A STUDY ON THE GLOBAL WEll-POSEDNESS FOR THE TWO-DIMENSIONAL BOUSSINESQ AND LANS-ALPHA MAGNETOHYDRODYNAMICS EQUATIONS

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We investigate the global (in time) regularity problem for two different models; generalized two-dimensional Boussinesq and Lans-alpha magnetohydrodynamics system. First, the global regularity of 2D incompressible generalized Euler-Boussinesq equations has been studied. We establish the global existence and uniqueness of solutions to the initial-value problem when the velocity field is “double logarithmically” more singular than the one given by the Biot-Savart law.This global regularity result goes beyond the critical case. Secondly, we consider the two-dimensional Navier-Stokes-Boussinesq equations with logarithmically super-critical dissipation. By implementing Besov space technique, the global well-posedness of initial value problem is established. These results improve the existing results of super-critical Boussinesq system of equations. Finally, we study the two-dimensional generalized Lans-alpha magnetohydrodynamics system. We mainly focus on Lans-alpha magnetohydrodynamics system of equations with logarithmically weaker dissipation than full dissipation together with zero diffusion or zero dissipation and logarithmically weaker diffusion than full diffusion. In both cases, we are successful to resolve global regularity issues.

TABLE OF CONTENTS

Chapter Page

CHAPTER 1

Introduction

The study of fluid dynamics is of great interest from both the mathematical as well as the physical point of view. A number of mathematical models have been proposed to describe the natural phenomena like atmospheric & oceanic flows, geophysical flows, and electrically conducting flows. Two commonly used models are the Boussinesq system of equations and magnetohydrodynamic equations.

Boussinesq system of equations is widely used to model large scale atmospheric & oceanic flows such as tornadoes, cyclones, and hurricanes. It describes the dynamics of fluid under the influence of gravitational force. This system is one of the well known models that describes the geophysical flows as well as other astrophysical situations where the stratification of the medium and the rotation of the earth play a dominant role.

The magnetohydrodynamic equations model electrically conducting fluid in the presence of magnetic field. These equations have been used to study various natural problems in geophysics and astrophysics. Mathematical analysis of the magnetohydrodynamic equations is extremely difficult due to the coupling between the velocity field and the magnetic field. In fact, whether the solutions of two-dimensional ideal MHD equations exist for all time or they blow-up in finite time is an outstanding open problem. Because of this difficulty, many regularized models have been proposed. In this dissertation, we will focus on one of such two-dimensional models, so called Lans-α magnetohydrodynamics model.

This dissertation is primarily focused on the global regularity issues of logarithmically supercritical two-dimensional Boussinesq equations and magnetohydrodynamics-α model. More precisely, in this dissertation we would like to answer the following interesting questions for the above system of nonlinear partial differential equations. Given a general smooth initial data, does there exist a solution on some time interval? Is it unique? Can the solution be extended for all time?

1.1 Boussinesq Equations

1.1.1 Background

The two-dimensional Boussinesq system of equations is a system of nonlinear partial differential equations that plays a very important role in the study of Raleigh-Bernard convection. This system acts as a lower dimensional model of the three-dimensional hydrodynamic equations. In fact, it is analogous to the $3D$ incompressible Euler and Navier-Stokes equations for axisymmetric swirling flow away from the symmetric axis, and it retains some key features of the $3D$ incompressible flow such as the vortex stretching mechanism. Thus, the qualitative behavior of the solutions for the two system of equations are expected to be identical. Better understanding of the $2D$ Boussinesq system may support some indications towards the regularity of $3D$ flows. The global regularity of two-dimensional Euler equations has been resolved [50]. However, the global regularity or finite time singularity for the three-dimensional Navier-Stokes equations is the most challenging open problem in fluid dynamics [50]. In fact, this is one million dollar prize problem announced by Clay Mathematics Institute [30]. Various efforts have been made by mathematicians, physicists, and engineers, but the mystery is still there. The global well-posedness of the 3D Navier-Stokes equation is extremely difficult because of the presence of vortex stretching term. There are a large volume of literature that provide the partial answer of the well-posedness of $3D$ incompressible flows.

The standard velocity formulation of the 2D Boussinesq equations with fractional dissipation and fractional thermal diffusion is given by

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu(-\Delta)^\alpha u = -\nabla p + \theta \mathbf{e}_2, \\ \nabla \cdot u = 0, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\beta \theta = 0, \end{cases} \tag{1.1}$$

with the corresponding vorticity $\omega = \nabla \times u$ satisfying

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \nu(-\Delta)^\alpha \omega = \partial_{x_1} \theta, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\beta \theta = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \omega, \end{cases} \tag{1.2}$$

where $(x, y) \in \mathbb{R}^2$, $t \geq 0$, $u = (u_1(x, y, t),\ u_2(x, y, t))$ denotes the 2D velocity field, $p = p(x, y, t)$ the pressure, $\theta = \theta(x, y, t)$ the temperature in the content of thermal convection and the density in the modeling of geophysical fluids, $\psi = \psi(x, y, t)$ the stream function, $\nu \geq 0$ the viscosity, $\kappa \geq 0$ the thermal diffusitivity, $\alpha \in (0, 1]$ and $\beta \in (0, 1]$ are real parameters, and $\mathbf{e}_2$ is the unit vector in the x_2-direction.

1.1.2 Existing results and open problems

The Boussinesq equations with full dissipation and full thermal diffusion, $\nu > 0$, $\kappa > 0$, with $\alpha = \beta = 1$, was first studied by Canon and DiBenedetto [6] in 1980. They found a unique, global in time, weak solution and improved the regularity of the solution when the initial data is smooth. In the case of inviscid Boussinesq system of equations, $\nu = 0$ and $\kappa = 0$, the global regularity issue of (1.1) is an outstanding open problem in mathematical fluid mechanics. In order to fulfill the gap between two extreme cases, the intermediate cases, when $\nu = 0$ or $\kappa = 0$ or the dissipation in horizontal direction only, or in the vertical direction only, have recently attracted considerable attention ([1, 2, 8, 12, 33, 36, 45, 72]). The main idea to study such anisotropic equations is to weaken either the dissipation or diffusion than a critical case and the regularity still holds. The global (in time) regularity, when $\nu > 0$ and $\kappa = 0$, or $\nu = 0$ and $\kappa > 0$ with $\alpha = 1$ and $\beta = 1$ have been established by Hou-Li [36] and Chae [12] independently. Global well-posedness for the anisotropic Boussinesq system with horizontal dissipation or thermal diffusion was first studied by Danchin and Paicu [25] and then further studied by the authors in [45] with more elementary approaches and milder assumptions. The global regularity issue with vertical dissipation and thermal diffusion has been resolved by Cao and Wu [8]. According to our knowledge, it is currently unknown if the global regularity still holds with vertical dissipation or thermal diffusion only. Recently, the result of global existence of smooth solutions to (1.1) is generalized to the cases when $\alpha = \frac{1}{2}$ or $\beta = \frac{1}{2}$. The global regularity result for the case $\nu > 0$, $\kappa = 0$, $\alpha = 1/2$ or $\nu = 0$, $\kappa > 0$, $\beta = 1/2$ have been obtained by Hmidi, Kerani and Rousset [35]. In 2011, Miao and Xue [52] established global well-posedness for the rough initial data when $\nu > 0$, $\kappa > 0$ with $0 < \beta < \alpha < 1/2$ and $\alpha + \beta > 1/2$. The authors in [21] also found unique global in time smooth solutions for the sufficiently smooth initial data when $\alpha, \beta \in (0, 1/2)$ and $\beta > \frac{1}{2+2\alpha}$. The global regularity of 2D Boussinesq system with supercritical cases (i.e. $\nu > 0, \kappa = 0, \alpha < 1/2$ Or $\nu = 0, \kappa > 0, \beta < 1/2$) is still unknown. In this direction, a few progress has been made by Hmidi [32], Chae &Wu [13] and Jiu, Miao, Wu, & Zhang [37]. Hmidi [32] relaxed the critical dissipation needed for global well-posedness by a logarithmic factor. More precisely, he considered the following Euler-Boussinesq system;

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \theta \mathbf{e}_2, \\ \nabla \cdot u = 0, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa \mathcal{L} \theta = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.3}$$

where $\mathcal{L} = \frac{|D|}{log^{\alpha}(e^4+|D|)}$ with $|D| = \sqrt{-\Delta}$. The operator $\mathcal{L}$ is defined by Fourier multiplier or by a nonlocal operator with a convolution kernel. Hmidi successfully resolved the global well-posedness for (1.3) when $\alpha = [0, \frac{1}{2}]$.

Chae and Wu [13] studied the generalized Boussinesq-Navier-Stokes system with a velocity field that is logarithmically more singular than the one determined by the vorticity through the 2D Biot-Savart law [50]. In fact, they studied the following system;

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \Lambda \omega = \partial_{x_1} \theta, \\ \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^{\perp} \psi = (-\partial_{x_2}, \partial_{x_1}), \ \ \Delta \psi = \Lambda^{\sigma} \log^{\gamma}(I - \Delta)\omega, \\ u(x,0) = u_0(x), \ \ \theta(x,0) = \theta_0(x), \end{cases} \tag{1.4}$$

where $\psi = \psi(x,t)$ is a scalar function of $x = (x_1, x_2) \in \mathbb{R}^2$ and $t \geq 0$. $\sigma \geq 0$ and $\gamma \geq 0$ are real parameters, $\Lambda = \sqrt{-\Delta}$, and Λ^{σ} are Fourier multiplier operator defined through Fourier transformation $\widehat{\Lambda^{\sigma}} f(\xi) = |\xi|^{\sigma} \hat{f}(\xi)$. The global well-posedness for (1.4) has been resolved for the special case when $\sigma = 0$ and $\gamma \geq 0$. It is worthy to mention that the global regularity result for (1.4) remains unknown for the more singular case when $\sigma > 0$.

1.1.3 Statement of problems and results

The 2D Boussinesq equations have recently attracted considerable attention. Consequently, many important results on the global well-posedness issue concerning the partial dissipation case have been established. Our main intention is to explore how far one can go beyond the critical dissipation or diffusion and still prove the global regularity for the Euler-Boussinesq equations and Navier-Stokes- Boussinesq equations.

With the motivation of the recent work of Chae and Wu [13] and Chae, Constantin and Wu [14], we first study the regularity issues of the generalized Euler-Boussinesq equations with a singular velocity [41]. More precisely, we focus on the existence and uniqueness of the solution of the following initial value problem (IVP):

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \partial_{x_1} \theta, \\ u = \nabla^{\perp} \psi, \quad \Delta \psi = \Lambda^{\sigma} P(\Lambda) \omega, \\ \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0 \\ \omega(x,0) = \omega_0(x), \quad \theta(x,0) = \theta_0(x), \quad x \in \mathbb{R}^2, \end{cases} \tag{1.5}$$

where $u = u(x,t)$ is a velocity field, $\theta = \theta(x,t)$, $\omega = \omega(x,t)$ and $\psi = \psi(x,t)$ are scalar functions, and $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$ and $\sigma \geq 0$ is a real parameter.

We will focus on IVP (1.5) with a very general class of symbol $P(\xi)$ for the operator $P(\Lambda)$ that is assumed to satisfy the following condition.

Condition 1.1 *The symbol $P(|\xi|)$ assumes the following properties:*

1. *P is continuous on $\mathbb{R}^2$ and $P \in C^{\infty}(\mathbb{R}^2 \setminus \{0\})$;*

2. *P is radially symmetric;*

3. *$P = P(|\xi|)$ is nondecreasing in $|\xi|$;*

4. *There exist two constants C and C_0 such that*

$$\sup_{2^{-1} \leq |\eta| \leq 2} \left| (I - \Delta_{\eta})^n \, P(2^j |\eta|) \right| \leq C \, P(C_0 \, 2^j)$$

for any integer j and $n = 1, 2$.

We remark that the fourth condition for P is very similar to the main condition in Mihlin-Hörmander Multiplier Theorem [61]. For notational convenience, we also assume that $P \geq 0$. Some special examples of P are

$$P(\xi) = \big(\log(1+|\xi|^2)\big)^\gamma \quad \text{with } \gamma \geq 0,$$
$$P(\xi) = \big(\log(1+\log(1+|\xi|^2))\big)^\gamma \quad \text{with } \gamma \geq 0,$$
$$P(\xi) = |\xi|^\beta \quad \text{with } \beta \geq 0,$$
$$P(\xi) = (\log(1+|\xi|^2))^\gamma\, |\xi|^\beta \quad \text{with } \gamma \geq 0 \text{ and } \beta \geq 0.$$

The goal here is to examine for what operator P obeying Condition 1.1, (1.5) is still globally well-posed. We note that when $P(\Lambda)$ is not the identity operator, (1.5) involves a velocity field that is more singular than the standard velocity determined through the Biot-Savart law [50]. The first major result (Theorem 3.1) asserts that if $P(\xi)$ obeying Condition 1.1 satisfies two more conditions, namely, $P(2^k) \leq C\sqrt{k}$ for a constant C and any large integer $k > 0$, and

$$\int_1^\infty \frac{1}{r\log(1+r)\,P(r)} dr = \infty.$$

Then the IVP (1.5) with $\sigma > 0$ and $(\omega_0, \theta_0) \in B^s_{q,\infty}(\mathbb{R}^2)$ for $q > 2$ and $s > 2$ has a unique global solution pair (ω, θ). As a consequence of this result, we establish the global well-posedness of (1.5) when P is a double logarithmic, namely

$$P(\xi) = \big(\log(1+\log(1+|\xi|^2))\big)^\gamma \quad \gamma \in [0,1]. \tag{1.6}$$

As we mentioned earlier that the main difficulty is to deal with the vortex stretching term $\partial_{x_1}\theta$, direct energy estimates do not yield the desired estimates. The idea is to combine the equation for ω and the equation for the Riesz transform of θ. First, we apply the Riesz transform to the θ equation to obtain

$$\partial_t \mathcal{R}\theta + u\cdot\nabla\mathcal{R}\theta + \Lambda\mathcal{R}\theta = -[\mathcal{R}, u\cdot\nabla]\theta, \qquad \mathcal{R} \equiv \Lambda^{-1}\partial_{x_1}.$$

Then we combine it with the ω equation and obtain

$$\partial_t G + u\cdot\ \nabla G = -[\mathcal{R}, u\cdot\nabla]\theta, \qquad G = \omega + \mathcal{R}\theta. \tag{1.7}$$

Although (1.7) hides the vortex stretching term $\partial_{x_1}\theta$, we need to obtain suitable bounds for the commutator. With the commutator estimates and the regularity of θ, we obtain the global bound for $\|\omega\|_{L^q}$, $\|\theta\|_{B^{0,P}_{\infty,2}}$ and consequently for $\|\omega\|_{L^\infty}$. Finally, we establish the desired global bounds for $\|\omega\|_{B^s_{q,\infty}}$ and $\|\theta\|_{B^s_{q,\infty}}$ for $s > 2$. In the first step, we obtain the global bounds for $\|\omega\|_{B^\beta_{q,\infty}}$ and $\|\theta\|_{B^\beta_{q,\infty}}$ for β in the range $\frac{2}{q} < \beta < 1$ by using the regularity of G in (1.7) and a logarithmic interpolation inequality bounding $\|\nabla u\|_{L^\infty}$ in terms of $\|\omega\|_{L^q\cap L^\infty}$ and $\|\omega\|_{B^\beta_{q,\infty}}$. In the second step, we take advantage of the regularity obtained in the first step and establish the bounds for the Besov index β_1 in the range $1 < \beta_1 < 2 - \frac{2}{q}$. A repetition of this step allows us to reach any index $s > 2$. Since the solution class is very regular, the uniqueness part can easily be obtained. The details are provided in chapter 3.

Our second work here is to study the global well-posedness of the Navier-Stokes-Boussinesq equations with logarithmically supercritical dissipation [42]. This work is motivated by the recent work of Chae and Wu [13] and Hmidi [32]. Attention here is focused on the following initial-value problem (IVP):

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega + \mathcal{L}\omega = \partial_{x_1}\theta, \\ \partial_t\theta + u\cdot\nabla\theta = 0, \\ u = \nabla^\perp\psi, \quad \Delta\psi = \omega, \\ \omega(x,0) = \omega_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.8}$$

where $\mathcal{L}$ is a nonlocal dissipation operator defined by

$$\mathcal{L}f(x) = \text{ p.v. } \int_{\mathbb{R}^2} \frac{f(x)-f(y)}{|x-y|^2} m(|x-y|)dy \tag{1.9}$$

and $m\colon (0,\infty) \to (0,\infty)$ is a smooth, positive, and non-increasing function that satisfies

(i) there exists $C_1 > 0$ such that

$$rm(r) \le C_1 \qquad \text{for all } r \le 1;$$

(ii) there exists $C_2 > 0$ such that

$$r|m'(r)| \le C_2 m(r) \qquad \text{for all } r > 0;$$

(iii) there exists $\beta > 0$ such that

$$r^\beta m(r) \text{ is non-increasing.}$$

$\mathcal{L}$ can be equivalently defined by a Fourier multiplier, namely

$$\widehat{\mathcal{L}f}(\xi) = P(|\xi|)\widehat{f}(\xi). \tag{1.10}$$

Throughout the chapter 4, we assume that $\mathcal{L}$ satisfies both (1.9) and (1.10) with $P(|\xi|) = m(\frac{1}{|\xi|})$ obeying the conditions stated above. Our main result is a global well-posedness theorem for the IVP (1.8) when $\mathcal{L}$ is slightly supercritical. More precisely, we prove that if $\mathcal{L}$ satisfies (1.9) and (1.10) and $a(\xi) = a(|\xi|) \equiv |\xi|/P(|\xi|)$ is positive, non-decreasing and satisfies

$$\lim_{|\xi|\to\infty} \frac{a(|\xi|)}{|\xi|^\sigma} = 0\,, \quad \forall\, \sigma > 0, \tag{1.11}$$

with the initial data (u_0, θ_0) in the class

$$u_0 \in H^1(\mathbb{R}^2), \quad \omega_0 \in L^q(\mathbb{R}^2) \cap B^0_{\infty,1}(\mathbb{R}^2)\,, \quad \theta_0 \in L^2(\mathbb{R}^2) \cap B^{0,a^2}_{\infty,1}(\mathbb{R}^2),$$

then, (1.8) has a unique global solution (u, θ) satisfying, for all $t > 0$,

$$u \in L^\infty_t H^1, \quad \omega \in L^\infty_t L^q \cap L^1_t B^0_{\infty,1}\,, \quad \theta \in L^\infty_t L^2 \cap L^\infty_t B^{0,a^2}_{\infty,1} \cap L^1_t B^{0,a}_{\infty,1}.$$

As a special consequence of this result, we will establish the global existence and uniqueness of classical solutions of (1.8) with logarithmically supercritical dissipation,

$$\widehat{\mathcal{L}u}(\xi) = P(|\xi|)\widehat{u}(\xi) \equiv \frac{|\xi|}{\log^\gamma(e + |\xi|)}\,\widehat{u}(\xi) \quad \text{for any } \gamma \ge 0.$$

We now explain the main difficulty that one encounters in the study of the global regularity of solutions to (1.8). Due to the vortex stretching term $\partial_{x_1}\theta$, a simple energy estimate will not lead to a global bound for $\|\omega\|_{L^2}$ unless $\mathcal{L}\omega$ is very dissipative. To overcome this difficulty, we consider a new quantity $\omega - \Lambda^{-1}\partial_{x_1}\theta$ to hide $\partial_{x_1}\theta$. Then the combined quantity $G = \omega - \mathcal{R}_a\theta \quad \text{with} \quad \mathcal{R}_a = \mathcal{L}^{-1}\partial_{x_1}$, satisfies

$$\partial_t G + u \cdot \nabla G + \mathcal{L}G = [\mathcal{R}_a, u \cdot \nabla]\theta.$$

After obtaining a general bound for the commutator $[\mathcal{R}_a, u \cdot \nabla]$, we get the global bounds for $\|G\|_{L^q}$ for $q \in (2,4)$. In order to show a global bound for $\|G\|_{L^q}$ and $\|\omega\|_{L^q}$ with $q \geq 4$, we first obtain a bound for the space-time norm of $\|G\|_{\widetilde{L}^r_t B^s_{q,1}}$ and consequently $\|G\|_{L^1_t B^{0,a}_{\infty,1}}$. Making use of bound $\|\theta\|_{L^1_t B^{0,a^2}_{\infty,1}}$ in terms of $\|\nabla u\|_{L^1_t L^\infty}$ algebraically, we establish global bounds for $\|\omega\|_{L^1_t B^{0,a}_{\infty,1}}$ and for $\|\theta\|_{L^1_t B^{0,a^2}_{\infty,1}}$, which, in turn, are sufficient for the global bound $\|\omega\|_{L^q}$ for any $q \geq 2$. These global bounds guarantee a global solution. To show the uniqueness, we consider the difference of two solutions $(u^{(1)}, \theta^{(1)})$ and $(u^{(2)}, \theta^{(2)})$ and show that the difference must vanish by controlling the velocity difference in $B^0_{2,\infty}$ and the difference $\theta^{(2)} - \theta^{(1)}$ in $B^{-1,a}_{2,\infty}$. The details are provided in chapter 4.

