. Let $Z = U'Y$ and partition Z as

$$Z' = [Z_1', Z_2', Z_2^{*\prime}, Z_3', Z_3^{*\prime}, Z_4']$$

with the dimensions of the partitions as follows:

1. Z_1 (1 x 1),

2. $Z_2(m_1 + b - t \times 1)$,

3. Z_2^* (p x 1) where $p = t - 1 - m_1$,

4. Z_3 (m_1 x 1),

5. Z_3^* (p x 1),

6. Z_4 (u x 1) where $u = M - b - t + 1$.

The partition of U as given in Chapter II is

$$U = [M^{-1/2}J_1^M, k^{-1/2}X_1P_{21}^*, k^{-1/2}X_1P_2, AP_3D_{A'A}^{-1/2}, P_4].$$

In order for the partition of Z to correspond to that of U, the partition

$$P_3 = [P_{31}^*, \tilde{P}_3]$$

is used. Hence,

$$U = [M^{-1/2}J_1^M, k^{-1/2}X_1P_{21}^*, k^{-1/2}X_1P_2, AP_{31}^*D_{A'A}^{-1/2}, A\tilde{P}_3D_{A'A}^{-1/2}, P_4]$$

and

$$U'Y = \begin{bmatrix} M^{-1/2}J_M^1\ Y \\ k^{-1}P_{21}^{*\prime}X_1'Y \\ k^{-1/2}P_2'X_1'Y \\ D_{A'A}^{-1/2}\ P_{31}^{*\prime}A'Y \\ D_{A'A}^{-1/2}\ \tilde{P}_3'A'Y \\ P_4'Y \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_2^* \\ Z_3 \\ Z_3^* \\ Z_4 \end{bmatrix} = Z.$$

It should be noted that

$$D_{A'A}^{-1/2} P_3' = D_{A'A}^{-1/2} \begin{bmatrix} P_{31}^{*\prime} \\ \tilde{P}_3' \end{bmatrix} = \begin{bmatrix} d_1^{-1/2} P_{31}^{*\prime} \\ d_2^{-1/2} P_{32}' \\ \vdots \\ d_s^{-1/2} P_{3s}' \end{bmatrix}$$

where the d_i $(i = 1, 2, \ldots, s)$ are the s distinct positive characteristic roots of $A'A$. Weeks and Graybill [4] have assumed that $d_1 = r$.

Next, partition P_2 as $P_2 = [P_{22}, P_{23}, \ldots, P_{2s}]$ to obtain

$$k^{-1/2} P_2' X_1' Y = \begin{bmatrix} k^{-1/2} P_{22}' X_1' Y \\ k^{-1/2} P_{23}' X_1' Y \\ \vdots \\ k^{-1/2} P_{2s}' X_1' Y \end{bmatrix} = \begin{bmatrix} Z_{22} \\ Z_{23} \\ \vdots \\ Z_{2s} \end{bmatrix} = Z_2^*$$

where the dimension of Z_{2i} is $m_i \times 1$ for $i = 2, \ldots, s$.

Also partition $\tilde{P}_3$ as $\tilde{P}_3 = [P_{32}, P_{33}, \ldots, P_{3s}]$ to obtain

$$D_{A'A}^{-1/2} \tilde{P}_3' A' Y = \begin{bmatrix} d_2^{-1/2} P_{32}' A' Y \\ d_3^{-1/2} P_{33}' A' Y \\ \vdots \\ d_s^{-1/2} P_{3s}' A' Y \end{bmatrix} = \begin{bmatrix} Z_{32} \\ Z_{33} \\ \vdots \\ Z_{3s} \end{bmatrix}$$

where the dimension of Z_{3i} is $m_i \times 1$ for $i = 2, \ldots s$. $U'Y$ can now be rewritten as

$$U'Y = \begin{bmatrix} M^{-1/2}J_M^1 Y \\ k^{-1/2}P_{21}^{*'} X_1'Y \\ k^{-1/2}P_{22}'X_1'Y \\ \cdot \\ \cdot \\ \cdot \\ k^{-1/2}P_{2s}'X_1'Y \\ r^{-1/2}P_{31}^{*}'A'Y \\ d_2^{-1/2}P_{32}'A'Y \\ \cdot \\ \cdot \\ \cdot \\ d_s^{-1/2}P_{3s}'A'Y \\ P_4'Y \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_{22} \\ \cdot \\ \cdot \\ \cdot \\ Z_{2s} \\ Z_3 \\ Z_{32} \\ \cdot \\ \cdot \\ \cdot \\ Z_{3s} \\ Z_4 \end{bmatrix} = Z.$$

From this partitioning of $U'Y = Z$ the following relationships exist:

1. $Z_1'Z_1 = M^{-1}Y'J_1^M J_M^1 Y = M s_1^2$
2. $Z_2'Z_2 = k^{-1}Y'X_1P_{21}^{*}P_{21}^{*}'X_1'Y = s_2$
3. $Z_{2i}'Z_{2i} = k^{-1}Y'X_1P_{2i}P_{2i}'X_1'Y = s_{5i} \quad i = 2, \ldots, s$
4. $Z_3'Z_3 = r^{-1}Y'AP_{31}^{*}P_{31}^{*}'A'Y = s_3$
5. $Z_{3i}'Z_{3i} = d_i^{-1}Y'AP_{3i}P_{3i}'A'Y = s_{6i} \quad i = 2, \ldots, s$
6. $Z_4'Z_4 = Y'P_4P_4'Y = s_4$
7. $Z_{2i}'Z_{3i} = (kd_i)^{-1/2}Y'X_1P_{2i}P_{3i}'A'Y = k_i s_{7i},$