1.2 Lans-α Magnetohydrodynamics System

1.2.1 Background

Electrically conducting fluids under the influence of magnetic field arise in important applications including plasma physics, geophysics and astronomy. Among many others, magnetohydrodynamics (MHD) flows are typical because of the more complex dynamics of the flow due to coupling of Maxwell's equations governing the magnetic field and the Navier-Stokes equations (NSE) governing the fluid motion via the Lorentz force and Ohm's law. The MHD equations are related to engineering problems such as plasma confinement, controlled thermonuclear fusion, liquid-metal cooling of nuclear reactors, and electromagnetic casting of metals. The standard form of the incompressible MHD equations is

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla\pi - \nu\Delta u = (b\cdot\nabla)b, \\ \partial_t b + (u\cdot\nabla)b - \eta\Delta b = (b\cdot\nabla)u, \\ \nabla\cdot u = \nabla\cdot b = 0 \quad (u,b)(x,0) = (u_0,b_0)(x). \end{cases} \tag{1.12}$$

Where $u(x,t)$ denotes the velocity vector field, $b(x,t)$ the magnetic vector field and π the pressure scalar field, ν the kinematic viscosity and η the magnetic diffusivity. A class of global weak solutions with finite energy and a class of local strong solutions for the system (1.12) in $2D$ and $3D$ have been successfully resolved by the authors in [26]. The smoothness and uniqueness of such weak solutions is an outstanding problem for $3D$.

An extension of the NSE, called Lans (Lagrangian-averaged Navier-Stokes)-α model (also known as the viscous Camassa-Holm equations), was introduced by the authors in [18]. This system is well-known for its remarkable performance as a closure model of turbulence in infinite channels and pipes as their solutions give excellent agreement with empirical data for a wide range of large Reynolds numbers. The authors in [19] also proposed the Leray-α model, the system (1.13) without $\sum_{k=1}^{2} v_k \nabla u_k$, and showed that it also compares successfully with empirical data from turbulent channel and pipe flows for a wide range of Reynold numbers.

The Lans-α MHD system with fractional dissipation and fractional diffusion is given by

$$\begin{cases} \partial_t v + (u\cdot\nabla)v + \sum_{k=1}^{2} v_k\nabla u_k + \nabla(\pi + \frac{1}{2}|b|^2) + \nu\Lambda^{2r_1}v = (b\cdot\nabla)b, \\ \partial_t b + (u\cdot\nabla)b - (b\cdot\nabla)u + \eta\Lambda^{2r_2}b = 0, \\ v = (1-\alpha^2\Delta)u,\ \nabla\cdot u = \nabla\cdot b = 0,\ (v,b)(x,0) = (v_0,b_0)(x). \end{cases} \tag{1.13}$$

Where $v(x,t)$ denotes the fluid velocity field, $u(x,t)$ the filtered velocity, $b(x,t)$ the magnetic vector field, and π the pressure scalar field. Here ν, $\eta \geq 0$ are the kinematic viscosity and magnetic diffusivity constants respectively and the fractional Laplacians Λ^{r_i} defined through the Fourier transform by

$$\widehat{\Lambda^{r_i} f}(\xi) = |\xi|^{r_i} \hat{f}(\xi), \quad i = 1, 2, \tag{1.14}$$

with their powers $r_i \geq 0$ and α denotes the length-scale parameter representing the width of the filters.

We may consider the α models as a numerical regularization of the underlying equation, which makes the nonlinearity milder, and hence the solutions of the modified equation are smoother. As α models are some sort of regularizing numerical schemes, they inherit some of the original properties of MHD equations. For the details of other α models including the Lans-α MHD we refer to [48].

1.2.2 Literature review

In a three-dimensional periodic domain, Linshiz and Titi [48] proved the existence of the unique weak solution pair to the system (1.13) at ν, $\eta > 0$ and $r_1 = r_2 = 1$. They also established the relation of the Lans-α MHD model to the MHD equations by proving some convergence result as $\alpha \to 0^+$. In fact, they showed that a subsequence of solutions of the Lans-α MHD equations converges to a certain solution (a Leray-Hopf solution) of the three-dimensional MHD equations. Later in 2011, the authors in [85] showed that when ν, $\eta > 0$ and $r_1 = r_2 = 1$, the global well-posedness result remains valid even in four-dimensional case [83]. In [29], the authors showed that for the system (1.13), given $(v_0, b_0) \in H^3(\mathbb{R}^2)$, the solution remains in such a space for all time in two cases: $\nu > 0$, $\eta = 0$, $r_1 = 1$ or $\nu = 0$, $\eta > 0$, $r_2 = 1$. Similar results on a closely related system is also shown in [86]. Very recently, Zhao and Zhu [82] proved the global regularity result in the case ν, $\eta > 0$, $r_1 = r_2 = \frac{n}{4}$, $n = 2, 3$ to the system (1.13). It is worth noticing that Yamazaki [76] gave an affirmative solution to one problem as remarked in Remark 1.1 of [82]. In particular, the author in [76] showed that the solution to the system (1.13) remains smooth for all time for any $r_1 + r_2 = 1$

with r_1, $r_2 \in (0,1)$ in case $n = 2$. For $n = 3$, the authors in [78] improved the result of Zhao and Zhu [82]. In fact, they proved that the system (1.13) has a unique global smooth solution in the case $r_1 + r_2 = \frac{3}{2}$, $n = 3$ with $\frac{3}{4} \le r_2 < \frac{3}{2}$. Many other related interesting results have been obtained ([11, 28, 43, 80, 84, 87]).

In the case $\alpha = 0$, the system (1.13) can be identified with the generalized MHD system which has been studied intensively by many authors ([10, 38, 65, 69, 71, 75, 79]). When $\alpha = 0$, $r_1 = r_2 = 1$, the system (1.13) reduces to the standard MHD equations. Mathematical analysis of the global regularity issue of the standard MHD equations can be found in [7, 9, 75, 77, 88] and references therein.

1.2.3 Statement of problems and results

The $2D$ MHD equations and Lans-α MHD equations have attracted many mathematicians, physicists and engineers and many results concerning the global existence and uniqueness in the partial dissipation case have been established. Our goal is to weaken the dissipation or the diffusion and still establish the global well-posedness for the $2D$ Lans-α MHD system. Motivated by the work of Wu [71], we focus on the global regularity of the solution of the following initial value problem:

$$\begin{cases} \partial_t v + (u\cdot\nabla)v + \sum_{k=1}^{2} v_k \nabla u_k + \nabla(\pi + \frac{1}{2}|b|^2) + \nu\mathcal{L}^2 v = (b\cdot\nabla)b, \\ \partial_t b + (u\cdot\nabla)b - (b\cdot\nabla)u + \eta\mathcal{L}^2 b = 0, \\ v = (1-\alpha^2\Delta)u,\ \ \nabla\cdot u = \nabla\cdot b = 0,\ \ (v,b)(x,0) = (v_0,b_0)(x). \end{cases} \tag{1.15}$$

Where $v(x,t)$ denotes the two-dimensional velocity vector field, $u(x,t)$ the filtered velocity, $b(x,t)$ the two-dimensional magnetic vector field and π the pressure scalar field. Here ν, $\eta \ge 0$ are the kinematic viscosity and magnetic diffusivity constants respectively. Finally, α denotes the length-scale parameter representing the width of the filters.

Global regularity results in the logarithmically supercritical regime was initiated by the author in [62] and [63] and inspired many others to extend these results to different models ([14], [32], [58], [71], [74], [75]). Our attention will be focused on (1.15) with a very general class of symbols $m(\xi)$ for the multiplier operator $\mathcal{L}$ defined through the Fourier transform by

$$\widehat{Lf}(\xi) = m(\xi)\hat{f}(\xi), \quad m(\xi) \geq \frac{|\xi|}{g(|\xi|)}.$$

The first result (Theorem 5.1) asserts that for $\nu > 0$, $\eta = 0$ and $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ a radially symmetric, non-decreasing function such that $g \geq 1$ and satisfies

$$\int_e^\infty \frac{d\tau}{g(\tau)\sqrt{\ln(\tau)}\tau} = \infty,$$

then for any v_0, $b_0 \in H^4(\mathbb{R}^2)$, there exists a unique classical solution pair to the system (1.15).

Our proof is inspired by the work in [71]; however, the proof in [71] does not seem to go through due to the fact that we are interested in the endpoint case while the results in [71] required both ν, $\eta > 0$. We have made some appropriate modifications on the proof of [75] to achieve our first result.

The second result (Theorem 5.2) we have proved says for $\nu = 0$, $\eta > 0$ and $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ a radially symmetric, non-decreasing function such that $g \geq 1$ and satisfies

$$\int_e^\infty \frac{d\tau}{g^4(\tau)\ln(\tau)\tau} = \infty,$$

then for any v_0, $b_0 \in H^4(\mathbb{R}^2)$, there exists a unique classical solution pair to the system (1.15).

In contrast to the classical MHD system, $\|v\|_{L^2}$ is not conserved for the solutions to (1.15). Moreover, due to the structure of (1.15), namely $(b \cdot \nabla)b$ and $(b \cdot \nabla)u$, which do not allow a simple commutator estimate, the a priori estimates for this system must be done in several steps. For the Lans-α MHD system, upon $\|v\|_{L^2}$-estimate, although $(u \cdot \nabla)v$ vanishes, $\sum_{k=1}^2 v_k \nabla u_k$ does not. This is precisely the problem stated in Remark 1.2 in [78]. To achieve our second result, we obtain a key estimate of $\|\mathcal{L}b\|_{L^2}$ (see Proposition 5.4.2). We believe that the method we have used to prove second result has further applications to other systems.

1.3 Organization of the Dissertation

This dissertation is organized as follows. The second chapter is dedicated to introduce standard notations, some definitions, functional spaces such as Besov spaces and generalized Besov spaces, and frequently used inequalities that we require for our study. The main work of this dissertation will be presented in chapters 3, 4, and 5.

Chapter three is divided into multiple sections. Section 3.2 is focused on establishing the global *a priori* bounds for the solution pair (ω, θ) for the generalized $2D$ Euler-Boussinesq equations with a singular velocity. In fact, global *a priori* bounds for $\|\omega\|_{L_t^\infty L^q}$, $\|\theta\|_{L_t^1 B^{0,P}_{\infty,2}}$ and $\|\omega\|_{L_t^\infty L^\infty}$ are obtained. Section 3.3 deals with the global bounds for $\|\omega\|_{B^s_{q,\infty}}$ and $\|\theta\|_{B^s_{q,\infty}}$ in two steps. In the first step, we obtain the global bounds for $\|\omega\|_{B^\beta_{q,\infty}}$ and $\|\theta\|_{B^\beta_{q,\infty}}$ in the range $\frac{2}{q} < \beta < 1$. The second step takes advantage of the regularity obtained in the first step and establishes the bounds for $\|\omega\|_{B^{\beta_1}_{q,\infty}}$ and $\|\theta\|_{B^{\beta_1}_{q,\infty}}$ in the range $\frac{2}{q} < \beta_1 < 2 - \frac{2}{q}$. A repetition of this step allows us to reach any index $s > 2$. Section 3.4 is devoted to show the existence and uniqueness of our main theorem (Theorem 3.1) of the third chapter.

Fourth chapter deals with the generalized $2D$ Navier-Stokes-Boussinesq equations with supercritical dissipation. It is divided into seven sections. First two sections are preliminaries and section 4.3 establishes a global *a priori* estimates for $\|G\|_{L^2}$ and consequently for $\|\omega\|_{B^{0,a-1}_{2,2}}$. Sections 4.4, 4.5 and 4.6 are dedicated to the global regularity results. In particular, we establish the global bounds for$\|G\|_{L^q}$ with $q \in (2, 4)$ followed by $\|G\|_{\widetilde{L}_t^r B^s_{q,1}}$ with $q \in [2, 4)$ and the global bounds for $\|\omega\|_{L_t^1 B^{0,a}_{\infty,1}}$ and $\|\omega\|_{L^q}$ for any $q \geq 2$. Section 4.7 presents the proof for the uniqueness of Theorem 4.1.

Fifth chapter is devoted to study the logarithmically extended global regularity results on two-dimensional Lans-α MHD system. This chapter is divided into different sections and subsections. Section 5.2 focuses on establishing the global a priori bounds for the case $\nu = 0$ and $\eta = 1$ whereas section 5.3 presents higher regularity results for Theorem 5.1. Section 5.4 is dedicated to a priori estimates in the case when $\nu = 1$ and $\eta = 0$. In subsection 5.4.1, we present our key estimates ($\|\mathcal{L}b\|_{L^2}^2$-estimate) that is essential to establish the global a priori bounds for Theorem 5.2. Section 5.5 details the higher regularity and Section 5.6 provides the proofs of Theorem 5.1 and Theorem 5.2.

CHAPTER 2

Preliminary

This chapter briefly presents some definitions, facts, and useful inequalities that we require for our study. We also introduce a very powerful set of function spaces called Besov spaces. With the notations and definitions of Besov spaces, we present Littlewood-Paley decomposition, Bony's para-products and commutator estimates.

Definition 2.1 *Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a domain. $L^p_{loc}(\Omega)$ is the space of Lebesgue measurable functions $f(x)$ on Ω such that*

$$\int_V |f(x)|^p dx < +\infty, \quad \textit{for any compact } V \subset \Omega$$

Remark 2.1 *If $p = \infty$, $\int_V |f(x)|^p dx$ is replaced by $ess \sup_{x \in V} |f(x)| < \infty$.*

Definition 2.2 *(L^p-space) Let $1 \leq p \leq \infty$. $L^p(\Omega)$ is the space of functions such that*

$$\|f\|_{L^p(\Omega)} = \begin{cases} (\int |f(x)|^p)^{\frac{1}{p}}, & \textit{if } 1 \leq p < \infty \\ ess \sup_{x \in \Omega} |f(x)|, & \textit{if } p = \infty \end{cases}$$

is finite.

Definition 2.3 *(l^p space) The space l^p consists of sequence $\{x_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$.*

Definition 2.4 *(Convolution) Let f and g be two Lebesgue measurable functions in $\mathbb{R}^n$. If $f(x-y)g(y)$ is integrable for almost every y then,*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

Definition 2.5 *(Weak derivative) Let $\Omega \subset \mathbb{R}^n$ be a domain and let α be a multi-index. Let $u \in L^1_{loc}(\Omega)$, we say v is α-weak derivative of u, denote $v = D^\alpha u$ if*

$$\int_\Omega u\, D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v\phi dx, \ \ \textit{for any} \ \ \phi \in C_0^\infty(\Omega)$$

Definition 2.6 *(Sobolev Space $W^{k,p}(\Omega)$) Let $k \geq 0$ be an integer and $1 \leq p \leq \infty$. $W^{k,p}(\Omega)$ consists of locally integrable functions f satisfying $D^\alpha f$ exists for all $|\alpha| \leq k$ and $\|D^\alpha f\|_{L^p} < +\infty$ for $|\alpha| \leq k$. More precisely,*

$$\|f\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\leq k} \|D^\alpha f\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}, & \textit{if } 1 \leq p < \infty \\ ess\sup_{|\alpha|\leq k} \|D^\alpha f\|_{L^\infty(\Omega)}, & \textit{if} \quad p = \infty \end{cases}$$

Definition 2.7 *(Distributions) $\mathcal{D}(\mathbb{R}^n)$ denotes the set of smooth functions with compact support ($C_0^\infty(\mathbb{R}^n)$) with a convergence notion.*
$\phi_k \in \mathcal{D}(\mathbb{R}^n)$ is said to converge to $\phi \in \mathcal{D}(\mathbb{R}^n)$ if

1. *There exists K (compact set in $\mathbb{R}^n$) such that $supp\, \phi_k \subset K$.*

2. *$max_{x\in K}|\phi_k(x) - \phi(x)| \longrightarrow 0$ as $k \longrightarrow \infty$.*

Distributions, denoted $\mathcal{D}'(\mathbb{R}^n)$, consists of all functionals f on $\mathcal{D}(\mathbb{R}^n)$ satisfying

1. *$< f, \lambda_1\phi_1 + \lambda_2\phi_2 >= \lambda_1 < f, \phi_1 > +\lambda_2 < f, \phi_2 >$ for any $\lambda_1, \lambda_2 \in \mathbb{R}$, $\phi_1, \phi_2 \in \mathcal{D}(\mathbb{R}^n)$.*

2. *If $\phi_k \longrightarrow \phi$ in $\mathcal{D}(\mathbb{R}^n)$, then $< f, \phi_k > \longrightarrow < f, \phi >$.*

Definition 2.8 *(Schwartz Space, $\mathcal{S}(\mathbb{R}^n)$) $\mathcal{S}(\mathbb{R}^n)$ consists of smooth functions ϕ satisfying*

$$\|\phi\|_{k,m} \equiv \sup_{x\in\mathbb{R}^n, |\alpha|\leq m} (1 + |x|)^k |D^\alpha \phi| < \infty$$

Definition 2.9 *(Tempered Distributions $\mathcal{S}'(\mathbb{R}^n)$) $\mathcal{S}'(\mathbb{R}^n)$ consists of all linear continuous functionals on $\mathcal{S}(\mathbb{R}^n)$.*

Definition 2.10 *(Inhomogeneous Sobolev Space $H^s(\mathbb{R}^n)$) For any $s \in \mathbb{R}$, the inhomogeneous Sobolev space $H^s(\mathbb{R}^n)$ consists of tempered distributions u such that $\hat{u}(\xi)$ is locally integrable and*

$$\|u\|_{H^s} = \left[\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}|^2 d\xi\right]^{\frac{1}{2}} < \infty$$

Definition 2.11 *(Homogeneous Sobolev Space $\mathring{H}^s(\mathbb{R}^n)$ For any $s \in \mathbb{R}$, $\mathring{H}^s(\mathbb{R}^n)$ consist of $u \in \mathcal{S}'$ such that $\hat{u} \in L^1_{loc}$ and*

$$\|u\|_{\mathring{H}^s} = \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}| d\xi\right)^{\frac{1}{2}} < \infty$$

We have the following properties;

1. If $s = 0$, $\mathring{H}^s = H^s = L^2$.

2. If $s = k$,where k is a positive integer $H^k \sim W^{k,2}$.

3. If $s > 0$, $H^s \subset \mathring{H}^s$.

4. If $s < 0$, $\mathring{H}^s \subset H^s$

2.1 Besov Space

This section provides the definitions of Besov spaces and related facts. In addition, it also introduces dyadic decomposition of unity as well as the Littlewood-Paley decomposition. We start with several notations.