   where $k_i = [d_i(r-d_i)]^{-1/2} \qquad i = 2, \ldots, s.$

In explanation of the seventh relationship it can be shown that

$$P_{2i}' = [k(r-d_i)]^{-1/2} P_{3i}' N.$$

Hence,

$$Z_{2i}' Z_{3i} = [k^2 d_i(r-d_i)]^{-1/2} Y'X_1NP_{3i}P_{3i}'A'Y$$

$$= [d_i(r-d_i)]^{-1/2} k^{-1} Y'X_1NP_{3i}P_{3i}'A'Y$$

$$= k_i s_{7i}.$$

The Joint Distribution of s_{5i}, s_{6i}, and s_{7i} $(i = 2, \ldots, s)$

In order to find the joint distribution of the $3s - 3$ statistics s_{5i}, s_{6i}, and s_{7i}, the covariance matrix of Z as well as $E(Z)$ must be found. These two matrices will be presented in correspondence with the partition

$$Z' = [Z_1', Z_2', Z_2^{*}{}', Z_3', Z_3^{*}{}', Z_4'] .$$

First,

$$E(Z') = [\mu M^{1/2}, \phi, \phi, \phi, \phi, \phi].$$

In explanation of the first element, we have

$$E(Z_1) = M^{-1/2} J_M^1 EY = \mu M^{-1/2} J_M^1 J_1^M = \mu M^{1/2} .$$

All the other elements of $E(Z)$ are zero since $E(Z) = (U' \bar{\mu}) = \mu U'J_1^M$ and all columns of U except the first add to zero.

The covariance matrix of Z [4] is given as

$$U' \Sigma U = \begin{bmatrix} G_1 & \phi & \phi & \phi & \phi & \phi \\ \phi & G_2 & \phi & \phi & \phi & \phi \\ \phi & \phi & G_2^{*} & \phi & G_{23}^{*} & \phi \\ \phi & \phi & \phi & G_3 & \phi & \phi \\ \phi & \phi & G_{23}^{*} & \phi & G_3^{*} & \phi \\ \phi & \phi & \phi & \phi & \phi & G_4 \end{bmatrix}$$

where

$$G_1 = \sigma^2 + k\sigma_1^2 + r\sigma_2^2,$$

$$G_2 = [\sigma^2 + k\sigma_1^2]\, I_{m_1+w},$$

$$G_2^* = [\sigma^2 + k\sigma_1^2]\, I_p + k^{-1}\sigma_2^2 D_{NN'},$$

$$G_{23}^* = k^{-1/2}\sigma_2^2 D_{NN'}^{1/2}\, \tilde{D}_{A'A}^{1/2},$$

$$G_3 = (\sigma^2 + r\sigma_2^2)\, I_{m_1},$$

$$G_3^* = \sigma^2 I_p + \sigma_2^2 \tilde{D}_{A'A},$$

and

$$G_4 = \sigma^2 I_u .$$

In explanation of the above notation, we have:

1. $$\tilde{D}_{A'A} = \begin{bmatrix} d_2 I_{m_2} & & \phi \\ & \ddots & \\ \phi & & d_s I_{m_s} \end{bmatrix}$$

where d_i and m_i $(i = 2, \ldots, s)$ are as defined in Chapter II.

2. $$D_{NN'} = \begin{bmatrix} k(r-d_2)I_{m_2} & & & \phi \\ & k(r-d_3)I_{m_3} & & \\ & & \ddots & \\ \phi & & & k(r-d_s)I_{m_s} \end{bmatrix}.$$

Expressing G_2^*, G_{23}^*, and G_3^* in terms of their respective diagonal elements, we have:

1. $$G_2^* = \begin{bmatrix} [\sigma^2+k\sigma_1^2+(r-d_2)\sigma_2^2]I_{m_2} & & \phi \\ & \ddots & \\ \phi & & [\sigma^2+k\sigma_1^2+(r-d_s)\sigma_2^2]I_{m_s} \end{bmatrix}.$$

2. $$G_{23}^* = \begin{bmatrix} [d_2(r-d_2)]^{1/2}\sigma_2^2 I_{m_2} & & \phi \\ & \ddots & \\ \phi & & [d_s(r-d_s)]^{1/2}\sigma_2^2 I_{m_s} \end{bmatrix} .$$

3. $$G_3^* = \begin{bmatrix} [\sigma^2 + d_2\sigma_2^2] I_{m_2} & & \phi \\ & \ddots & \\ \phi & & [\sigma^2 + d_s\sigma_2^2] I_{m_s} \end{bmatrix} .$$

From multivariate normal theory a necessary and sufficient condition for the subvectors of Z to be jointly independent is that corresponding submatrices of $U' \Sigma U$ be equal to the null matrix. Hence, from the covariance matrix of $U'Y = Z$ we have that Z_1, Z_2, Z_3, and Z_4 are mutually independent and each jointly independent of Z_2^* and Z_3^*. The only dependency is between the subvectors Z_2^* and Z_3^*. From these considerations the independence of four of the $3s + 1$ sufficient statistics can be determined.

Writing $s_1 = M^{-1/2} Z_1$, $s_2 = Z_2' Z_2$, $s_3 = Z_3' Z_3$, and $s_4 = Z_4' Z_4$ we can conclude that the $s_j (j = 1, \ldots, 4)$ are mutually independent due to the mutual independence of the Z_j $(j = 1, \ldots, 4)$.

Applying still another theorem from multivariate normal theory [1] we have the result that

$$Z^* \sim MVN(\phi, \Sigma^{**})$$

where

$$Z^{*\prime} = [Z_2^{*\prime}, Z_3^{*\prime}]_{1 \times 2p}$$

and

$$\not{\Sigma}^{**} = \begin{bmatrix} G_2^{*} & G_{23}^{*} \\ G_{23}^{*\prime} & G_3^{*} \end{bmatrix}_{2p \times 2p} .$$

By partitioning Z_2^{*} and Z_3^{*} as previously defined,

$$(\text{i.e., } Z^{*\prime} = [\, Z_{22}, \ldots, Z_{2s}, Z_{32}, \ldots, Z_{3s}] \,)$$

we see that the covariance matrix of Z^{*} is

$$\operatorname{cov}(Z^{*}) = \begin{bmatrix} c_2^{*} I_{m_2} & & \phi & d_2^{*} I_{m_2} & & \phi \\ & \ddots & & & \ddots & \\ \phi & & c_s^{*} I_{m_s} & \phi & & d_s^{*} I_{m_s} \\ d_2^{*} I_{m_2} & & \phi & e_2^{*} I_{m_2} & & \phi \\ & \ddots & & & \ddots & \\ \phi & & d_s^{*} I_{m_s} & \phi & & e_s^{*} I_{m_s} \end{bmatrix}$$

where

$$c_i^{*} = \sigma^2 + k\sigma_1^{\,2} + (r - d_i)\,\sigma_2^{\,2},$$

$$d_i^{*} = [\, d_i(r - d_i)]^{1/2}\, \sigma_2^{\,2},$$

and

$$e_i^{*} = \sigma^2 + d_i\, \sigma_2^{\,2}$$

for $i = 2, \ldots, s$. For the subvector Z^{*} the only dependencies between Z_{2i} and Z_{3j} are when $i = j$ $(i, j = 2, \ldots, s)$. Hence,

$$\begin{bmatrix} Z_{2i} \\ Z_{3i} \end{bmatrix} \sim \text{MVN}\left(\begin{bmatrix} \phi \\ \phi \end{bmatrix}, \begin{bmatrix} c_i^{*} I_{m_i} & d_i^{*} I_{m_i} \\ d_i^{*} I_{m_i} & e_i^{*} I_{m_i} \end{bmatrix} \right)$$

for $i = 2, \ldots, s$, where Z_{2i} and Z_{3i} have dimension $m_i \times 1$.

Let $\begin{bmatrix} Z_{2i} \\ Z_{3i} \end{bmatrix}$ be denoted by Z_{23i}^{*}. Then, $\operatorname{cov}(Z_{23i}^{*}, Z_{23j}^{*}) = \phi$

for $i \neq j$, $(i, j = 2, \ldots, s)$. Now, partition Z_{2i} and Z_{3i} as

$$Z_{2i}' = [z_{2i_1}, z_{2i_2}, \ldots, z_{2i_{m_i}}]$$

and

$$Z_{3i}' = [z_{3i_1}, z_{3i_2}, \ldots, z_{3i_{m_i}}].$$

Again, from the covariance matrix of Z_{23i}^* we see that the only dependencies between z_{2i_α} and $z_{3i_{\alpha'}}$ are when $\alpha = \alpha'$, $(\alpha, \alpha' = 1, 2, \ldots, m_i)$.

Thus, denoting $\begin{bmatrix} z_{2i_\alpha} \\ z_{3i_\alpha} \end{bmatrix}$ as Q_{i_α}, we have

$$Q_{i_\alpha} \sim \text{BVN} \left(\binom{0}{0} ; \begin{bmatrix} c_i^* & d_i^* \\ d_i^* & e_i^* \end{bmatrix} \right), \text{ for } \alpha = 1, 2, \ldots, m_i.$$

Also,

$$\text{cov}\,(Q_{i_\alpha}, Q_{i_{\alpha'}}) = \phi \quad \text{if } \alpha \neq \alpha'.$$

Recalling the theorem in Chapter III as given by Anderson [1] we have that

$$A_i = \sum_{\alpha=1}^{m_i} Q_{i_\alpha} Q_{i_\alpha}'$$

is distributed as a Wishart with parameters m_i and

$$\Sigma_i = \begin{bmatrix} c_i^* & d_i^* \\ d_i^* & e_i^* \end{bmatrix}.$$

Now, writing

$$\sum_{\alpha=1}^{m_i} Q_{i_\alpha} Q_{i_\alpha}'$$

as a matrix we have

$$A_i = \sum_{\alpha=1}^{m_i} \begin{bmatrix} z_{2i_\alpha} \\ z_{3i_\alpha} \end{bmatrix} [z_{2i_\alpha}, z_{3i_\alpha}] = \sum_{\alpha=1}^{m_i} \begin{bmatrix} z_{2i_\alpha}^2 & z_{2i_\alpha} z_{3i_\alpha} \\ z_{3i_\alpha} z_{2i_\alpha} & z_{3i_\alpha}^2 \end{bmatrix}$$

or,

$$A_i = \begin{bmatrix} \Sigma z_{2i_\alpha}^2 & \Sigma z_{2i_\alpha} z_{3i_\alpha} \\ \Sigma z_{3i_\alpha} z_{2i_\alpha} & \Sigma z_{3i_\alpha}^2 \end{bmatrix}$$

$$= \begin{bmatrix} Z_{2i}' Z_{2i} & Z_{2i}' Z_{3i} \\ Z_{3i}' Z_{2i} & Z_{3i}' Z_{3i} \end{bmatrix}$$

$$= \begin{bmatrix} s_{5i} & k_i s_{7i} \\ k_i s_{7i} & s_{6i} \end{bmatrix} .$$

Hence,

$$A_i \sim W(\not\Sigma_i, m_i) \quad (i = 2, \ldots, s)$$

or,

$$h_i(s_{5i}, s_{6i}, s_{7i}) = \frac{|A_i|^{(1/2)(m_i - 3)} \exp - \frac{1}{2} \operatorname{tr} A_i \not\Sigma_i}{2^{m_i} \pi^{1/2} |\not\Sigma_i|^{m_i/2} \Gamma(\frac{m_i}{2}) \Gamma(\frac{m_i - 1}{2})} .$$

Now, since A_i and A_j are independent for $i \neq j$, we have

$$h_1(s_{52}, \ldots s_{5s}, s_{62}, \ldots s_{6s}, s_{72}, \ldots, s_{7s})$$

$$= \prod_{i=2}^{s} h_i(s_{5i}, s_{6i}, s_{7i}) .$$

Hence, the joint distribution is the product of $s - 1$ independent Wisharts.