We denote by $\mathcal{S}(\mathbb{R}^n)$ the usual Schwarz class and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions. We define $\mathcal{S}_0$ to be the following subspace of $\mathcal{S}$.

$$\mathcal{S}_0 = \left\{\phi \in \mathcal{S} : \int_{\mathbb{R}^d} \phi(x)\, x^\gamma\, dx = 0,\ |\gamma| = 0, 1, 2, \cdots\right\}$$

Its dual $\mathcal{S}_0'$ is given by

$$\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^{\perp} = \mathcal{S}'/\mathcal{P}$$

where $\mathcal{P}$ denotes the space of multinomials. In other words, two distributions in $\mathcal{S}'$ are identified as the same in $\mathcal{S}_0'$ if their difference is a polynomial. For $j \in \mathbb{Z}$, we define

$$A_j = \left\{ \xi \in \mathbb{R}^n : \; 2^{j-1} \leq |\xi| < 2^{j+1} \right\}.$$

Then the Littlewood-Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}} \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\mathrm{supp}\widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jn}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1 \;\;, & \text{if } \xi \in \mathbb{R}^n \setminus \{0\}, \\ 0 \;\;, & \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\}.$$

In addition, if $\psi \in \mathcal{S}_0$, then

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for any } \xi \in \mathbb{R}^n.$$

That is, for $\psi \in \mathcal{S}_0$,

$$\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi.$$

As a consequence, for any $f \in \mathcal{S}_0'$

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f$$

in the sense of weak-$*$ topology of $\mathcal{S}_0'$. For notational convenience, we define

$$\mathring{\Delta}_j f = \Phi_j * f, \qquad j \in \mathbb{Z}.$$

Definition 2.12 *(Homogeneous Besov Space $\mathring{B}^s_{p,q}$) For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the homogeneous Besov space $\mathring{B}^s_{p,q}$ consists of $f \in \mathcal{S}'_0$ with the norm defined by*

$$\|f\|_{\mathring{B}^s_{p,q}} \equiv \|2^{js}\|\mathring{\Delta}_j f\|_{L^p}\|_{l^q} = \begin{cases} \left(\sum_{j=-\infty}^{\infty} 2^{jsq}[\int_{\mathbb{R}^n} |\mathring{\Delta}_j f|^p]^{\frac{q}{p}}\right)^{\frac{1}{q}} & \text{for } q < \infty \\ \sup_j 2^{js}[\int_{\mathbb{R}^n} |\mathring{\Delta}_j f|^p]^{\frac{1}{p}} & \text{for } q = \infty \end{cases}$$

is finite.

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^n.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \le -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \cdots. \end{cases}$$

Definition 2.13 *The inhomogeneous Besov space $B^s_{p,q}$ with $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'$ satisfying*

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q} = \begin{cases} \|\Delta_{-1} f\|_{L^p} + (\sum_{j=0}^{\infty} 2^{jsq}[\int_{\mathbb{R}^n} |\Delta_j f|^p]^{\frac{q}{p}})^{\frac{1}{q}} & \text{for } q < \infty \\ \|\Delta_{-1} f\|_{L^p} + \sup_{0 \le j < \infty} 2^{js}[\int_{\mathbb{R}^n} |\Delta_j f|^p]^{\frac{1}{p}} & \text{for } q = \infty \end{cases}$$

is finite.

The Besov spaces $\mathring{B}^s_{p,q}$ and $B^s_{p,q}$ with $s \in (0,1)$ and $1 \le p, q \le \infty$ can alternatively be defined by the norms

$$\|f\|_{\mathring{B}^s_{p,q}} = \left(\int_{\mathbb{R}^n} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{n+sq}} dt\right)^{1/q},$$

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \left(\int_{\mathbb{R}^n} \frac{(\|f(x+t)-f(x)\|_{L^p})^q}{|t|^{n+sq}} dt\right)^{1/q}.$$

When $q=\infty$, the expressions are defined in the usual way.

Definition 2.14 *For $t>0$, $s\in\mathbb{R}$ and $1\le p,q,r\le\infty$, the space-time spaces $\widetilde{L}^r_t\mathring{B}^s_{p,q}$ and $\widetilde{L}^r_t B^s_{p,q}$ are defined through the norms*

$$\|f\|_{\widetilde{L}^r_t\mathring{B}^s_{p,q}} \equiv \|2^{js}\|\mathring{\Delta}_j f\|_{L^r_t L^p}\|_{l^q},$$

$$\|f\|_{\widetilde{L}^r_t B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^r_t L^p}\|_{l^q}$$

These spaces are related to the classical space-time spaces $L^r_t\mathring{B}^s_{p,q}$, $L^r_t B^s_{p,q}$ via the Minkowski inequality.

Many frequently used function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations.

Proposition 2.1.1 *The relation between the Besov spaces and other spaces and frequently used embeeding relations are given by*

1. *For any $s\in\mathbb{R}$,*

$$\mathring{H}^s \sim \mathring{B}^s_{2,2}, \quad H^s \sim B^s_{2,2}.$$

2. *For any $s\in\mathbb{R}$ and $1<q<\infty$,*

$$\mathring{B}^s_{q,\min\{q,2\}} \hookrightarrow \mathring{W}^{s,q} \hookrightarrow \mathring{B}^s_{q,\max\{q,2\}}.$$

   In particular, $\mathring{B}^0_{q,\min\{q,2\}} \hookrightarrow L^q \hookrightarrow \mathring{B}^0_{q,\max\{q,2\}}$.

3. *For any $s>0$*

$$B^s_{p,q} \subset \mathring{B}^s_{p,q}$$

4. *For $s_1\le s_2$*

$$B^{s_2}_{p,q} \subset B^{s_1}_{p,q}$$

5. *If* $1 \le q_1 \le q_2 \le \infty$

$$\mathring{B}^s_{p,q_1} \subset \mathring{B}^s_{p,q_2}, \qquad B^s_{p,q_1} \subset B^s_{p,q_2}$$

6. *If* $1 \le p_1 \le p_2 \le \infty, 1 \le r_1 \le r_2 \le \infty$ *and* $s_1 \ge s_2, s_1, s_2 \in \mathbb{R}$*, satisfy*

$$s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}$$

   Then

$$\mathring{B}^{s_1}_{p_1,r_1}(\mathbb{R}^n) \hookrightarrow \mathring{B}^{s_2}_{p_2,r_2}(\mathbb{R}^n), \quad B^{s_1}_{p_1,r_1}(\mathbb{R}^n) \hookrightarrow B^{s_2}_{p_2,r_2}(\mathbb{R}^n)$$

7. *Let* p *and* $q \in [1, \infty]$*, and* $p \le q$ *then,* $\mathring{B}^{n(\frac{1}{p}-\frac{1}{q})}_{p,1}(\mathbb{R}^n)$ *is continuously embedded in* $L^q(\mathbb{R}^n)$*.*

8. *If* $p \in [0, 1)$*, then* $\mathring{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ *is continuously embedded in* C_0*, where* C_0 *is the space of bounded continuous functions that vanish at* ∞*.*

9. *Let* $p \in [1, \infty]$*, then* $L^p(\mathbb{R}^n)$ *is continuously embedded in* $\mathring{B}^0_{p,\infty}(\mathbb{R}^n)$*.*

10. *For* $p \in [1, 2]$*,* L^p *is continuously embedded in* $\mathring{B}^0_{p,p}$*.*

11. *For* $p \in [2, \infty]$*,* $\mathring{B}^0_{p,2}$ *is continuously embedded in* $L^p(\mathbb{R}^n)$*; For* $p \in (1, 2]$*,* $L^p(\mathbb{R}^n)$ *is continuously embedded in* $\mathring{B}^0_{p,2}$*.*

2.2 Generalized Besov Space

This section first introduces a very general class of operator $P(\xi)$ for the Fourier multiplier operator $P(\Lambda)$, that is defined through the Fourier transform $\widehat{P(\Lambda)}f(\xi) = P(\xi)\hat{f}(\xi)$. It then presents the definition of the generalized Besov space by using some properties of this operator.

Definition 2.15 *For* $p, q \in [1, \infty]$ *and* $s, \gamma \in \mathbb{R}$*, the generalized Besov space* $B^{s,\gamma}_{p,q}$ *is defined by*

$$\|f\|_{B^{s,\gamma}_{p,q}} \equiv \|2^{js}(1 + |j|)^{\gamma} \, \|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

Definition 2.16 *Let $P = P(|x|) : (0,\infty) \to (0,\infty)$ be a non-decreasing function satisfying Condition 1.1 and*

$$\lim_{|x|\to\infty} \frac{P(|x|)}{|x|^\epsilon} = 0\,, \quad \forall\, \epsilon > 0.$$

For $s \in \mathbb{R}$ and $1 \le p,q \le \infty$, the generalized Besov spaces $\mathring{B}^{s,P}_{p,q}$ and $B^{s,P}_{p,q}$ are defined through the norms

$$\|f\|_{\mathring{B}^{s,P}_{p,q}} \equiv \|2^{js} P(2^j)\, \|\mathring{\Delta}_j f\|_{L^p}\|_{l^q} < \infty,$$

$$\|f\|_{B^{s,P}_{p,q}} \equiv \|2^{js} P(2^j)\, \|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

As in the case of Besov spaces, we can similarly define the space-time generalized Besov spaces.

Definition 2.17 *For $t > 0$, $s \in \mathbb{R}$ and $1 \le p,q,r \le \infty$, the space-time spaces $\widetilde{L}^r_t \mathring{B}^{s,P}_{p,q}$ and $\widetilde{L}^r_t B^{s,P}_{p,q}$ are defined through the norms*

$$\|f\|_{\widetilde{L}^r_t \mathring{B}^{s,P}_{p,q}} \equiv \|2^{js}\, P(2^j) \|\mathring{\Delta}_j f\|_{L^r_t L^p}\|_{l^q},$$

$$\|f\|_{\widetilde{L}^r_t B^{s,P}_{p,q}} \equiv \|2^{js}\, P(2^j) \|\Delta_j f\|_{L^r_t L^p}\|_{l^q}$$

These spaces are related to the classical space-time spaces $L^r_t \mathring{B}^{s,P}_{p,q}$ and $L^r_t B^{s,P}_{p,q}$ via the Minkowski inequality.

We will need a Bernstein type inequality for fractional derivatives. These type of inequalities trade integrability for derivatives and the proof can be found, for example, in [47]. The following proposition provides Bernstein type inequalities for fractional derivatives.

Proposition 2.2.1 *Let $\alpha \ge 0$. Let $1 \le p \le q \le \infty$.*

1) If f satisfies

$$supp\, \widehat{f} \subset \{\xi \in \mathbb{R}^n :\ |\xi| \le K 2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^n)} \le C_1\, 2^{2\alpha j + jn(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^n)}.$$

2) If f satisfies

$$supp\,\widehat{f} \subset \{\xi \in \mathbb{R}^n : \; K_1 2^j \leq |\xi| \leq K_2 2^j\}$$

for some integer j and constants $0 < K_1 \leq K_2$, then

$$C_1\, 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^n)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^n)} \leq C_2\, 2^{2\alpha j + jn(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^n)},$$

where C_1 and C_2 are constants depending on α, p and q only.

2.3 Para-products

This section presents the basic tool of the para-differential calculus which is Bony's decomposition. Of course, we will use the dyadic decomposition in the frequency space as well as the characterizations of Besov spaces.

The idea is as follows. Given two tempered distributions f and g, we write

$$f = \sum_j \Delta_j f \quad \text{and} \quad g = \sum_j \Delta_j g.$$

The product, when it exists, may be written as

$$fg = \sum_{j,k} \Delta_j f\, \Delta_k g.$$

Bony's decomposition, distinguishes three parts in the product fg: the first one is related to the terms where the frequencies of f are large compared with those of g, the second one concerns the terms where the frequencies of g are large compared with those of f and finally the third part where the frequencies of f and g are of the same size. More precisely, we have the following definitions.

Definition 2.18 *The bilinear operator $T_f g$ is called the para-product of g by f and is defined by*

$$T_f g = \sum_j \sum_{k \leq j-2} \Delta_k f \Delta_j g.$$

The symmetric bilinear operator $R(f,g)$ is called the remainder of the product fg and is defined by

$$R(f,g) = \sum_{|j-k|\le 1} \Delta_j f \Delta_k g.$$

Hence from the definition of the para-product and remainder operators, we have

$$fg = T_f g + T_g f + R(f,g).$$

We need to note that the paraproduct is always defined for two compactly supported distributions and the regularity of $T_f g$ is mostly determined by that of g

We now introduce a useful notation for the partial sum or low frequency cut-off S_j. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k$$

Thus, for any $f \in \mathcal{S}'$,

$$S_j f = \sum_{k=-1}^{j-1} \Delta_k f.$$

The Fourier transform of $S_j f$ is supported on the ball of radius 2^j. With this notation, we have the following definition.

Definition 2.19 *(Paraproduct decomposition) The product of two distributions f and g is defined as*

$$f\cdot g = \sum_j S_{j-1} f \Delta_j g + \sum_j \Delta_j f S_{j-1} g + \sum_j \Delta_j f \tilde{\Delta}_j g,$$

where

$$\tilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}, \quad j = 0,1,2,\cdots$$

We now point out several simple facts concerning the operator Δ_j:

$$\Delta_j \Delta_k = 0, \quad \text{if} \quad |j-k| \ge 2:$$

$$S_j = \sum_{k=-\infty}^{j} \Delta_k \to I, \quad \text{as} \quad j \to \infty;$$

$$\Delta_k(S_{j-1}f\Delta_j f) = 0, \quad \text{if} \quad |j-k| \geq 4.$$

The following lemma from [14] provides the fundamental estimates for $\|\Delta_j \nabla u\|_{L^p}$ and $\|S_N \nabla u\|_{L^p}$.

Lemma 2.1 *Assume that the symbol Q satisfies Condition 1.1 and that u and ω are related through*

$$u = \nabla^{\perp}\Delta^{-1}Q(\Lambda)\omega.$$

Then, for any integer $j \geq 0$ and $N \geq 0$,

$$\begin{aligned} \|S_N \nabla u\|_{L^p} &\leq C_p\, Q(C_0 2^N)\, \|S_N \omega\|_{L^p}, \quad 1 < p < \infty, \\ \|\Delta_j \nabla u\|_{L^q} &\leq C\, Q(C_0 2^j)\, \|\Delta_j \omega\|_{L^q}, \quad 1 \leq q \leq \infty, \end{aligned}$$

where C_p is a constant depending on p only, C_0 and C are pure constants.

2.4 Commutator

The main difficulty to show a global a priori bounds for the solution to $2D$ Boussinesq system arises due to the vortex stretching term $\partial_{x_1}\theta$. To hide this vortex stretching term, we introduce the notion of commutator. The trade-off is that we need to obtain suitable bounds for the commutator. We will establish in this section some commutator estimates.

Definition 2.20 *The commutator of two functions f and g is defined as*

$$[f, g] = fg - gf$$

In particular, for three functions f, g and h, it can be expressed as

$$[f, g]h = f[g(h)] - g[f(h)]$$

In the subsequent chapters, we will use an estimate for the commutator $[\mathcal{R}, u\cdot\nabla]\theta = \mathcal{R}(u\cdot\nabla\theta) - u\cdot\nabla(\mathcal{R}\theta)$, where $\mathcal{R} \equiv \Lambda^{-1}\partial_{x_1}$. We first state a fact given by the following lemma due to [13] and [34].

Lemma 2.2 *Consider two different cases: $\delta \in (0,1)$ and $\delta = 1$.*

1. *Let $\delta \in (0,1)$ and $q \in [1,\infty]$. If $|x|^\delta h \in L^1$, $f \in \mathring{B}^\delta_{q,\infty}$ and $g \in L^\infty$, then*

$$\|h * (fg) - f(h * g)\|_{L^q} \le C \, \||x|^\delta h\|_{L^1} \, \|f\|_{\mathring{B}^\delta_{q,\infty}} \|g\|_{L^\infty},$$

   where C is a constant independent of f, g and h.

2. *Let $\delta = 1$. Let $q \in [1,\infty]$. Let $r_1 \in [1,q]$ and $r_2 \in [1,\infty]$ satisfying $\frac{1}{r_1} + \frac{1}{r_2} = 1$. Then*

$$\|h * (fg) - f(h * g)\|_{L^q} \le C \, \||x|h\|_{L^{r_1}} \, \|\nabla f\|_{L^q} \, \|g\|_{L^{r_2}},$$

We now state and prove the commutator estimate.

Proposition 2.4.1 *Let $\mathcal{R} = \Lambda^{-1}\partial_{x_1}$ denote the Riesz transform. Assume that the symbol P satisfies Condition 1.1 and*

$$\text{for any } \epsilon > 0, \quad \lim_{|\xi|\to\infty} \frac{P(|\xi|)}{|\xi|^\epsilon} = 0. \tag{2.1}$$

Assume that u and ω are related by

$$u = \nabla^\perp \Delta^{-1} \Lambda^\sigma P(\Lambda)\omega$$

with $\sigma \in [0,1)$. Then, for any $p \in (1,\infty)$ and $r \in [1,\infty]$,

$$\|[\mathcal{R}, u\cdot\nabla]\theta\|_{B^0_{p,r}} \le C\|\omega\|_{L^p}\|\theta\|_{B^{\sigma,P}_{\infty,r}} + C\,\|\omega\|_{L^p}\,\|\theta\|_{L^p} \tag{2.2}$$

and, for any $r \in [1,\infty]$, $q \in (1,\infty)$ and any $\epsilon > 0$,

$$\|[\mathcal{R}, u\cdot\nabla]\theta\|_{B^0_{\infty,r}} \le C(\|\omega\|_{L^q} + \|\omega\|_{L^\infty})\|\theta\|_{B^{\sigma+\epsilon}_{\infty,r}} + C\,\|\omega\|_{L^q}\,\|\theta\|_{L^q} \tag{2.3}$$

for some constant C.