For the joint distribution of the $3s + 1$ statistics, it remains to find $f_1(s_1)$, $f_2(s_2)$, $f_3(s_3)$, and $f_4(s_4)$ since each of s_1, s_2, s_3, and s_4 is mutually independent of all the other statistics.

The Joint Distribution of s_1, s_2, s_3, and s_4

As was noted in the previous section $h(s_1, s_2, s_3, s_4) = f_1(s_1)f(s_2)f(s_3)f(s_4)$ due to the independence of the four statistics involved. In this section the distributions will be found for each of the four statistics s_1, s_2, s_3, and s_4.

Distribution of $s_1 = \bar{y}_{\cdot\cdot\cdot\cdot}$

If $Y \sim MVN(\bar{\mu}, \not\Sigma)$, then $BY \sim MVN(B\bar{\mu}, B \not\Sigma B')$ where B is a $q \times M$ matrix of rank $q \leq M$. [1] Hence, expressing s_1 as $s_1 = \frac{1}{M} J_M^1 Y$, we have that

$$s_1 \sim N\left(\frac{1}{M} J_M^1 \bar{\mu}, \frac{1}{M^2} J_M^1 \not\Sigma J_1^M\right).$$

But $\bar{\mu} = \mu J_1^M$, where μ is a scalar, and

$$\not\Sigma = [X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + \sigma^2 I_M].$$

Therefore,

$$E(s_1) = \frac{1}{M} J_M^1 J_1^M \mu = \mu$$

and

$$\begin{aligned} \text{var}(s_1) &= M^{-2}[J_M^1 X_1 X_1' J_1^M \sigma_1^2 + J_M^1 X_2 X_2' J_1^M \sigma_2^2 \\ &\quad + J_M^1 J_1^M \sigma^2] \\ &= M^{-2}[k^2 J_b^1 J_1^b \sigma_1^2 + r^2 J_t^1 J_1^t \sigma_2^2 + M\sigma^2] \\ &= M^{-1}[k\sigma_1^2 + r\sigma_2^2 + \sigma^2]. \end{aligned}$$

Thus,

$$s_1 \sim N(\mu, M^{-1}[k\sigma_1^2 + r\sigma_2^2 + \sigma^2]).$$

Let the functional form of this distribution be denoted by $f_1(s_1)$.

Before proceeding to the other distributions, it will be helpful to make use of a theorem given by Graybill [2] which states that if $Y \sim N(\mu, \Sigma)$, then $Y'BY \sim \chi'^2(k, \lambda = \frac{1}{2}\mu' B \mu)$ if B is of rank k and $B\Sigma$ is idempotent. This theorem will be used in finding the distributions of s_2, s_3, and s_4.

<u>Distribution of $s_2 = k^{-1} Y' X_1 P_{21}^* P_{21}^{*\prime} X_1' Y$</u>

Let $B_2 = k^{-1} X_1 P_{21}^* P_{21}^{*\prime} X_1'$. Then

$$B_2 \Sigma B_2 = k^{-2} X_1 P_{21}^* P_{21}^{*\prime} X_1' [X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + \sigma^2 I_M] X_1 P_{21}^* P_{21}^{*\prime} X_1'$$

$$= k^{-2} X_1 P_{21}^* P_{21}^{*\prime} [\sigma_1^2 k^2 I_b + \sigma_2^2 N'N + \sigma^2 k I_b] P_{21}^* P_{21}^{*\prime} X_1'.$$

But

$$P_{21}^{*\prime} N'N P_{21}^* = \phi_{m_1+w}$$

and

$$P_{21}^{*\prime} P_{21}^* = I_{m_1+w}.$$

Therefore,

$$B_2 \Sigma B_2 = k^{-2} X_1 P_{21}^* [(\sigma_1^2 k^2 + \sigma^2 k) I_{m_1+w}] P_{21}^{*\prime} X_1'$$

$$= k^{-1}(\sigma_1^2 k + \sigma^2) X_1 P_{21}^* P_{21}^{*\prime} X_1'$$

$$= (\sigma^2 + k\sigma_1^2)\ B_2.$$

Let $B_2^* = (\sigma^2 + k\sigma_1^2)^{-1} B_2$. Then, $B_2^* \Sigma B_2^* = B_2^*$ and

$$Y' B_2^* Y \sim \chi'^2(\rho(B_2^*), \tfrac{1}{2}\bar{\mu}' B_2^* \bar{\mu}),$$

where $\rho(B_2^*)$ denotes the rank of B_2^*.

Now $\rho(B_2^*) = \rho(B_2)$. However, it can be shown that B_2 is idempotent. Hence, $\rho(B_2) = tr(B_2)$ and

$$tr(B_2) = k^{-1} tr(X_1 P_{21}^* P_{21}^{*\prime} X_1^{\prime})$$
$$= tr(P_{21}^{*\prime} P_{21}^*)$$
$$= m_1 + w.$$

Also,

$$\frac{1}{2}\bar{\mu}' B_2^* \bar{\mu} = \frac{1}{2}\mu^2 [k(\sigma^2 + k\sigma_1^2)]^{-1} J_M^1 X_1 P_{21}^* P_{21}^{*\prime} X_1^{\prime} J_1^M .$$

But $J_M^1 X_1 P_{21}^* = k J_b^1 P_{21}^*$ and this operation is adding the elements of each column of P_{21}^*. Since the first column of the orthogonal matrix P_2^*, of which P_{21}^* is a partition, is a vector of the form $b^{-1/2} J_1^b$, the other columns of P_2^* must add to zero. Hence, $J_b^1 P_{21}^* = \phi$ and $\frac{1}{2}\bar{\mu}' B_2^* \bar{\mu} = 0$.

It has now been shown that

$$(\sigma^2 + k\sigma_1^2)^{-1} Y' B_2 Y \sim \chi^2(m_1 + w)$$

or,

$$(\sigma^2 + k\sigma_1^2)^{-1} s_2 \sim \chi^2(m_1 + w).$$

Let the density of s_2 be denoted by $f_2(s_2)$.

<u>Distribution of $s_3 = r^{-1} Y' AP_{31}^* P_{31}^{*\prime} A' Y$</u>

Let $B_3 = r^{-1} AP_{31}^* P_{31}^{*\prime} A'$. Then,

$$B_3 \not\Sigma B_3 = r^{-2} AP_{31}^* P_{31}^{*\prime} A' [X_1 X_1' \sigma_1^2 + X_2 X_2' \sigma_2^2 + \sigma^2 I_M] AP_{31}^* P_{31}^{*\prime} A' .$$

But $A'X_1 = \phi$ and $P_{31}^{*\prime} A X_2 X_2' AP_{31}^* = P_{31}^{*\prime} A'AA'AP_{31}^* = r^2 I_{m_1}$.

Therefore,

$$B_3 \not\Sigma B_3 = r^{-2} AP_{31}^* [\sigma_2^2 r^2 I_{m_1} + \sigma^2 r I_{m_1}] P_{31}^{*\prime} A'$$

$$B_3 \Sigma B_3 = r^{-1}(\sigma_2^{\,2} r + \sigma^2) A P_{31}^{\;*} P_{31}^{\;*\prime} A'$$

$$= (\sigma^2 + r\sigma_2^{\,2}) B_3.$$

Let $B_3^{\;*} = (\sigma^2 + r\sigma_2^{\,2})^{-1} B_3$. Then $B_3^{\;*} \Sigma B_3^{\;*} = B_3^{\;*}$ and

$$Y' B_3^{\;*} Y \sim \chi'^2 \left(\rho\,(B_3^{\;*}),\ \tfrac{1}{2}\,\bar{\mu}' B_3^{\;*} \bar{\mu}\right).$$

Next, $\rho(B_3^{\;*}) = \rho\,(B_3)$. But it is easily shown that B_3 is idempotent. Hence,

$$\rho\,(B_3^{\;*}) = \operatorname{tr}\,(B_3) = r^{-1} \operatorname{tr}(P_{31}^{\;*\prime} A' A\, P_{31}^{\;*}) = m_1.$$

Also,

$$\tfrac{1}{2}\,\bar{\mu}' B_3^{\;*} \bar{\mu} = \tfrac{1}{2}\,\mu^2 (\sigma^2 + r\,\sigma_2^{\,2})^{-1} J_M^1 A P_{31}^{\;*} P_{31}^{\;*\prime} A' J_1^M = 0$$

since $J_M^1 A = \phi$. Thus,

$$Y' B_3^{*} Y = (\sigma^2 + r\sigma_2^{\,2})^{-1} Y' B_3 Y = (\sigma^2 + r\sigma_2^{\,2})^{-1} s_3$$

and

$$(\sigma^2 + r\,\sigma_2^{\,2})^{-1}\ s_3 \sim \chi^2\,(m_1).$$

Let the density of s_3 be denoted by $f_3(s_3)$.

<u>Distribution of $s_4 = Y' P_4 P_4' Y$</u>

Let $B_4 = P_4 P_4'$. Then,

$$B_4 \Sigma B_4 = P_4 P_4' \left[\sigma_1^{\,2} X_1 X_1' + \sigma_2^{\,2} X_2 X_2' + \sigma^2 I_M\right] P_4 P_4'.$$

Weeks and Graybill [4] have shown that $P_4' X_1 = \phi$ and $P_4' X_2 = \phi$. Hence,

$$B_4 \Sigma B_4 = P_4 [\sigma^2 P_4' P_4] P_4' = \sigma^2 P_4 I_u P_4' = \sigma^2 P_4 P_4'.$$

Let $B_4^{\;*} = \sigma^{-2} B_4$. Then,

$$B_4^{\;*} \Sigma B_4^{\;*} = B_4^{\;*}$$

and

$$Y'B_4^* Y \sim \chi'^2(\rho(B_4^*), \tfrac{1}{2}\bar{\mu}'B_4^*\bar{\mu}).$$

But $\rho(B_4^*) = \rho(B_4)$ and B_4 is idempotent. Hence,

$$\rho(B_4^*) = \text{tr } B_4 = \text{tr}(P_4'P_4) = \text{tr } I_u = u$$

where $u = M - b + t + 1$. Also,

$$\frac{1}{2}\bar{\mu}' B_4^* \bar{\mu} = \frac{1}{2}\mu^2 J_M^1 P_4 P_4' J_1^M = 0$$

since the elements in each column of P_4 add to zero. Thus,

$$Y' B_4^* Y = \sigma^{-2} Y'B_4 Y = \sigma^{-2} s_4$$

and

$$\sigma^{-2} s_4 \sim \chi^2(u).$$

Let the density of s_4 be denoted by $f_4(s_4)$.