Proof. [Proof of Proposition 2.4.1] By the definition of $B^0_{p,r}$,

$$\|[\mathcal{R}, u\cdot\nabla]\theta\|_{B^0_{p,r}} = \left[\sum_{j=-1}^{\infty} \|\Delta_j[\mathcal{R}, u\cdot\nabla]\theta\|^r_{L^p}\right]^{\frac{1}{r}}.$$

First we decompose $\Delta_j[\mathcal{R}, u\cdot\nabla]\theta$ into three parts using the notion of paraproducts:

$$\Delta_j[\mathcal{R}, u\cdot\nabla]\theta \;=\; J_1 + J_2 + J_3,$$

where

$$\begin{aligned}
J_1 &= \sum_{|k-j|\le 2} \Delta_j(\mathcal{R}(S_{k-1}u\cdot\nabla\Delta_k\theta) - S_{k-1}u\cdot\nabla\mathcal{R}\Delta_k\theta),\\
J_2 &= \sum_{|k-j|\le 2} \Delta_j(\mathcal{R}(\Delta_k u\cdot\nabla S_{k-1}\theta) - \Delta_k u\cdot\nabla\mathcal{R}S_{k-1}\theta),\\
J_3 &= \sum_{k\ge j-1} \Delta_j(\mathcal{R}(\Delta_k u\cdot\nabla\tilde{\Delta}_k\theta) - \Delta_k u\cdot\nabla\mathcal{R}\tilde{\Delta}_k\theta)
\end{aligned}$$

with $\tilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. The Fourier transform of $S_{k-1}u\cdot\nabla\Delta_k\theta$ is supported in the annulus $2^k A$, where A denotes a fixed annulus. $\mathcal{R}$ acting on this term can be represented as a convolution with the kernel $h_k(x) = 2^{nk}h(2^k x)$ with $n = 2$, where h is a smooth function with compact support. That is,

$$\begin{aligned}
&\mathcal{R}(S_{k-1}u\cdot\nabla\Delta_k\theta) - S_{k-1}u\cdot\nabla\mathcal{R}\Delta_k\theta\\
&\qquad = h_k * (S_{k-1}u\cdot\nabla\Delta_k\theta) - S_{k-1}u\cdot\nabla(h_k * \Delta_k\theta).
\end{aligned}$$

Therefore, according to Lemma 2.2,

$$\|J_1\|_{L^p} \le C\,\||x|h_j\|_{L^1}\,\|\nabla S_{j-1}u\|_{L^p}\,\|\nabla\Delta_j\theta\|_{L^\infty}. \tag{2.4}$$

Applying Lemma 2.1, Bernstein's inequality and the equality

$$\||x|h_j\|_{L^1} = 2^{-j}\,\||x|h(x)\|_{L^1} = C\,2^{-j},$$

we obtain

$$\begin{aligned}
\|J_1\|_{L^p} &\le C\,2^{\sigma j}\,P(2^j)\|S_{j-1}\omega\|_{L^p}\,\|\Delta_j\theta\|_{L^\infty}\\
&\le C\,2^{\sigma j}\,P(2^j)\,\|\omega\|_{L^p}\,\|\Delta_j\theta\|_{L^\infty}.
\end{aligned}$$

Similarly,

$$\begin{aligned} \|J_2\|_{L^p} &\le C\,2^{-j}\,2^{\sigma j}\,P(2^j)\|\Delta_j\omega\|_{L^p}\,\|\nabla S_{j-1}\theta\|_{L^\infty} \\ &\le C\,2^{-(1-\sigma)j}\,P(2^j)\,\|\Delta_j\omega\|_{L^p}\sum_{m\le j-1} 2^m\|\Delta_m\theta\|_{L^\infty} \\ &\le C\,\|\Delta_j\omega\|_{L^p}\sum_{m\le j-1}\frac{2^{(1-\sigma)m}P(2^j)}{2^{(1-\sigma)j}P(2^m)}\,2^{\sigma m}P(2^m)\,\|\Delta_m\theta\|_{L^\infty}. \end{aligned}$$

The estimate of $\|J_3\|_{L^p}$ is different. We need to distinguish between low frequency and high frequency terms. For the high frequency terms, we do not need the commutator structure. For $j = 0, 1$, the terms in J_3 with $k = -1, 0, 1$ have Fourier transforms containing the origin in their support and the lower bound part of Bernstein's inequality does not apply. To deal with these low frequency terms, we take advantage of the commutator structure and bound them by Lemma 2.2. More precisely, for $j = 0, 1$ and $k = -1, 0, 1$,

$$\begin{aligned} &\|\Delta_j(\mathcal{R}(\Delta_k u\cdot\nabla\tilde{\Delta}_k\theta) - \Delta_k u\cdot\nabla\mathcal{R}\tilde{\Delta}_k\theta)\|_{L^p} \\ &\qquad \le C\,\|\nabla\Delta_k u\|_{L^p}\,\|\Delta_k\theta\|_{L^p} \le C\,\|\omega\|_{L^p}\,\|\theta\|_{L^p}. \end{aligned}$$

For higher frequency terms, we first apply Bernstein's inequality to obtain

$$\begin{aligned} \|J_3\|_{L^p} &\le C\sum_{k\ge j-1} 2^j\|\mathcal{R}(\Delta_k u\cdot\tilde{\Delta}_k\theta)\|_{L^p} + C\sum_{k\ge j-1} 2^j\|\Delta_k u\cdot\mathcal{R}\tilde{\Delta}_k\theta\|_{L^p} \\ &\le C\sum_{k\ge j-1} 2^{j-k}\,\|\nabla\Delta_k u\|_{L^p}\,\|\Delta_k\theta\|_{L^\infty} \\ &\le C\sum_{k\ge j-1} 2^{j-k}\,\|\Delta_k\omega\|_{L^p}2^{\sigma k}P(2^k)\|\Delta_k\theta\|_{L^\infty}. \end{aligned}$$

Thanks to $\sigma \in [0, 1)$ and the assumption on P in (2.1), we obtain, by Young's inequality for series convolution,

$$\begin{aligned} \|[\mathcal{R}, u\cdot\nabla]\theta\|_{B^0_{p,r}} &= C\left[\sum_{j=-1}^{\infty}(\|J_1\|^r_{L^p} + \|J_2\|^r_{L^p} + \|J_3\|^r_{L^p})\right]^{\frac{1}{r}} \\ &= C\,\|\omega\|_{L^p}\|\theta\|_{B^{\sigma,P}_{\infty,r}} + C\,\|\omega\|_{L^p}\,\|\theta\|_{L^p}. \end{aligned}$$

This completes the proof of (2.2). We now prove (2.3). We shall only provide those estimates that are different from the previous ones. As in (2.4), we still have

$$\|J_1\|_{L^\infty} \le C\, \||x|h_j\|_{L^1}\, \|\nabla S_{j-1}u\|_{L^\infty}\, \|\nabla\Delta_j\theta\|_{L^\infty}.$$

But $\|\nabla S_{j-1}u\|_{L^\infty}$ is bounded differently here. By Lemma 2.2 and the assumption in (2.1), we obtain, for $\sigma \in [0,1)$ and for any $\epsilon > 0$,

$$\begin{aligned} \|\nabla S_{j-1}u\|_{L^\infty} &\le \|\nabla\Delta_{-1}u\|_{L^\infty} + \sum_{0\le m\le j-2} \|\Delta_m \nabla u\|_{L^\infty} \\ &\le C\|w\|_{L^q} + \sum_{0\le m\le j-2} 2^{\sigma m}\, P(2^m)\, \|\Delta_m\omega\|_{L^\infty} \\ &\le C\|w\|_{L^q} + C\, 2^{(\sigma+\epsilon)j}\|\omega\|_{L^\infty}. \end{aligned}$$

Consequently,

$$\|J_1\|_{L^\infty} \le C\,(\|w\|_{L^q} + \|\omega\|_{L^\infty}) 2^{(\sigma+\epsilon)j}\|\Delta_j\theta\|_{L^\infty}.$$

The bounds for J_2 and J_3 can be obtained by simply setting $p=\infty$ in the corresponding bounds for $\|J_2\|_{L^p}$ and $\|J_3\|_{L^p}$ above. This completes the proof of Proposition 2.4.1. ■

Proposition 2.4.2 *Let a and $\mathcal{R}_a$ be defined by $a(|\xi|) \equiv \frac{|\xi|}{P(|\xi|)}$ and $\mathcal{R}_a = \mathcal{L}^{-1}\partial_{x_1}$. Assume*

$$p \in [2,\infty), \quad q \in [1,\infty], \quad 0 < s < \delta.$$

Let $[\mathcal{R}_a, u]F = \mathcal{R}_a(uF) - u\mathcal{R}_a F$ be a standard commutator. Then

$$\|[\mathcal{R}_a, u]F\|_{B^{s,a}_{p,q}} \le C\,(\|u\|_{\mathring{B}^{\delta}_{p,\infty}}\, \|F\|_{B^{s-\delta,\, a^2}_{\infty,q}} + \|u\|_{L^2}\, \|F\|_{L^2}),$$

where C denotes a constant independent of a and $\mathcal{R}_a$.

Proof. [Proof of Proposition 2.4.2] Let $j \ge -1$ be an integer. Using the notion of paraproducts, we decompose $\Delta_j[\mathcal{R}_a, u]F$ into three parts,

$$\Delta_j[\mathcal{R}_a, u]F \;=\; I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \sum_{|k-j|\le 2} \Delta_j(\mathcal{R}_a(S_{k-1}u\cdot\Delta_k F) - S_{k-1}u\cdot\mathcal{R}_a\Delta_k F), \\ I_2 &= \sum_{|k-j|\le 2} \Delta_j(\mathcal{R}_a(\Delta_k u\cdot S_{k-1}F) - \Delta_k u\cdot\mathcal{R}_a S_{k-1}F), \\ I_3 &= \sum_{k\ge j-1} \Delta_j(\mathcal{R}_a(\Delta_k u\cdot\tilde{\Delta}_k F) - \Delta_k u\mathcal{R}_a\cdot\tilde{\Delta}_k F). \end{aligned}$$

When the operator $\mathcal{R}_a$ acts on a function whose Fourier transform is supported on an annulus, it can be represented as a convolution kernel. Since the Fourier transform of $S_{k-1}u\cdot\Delta_k F$ is supported on an annulus around the radius of 2^k, we can write

$$h_k\star(S_{k-1}u\cdot\Delta_k F) - S_{k-1}u\cdot(h_k\star\Delta_k F),$$

where h_k is given by the inverse Fourier transform of $i\xi_1 P^{-1}(|\xi|)\,\widetilde{\Phi}_k(\xi)$, namely

$$h_k(x) = \Big(i\xi_1 P^{-1}(|\xi|)\,\widetilde{\Phi}_k(\xi)\Big)^{\vee}(x).$$

Here $\widetilde{\Phi}_k(\xi)\in C_0^\infty(\mathbb{R}^2)$, $\widetilde{\Phi}_k(\xi)$ is also supported on an annulus around the radius of 2^k and is identically equal to 1 on the support of $S_{k-1}u\cdot\Delta_k F$. Therefore, recalling (4.9), we can write

$$i\xi_1 P^{-1}(|\xi|)\,\widetilde{\Phi}_k(\xi) = i\frac{\xi_1}{|\xi|}\tilde{\Phi}_0(2^{-k}\xi)\,a(|\xi|).$$

Therefore,

$$h_k(x) = 2^{2k}\,h_0(2^k x)*a^{\vee}(x), \quad h_0(x) = \left(\frac{\xi_1}{|\xi|}\tilde{\Phi}_0(\xi)\right)^{\vee}.$$

By Lemma 2.2,

$$\begin{aligned} \|I_1\|_{L^p} &\le C\,\||x|^\delta h_j\|_{L^1}\|S_{j-1}u\|_{\mathring{B}^\delta_{p,\infty}}\,\|\Delta_j F\|_{L^\infty} \\ &\le C\,2^{-\delta j}\,a(2^j)\,\|S_{j-1}u\|_{\mathring{B}^\delta_{p,\infty}}\,\|\Delta_j F\|_{L^\infty}. \end{aligned}$$

I_2 in L^p can be estimated as follows.

$$\begin{aligned}
\|I_2\|_{L^p} &\le C\, 2^{-\delta j}\, a(2^j)\, \|S_{j-1}F\|_{L^\infty} \|\Delta_j u\|_{\mathring{B}^\delta_{p,\infty}} \\
&\le C\, 2^{-\delta j}\, a(2^j) \sum_{m\le j-1} \|\Delta_m F\|_{L^\infty}\, \|\Delta_j u\|_{\mathring{B}^\delta_{p,\infty}} \\
&= C\, 2^{-sj} a^{-1}(2^j) \sum_{m\le j-1} 2^{(s-\delta)(j-m)} \frac{a^2(2^j)}{a^2(2^m)}\, 2^{(s-\delta)m} a^2(2^m)\, \|\Delta_m F\|_{L^\infty} \|\Delta_j u\|_{\mathring{B}^\delta_{p,\infty}}.
\end{aligned}$$

The estimate of $\|I_3\|_{L^p}$ is different. We need to distinguish between low frequency and high frequency terms. For $j = 0, 1$, the terms in I_3 with $k = -1, 0, 1$ have Fourier transforms containing the origin in their support and the lower bound part of Bernstein's inequality does not apply. To deal with these low frequency terms, we take advantage of the commutator structure and bound them by Lemma 2.2. The kernel h corresponding to $\mathcal{R}_a$ still satisfies, for any $r_1 \in (1, \infty)$,

$$\| |x|\, h\|_{L^{r_1}} \le C.$$

Therefore, by Lemma 2.2 and Bernstein's inequality, for $j = 0, 1$ and $k = -1, 0, 1$,

$$\begin{aligned}
\|\Delta_j(\mathcal{R}_a(\Delta_k u \cdot \tilde{\Delta}_k F) - \Delta_k u \cdot \mathcal{R}_a \tilde{\Delta}_k F)\|_{L^p} &\le C\, \| |x| h\|_{L^{r_1}}\, \|\nabla \Delta_k u\|_{L^p}\, \|\Delta_k F\|_{L^{r_2}} \\
&\le C\, \|u\|_{L^2}\, \|F\|_{L^2}.
\end{aligned}$$

where $\frac{1}{r_1} + \frac{1}{r_2} = 1$. For the high frequency terms, we do not need the commutator structure. By Lemma 2.1 and Hölder's inequality,

$$\begin{aligned}
\|I_{31}\|_{L^p} \equiv \left\| \sum_{k\ge j-1} \Delta_j(\mathcal{R}_a(\Delta_k u \cdot \tilde{\Delta}_k F)) \right\|_{L^p} &\le \sum_{k\ge j-1} C\, a(2^j)\, \|\Delta_k u\|_{L^p}\, \|\Delta_k F\|_{L^\infty} \\
&\le C\; a(2^j) \sum_{k\ge j-1} 2^{-\delta k}\, 2^{\delta k} \|\Delta_k u\|_{L^p}\, \|\Delta_k F\|_{L^\infty} \\
&\le C\, 2^{-sj}\, a^{-1}(2^j)\, \|u\|_{\mathring{B}^\delta_{p,\infty}} \sum_{k\ge j-1} 2^{s(j-k)}\, \frac{a^2(2^j)}{a^2(2^k)}\, 2^{(s-\delta)k} a^2(2^k)\; \|\Delta_k F\|_{L^\infty}.
\end{aligned}$$

$I_{32} \equiv \sum_{k\ge j-1} \Delta_k u \cdot \mathcal{R}_a \tilde{\Delta}_k F$ admits the same bound. Therefore, by the definition of generalized Besov space norm,

$$
\begin{aligned}
\|[\mathcal{R}_a, u]F\|_{B^{s,a}_{p,q}} \leq & \left[\sum_{j\geq -1} 2^{qsj} a^q(2^j)\|I_1\|^q_{L^p}\right]^{\frac{1}{q}} + \left[\sum_{j\geq -1} 2^{qsj} a^q(2^j)\|I_2\|^q_{L^p}\right]^{\frac{1}{q}} \\
& + \left[\sum_{j\geq -1} 2^{qsj} a^q(2^j)(\|I_{31}\|^q_{L^p} + \|I_{32}\|^q_{L^p})\right]^{\frac{1}{q}} + C\,\|u\|_{L^2}\,\|F\|_{L^2}.
\end{aligned}
$$

The first term on the right is clearly bounded by

$$
C\,\|u\|_{\mathring{B}^{\delta}_{p,\infty}} \left[\sum_{j\geq -1} 2^{q(s-\delta)j} a^{2q}(2^j)\|\Delta_j F\|^q_{L^\infty}\right]^{\frac{1}{q}} = C\,\|u\|_{\mathring{B}^{\delta}_{p,\infty}}\,\|F\|_{B^{s-\delta,\,a^2}_{\infty,q}}.
$$

Due to $s<\delta$, (1.11) and a convolution inequality for series,

$$
\left[\sum_{j\geq -1} 2^{qsj} a^q(2^j)\|I_2\|^q_{L^p}\right]^{\frac{1}{q}} \leq C\,\|u\|_{\mathring{B}^{\delta}_{p,\infty}}\,\|F\|_{B^{s-\delta,\,a^2}_{\infty,q}}.
$$

Thanks to $0<s$, (1.11) and a convolution inequality for series,

$$
\left[\sum_{j\geq -1} 2^{qsj} a^q(2^j)\|I_{31}\|^q_{L^p}\right]^{\frac{1}{q}} \leq C\,\|u\|_{\mathring{B}^{\delta}_{p,\infty}}\,\|F\|_{B^{s-\delta,\,a^2}_{\infty,q}}.
$$

This completes the proof of Proposition 2.4.2. ■

2.5 Frequently Used Inequalities

To estimate the certain norms of a function, we often use the following inequalities.