In summary the distributions of the four statistics are

$$s_1 \sim N(\mu, M^{-1}[\sigma^2 + k\sigma_1^2 + r\sigma_2^2]),$$

$$s_2 \sim (\sigma^2 + k\sigma_1^2)\chi^2(m_1 + w),$$

$$s_3 \sim (\sigma^2 + r\sigma_2^2)\chi^2(m_1),$$

and

$$s_4 \sim \sigma^2\chi^2(u).$$

Thus, due to the independence property, it can be stated that

$$h(s_1, s_2, s_3, s_4) = f_1(s_1)f_2(s_2)f_3(s_3)f_4(s_4).$$

This joint distribution is the product of a normal distribution and three independent chi-square distributions.

The Joint Distribution of the 3s + 1 Statistics of the Minimal Set

Knowing the joint distributions h of s_1, s_2, s_3, and s_4 and h_1 of s_{5i}, s_{6i}, and s_{7i}, $(i = 2, \ldots, s)$, and since h and h_1 involve independent variables, we can now define the joint distribution h_o (say) of all the statistics in the minimal set for a general class of designs. Hence,

$$h_o(s_1, s_2, s_3, s_4, s_{52}, \ldots, s_{5s}, s_{62}, \ldots, s_{6s}, s_{72}, \ldots, s_{7s})$$

$$= h(s_1, s_2, s_3, s_4) h_1(s_{52}, \ldots, s_{5s}, s_{62}, \ldots, s_{6s}, s_{72}, \ldots, s_{7s})$$

$$= f_1(s_1) f_2(s_2) f_3(s_3) f_4(s_4) \prod_{i=2}^{s} h_i(s_{5i}, s_{6i}, s_{7i}).$$

With the knowledge of this distribution we are now in a position to give the expected values of the 3s + 1 statistics and the corresponding covariance matrix. Since s_2, s_3, and s_4 are constants times χ^2 variables their means and variances are easily obtained by knowing their respective distributions. The means and variances of the statistics involved in the s - 1 independent Wisharts are also readily accessible by the same procedure used in Chapter IV. These derivations are the content of the next section.

Expected Values and Variances of the 3s + 1 Statistics

Expected Value and Variance of s_1

Since s_1 is a normal variate whose distribution is known, we have $E(s_1) = \mu$ and $\text{var}(s_1) = M^{-1}(\sigma^2 + k\sigma_1^2 + r\sigma_2^2)$.

Expected Value and Variance of s_2

It has been shown that $(\sigma^2 + k\sigma_1^2)^{-1} s_2 \sim \chi^2 (m_1 + w)$. Thus, from the distributional properties of a χ^2 variable we have that

$$E [(\sigma^2 + k\sigma_1^2)^{-1} s_2] = m_1 + w.$$

Hence,

$$E(s_2) = (\sigma^2 + k\sigma_1^2)(m_1 + w).$$

Also,

$$\text{var} [(\sigma^2 + k\sigma_1^2)^{-1} s_2] = 2(m_1 + w).$$

Hence,

$$\text{var} (s_2) = 2 (\sigma^2 + k\sigma_1^2)^2(m_1+w).$$

Expected Value and Variance of s_3

Since $(\sigma^2 + r \sigma_2^2)^{-1} s_3 \sim \chi^2(m_1)$, we have, by the same method previously used, $E(s_3) = \sigma^2 + r\sigma_2^2)m_1$ and $\text{var} (s_3)=2(\sigma^2+r\sigma_2^2)^2 m_1$.

Expected Value and Variances of s_4

Since $\sigma^{-2} s_4 \sim \chi^2(u)$, we have $E(s_4) = \sigma^2 u$ and $\text{var} (s_4) = 2\sigma^4 u$.

Expected Values of s_{5i}, s_{6i}, and s_{7i} $(i = 2, . . . , s)$

It was previously shown in this chapter that

$$A_i = \sum_{\alpha=1}^{m_i} Q_{i\alpha} Q_{i\alpha}'$$

is distributed as a Wishart with covariance matrix

$$\not\Sigma_i = \begin{bmatrix} c_i^* & d_i^* \\ d_i^* & e_i^* \end{bmatrix}$$

where

$$c_i^* = \sigma^2 + k\,\sigma_1^2 + (r-d_i)\,\sigma_2^2 ,$$

$$d_i^* = [d_i(r-d_i)]^{1/2}\,\sigma_2^2 ,$$

and

$$e_i^* = \sigma^2 + d_i'\,\sigma_2^2 .$$

But

$$A_i = \begin{bmatrix} s_{5i} & k_i s_{7i} \\ k_i s_{7i} & s_{6i} \end{bmatrix}$$

where $k_i = [d_i(r-d_i)]^{-1/2}$. Hence, by the same procedure used in Chapter IV, the expected values and variances of the elements of A_i are easily found.

The expected value of the (p, q)-th element of the matrix A_i is m_i times the corresponding element of Σ_i. Using this fact, the expected values of s_{5i}, s_{6i}, and s_{7i} are as follows:

1. $E(s_{5i}) = m_i c_i^* = m_i[\sigma^2 + k\sigma_1^2 + (r-d_i)\sigma_2^2]$.

2. $E(s_{6i}) = m_i\, e_i^* = m_i[\sigma^2 + d_i\sigma_2^2]$.

3. $E(s_{7i}) = m_i k_i^{-1}\, d_i^*$

$$= m_i[d_i(r-d_i)]^{1/2}[d_i(r-d_i)]^{1/2}\sigma_2^2$$

$$= m_i d_i(r-d_i)\,\sigma_2^2 .$$

These three expected values hold for $i = 2, \ldots, s$.