Lemma 2.3 *(Hölder's Inequality) Let $1\leq p_1,\cdots,p_m\leq\infty$, with $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=1$ and assume that $f_k\in L^{p_k}$ for $k=1,\cdots,m$, then*

$$
\|f_1 f_2\cdots f_m\|_{L^1} \leq \prod_{k=1}^{m}\|f_k\|_{L^{p_k}}.
$$

An extension is given by

$$
\|fg\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q} \quad \textit{with} \quad \frac{1}{r}=\frac{1}{p}+\frac{1}{q}.
$$

Lemma 2.4 *(Minkowski Inequality) For* $1 \le p \le \infty$

$$\left(\int \left|\int f(x,y)dy\right|^p dx\right)^{\frac{1}{p}} \le \int \left(\int |f(x,y)|^p dx\right)^{\frac{1}{p}} dy.$$

More generally, for $1 \le q \le p \le \infty$

$$\|f\|_{L_x^p L_y^q} \le \|f\|_{L_y^q L_x^p}.$$

Lemma 2.5 *(Young's Inequality for Product) Let* a *and* b *be non-negative real numbers. For* $1 < p, q < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Lemma 2.6 *(Young's Inequality for Convolution) If* $f \in L^p$, $g \in L^q$, $1 \le p, q, r \le \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, *then* $f * g \in L^r$ *and*

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}.$$

Lemma 2.7 *For* $0 \le p < \infty$ *and* $a,\ b > 0$,

$$(a+b)^p \le 2^p (a^p + b^p).$$

Lemma 2.8 *(cf. [15]) Let* f *be divergence-free vector field such that* $\nabla f \in L^p$, $p \in (1, \infty)$. *Then*

$$\|\nabla f\|_{L^p} \le c \frac{p^2}{p-1} \|\nabla \times f\|_{L^p}$$

Lemma 2.9 *(cf. [39]) Let* $f \in \dot{W}^{\delta,p_1} \cap L^{q_2}$, $g \in \dot{W}^{\delta,p_2} \cap L^{q_1}$, $\delta \ge 0$, $1 < p_k < \infty$, $1 < q_k \le \infty$, $\frac{1}{p_k} + \frac{1}{q_k} = \frac{1}{p}$, $k = 1,\ 2$. *Then*

$$\|fg\|_{\dot{W}^{\delta,p}} \lesssim (\|f\|_{\dot{W}^{\delta,p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{q_2}} \|g\|_{\dot{W}^{\delta,p_2}}).$$

Lemma 2.10 *(cf. [40]) Let* $f,\ g$ *be smooth such that* $\nabla f \in L^{p_1}$, $\Lambda^{s-1} g \in L^{p_2}$, $\Lambda^s f \in L^{p_3}$, $g \in L^{p_4}$, $p \in (1, \infty)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, $p_2,\ p_3 \in (1, \infty)$, $s > 0$. *Then*

$$\|\Lambda^s (fg) - f \Lambda^s g\|_{L^p} \lesssim (\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}})$$

Lemma 2.11 *(Gagliardo-Nirenberg Inequality) For $1 \leq q, r \leq \infty$, and given integers α and β satisfying $0 \leq \alpha < \beta$, we have for a smooth function $f : \mathbb{R}^n \to \mathbb{R}$*

$$\|\Lambda^\alpha f\|_{L^p} \leq C\|\Lambda^\beta f\|_{L^q}^a \|f\|_{L^r}^{1-a},$$

where $\left(\frac{1}{p} - \frac{\alpha}{n}\right) = a\left(\frac{1}{q} - \frac{\beta}{n}\right) + (1-a)\frac{1}{r}$

Lemma 2.12 *(Calderón-Zygmund Inequality) For the Riesz transform $\mathcal{R}$, we have*

$$\|\mathcal{R}f\|_{L^p} \leq \|f\|_{L^p}, \qquad for \quad 1 < p < \infty.$$

This inequality does not hold for $p = \infty$. However, when $\text{supp}(\hat{f})$ does not contain the origin, the inequality holds for the case $p = \infty$ as well.

Lemma 2.13 *(Brezis-Wainger Inequality)(cf. [5]) Let $f \in L^2(\mathbb{R}^2) \cap W^{s,p}(\mathbb{R}^2)$ where $s \in \mathbb{R}$ such that $p \in [2, \infty)$, $\frac{2}{p} < s$. Then*

$$\|f\|_{L^\infty(\mathbb{R}^2)} \lesssim_{s,p} (\|f\|_{L^2(\mathbb{R}^2)} + \|f\|_{H^1(\mathbb{R}^2)} \log_2(2 + \|f\|_{W^{s,p}(\mathbb{R}^2)}) + 1).$$

Lemma 2.14 *(Grönwall Inequality)*

1. *(Differential form) Let $f(t)$ be a non-negative continuous function on an interval $[0, T]$ with*

$$f'(t) \leq g(t)f(t) + h(t),$$

where $g(t)$ and $h(t)$ are non-negative and continuous on [0,T]. Then

$$f(t) \leq exp\left(\int_0^t g(s)ds\right)[f(0) + \int_0^t h(s)ds] \quad \forall \quad t \in [0, T].$$

2. *(Integral form) If f, g and $h \geq 0$ are continuous on $[0, T]$, h is differentiable, and*

$$g(t) \leq h(t) + \int_0^t f(s)g(s)ds, \qquad t \in [0, T]$$

then

$$g(t) \leq h(0)exp\left(\int_0^t f(s)ds\right) + \int_0^t h'(s)\left[exp\int_s^t f(\tau)d\tau\right]ds, \quad t \in [0, T].$$

Lemma 2.15 *(Osgood Inequality) Let $\alpha(t) > 0$ be a locally integrable function. Assume $\omega(t) \geq 0$ be a continuous and nondecreasing function on $(0, \infty)$ satisfying*

$$\int_1^\infty \frac{1}{\omega(r)} dr = \infty.$$

Suppose that $\rho(t) > 0$ satisfies

$$\rho(t) \leq a + \int_{t_0}^t \alpha(s)\omega(\rho(s))ds$$

for some constant $a \geq 0$. Then if $a = 0$, then $\rho \equiv 0$; if $a > 0$, then

$$-\Omega(\rho(t)) + \Omega(a) \leq \int_{t_0}^t \alpha(\tau)d\tau,$$

where

$$\Omega(x) = \int_x^1 \frac{dr}{\omega(r)}.$$

CHAPTER 3

Generalized $2D$ Euler-Boussinesq Equations with a Singular Velocity

This chapter studies the global (in time) regularity problem concerning a system of equations generalizing the two-dimensional incompressible Boussinesq equations of the form

$$\begin{cases} \partial_t v + u\cdot\nabla v - \sum_{j=1}^{2} u_j \nabla v_j = -\nabla p + \theta \mathbf{e}_2, \\ \nabla\cdot v = 0, \quad u = \Lambda^\sigma P(\Lambda) v, \\ \partial_t \theta + u\cdot\nabla\theta + \Lambda\theta = 0, \end{cases} \tag{3.1}$$

where $v = v(x,t)$ and $u = u(x,t)$ are 2D vector fields depending on $x = (x_1, x_2) \in \mathbb{R}^2$ and $t \geq 0$, $p = p(x,t)$ and $\theta = \theta(x,t)$ are scalar functions, $\mathbf{e}_2$ is the unit vector in the x_2-direction and $\sigma \geq 0$ is a real parameter. Here the Zygmund operator $\Lambda = (-\Delta)^{1/2}$, Λ^σ and the Fourier multiplier operator $P(\Lambda)$ are defined through the Fourier transform, namely

$$\widehat{\Lambda^\sigma f}(\xi) = |\xi|^\sigma \widehat{f}(\xi) \quad \text{and} \quad \widehat{P(\Lambda) f}(\xi) = P(|\xi|)\, \widehat{f}(\xi).$$

We remark that (3.1) can be reformulated in terms of the vorticity $\omega = \nabla\times v$ as

$$\begin{cases} \partial_t \omega + u\cdot\nabla\omega = \partial_{x_1}\theta, \\ u = \nabla^\perp \psi, \quad \Delta\psi = \Lambda^\sigma P(\Lambda)\omega, \\ \partial_t \theta + u\cdot\nabla\theta + \Lambda\theta = 0, \end{cases} \tag{3.2}$$

where $\omega = \omega(x,t)$ and $\psi = \psi(x,t)$ are scalar functions and $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. We will mostly work with the vorticity formulation (3.2) rather than the velocity formulation

(3.1) together with the initial conditions

$$\omega(x,0) = \omega_0(x), \quad \theta(x,0) = \theta_0(x), \quad x \in \mathbb{R}^2. \tag{3.3}$$

The generalized Euler-Boussinesq system in (3.1) or (3.2) reduces to the generalized 2D Euler equations studied by [14] when $\theta = 0$. Furthermore, (3.1) or (3.2) generalizes the 2D incompressible Boussinesq equations given by

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu \Lambda^\alpha u = -\nabla p + \theta \mathbf{e}_2, \\ \nabla \cdot u = 0, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa \Lambda^\beta \theta = 0 \end{cases} \tag{3.4}$$

with the corresponding vorticity $\omega = \nabla \times u$ satisfying

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \nu \Lambda^\alpha \omega = \partial_{x_1} \theta, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \omega, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa \Lambda^\beta \theta = 0, \end{cases} \tag{3.5}$$

where $\nu \geq 0$, $\kappa \geq 0$, $\alpha \in (0,2]$ and $\beta \in (0,2]$ are real parameters. Clearly (3.2) with $\sigma = 0$ and $P(\Lambda) = I$ is simply (3.5) with $\nu = 0$ and $\beta = 1$, where I denotes the identity operator.

We point out that ω in (3.2) is not same as the vorticity given in (3.5). We start with vorticity ω given in (3.2) and inversely defined the velocity by

$$u = \nabla^\perp \Delta^{-1} \Lambda^\sigma P(\Lambda) \omega$$

The main goal of this chapter is to examine for what operator P obeying Condition 1.1, (3.1) or (3.2) is still globally well-posed. The following theorem, which is the main theorem of this chapter, provides the answer of this question.

Theorem 3.1 *Let $\sigma = 0$. Assume the symbol $P(|\xi|)$ obeys Condition 1.1 and*

$$P(2^k) \leq C\sqrt{k} \quad \textit{for a constant } C \textit{ and any large integer } k > 0, \tag{3.6}$$

$$\int_1^\infty \frac{1}{r\log(1+r)\,P(r)}dr = \infty. \tag{3.7}$$

Let $q > 2$ and let $s > 2$. Consider the IVP (3.2) and (3.3) with $\omega_0 \in B^s_{q,\infty}(\mathbb{R}^2)$ and $\theta_0 \in B^s_{q,\infty}(\mathbb{R}^2)$. Then the IVP (3.2) and (3.3) has a unique global solution (ω, θ) satisfying, for any $T > 0$ and $t \le T$,

$$\omega \in C([0,T]; B^s_{q,\infty}(\mathbb{R}^2)), \quad \theta \in C([0,T]; B^s_{q,\infty}(\mathbb{R}^2) \cap L^1([0,T]; B^{s+1}_{q,\infty}(\mathbb{R}^2)). \tag{3.8}$$

An example of $P(|\xi|)$ that verifies Condition 1.1, (3.6) and (3.7) is the double logarithmic function

$$P(|\xi|) = (\log(1 + \log(1 + |\xi|^2)))^\gamma, \quad \gamma \in [0,1]. \tag{3.9}$$

As a special consequence of Theorem 3.1, the global well-posedness of (3.2) with $P(|\xi|)$ given by (3.9) is given in the following corollary.

Corollary 3.1 *Let $q > 2$ and let $s > 2$. Let $\omega_0 \in B^s_{q,\infty}(\mathbb{R}^2)$ and $\theta_0 \in B^s_{q,\infty}(\mathbb{R}^2)$. Assume that $\sigma = 0$ and $P(|\xi|)$ is given by (3.9). Then the IVP (3.2) and (3.3) has a unique global solution.*

When $\theta \equiv 0$, the result in Corollary 3.1 reduces to one of the theorems for the generalized 2D Euler in [14]. On the other hand, when P is the identity operator, we reproduce the global well-posedness for one of the critical Boussinesq equations [34].

3.1 Global *a priori* Bounds for $\|\omega\|_{L^\infty_t L^q}$, $\|\theta\|_{L^1_t B^{0,P}_{\infty,2}}$ and $\|\omega\|_{L^\infty_t L^\infty}$

This section establishes the global bounds for $\|\omega\|_{L^\infty_t L^q}$, $\|\theta\|_{L^1_t B^{0,P}_{\infty,2}}$ and $\|\omega\|_{L^\infty_t L^\infty}$. In order to obtain these bounds, we have to set $\sigma = 0$ and assume P satisfies Condition 1.1 and also (3.6), eventhough some of the intermediate results in this section hold for $\sigma \ge 0$ and P satisfies milder conditions.

The following proposition provides a logarithmic type interpolation inequality that bounds $\|\nabla u\|_{L^\infty}$.

Proposition 3.1.1 *Assume that the symbol Q satisfies Condition 1.1 and (3.6). Let u and ω be related through*

$$u = \nabla^\perp \Delta^{-1} Q(\Lambda)\omega.$$

Then, for any $1 \le q \le \infty$, $\beta > 2/q$, and $1 < p < \infty$,

$$\|\nabla u\|_{L^\infty} \le C\,(1 + \|\omega\|_{L^p}) + C\,\|\omega\|_{L^\infty}\, \log(1 + \|\omega\|_{B^\beta_{q,\infty}})\, Q\left(\|\omega\|^{\frac{2q}{q\beta-2}}_{B^\beta_{q,\infty}}\right),$$

where C's are constants that depend on p, q and β only.

Proof. [Proof of Proposition 3.1.1] For any integer $N \ge 0$, we have

$$\|\nabla u\|_{L^\infty} \le \|\Delta_{-1}\nabla u\|_{L^\infty} + \sum_{k=0}^{N-1} \|\Delta_k \nabla u\|_{L^\infty} + \sum_{k=N}^{\infty} \|\Delta_k \nabla u\|_{L^\infty}.$$

By Bernstein's inequality and Lemma 2.1, we have

$$\|\nabla u\|_{L^\infty} \;\; \le \;\; C\,\|\omega\|_{L^p} + C\,N\,Q(2^N)\,\|\omega\|_{L^\infty} + C\,\sum_{k=N}^{\infty} (2^k)^{\frac{2}{q}}\,\|\nabla \Delta_k u\|_{L^q}.$$

By Lemma 2.1,

$$\|\nabla u\|_{L^\infty} \;\; \le \;\; C\,\|\omega\|_{L^p} + C\,N\,Q(2^N)\,\|\omega\|_{L^\infty} + \,C\,\sum_{k=N}^{\infty} (2^k)^{\frac{2}{q}}\,Q(2^k)\|\Delta_k \omega\|_{L^q}.$$

By the definition of Besov space $B^\beta_{q,\infty}$,

$$\|\Delta_k \omega\|_{L^q} \le 2^{-\beta\,k}\|\omega\|_{B^\beta_{q,\infty}}.$$

Therefore,

$$\|\nabla u\|_{L^\infty} \;\; \le \;\; C\,\|\omega\|_{L^p} + C\,N\,Q(2^N)\,\|\omega\|_{L^\infty} + \,C\,\|\omega\|_{B^\beta_{q,\infty}} \sum_{k=N}^{\infty} (2^k)^{(\frac{2}{q}-\beta)}\;Q(2^k).$$

Due to $\frac{2}{q} - \beta < 0$ and (3.6), we can choose $\epsilon > 0$ such that

$$\epsilon + \frac{2}{q} - \beta < 0 \quad \text{and} \quad Q(2^N) \le 2^{\epsilon N}.$$

Especially, we take $\epsilon = \frac{1}{2}(\beta - \frac{2}{q})$ to get

$$\|\nabla u\|_{L^\infty} \;\; \le \;\; C\,\|\omega\|_{L^p} + C\,N\,Q(2^N)\,\|\omega\|_{L^\infty} + \,C\,\|\omega\|_{B^\beta_{q,\infty}}\,(2^N)^{(\frac{1}{q}-\frac{\beta}{2})}.$$

If we choose N to be the largest integer satisfying

$$N \le \frac{1}{\frac{\beta}{2} - \frac{1}{q}} \log_2 \left(1 + \|\omega\|_{B^\beta_{q,\infty}}\right),$$

we then obtain the desired result in Proposition 3.1.1. ■

We now prove two lemmas that are needed to prove the main result of this section.

Lemma 3.1 *Let $\sigma \in [0,1)$. Assume that the symbol P satisfies Condition 1.1 and (2.1). Let (ω,θ) be a smooth solution of (3.2). Then, for any $q \in [2,\infty)$ and for any $t > 0$,*

$$\|\omega(t)\|_{L^q} \le C\left(\|\omega_0\|_{L^q} + \|\theta_0\|_{L^q}\right) e^{C\, t \|\theta_0\|_{L^q}}\, e^{C\int_0^t \|\theta(\tau)\|_{B^{\sigma,P}_{\infty,2}}\, d\tau}, \tag{3.10}$$

where C's are pure constants.

Proof. [Proof of Lemma 3.1] We start with the equations satisfied by G and $\mathcal{R}\theta$,

$$\begin{aligned} &\partial_t G + u\cdot\ \nabla G = -[\mathcal{R}, u\cdot\nabla]\theta, \\ &\partial_t \mathcal{R}\theta + u\cdot\nabla\mathcal{R}\theta + \Lambda\mathcal{R}\theta = -[\mathcal{R}, u\cdot\nabla]\theta. \end{aligned} \tag{3.11}$$

By the embedding $B^0_{q,2} \hookrightarrow L^q$ for $q \ge 2$ and Lemma 2.4.2,

$$\begin{aligned} \|\omega(t)\|_{L^q} &\le \|G_0\|_{L^q} + \|\mathcal{R}\theta_0\|_{L^q} + 2\int_0^t \|[\mathcal{R}, u\cdot\nabla]\theta\|_{L^q} d\tau \\ &\le \|G_0\|_{L^q} + \|\mathcal{R}\theta_0\|_{L^q} + 2\int_0^t \|[\mathcal{R}, u\cdot\nabla]\theta\|_{B^0_{q,2}} d\tau \\ &\le \|G_0\|_{L^q} + \|\theta_0\|_{L^q} + C\int_0^t \left[\|\omega(\tau)\|_{L^q}(\|\theta(\tau)\|_{B^{\sigma,P}_{\infty,2}} + \|\theta_0\|_{L^q})\right] d\tau, \end{aligned}$$

which implies (3.10), by Gronwall's inequality. ■

The second lemma makes use of the dissipation in the θ-equation,

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \Lambda\theta = 0, \\ u = \nabla^\perp\psi, \quad \Delta\psi = \Lambda^\sigma P(\Lambda)\omega, \\ \theta(x,0) = \theta_0(x). \end{cases} \tag{3.12}$$

Lemma 3.2 *Let $\sigma \in [0,1)$. Assume that the symbol P satisfies Condition 1.1 and (2.1). Let $q \in (1,\infty)$. Then, any smooth solution (ω,θ) solving (3.12) satisfies, for each integer $j \geq 0$,*

$$2^{j(1-\sigma)}\|\Delta_j\theta\|_{L^1_t L^q} \leq 2^{-j\sigma}\|\Delta_j\theta_0\|_{L^q} + C\,P(2^j)\,\|\theta_0\|_{L^\infty}\int_0^t \|\omega(\tau)\|_{L^q}d\tau, \tag{3.13}$$

where C is a pure constant.

Proof. [Proof of Lemma 3.2] Letting $j \geq 0$ and applying Δ_j to (3.12), multiplying by $\Delta_j\theta|\Delta_j\theta|^{q-2}$ and integrating over $\mathbb{R}^2$, we obtain, after integrating by parts,

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j\theta\|^q_{L^q} + \int \Delta_j\theta|\Delta_j\theta|^{q-2}\Lambda\Delta_j\theta\,dx = -\int \Delta_j\theta|\Delta_j\theta|^{q-2}\Delta_j(u\cdot\nabla\theta)\,dx.$$

Due to the lower bound (see, e.g., [17, 70])

$$\int \Delta_j\theta|\Delta_j\theta|^{q-2}\Lambda\Delta_j\theta\,dx \geq C2^j\|\Delta_j\theta\|^q_{L^q}$$

and the decomposition of $[\Delta_j,\, u\cdot\nabla]\theta$ into five parts,

$$\Delta_j(u\cdot\nabla\theta) = J_1 + J_2 + J_3 + J_4 + J_5$$

with

$$\begin{aligned}
J_1 &= \sum_{|j-k|\leq 2}[\Delta_j,\, S_{k-1}u\cdot\nabla]\Delta_k\theta,\\
J_2 &= \sum_{|j-k|\leq 2}(S_{k-1}u - S_ju)\cdot\nabla\Delta_j\Delta_k\theta,\\
J_3 &= S_ju\cdot\nabla\Delta_j\theta,\\
J_4 &= \sum_{|j-k|\leq 2}\Delta_j(\Delta_k u\cdot\nabla S_{k-1}\theta),\\
J_5 &= \sum_{k\geq j-1}\Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k\theta),
\end{aligned}$$

we obtain, by Hölder's inequality,

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j\theta\|^q_{L^q} + C2^j\|\Delta_j\theta\|^q_{L^q} \leq \|\Delta_j\theta\|^{q-1}_{L^q}\left(\|J_1\|_{L^q} + \|J_2\|_{L^q} + \|J_4\|_{L^q} + \|J_5\|_{L^q}\right).$$

The integral involving J_3 becomes zero due to the divergence-free condition $\nabla \cdot S_j u = 0$. The terms on the right can be bounded as follows. To bound $\|J_1\|_{L^q}$, we write $[\Delta_j,\, S_{k-1}u \cdot \nabla]\Delta_k\theta$ as an integral,

$$[\Delta_j,\, S_{k-1}u \cdot \nabla]\Delta_k\theta = \int \Phi_j(x-y)(S_{k-1}u(y) - S_{k-1}u(x)) \cdot \nabla \Delta_k\theta(y)dy,$$

where Φ_j is the kernel associated with the operator Δ_j. By Lemma 2.2 and the inequality

$$\|\Phi_j(x)|x|^{1-\sigma}\|_{L^1} \le 2^{-j(1-\sigma)}\,\|\Phi_0(x)|x|^{1-\sigma}\|_{L^1} \le C\,2^{-j(1-\sigma)},$$

we have

$$\begin{aligned} \|J_1\|_{L^q} &\le \sum_{|j-k|\le 2} \|\Phi_j(x)|x|^{1-\sigma}\|_{L^1}\,\|S_{k-1}u\|_{B^{1-\sigma}_{q,\infty}}\,\|\nabla\Delta_k\theta\|_{L^\infty} \\ &\le C \sum_{|j-k|\le 2} 2^{-j(1-\sigma)}\,\|S_{k-1}u\|_{B^{1-\sigma}_{q,\infty}}\,2^k\|\Delta_k\theta\|_{L^\infty}. \end{aligned}$$

Recalling that $\Lambda^{1-\sigma}u = \nabla^\perp\Delta^{-1}\Lambda P(\Lambda)\omega$ and applying Lemma 2.1, we obtain

$$\|S_{k-1}u\|_{B^{1-\sigma}_{q,\infty}} \le C\,\|\Lambda^{1-\sigma}S_{k-1}u\|_{L^q} \le C\,P(2^j)\,\|S_{k-1}\omega\|_{L^q} \le C\,P(2^j)\,\|\omega\|_{L^q}.$$

Therefore,

$$\|J_1\|_{L^q} \le C2^{j\sigma}\,P(2^j)\,\|\omega\|_{L^q}\|\Delta_j\theta\|_{L^\infty}.$$

By Bernstein's inequality,

$$\begin{aligned} \|J_2\|_{L^q} &\le \sum_{|j-k|\le 2} \|S_j u - S_{k-1}u\|_{L^q}\,\|\nabla\Delta_j\theta\|_{L^\infty} \le C\|\Delta_j u\|_{L^q}2^j\|\Delta_j\theta\|_{L^\infty} \\ &\le C\,\|\nabla\Delta_j u\|_{L^q}\|\Delta_j\theta\|_{L^\infty} \\ &\le C2^{j\sigma}\,P(2^j)\,\|\Delta_j\omega\|_{L^q}\|\Delta_j\theta\|_{L^\infty}. \end{aligned}$$

We remark that we have applied the lower bound part of Bernstein's inequality in the second inequality above. This is valid for $j \ge 0$. Similarly,

$$\begin{aligned} \|J_4\|_{L^q} &\le C\,\|\Delta_j u\|_{L^q}\|\nabla S_{j-1}\theta\|_{L^\infty} \le C\,\|\Delta_j u\|_{L^q}2^j\|S_j\theta\|_{L^\infty} \\ &\le C\,\|\nabla\Delta_j u\|_{L^q}\|\theta\|_{L^\infty} \le C\,2^{j\sigma}\,P(2^j)\,\|\Delta_j\omega\|_{L^q}\|\theta\|_{L^\infty}. \end{aligned}$$

Using $\sigma \in [0,1)$ and the condition on P in (2.1),

$$\begin{aligned}
\|J_5\|_{L^q} &\leq C \sum_{k\geq j-1} 2^j \|\Delta_k u\|_{L^q} \|\widetilde{\Delta}_k \theta\|_{L^\infty} \\
&\leq C \sum_{k\geq j-1} 2^{j-k} \|\nabla \Delta_k u\|_{L^q} \, \|\Delta_k \theta\|_{L^\infty} \\
&\leq 2^{j\sigma} \sum_{k\geq j-1} 2^{(j-k)(1-\sigma)} \, P(2^k) \, \|\Delta_k \omega\|_{L^q} \|\Delta_k \theta\|_{L^\infty} \\
&\leq C 2^{j\sigma} \, P(2^j) \, \|\omega\|_{L^q} \|\theta\|_{L^\infty}.
\end{aligned}$$

Collecting the estimates above, we obtain

$$\frac{d}{dt}\|\Delta_j \theta\|_{L^q} + C\, 2^j \|\Delta_j \theta\|_{L^q} \leq C 2^{j\sigma} \, P(2^j) \, \|\omega\|_{L^q} \|\theta_0\|_{L^\infty}.$$

Integrating with respect to time yields

$$\|\Delta_j \theta(t)\|_{L^q} \leq e^{-C2^j t} \, \|\Delta_j \theta_0\|_{L^q} + C 2^{j\sigma} \, P(2^j) \, \|\theta_0\|_{L^\infty} \int_0^t e^{-C2^j(t-\tau)} \, \|\omega(\tau)\|_{L^q} \, d\tau.$$

We further take the L^1-norm in time to obtain

$$2^j \|\Delta_j \theta\|_{L^1_t L^q} \leq \|\Delta_j \theta_0\|_{L^q} + C 2^{j\sigma} \, P(2^j) \, \|\theta_0\|_{L^\infty} \int_0^t \|\omega(\tau)\|_{L^q} d\tau,$$

which is the desired result. This completes the proof of Lemma 3.2. ■

With these two lemmas, we are now ready to prove a proposition that provides the global bounds for$\|\omega\|_{L^\infty_t L^q}$, $\|\theta\|_{L^1_t B^{0,P}_{\infty,2}}$ and $\|\omega\|_{L^\infty_t L^\infty}$.

Proposition 3.1.2 *Let $\sigma = 0$ and $q > 2$. Assume the symbol P satisfies Condition 1.1 and (3.6). Let (ω, θ) be a smooth solution of (3.2) with $\omega_0 \in B^s_{q,\infty}$ and $\theta_0 \in B^s_{q,\infty}$. Then, for any $T > 0$ and $0 < t \leq T$,*

$$\|\omega(t)\|_{L^q} \leq C(T), \qquad \|\theta\|_{L^1_t B^{0,P}_{\infty,2}} \leq C(T), \qquad \|\omega(t)\|_{L^\infty} \leq C(T)$$

for some constant C depending T and the initial norms of ω_0 and θ_0.

Proof. [Proof of Proposition 3.1.2] The proof uses the bounds in Lemmas 3.1 and 3.2 with $\sigma = 0$. By the definition of $B^{0,P}_{\infty,2}$ and the embedding $B^{0,P}_{\infty,1} \hookrightarrow B^{0,P}_{\infty,2}$,

$$\|\theta\|_{L^1_t B^{0,P}_{\infty,2}} \leq \int_0^t \left[\sum_{j=-1}^{N-1} (P(2^j))^2 \|\Delta_j\theta\|^2_{L^\infty}\right]^{\frac{1}{2}} d\tau + \int_0^t \sum_{j=N}^{\infty} P(2^j)\|\Delta_j\theta\|_{L^\infty}\, d\tau.$$

Thanks to the condition on P in (3.6),

$$\|\theta\|_{L^1_t B^{0,P}_{\infty,2}} \leq t\|\theta_0\|_{L^\infty} N + \sum_{j\geq N} P(2^j)\|\Delta_j\theta\|_{L^1_t L^\infty}. \tag{3.14}$$

Since $q \in (2,\infty)$ and P satisfies (3.6), we choose $\epsilon > 0$ such that

$$-1 + \epsilon + \frac{2}{q} < 0, \qquad (P(2^j))^2\, 2^{-j\epsilon} \leq 1.$$

By Bernstein's inequality and Lemma 3.2 with $\sigma = 0$,

$$\begin{aligned}
\sum_{j\geq N} P(2^j)\, \|\Delta_j\theta\|_{L^1_t L^\infty} &\leq \sum_{j\geq N} P(2^j)\, 2^{j\frac{2}{q}} \|\Delta_j\theta\|_{L^1_t L^q} \\
&\leq C \sum_{j\geq N} (P(2^j))^2\, 2^{j(\frac{2}{q}-1)} (\|\theta_0\|_{L^q} + \|\theta_0\|_{L^\infty}\|\omega\|_{L^1_t L^q}) \\
&\leq C \sum_{j\geq N} 2^{j(\frac{2}{q}+\epsilon-1)} (\|\theta_0\|_{L^q} + \|\theta_0\|_{L^\infty}\|\omega\|_{L^1_t L^q}) \\
&\leq C\,\|\theta_0\|_{L^q} + C\, 2^{N(-1+\epsilon+\frac{2}{q})}\|\theta_0\|_{L^\infty}\|\omega\|_{L^1_t L^q}.
\end{aligned}$$

Inserting the estimates above in (3.14) and choosing N to be the largest integer satisfying

$$N \leq \frac{\log(1 + \|\omega\|_{L^1_t L^q})}{(1 - \epsilon - \frac{2}{q})} + 1$$

leads to

$$\|\theta\|_{L^1_t B^{0,P}_{\infty,2}} \leq C\,\|\theta_0\|_{L^\infty\cap L^q} + C\,\|\theta_0\|_{L^\infty} t\, \log\left(1 + \int_0^t \|\omega(\tau)\|_{L^q}\, d\tau\right).$$

It then follows from this estimate and (3.10) with $\sigma = 0$ that

$$\|\theta\|_{L^1_t B^{0,P}_{\infty,2}} \leq C\, t \log(1 + C\, t) + C\, t\|\theta\|_{L^1_t B^{0,P}_{\infty,2}}, \tag{3.15}$$

where C's are constants depending on $\|\theta_0\|_{L^q}$ and $\|\theta_0\|_{L^\infty}$. This inequality allows us to conclude that, for any $T > 0$ and $t \le T$,

$$\|\theta\|_{L^1_t B^{0,P}_{\infty,2}} \le C(T, \|\omega_0\|_{L^q}, \|\theta_0\|_{L^q \cap L^\infty}). \tag{3.16}$$

In fact, (3.16) is first obtained on a finite-time interval and the global bound is then obtained through an iterative process. Finally we prove the global bound for $\|\omega\|_{L^\infty}$. By (3.13) with $\sigma = 0$ and (3.6), we have, for any integer $j \ge 0$ and any $\epsilon > 0$,

$$2^{j(1-\epsilon)}\|\Delta_j\theta\|_{L^1_t L^q} \le \|\theta_0\|_{L^q} + C\|\theta_0\|_{L^\infty}\int_0^t \|\omega(\tau)\|_{L^q} d\tau \le C(T). \tag{3.17}$$

Since $q \in (2, \infty)$, we can choose $\epsilon > 0$ such that

$$2\epsilon + \frac{2}{q} - 1 < 0.$$

By Bernstein's inequality,

$$\|\theta\|_{B^\epsilon_{\infty,1}} \le \sum_{j\ge -1} 2^{(2\epsilon+\frac{2}{q}-1)j} 2^{(1-\epsilon)j}\|\Delta_j\theta\|_{L^q} \le C \sup_{j\ge -1} 2^{j(1-\epsilon)}\|\Delta_j\theta\|_{L^q}.$$

It then follows from (3.17) that, for any $t \le T$,

$$\|\theta\|_{L^1_t B^\epsilon_{\infty,1}} \le C(T). \tag{3.18}$$

Starting with the equations of G and $\mathcal{R}\theta$, namely (3.11), and applying Lemma 2.4.2, we have, for any $\epsilon > 0$,

$$\begin{aligned}
\|G\|_{L^\infty} + \|\mathcal{R}\theta\|_{L^\infty} &\le \|G_0\|_{L^\infty} + \|\mathcal{R}\theta_0\|_{L^\infty} + 2\int_0^t \|[\mathcal{R},\ \ u\cdot\nabla]\theta\|_{B^0_{\infty,1}} d\tau \\
&\le \|G_0\|_{L^\infty} + \|\mathcal{R}\theta_0\|_{L^\infty} \\
&\qquad + \int_0^t ((\|\omega\|_{L^q} + \|\omega\|_{L^\infty})\|\theta\|_{B^\epsilon_{\infty,1}} + \|\omega\|_{L^q}\|\theta\|_{L^q}) d\tau \\
&\le \|G_0\|_{L^\infty} + \|\mathcal{R}\theta_0\|_{L^\infty} + \int_0^t (\|G\|_{L^\infty} + \|\mathcal{R}\theta\|_{L^\infty})\|\theta\|_{B^\epsilon_{\infty,1}} d\tau \\
&\qquad + \int_0^t (\|\omega\|_{L^q}\|\theta\|_{B^\epsilon_{\infty,1}} + \|\omega\|_{L^q}\|\theta\|_{L^q})\, d\tau.
\end{aligned}$$

By Gronwall's inequality, (3.18) and the global bound for $\|\omega\|_{L^q}$, we have

$$\|\omega\|_{L^\infty} \le \|G\|_{L^\infty} + \|\mathcal{R}\theta\|_{L^\infty} \le C(T).$$

This completes the proof of Proposition 3.1.2. ■

3.2 Global Bounds for $\|\omega\|_{B^s_{q,\infty}}$ and $\|\theta\|_{B^s_{q,\infty}}$

This section is devoted to establish the global bounds for $\|\omega\|_{B^s_{q,\infty}}$ and $\|\theta\|_{B^s_{q,\infty}}$. It can be achieved in two steps. In the first step, we obtain the global bounds for $\|\omega\|_{B^\beta_{q,\infty}}$ and $\|\theta\|_{B^\beta_{q,\infty}}$ in the range $\frac{2}{q} < \beta < 1$. The restriction of β in this range is due to one of the para-products decomposed from the nonlinear terms. The second step takes advantage of the regularity obtained in the first step and establishes the bounds for $\|\omega\|_{B^{\beta_1}_{q,\infty}}$ and $\|\theta\|_{B^{\beta_1}_{q,\infty}}$ in the range $\frac{2}{q} < \beta_1 < 2 - \frac{2}{q}$. A repetition of this step allows us to reach any index $s > 2$. The following proposition provides the detail.

Proposition 3.2.1 *Assume that $\sigma = 0$ and the symbol $P(|\xi|)$ obeys Condition 1.1, (3.6) and (3.7). Let $q > 2$ and let $s > 2$. Consider the IVP (3.2) and (3.3) with $\omega_0 \in B^s_{q,\infty}(\mathbb{R}^2)$ and $\theta_0 \in B^s_{q,\infty}(\mathbb{R}^2)$. Let (ω, θ) be a smooth solution of (3.2). Then (ω, θ) admits a global a priori bound. More precisely, for any $T > 0$ and $t \le T$,*

$$\|(\omega(t), \theta(t))\|_{B^s_{q,\infty}} \le C(s, q, T, \|(\omega_0, \theta_0)\|_{B^s_{q,\infty}}),$$

where C is a constant depending on s, q, T and the initial norm.