<u>Variance of s_{5i}, s_{6i}, and s_{7i} $(i = 2, \ldots, s)$</u>

If $A_{i_{pq}}$ denotes the (p, q)-th element of A_i and σ_{pq} denotes the (p, q)-th element of Σ_i, then the general expression for the covariance of any two elements of A_i is

$$\operatorname{cov}(A_{i_{pq}}, A_{i_{p'q'}}) = m_i(\sigma_{pp'}\sigma_{qq'} + \sigma_{pq'}\sigma_{qp'}).$$

Using this general expression, the variances of s_{5i}, s_{6i}, and s_{7i} are as follows:

1. $\operatorname{var} s_{5i} = 2m_i c_i^{*2} = 2m_i[\sigma^2 + k\sigma_1^2 + (r-d_i)\sigma_2^2]^2$

2. $\operatorname{var} s_{6i} = 2m_i e_i^{*2} = 2m_i[\sigma^2 + d_i\sigma_2^2]^2$

3. $\operatorname{var} s_{7i} = k_i^{-2} m_i[d_i^{*2} + c_i^* e_i^*]$

$$= d_i(r-d_i)m_i[\sigma^4 + 2d_i(r-d_i)\sigma_2^4 + k\sigma^2\sigma_1^2 + r\sigma^2\sigma_2^2 + kd_i\sigma_1^2\sigma_2^2].$$

These three variances hold for $i = 2, \ldots, s$.

<u>Covariance of (s_{5i}, s_{6i}); (s_{5i}, s_{7i}); and (s_{6i}, s_{7i}) $(i = 2, \ldots, s)$</u>

Again using the general expression for the covariance of two elements of A_i as previously given, the covariances of s_{5i}, s_{6i}, and s_{7i} are as follows:

1. $\operatorname{cov}(s_{5i}, s_{6i}) = 2m_i d_i^{*2} = 2m_i d_i(r-d_i)\sigma_2^4.$

2. $\operatorname{cov}(s_{5i}, s_{7i}) = 2m_i k_i^{-2} c_i^* d_i^*$

$$= 2m_i d_i(r-d_i)[(r-d_i)\sigma_2^4 + \sigma^2\sigma_2^2 + k\sigma_1^2\sigma_2^2].$$

3. $\operatorname{cov}(s_{6i}, s_{7i}) = 2m_i k_i^{-2} d_i^* e_i^*$

$$= 2m_i d_i(r-d_i)[d_i\sigma_2^4 + \sigma^2\sigma_2^2].$$

As before, the three covariances hold for $i = 2, \ldots, s$.

Table X summarizes the preceding drivations and gives the expected values and the covariance matrix of the $3s + 1$ statistics. Using the results of this chapter, one could now find unbiased estimators of the different variance components based on the minimal set of sufficient statistics and the respective variances of the estimators. Certainly, as in the special cases of the BIB designs of Chapter IV, any variance of an estimator will be a function of the true variances themselves. This fact extremely complicates the search for an estimator with the minimum variance property. Excluding special cases of the general class of designs, the matter is further complicated by the lack of knowledge of the characteristic roots of $A'A$ and their respective multiplicities. These statements are obvious upon examination of the variances and covariances given in Table X.

TABLE X

EXPECTED VALUES AND COVARIANCE MATRIX OF THE MINIMAL SUFFICIENT STATISTICS FOR A GENERAL CLASS OF DESIGNS

Statistic	Expected Value	Covariance Matrix
s_1	μ	H_1
s_2	$(m_1+w)(\sigma^2+k\sigma_1^2)$	H_2 $\qquad\qquad \phi$
s_3	$m_1(\sigma^2 + r\,\sigma_2^2)$	H_3
s_4	$u\,\sigma^2$	H_4
s_{52}	$m_2[\sigma^2+k\sigma_1^2+(r-d_2)\sigma_2^2]$	$H_{52}\ H_{562}\ H_{572}$
s_{62}	$m_2[\sigma^2 + d_2\sigma_2^2]$	$H_{562}\ H_{62}\ H_{672}$
s_{72}	$m_2 d_2(r-d_2)\sigma_2^2$	$H_{572}\ H_{672}\ H_{72}$
.	.	.
.	.	.
.	.	.
s_{5s}	$m_s[\sigma^2+k\sigma_1^2+(r-d_s)\sigma_2^2]$	$H_{5s}\ H_{56s}\ H_{57s}$
s_{6s}	$m_s[\sigma^2 + d_s\sigma_2^2]$	$\phi \qquad\qquad H_{56s}\ H_{6s}\ H_{67s}$
s_{7s}	$m_s d_s(r-d_s)\sigma_2^2$	$H_{57s}\ H_{67s}\ H_{7s}$

$$H_1 = M^{-1}(\sigma^2+k\sigma_1^2+r\sigma_2^2)$$

$$H_2 = 2(m_1 + w)(\sigma^2 + k\sigma_1^2)^2 \quad \text{where } w = b - t$$

$$H_3 = 2m_1(\sigma^2 + r\,\sigma_2^2)^2$$

$$H_4 = 2u\,\sigma^4 \quad \text{where } u = M - b - t + 1$$

$$H_{5i} = 2m_i[\sigma^2 + k\sigma_1^2 + (r-d_i)\sigma_2^2]^2 \quad i = 2, \ldots, s$$

$$H_{6i} = 2m_i[\sigma^2 + d_i\sigma_2^2]^2 \quad i = 2, \ldots, s$$

$$H_{7i} = m_i d_i(r-d_i)[\sigma^4+2d_i(r-d_i)\sigma_2^4+k\sigma^2\sigma_1^2+r\sigma^2\sigma_2^2+kd_i\,\sigma_1^2\sigma_2^2] \quad i = 2, \ldots, s$$

$$H_{56i} = 2m_i\,d_i\,(r-d_i)\sigma_2^4 \quad i = 2, \ldots, s$$

$$H_{57i} = 2m_i d_i(r-d_i)[(r-d_i)\sigma_2^4 + \sigma^2\sigma_2^2 + k\,\sigma_1^2\sigma_2^2]$$

$$H_{67i} = 2m_i d_i(r-d_i)[d_i\sigma_2^4 + \sigma^2\sigma_2^2]$$

CHAPTER VI

SUMMARY AND EXTENSIONS

A set of minimal sufficient statistics has been given by Weeks and Graybill [4] for a general class of designs assuming an Eisenhart Model II. All the statistics of the minimal set, however, are not independent. This fact complicates the search for unbiased estimators of the variance components unless the distribution of the minimal set of sufficient statistics is known. The complication is encountered when using an estimator which is a function of dependent statistics of the minimal set. If the distribution of the minimal set is known, then the variance of any estimator based on the set of minimal sufficient statistics may be found and variances of different estimators of the same variance component can be compared.

As a special case of the general two-way classification model, the joint distribution of a set of minimal sufficient statistics for the BIB design has been derived in Chapter III. This derivation was undertaken on the premise that the minimal set contained six statistics. This condition is equivalent to imposing the restriction that $b > t$. If $b = t$, the minimal set contains only five statistics.

Knowing the distribution of the six statistics of the minimal set it was possible to find their respective variances and covariances. Then, using different linear functions of statistics from the minimal sufficient set, several unbiased estimators of each of the variance components σ^2, σ_1^2, and σ_2^2 were chosen and their variances compared.

Under certain conditions on one of the classifications (say blocks) of the two-way classification model, it has been shown for some particular BIB designs that the variance of one estimator of an individual variance component is uniformly smaller than the variances of other chosen estimators.

Other special cases of BIB designs which fail to conform to the given conditions are considered in more detail. For these designs comparisons of variances of the different estimators are presented in graphical form showing those regions for which the variances differ in magnitude. These regions are functions of the particular BIB design under consideration and the ratios of the true variance components which were assumed in the model.

Chapter V pertains to the extension of the derivation of the distribution of the set of minimal sufficient statistics to a general class of designs. For this general class the minimal set contains $3s + 1$ statistics where s is the number of distinct positive characteristic roots of $A'A$ and $A'A$ is the coefficient matrix of $\hat{\tau}$ in the system $A'A\hat{\tau} = A'Y$. The restriction $b > t$ is not imposed in the general class of designs.

The joint distribution of the statistics of the minimal set for the general case is found to be the product of a normal, three independent chi-squares, and $s-1$ independent Wisharts. The expected values and covariance matrix of the $3s + 1$ statistics are also given.

In Chapter IV, three systems of equations, each having an infinite number of solutions, were given for finding unbiased estimators of σ^2, σ_1^2, and σ_2^2. In that chapter, certain solutions of these systems were chosen that yielded unbiased estimators which might normally be selected when considering the expected values of the statistics in the minimal sufficient set.

In extension of the results obtained in this thesis, other estimators and their variances could be investigated as a function of the unknowns (g_1, . . . , g_6) in the three systems. That is, solutions of each system are functions of two arbitrarily chosen g_i and any estimator and its variance could be expressed in terms of these g values. Therefore, by incrementing the g values in some systematic manner, a sequence of unbiased estimators and their variances could be obtained, thereby gaining insight into the search for minimum variance unbiased estimators.

SELECTED BIBLIOGRAPHY

[1] Anderson, T. W. *An Introduction to Multivariate Statistical Analysis*, New York: John Wiley and Sons, Inc., 1958.

[2] Graybill, Franklin A. *Introduction to Linear Statistical Models*, *Vol. 1*, New York: McGraw-Hill Publishing Co., 1961.

[3] Weeks, David L. "Variance Components in Two-way Classification Models," (unpublished Ph.D. thesis, Oklahoma State University, 1959).

[4] Weeks, David L., and Franklin A. Graybill. "A Minimal Sufficient Statistic for a General Class of Designs." *Sankhya: The Indian Journal of Statistics*, Series A, Vol. 24, Part 4, 1962, 339-354.

VITA

David Elliott Bee

Candidate for the Degree of

Doctor of Philosophy

Thesis: ESTIMATION OF VARIANCE COMPONENTS IN INCOMPLETE BLOCK DESIGNS

Major Field: Mathematics and Statistics

Biographical:

Personal Data: Born in McAlester, Oklahoma, July 15, 1936, the son of Ben Vernon and Margaret E. Bee.

Education: Attended grade school in McAlester, Oklahoma; graduated from McAlester High School in 1954; attended Western Kentucky State College from September, 1954, to January, 1956; attended Eastern Oklahoma Agriculture and Mechanical College from January, 1956, to May, 1956; attended night school at Central Christian College from January, 1958, to May, 1958; received the Bachelor of Science degree from Oklahoma State University, with a major in Mathematics, in May, 1961; received the Master of Science degree from Oklahoma State University, with a major in Mathematics, in August, 1962; completed requirements for the Doctor of Philosophy degree in August, 1965.

Professional experience: Employed by Phillips Petroleum Company, Comptrollers Department, from August, 1956 to September, 1958; employed by Oklahoma State University Statistical Laboratory, Department of Mathematics and Statistics in the following capacities:
Student Assistant from June, 1960, to March, 1961,
Research Assistant from March, 1961, to August, 1961,
Staff Assistant from June, 1962, to August, 1962,
Graduate Assistant from June, 1963, to August, 1963;
employed by Kansas State University, Department of Statistics, as a visiting assistant professor from June, 1964, to August, 1964; employed by Oklahoma State University, Department of Mathematics and Statistics, as a graduate assistant from September, 1964 to May, 1965.