Proof. [Proof of Proposition 3.2.1] Let $j \ge -1$ be an integer. Applying Δ_j to the equation of G, namely (3.11), multiplying by $\Delta_j G|\Delta_j G|^{q-2}$ and integrating over $\mathbb{R}^2$, we obtain, after integrating by parts,

$$\begin{aligned} \frac{1}{q}\frac{d}{dt}\|\Delta_j G\|^q_{L^q} &= -\int \Delta_j G|\Delta_j G|^{q-2}\Delta_j(u\cdot\nabla G)\,dx \\ &\quad -\int \Delta_j[\mathcal{R}, u\cdot\nabla]\theta\,\Delta_j G|\Delta_j G|^{q-2}\,dx. \end{aligned}$$

Following the notion of paraproducts, we decompose $\Delta_j(u \cdot \nabla G)$ into five parts,

$$\Delta_j(u\cdot\nabla G) = J_1 + J_2 + J_3 + J_4 + J_5$$

with

$$
\begin{aligned}
J_1 &= \sum_{|j-k|\le 2} [\Delta_j,\, S_{k-1}u\cdot\nabla]\Delta_k G,\\
J_2 &= \sum_{|j-k|\le 2} (S_{k-1}u - S_j u)\cdot\nabla\Delta_j\Delta_k G,\\
J_3 &= S_j u\cdot\nabla\Delta_j G,\\
J_4 &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u\cdot\nabla S_{k-1}G),\\
J_5 &= \sum_{k\ge j-1} \Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k G).
\end{aligned}
$$

By Hölder's inequality,

$$
\frac{1}{q}\frac{d}{dt}\|\Delta_j G\|_{L^q}^q \le \|\Delta_j G\|_{L^q}^{q-1}\left(\|J_1\|_{L^q}+\|J_2\|_{L^q}+\|J_4\|_{L^q}+\|J_5\|_{L^q}+\|J_6\|_{L^q}\right),
$$

where $J_6 = \Delta_j[\mathcal{R}, u\cdot\nabla]\theta$. The integral involving J_3 becomes zero due to the divergence-free condition $\nabla\cdot S_j u = 0$. The terms on the right can be bounded as follows. To bound $\|J_1\|_{L^q}$, we write $[\Delta_j,\, S_{k-1}u\cdot\nabla]\Delta_k G$ as an integral,

$$
[\Delta_j,\, S_{k-1}u\cdot\nabla]\Delta_k G = \int \Phi_j(x-y)(S_{k-1}u(y) - S_{k-1}u(x))\cdot\nabla\Delta_k G(y)dy,
$$

where Φ_j is the kernel associated with the operator Δ_j. By a standard commutator estimate ([14, 70]).

$$
\|J_1\|_{L^q} \le C\sum_{|j-k|\le 2}\|\nabla S_{k-1}u\|_{L^\infty}\|\Delta_k G\|_{L^q}.
$$

By Hölder's and Bernstein's inequalities,

$$
\|J_2\|_{L^q} \le C\,\|\nabla\Delta_j u\|_{L^\infty}\,\|\Delta_j G\|_{L^q}.
$$

We have especially applied the lower bound part in Bernstein's inequalities (see Proposition 2.2.1). The purpose is to shift the derivative ∇ from G to u. It is worth pointing out that the lower bound does not apply when $j=-1$. In the case when $j=-1$, J_2 involves only low modes and there is no need to shift the derivative from G to u. J_2 is bounded differently. When $j=-1$, J_2 becomes

$$
J_2 = -S_0(u)\cdot\nabla\Delta_1\Delta_{-1}G = -\Delta_{-1}u\cdot\nabla\Delta_1\Delta_{-1}G,
$$

whose L^q-norm can be bounded by

$$\|J_2\|_{L^q} \le C\|\Delta_{-1}u\|_{L^\infty}\,\|\Delta_{-1}G\|_{L^q} \le C\|\omega\|_{L^q}\,\|G\|_{L^q}.$$

For J_4 and J_5, we have, by Bernstein's inequality,

$$\begin{aligned}
\|J_4\|_{L^q} &\le C \sum_{|j-k|\le 2} \|\Delta_k u\|_{L^\infty}\,\|\nabla S_{k-1}G\|_{L^q} \\
&\le C \sum_{|j-k|\le 2} \|\nabla\Delta_k u\|_{L^\infty} \sum_{m\le k-1} 2^{m-k}\|\Delta_m G\|_{L^q}, \\
\|J_5\|_{L^q} &\le C \sum_{k\ge j-1} 2^j\,\|\Delta_k u\|_{L^\infty}\|\widetilde{\Delta}_k G\|_{L^q} \\
&\le C \sum_{k\ge j-1} 2^{j-k}\,\|\nabla\Delta_k u\|_{L^\infty}\,\|\widetilde{\Delta}_k G\|_{L^q}.
\end{aligned}$$

Furthermore, for any $\beta\in\mathbb{R}$,

$$\begin{aligned}
\|J_1\|_{L^q} &\le C \sum_{|j-k|\le 2} \|\nabla u\|_{L^\infty} 2^{-\beta(k+1)}\,2^{\beta(k+1)}\|\Delta_k G\|_{L^q} && (3.19)\\
&\le C\,2^{-\beta(j+1)}\,\|G\|_{B^\beta_{q,\infty}}\,\|\nabla u\|_{L^\infty} \sum_{|j-k|\le 2} 2^{\beta(j-k)} && (3.20)\\
&\le C\,2^{-\beta(j+1)}\,\|G\|_{B^\beta_{q,\infty}}\,\|\nabla u\|_{L^\infty}, && (3.21)
\end{aligned}$$

where C is a constant depending on β only. It is clear that $\|J_2\|_{L^q}$ admits the same bound. For any $\beta<1$, we have

$$\begin{aligned}
\|J_4\|_{L^q} &\le C\,\|\nabla u\|_{L^\infty} \sum_{|j-k|\le 2}\,\sum_{m<k-1} 2^{m-k}\,2^{-\beta(m+1)}\,2^{\beta(m+1)}\,\|\Delta_m G\|_{L^q} \\
&\le C\,\|\nabla u\|_{L^\infty}\,\|G\|_{B^\beta_{q,\infty}} \sum_{|j-k|\le 2}\,\sum_{m<k-1} 2^{m-k}\,2^{-\beta(m+1)} \\
&= C\,2^{-\beta(j+1)}\,\|G\|_{B^\beta_{q,\infty}}\,\|\nabla u\|_{L^\infty} \sum_{|j-k|\le 2} 2^{\beta(j-k)} \sum_{m<k-1} 2^{(m-k)(1-\beta)} \\
&\le C\,2^{-\beta(j+1)}\,\|G\|_{B^\beta_{q,\infty}}\,\|\nabla u\|_{L^\infty},
\end{aligned}$$

where C is a constant depending on β only and the condition $\beta<1$ is used to guarantee that $(m-k)(1-\beta)<0$. For any $\beta>-1$,

$$\begin{aligned}
\|J_5\|_{L^q} &\le C\,\|\nabla u\|_{L^\infty}\,2^{-\beta(j+1)} \sum_{k\ge j-1} 2^{(\beta+1)(j-k)}\,2^{\beta(k+1)}\,\|\widetilde{\Delta}_k G\|_{L^q} \\
&\le C\,2^{-\beta(j+1)}\,\|G\|_{B^\beta_{q,\infty}}\,\|\nabla u\|_{L^\infty}.
\end{aligned}$$

$\|J_6\|_{L^q} = \|\Delta_j[\mathcal{R}, u\cdot\nabla]\theta\|_{L^q}$ can be estimated as in the proof of Proposition 2.4.2,

$$\|J_6\|_{L^q} \le C\,(\|\omega\|_{L^q} + \|\omega\|_{L^\infty})2^{\epsilon j}\|\Delta_j\theta\|_{L^q}$$

for any fixed $\epsilon > 0$, where C is a constant depending on ϵ. For the purpose to be specified later, we choose

$$\epsilon > 0, \quad \beta + \epsilon < 1.$$

Collecting these estimates and invoking the global bounds for $\|\omega\|_{L^q\cap L^\infty}$, we obtain, for any $-1 < \beta < 1$,

$$\frac{d}{dt}\|\Delta_j G\|_{L^q} \le C\,2^{-\beta(j+1)}\,\|G\|_{B^\beta_{q,\infty}}\,\|\nabla u\|_{L^\infty} + C\,2^{\epsilon j}\|\Delta_j\theta\|_{L^q} + C.$$

Let $\widetilde{\beta} = \beta + \epsilon < 1$. By applying the process above to the equation for θ and making use of the fact that

$$\int \Delta_j\theta|\Delta_j\theta|^{q-2}\Lambda\Delta_j\theta\,dx \ge 0,$$

we obtain

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^q} \le C\,2^{-\widetilde{\beta}(j+1)}\,\|\theta\|_{B^{\widetilde{\beta}}_{q,\infty}}\,\|\nabla u\|_{L^\infty}.$$

Integrating the inequalities in time and adding them up, we obtain

$$X(t) \le C + X(0) + C\int_0^t (1 + \|\nabla u(\tau)\|_{L^\infty})X(\tau)\,d\tau. \tag{3.22}$$

where we have set

$$X(t) \equiv \|G(t)\|_{B^\beta_{q,\infty}} + \|\theta(t)\|_{B^{\widetilde{\beta}}_{q,\infty}}.$$

By Proposition 3.1.1, for any $\frac{2}{q} < \beta$,

$$\begin{aligned}
\|\nabla u\|_{L^\infty} &\le C\,(1 + \|\omega\|_{L^p}) + C\,\|\omega\|_{L^\infty}\,P\left(\|\omega\|^{\frac{2q}{q\beta-2}}_{B^\beta_{q,\infty}}\right)\,\log(1 + \|\omega\|_{B^\beta_{q,\infty}}) \\
&\le C\,(1 + \|\omega\|_{L^p}) + C\,\|\omega\|_{L^\infty}\,P\left(X(t)^{\frac{2q}{q\beta-2}}\right)\,\log(1 + X(t)).
\end{aligned}$$

Inserting this inequality in (3.22) and applying Osgood's inequality, we obtain desired bound, for $t \le T$,

$$\|\omega(t)\|_{B^\beta_{q,\infty}} \le \|G(t)\|_{B^\beta_{q,\infty}} + \|\theta(t)\|_{B^{\widetilde{\beta}}_{q,\infty}} = X(t) \le C(T).$$

We now proceed to show that, for any $t \le T$,

$$\|\omega(t)\|_{B^{\beta_1}_{q,\infty}} \le C(T) \quad \text{for any } \beta_1 \text{ satisfying } 1 < \beta_1 < 2 - \tfrac{2}{q}.$$

In order to get this global bound, we first obtain the global bound for $\|\theta(t)\|_{B^{\beta_1}_{q,\infty}}$ from the equation for θ and then use it to get the global bound for $\|G\|_{B^{\beta_1}_{q,\infty}}$.

As we have seen from the previous part, J_4 is the only term that requires $\beta < 1$. In the process of estimating $\|\theta(t)\|_{B^{\beta_1}_{q,\infty}}$, the corresponding terms $\widetilde{J}_1$, $\widetilde{J}_2$, $\widetilde{J}_5$ can be bounded the same way as before, namely

$$\|\widetilde{J}_1\|_{L^q},\ \|\widetilde{J}_2\|_{L^q},\ \|\widetilde{J}_5\|_{L^q} \le C\, 2^{-\beta_1(j+1)}\, \|\theta\|_{B^{\beta_1}_{q,\infty}}\, \|\nabla u\|_{L^\infty}. \tag{3.23}$$

$\|\widetilde{J}_4\|_{L^q}$ is estimated differently. We start with the basic bound

$$\|\widetilde{J}_4\|_{L^q} \le C \sum_{|j-k|\le 2} \|\nabla \Delta_k u\|_{L^\infty} \sum_{m<k-1} 2^{m-k} \|\Delta_m \theta\|_{L^q}.$$

Since $\beta_1 + \frac{2}{q} < 2$, we can choose $\frac{2}{q} < \beta < 1$ and $\epsilon > 0$ such that

$$\beta_1 + \frac{2}{q} + \epsilon < 2\beta. \tag{3.24}$$

By Berntsein's inequality and Lemma 2.1,

$$\begin{aligned}
\|\nabla \Delta_k u\|_{L^\infty} &\le C\, 2^{\frac{2k}{q}}\, \|\nabla \Delta_k u\|_{L^q} \le C\, 2^{\frac{2k}{q}}\, P(2^k) \|\Delta_k \omega\|_{L^q} \\
&\le C\, 2^{k(\frac{2}{q}+\epsilon)} \|\Delta_k \omega\|_{L^q} \le C\, 2^{k(\frac{2}{q}+\epsilon-\beta)}\, \|\omega\|_{B^{\beta}_{q,\infty}}.
\end{aligned}$$

Clearly, for any $\beta < 1$,

$$\begin{aligned}
\sum_{m<k-1} 2^{m-k}\, \|\Delta_m \theta\|_{L^q} &= 2^{-\beta k} \sum_{m<k-1} 2^{(m-k)(1-\beta)} 2^{\beta m}\, \|\Delta_m \theta\|_{L^q} \\
&\le C\, 2^{-\beta k} \|\theta\|_{B^{\beta}_{q,\infty}}.
\end{aligned}$$

Therefore, according to (3.24) and the global bound in the first step,

$$\|\widetilde{J}_4\|_{L^q} \le C\, 2^{-\beta_1(j+1)}\, \|\omega\|_{B^{\beta}_{q,\infty}}\, \|\theta\|_{B^{\beta}_{q,\infty}}\, 2^{(\beta_1+\frac{2}{q}+\epsilon-2\beta)j} \le C\, 2^{-\beta_1(j+1)}. \tag{3.25}$$

Collecting the estimates in (3.23) and (3.25), we have

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^q} \le C\,2^{-\beta_1(j+1)}\,\|\theta\|_{B^{\beta_1}_{q,\infty}}\,\|\nabla u\|_{L^\infty} + C\,2^{-\beta_1(j+1)}.$$

Bounding $\|\nabla u\|_{L^\infty}$ by the interpolation inequality in Proposition 3.1.1 and applying Osgood inequality lead to the desired global bound for $\|\theta\|_{B^{\beta_1}_{q,\infty}}$. With this bound at our disposal, we then obtain a global bound for $\|G\|_{B^{\beta_1}_{q,\infty}}$ by going through a similar process on the equation of G. Therefore, for any $t \le T$,

$$\|\omega\|_{B^{\beta_1}_{q,\infty}} \le \|\theta\|_{B^{\beta_1}_{q,\infty}} + \|G\|_{B^{\beta_1}_{q,\infty}} \le C(T).$$

If necessary, we can repeat the second step a few times to achieve the global bound for ω and θ in $B^s_{q,\infty}$ for any $s > 2$. This completes the proof of Proposition 3.2.1. ■

3.3 Existence and Uniqueness of Theorem 3.1

This section provides the existence and uniqueness of Theorem 3.1.

Proof. [Proof of Theorem 3.1]

Due to the high regularity in the class $\omega \in C([0,T]; B^s_{q,\infty}(\mathbb{R}^2))$, $\theta \in C([0,T]; B^s_{q,\infty}(\mathbb{R}^2) \cap L^1([0,T]; B^{s+1}_{q,\infty}(\mathbb{R}^2))$ of solutions (ω, θ), the uniqueness of solutions is obvious. We will focus on establishing the existence of solutions. The first step is to obtain a local (in time) solution and then extend it into a global solution through the global *a priori* bounds obtained in the previous section.

The local solution can be constructed through the method of successive approximation. That is, we consider a successive approximation sequence $\{(\omega^{(n)}, \theta^{(n)})\}$ solving

$$
\begin{cases}
\omega^{(1)} = S_2\omega_0, \quad \theta^{(1)} = S_2\theta_0, \\
u^{(n)} = \nabla^{\perp}\Delta^{-1}P(\Lambda)\,\omega^{(n)}, \\
\partial_t\omega^{(n+1)} + u^{(n)}\cdot\nabla\omega^{(n+1)} = \partial_{x_1}\theta^{(n+1)}, \\
\partial_t\theta^{(n+1)} + u^{(n)}\cdot\nabla\theta^{(n+1)} + \Lambda\theta^{(n+1)} = 0, \\
\omega^{(n+1)}(x,0) = S_{n+2}\omega_0(x), \quad \theta^{(n+1)}(x,0) = S_{n+2}\theta_0(x).
\end{cases} \tag{3.26}
$$

In order to show that $\{(\omega^{(n)},\theta^{(n)})\}$ converges to a solution of (3.2), it suffices to prove that $\{(\omega^{(n)},\theta^{(n)})\}$ obeys the following properties:

(1) There exists a time interval $[0,T_1]$ over which $\{(\omega^{(n)},\theta^{(n)})\}$ are bounded uniformly in terms of n. More precisely, we show that

$$\|(\omega^{(n)},\theta^{(n)})\|_{B^s_{q,\infty}} \le C(T_1, \|(\omega_0,\theta_0)\|_{B^s_{q,\infty}}),$$

for a constant depending on T_1 and the initial norm only.

(2) There exists $T_2 > 0$ such that $\omega^{(n+1)} - \omega^{(n)}$ and $\theta^{(n+1)} - \theta^{(n)}$ are Cauchy sequence in $B^{s-1}_{q,\infty}$, namely

$$\|\omega^{(n+1)} - \omega^{(n)}\|_{B^{s-1}_{q,\infty}} \le C(T_2)\,2^{-n}, \quad \|\theta^{(n+1)} - \theta^{(n)}\|_{B^{s-1}_{q,\infty}} \le C(T_2)\,2^{-n}$$

for any $t \in [0,T_2]$, where $C(T_2)$ is independent of n.

If the properties stated in (1) and (2) hold, then there exists (ω,θ) satisfying, for $T = \min\{T_1,T_2\}$,

$$\omega(\cdot,t) \in B^s_{q,\infty}, \quad \theta(\cdot,t) \in B^s_{q,\infty} \quad \text{for} \quad 0 \le t \le T,$$

$$\omega^{(n)}(\cdot,t) \to \omega(\cdot,t) \quad \text{in} \quad B^{s-1}_{q,\infty}, \qquad \theta^{(n)}(\cdot,t) \to \theta(\cdot,t) \quad \text{in} \quad B^{s-1}_{q,\infty}.$$

It is then easy to show that (ω,θ) solves (3.2) and we thus obtain a local solution and the global bounds in Sections 3.1 and 3.2 allow us to extend it into a global solution.

It then remains to verify the properties stated in (1) and (2). Property (1) can be shown as in Sections 3.1 and 3.2. To verify property (2), we consider the equations for the differences $\omega^{(n+1)} - \omega^{(n)}$ and $\theta^{(n+1)} - \theta^{(n)}$ and prove it inductively in n. The bounds can be achieved in a similar fashion in Sections 3.1 and 3.2. We thus omit further detail. This completes the proof of Theorem 3.1. ■

CHAPTER 4

Logarithmically Supercritical Boussinesq-Navier-Stokes Equations

4.1 Introduction

This chapter deals with the establishment of the global regularity of the two dimensional Boussinesq-Navier-Stokes equations with logarithmically supercritical dissipation. We consider the initial value problem (IVP) of the form

$$\begin{cases} \partial_t u + u \cdot \nabla u + \mathcal{L} u = -\nabla p + \theta \mathbf{e}_2, \\ \partial_t \theta + u \cdot \nabla \theta = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{4.1}$$

where $u : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field denoting the velocity, $\theta : \mathbb{R}^2 \to \mathbb{R}$ is a scalar function, $\mathbf{e}_2$ is the unit vector in the x_2 direction, and $\mathcal{L}$ is a nonlocal dissipation operator defined by

$$\mathcal{L} f(x) = \text{ p.v. } \int_{\mathbb{R}^2} \frac{f(x) - f(y)}{|x-y|^2} m(|x-y|) dy \tag{4.2}$$

and $m \colon (0,\infty) \to (0,\infty)$ is a smooth, positive, non-increasing function, which obeys

(i) there exists $C_1 > 0$ such that

$$r m(r) \le C_1 \qquad \text{for all } r \le 1;$$

(ii) there exists $C_2 > 0$ such that

$$r |m'(r)| \le C_2 m(r) \qquad \text{for all } r > 0;$$

(iii) there exists $\beta > 0$ such that

$$r^{\beta} m(r) \text{ is non-increasing.}$$

As pointed out in [23], $\mathcal{L}$ can be equivalently defined by a Fourier multiplier, namely

$$\widehat{\mathcal{L}f}(\xi) = P(|\xi|)\widehat{f}(\xi) \tag{4.3}$$

for $P(|\xi|) = m(\frac{1}{|\xi|})$ when $P(\xi)$ satisfies the following conditions:

1. P satisfies the doubling condition: for any $\xi \in \mathbb{R}^2$,

$$P(2|\xi|) \leq c_D P(|\xi|)$$

   with constant $c_D \geq 1$;

2. P satisfies the Hormander-Mikhlin condition (see [61]): for any $\xi \in \mathbb{R}^2$,

$$|\xi|^{|k|} \, |\partial_\xi^k P(|\xi|)| \leq c_H P(|\xi|)$$

   for some constant $c_H \geq 1$, and for all multi-indices $k \in \mathbb{Z}^d$ with $|k| \leq N$, with N only depending on c_D;

3. P has sub-quadratic growth at ∞, i.e.

$$\int_0^1 P(|\xi|^{-1})|\xi| d|\xi| < \infty$$

4. P satisfies

$$(-\Delta)^2 P(|\xi|) \geq c_H^{-1} P(\xi)|\xi|^{-4}$$

   for all $|\xi|$ sufficiently large.

Throughout the rest of this paper we assume that $\mathcal{L}$ satisfies both (4.2) and (4.3) with $P(|\xi|) = m(\frac{1}{|\xi|})$ obeying the conditions stated above. Some examples of $m(r)$

are given below:

$$
\begin{aligned}
m(r) &= \frac{1}{r^\alpha} \qquad \text{for } r>0 \text{ and } \alpha\in(0,1], \text{ which yields } \mathcal{L}=\Lambda^\alpha;\\
m(r) &= \frac{1}{r\log^\gamma(e+1/r)} \qquad \text{for } r>0, \gamma\geq 0;\\
m(r) &= \frac{1}{r\log\log(e^2+1/r)} \qquad \text{for } r>0,
\end{aligned}
$$

We remark that (4.1) can be reformulated in terms of the vorticity $\omega = \nabla\times u$ as follows:

$$
\begin{cases}
\partial_t\omega + u\cdot\nabla\omega + \mathcal{L}\omega = \partial_{x_1}\theta,\\
\partial_t\theta + u\cdot\nabla\theta = 0,\\
u = \nabla^\perp\psi, \quad \Delta\psi = \omega,\\
\omega(x,0) = \omega_0(x), \quad \theta(x,0) = \theta_0(x),
\end{cases}
\tag{4.4}
$$

where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ and ψ denotes the stream function. Our main result is a global well-posedness theorem for the IVP (4.1) or (4.4) when $\mathcal{L}$ is slightly supercritical. More precisely, we have the following theorem.

Theorem 4.1 *Consider the IVP (4.1) and assume that $\mathcal{L}$ satisfies (4.2) and (4.3) with $P(|\xi|) = m(\frac{1}{|\xi|})$ obeying the aforementioned conditions. We further assume that $a(\xi) = a(|\xi|) \equiv |\xi|/P(|\xi|)$ is positive, non-decreasing and satisfies*

$$
\lim_{|\xi|\to\infty} \frac{a(|\xi|)}{|\xi|^\sigma} = 0\,, \quad \forall\,\sigma>0. \tag{4.5}
$$

Let $q>2$ and let the initial data (u_0,θ_0) be in the class

$$
u_0\in H^1(\mathbb{R}^2), \quad \omega_0 \in L^q(\mathbb{R}^2)\cap B^0_{\infty,1}(\mathbb{R}^2)\,, \quad \theta_0\in L^2(\mathbb{R}^2)\cap B^{0,a^2}_{\infty,1}(\mathbb{R}^2),
$$

where $\omega_0 = \nabla\times u_0$ is the initial vorticity. Then (4.1) has a unique global solution (u,θ) satisfying, for all $t>0$,

$$
u\in L^\infty_t H^1, \quad \omega\in L^\infty_t L^q \cap L^1_t B^0_{\infty,1}\,, \quad \theta \in L^\infty_t L^2 \cap L^\infty_t B^{0,a^2}_{\infty,1}\cap L^1_t B^{0,a}_{\infty,1}.
$$

As an application of Theorem 4.1, we establish the global existence and uniqueness of classical solutions of (4.1) with logarithmically supercritical dissipation,

$$\widehat{\mathcal{L}u}(\xi) = P(|\xi|)\widehat{u}(\xi) \equiv \frac{|\xi|}{\log^{\gamma}(e+|\xi|)}\,\widehat{u}(\xi) \quad \text{for any } \gamma \geq 0. \tag{4.6}$$

Corollary 4.1 *Consider the IVP (4.1) with $\mathcal{L}$ given by (4.6). Assume that $(u_0, \theta_0) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ with $s > 1$. Then IVP (4.1) with $\mathcal{L}$ given by (4.6) has a unique global solution $(u, \theta) \in L^\infty([0,T]; H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2))$ for any $T > 0$.*

We now explain the main difficulty that one encounters in the study of the global regularity of solutions to (4.1). Due to the "vortex stretching" term $\partial_{x_1}\theta$, a simple energy estimate will not lead to a global bound for $\|\omega\|_{L^2}$ unless $\mathcal{L}\omega$ is very dissipative. To overcome this difficulty, we consider a new quantity $\omega - \Lambda^{-1}\partial_{x_1}\theta$ to hide $\partial_{x_1}\theta$. Then the combined quantity

$$G = \omega - \mathcal{R}_a\theta \quad \text{with} \quad \mathcal{R}_a = \mathcal{L}^{-1}\partial_{x_1}, \tag{4.7}$$

satisfies

$$\partial_t G + u\cdot\nabla G + \mathcal{L}G = [\mathcal{R}_a, u\cdot\nabla]\theta. \tag{4.8}$$

The trade-off is now to deal with the commutator $[\mathcal{R}_a, u\cdot\nabla]\theta$ instead of $\partial_{x_1}\theta$.

4.2 Preliminary Estimates

This section provides several estimates to be used throughout this chapter. We recall that $\mathcal{L}$ denotes the operator defined by both (4.2) and (4.3). In addition,

$$a(|\xi|) \equiv \frac{|\xi|}{P(|\xi|)}, \qquad \mathcal{R}_a = \mathcal{L}^{-1}\partial_{x_1}. \tag{4.9}$$

The first two lemmas provide lower bounds involving $\mathcal{L}$. These bounds are useful when we estimate the L^p-norms of the solution.

Lemma 4.1 *Let $\mathcal{L}$ be the operator defined by (4.2). Then, for $p > 1$,*

$$|f(x)|^{p-2}f(x)(\mathcal{L}f(x)) \geq \frac{1}{p}\mathcal{L}(|f|^p).$$

Proof. By (4.2),

$$\mathcal{L}f(x) = \text{ p.v. } \int \frac{f(x)-f(y)}{|x-y|^d} m(|x-y|)dy$$

and thus

$$|f(x)|^{p-2}f(x)\mathcal{L}f(x) = \text{ p.v. } \int \frac{|f(x)|^p - |f(x)|^{p-2}f(x)f(y)}{|x-y|^d} m(|x-y|)dy.$$

By Young's inequality,

$$|f(x)|^{p-2}f(x)f(y) \le |f(x)|^{p-1}|f(y)| \le \frac{p-1}{p}|f(x)|^p + \frac{1}{p}|f(y)|^p$$

Therefore,

$$\begin{aligned} &|f(x)|^{p-2}f(x)\mathcal{L}f(x) \\ &\ge \frac{1}{p} \text{ p.v. } \int \frac{p|f(x)|^p - (p-1)|f(x)|^p - |f(y)|^p}{|x-y|^d} m(|x-y|)dy \\ &\ge \frac{1}{p}\mathcal{L}(|f|^p). \end{aligned}$$

This completes the proof of Lemma 4.1. ■

Lemma 4.2 *Let $\mathcal{L}$ be the operator defined by (4.2). Then, for $p \ge 2$,*

$$\int |f|^{p-2}f(\mathcal{L}f)dx \ge \frac{2}{p}\int \left|\mathcal{L}^{\frac{1}{2}}(|f|^{\frac{p}{2}})\right|^2 dx.$$

Proof. The $p=2$ case is trivial. For $p>2$, let $\beta = \frac{p}{2}-2$. By Lemma 4.1,

$$\begin{aligned} \int |f|^{p-2}f(\mathcal{L}f)dx &= \int |f|^{\frac{p}{2}}|f|^{\beta}f(\mathcal{L}f)dx \\ &\ge \int |f|^{\frac{p}{2}}\frac{2}{p}(\mathcal{L}(|f|^{\frac{p}{2}})dx \\ &= \frac{2}{p}\int \left|\mathcal{L}^{\frac{1}{2}}(|f|^{\frac{p}{2}})\right|^2 dx. \end{aligned}$$

This completes the proof of Lemma 4.2. ■

The following lemma is a generalized version of the Bernstein type inequality associated with the operator $\mathcal{L}$.

Lemma 4.3 *Let $j \geq 0$ be an integer and $p \in [2, \infty)$. Let $\mathcal{L}$ be defined by (4.2) and (4.3). Then, for any $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$P(2^j)\|\Delta_j f\|^p_{L^p(\mathbb{R}^d)} \leq C \int_{\mathbb{R}^d} |\Delta_j f|^{p-2} \Delta_j f \mathcal{L} \Delta_j f \, dx, \tag{4.10}$$

where C is a constant depending on p and d only.

Proof. The case when $p = 2$ simply follows from Plancherel's theorem. Now we assume $p > 2$. The proof modifies the corresponding ones in [17, 32]. Let $N > 0$ be an integer to be specified later. Clearly,

$$\|\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2} \leq \|S_N \Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2} + \|(Id - S_N)\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2} \equiv I_1 + I_2.$$

By the standard Bernstein inequality, for $s > 0$,

$$I_2 \leq C 2^{-Ns} \||\Delta_j f|^{\frac{p}{2}}\|_{B^{1+s}_{2,2}}.$$

Applying Lemma 3.2 of [17], we have, for $s \in (0, \min(\frac{p}{2} - 1, 2))$,

$$\||\Delta_j f|^{\frac{p}{2}}\|_{B^{1+s}_{2,2}} \leq C \|\Delta_j f\|^{\frac{p}{2}-1}_{B^0_{p,2}} \|\Delta_j f\|_{B^{1+s}_{p,2}} \leq C 2^{j(1+s)} \|\Delta_j f\|^{\frac{p}{2}}_{L^p}.$$

Therefore,

$$I_2 \leq C 2^{-Ns} 2^{j(1+s)} \|\Delta_j f\|^{\frac{p}{2}}_{L^p}.$$

By Lemma 2.1,

$$I_1 = \|S_N \Lambda \mathcal{L}^{-\frac{1}{2}} \mathcal{L}^{\frac{1}{2}}(|\Delta_j f|^{\frac{p}{2}})\|_{L^2} \leq C \, 2^N \, (P(2^N))^{-\frac{1}{2}} \, \|\mathcal{L}^{\frac{1}{2}}(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}.$$

Combining the estimates leads to

$$\|\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2} \leq C 2^{-Ns} 2^{j(1+s)} \|\Delta_j f\|^{\frac{p}{2}}_{L^p} \, + \, C \, 2^N (P(2^N))^{-\frac{1}{2}} \|\mathcal{L}^{\frac{1}{2}}(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}.$$

By the generalized Bernstein inequality for Λ in [17],

$$2^j \|\Delta_j f\|^{\frac{p}{2}}_{L^p} \leq C \, \|\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}.$$

Therefore,

$$2^j\|\Delta_j f\|_{L^p}^{\frac{p}{2}} \le C2^{-Ns}2^{j(1+s)}\|\Delta_j f\|_{L^p}^{\frac{p}{2}} + C\,2^N(P(2^N))^{-\frac{1}{2}}\|\mathcal{L}^{\frac{1}{2}}(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}. \quad (4.11)$$

We now choose $j < N \le j + N_0$ with N_0 independent of j such that

$$C\,2^{-(N-j)s} \le \frac{1}{2}.$$

(4.10) then follows from (4.11). This completes the proof of Lemma 4.3. ■

4.3 Global Bound for $\|\omega\|_{B^{0,a-1}_{2,2}}$

This section establishes a global *a priori* estimates for $\|G\|_{L^2}$ and consequently for $\|\omega\|_{B^{0,a-1}_{2,2}}$.

Proposition 4.3.1 *Assume that the initial data (u_0, θ_0) satisfies the conditions in Theorem 4.1. Let (u, θ) be the corresponding solution and let $\omega = \nabla \times u$ be the vorticity. Let*

$$G = \omega - \mathcal{R}_a\theta, \qquad \mathcal{R}_a = \mathcal{L}^{-1}\partial_{x_1}. \quad (4.12)$$

Then, for any $t \ge 0$,

$$\|G\|_{L^2}^2 + \int_0^t \|\mathcal{L}^{\frac{1}{2}}G(\tau)\|_{L^2}^2\,d\tau \le B(t)$$

and consequently

$$\|\omega(t)\|_{B^{0,\,a-1}_{2,2}} \le B(t),$$

where $B(t)$ is integrable on any finite-time interval $[0, T]$.

Proof. Trivially u and θ obey the following global *a priori* bounds

$$\|\theta(t)\|_{L^2\cap L^\infty} \le \|\theta_0\|_{L^2\cap L^\infty}, \quad \|u(t)\|_{L^2} \le \|u_0\|_{L^2} + t\|\theta_0\|_{L^2}. \quad (4.13)$$

It is easy to check that G satisfies

$$\partial_t G + u\cdot\nabla G + \mathcal{L}G = [\mathcal{R}_a, u\cdot\nabla]\theta. \quad (4.14)$$

Taking the inner product with G leads to

$$\frac{1}{2}\frac{d}{dt}\|G\|_{L^2}^2 + \int G\,\mathcal{L}G\,dx = \int G\nabla\cdot[\mathcal{R}_a, u]\theta\,dx. \tag{4.15}$$

By the Hölder inequality and the boundedness of Riesz transforms on L^2,

$$\left|\int G\nabla\cdot[\mathcal{R}_a, u]\theta dx\right| \le \|\mathcal{L}^{\frac{1}{2}}G\|_{L^2}\,\|\mathcal{L}^{-\frac{1}{2}}\Lambda[\mathcal{R}_a, u]\theta\|_{L^2}.$$

Inserting this estimate in (4.15) and applying Young's inequality, we obtain

$$\frac{d}{dt}\|G\|_{L^2}^2 + \|\mathcal{L}^{\frac{1}{2}}G\|_{L^2}^2 \le \|\mathcal{L}^{-\frac{1}{2}}\Lambda[\mathcal{R}_a, u]\theta\|_{L^2}^2. \tag{4.16}$$

By the definition of the norm of generalized Besov space, $\|\mathcal{L}^{-\frac{1}{2}}\Lambda f\|_2 \le \|f\|_{B^{\frac{1}{2},\frac{a}{2}}_{2,2}}$. Applying Proposition 2.4.2 with $\delta > \frac{1}{2}$ and $p = q = 2$, we obtain

$$\|[\mathcal{R}_a, u]\theta\|_{B^{\frac{1}{2},\frac{a}{2}}_{2,2}} \le C\|u\|_{B^{\delta}_{2,\infty}}\,\|\theta\|_{B^{\frac{1}{2}-\delta,\,\frac{a^2}{4}}_{\infty,2}} + C\,\|u\|_{L^2}\,\|\theta\|_{L^2}.$$

Since $u = \nabla^{\perp}\Delta^{-1}\omega$,

$$\begin{aligned}\|u\|_{B^{\delta}_{2,\infty}} &= \sup_{j\ge -1} 2^{\delta j}\,\|\Delta_j u\|_{L^2} \le \|\Delta_{-1}u\|_{L^2} + \sup_{j\ge 0} 2^{\delta j}\,\|\Delta_j\nabla^{\perp}\Delta^{-1}\omega\|_{L^2}\\ &\le \|u\|_{L^2} + \sup_{j\ge 0} 2^{(\delta-1)j}\|\Delta_j\omega\|_{L^2} \le \|u\|_{L^2} + \|\omega\|_{B^{0,a-1}_{2,2}}.\end{aligned}$$

For $\delta > \frac{1}{2}$, $\|\theta\|_{B^{\frac{1}{2}-\delta,\,\frac{a^2}{4}}_{\infty,2}} \le \|\theta\|_{L^\infty}$. Therefore,

$$\|\mathcal{L}^{-\frac{1}{2}}\Lambda[\mathcal{R}_a, u]\theta\|_{L^2} \le \|[\mathcal{R}_a, u]\theta\|_{B^{\frac{1}{2},\frac{a}{2}}_{2,2}} \le C\,\|u\|_{L^2}\,\|\theta\|_{L^2\cap L^\infty} + \|\omega\|_{B^{0,a-1}_{2,2}}\|\theta\|_{L^\infty}. \tag{4.17}$$

We can bound the $\|\omega\|_{B^{0,a-1}_{2,2}}$ by

$$\|\omega\|_{B^{0,a-1}_{2,2}} \le \|G\|_{B^{0,a-1}_{2,2}} + \|\mathcal{R}_a\theta\|_{B^{0,a-1}_{2,2}} \le \|G\|_2 + \|\theta\|_2. \tag{4.18}$$

Since $\|u\|_{L^2}$ and $\|\theta\|_{L^2\cap L^\infty}$ are bounded by (4.13), we combine (4.16), (4.17) and (4.18) to obtain the desired result. This completes the proof of Proposition 4.3.1. ■

4.4 Global Bound for $\|G\|_{L^q}$ with $q \in (2,4)$

This section establishes a global *a priori* bounds for $\|G\|_{L^q}$ with $q \in (2,4)$.

Proposition 4.4.1 *Assume that the initial data (u_0, θ_0) satisfies the conditions stated in Theorem 4.1. Let (u, θ) be the corresponding solution and G be defined as in (4.12). Then, for any $q \in (2,4)$, G obeys the global bound, for any $T > 0$ and $t \le T$,*

$$\|G(t)\|_{L^q}^q + C\int_0^t \int \left|\mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})\right|^2 dxdt + C\int_0^t \|G\|^q_{L^{\frac{2q}{1+\epsilon}}}\, d\tau \le B(t), \tag{4.19}$$

where C is a constant depending on q only and $B(t)$ is integrable on any finite time interval. A special consequence is that, for any small $\epsilon > 0$,

$$\|\omega(t)\|_{B^{-\epsilon}_{q,\infty}} \le B(t). \tag{4.20}$$

Proof. Multiplying (4.14) by $G|G|^{q-2}$ and integrating with respect to x, we obtain

$$\frac{1}{q}\frac{d}{dt}\|G\|_{L^q}^q + \int G|G|^{q-2}\mathcal{L}G dx = -\int G|G|^{q-2}\nabla\cdot[\mathcal{R}_a, u]\theta\, dx.$$

By Lemma 4.2,

$$\int G|G|^{q-2}\mathcal{L}Gdx \ge C\int |\mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})|^2\, dx.$$

Set $\epsilon > 0$ to be small, say, for $q \in (2,4)$,

$$(1+\epsilon)\left(1 - \frac{2}{q}\right) < \frac{1}{2}.$$

Thanks to the condition in (4.5) and by a Sobolev embedding,

$$\begin{aligned}
\|\mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})\|_{L^2}^2 &= \sum_{j\ge -1}\|\Delta_j \mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})\|_{L^2}^2 \\
&= \sum_{j\ge -1} 2^j a^{-1}(2^j)\|\Delta_j(|G|^{\frac{q}{2}})\|_{L^2}^2 \\
&\ge C\sum_{j\ge -1} 2^{(1-\epsilon)j}\|\Delta_j(|G|^{\frac{q}{2}})\|_{L^2}^2 \\
&= C\,\|\Lambda^{\frac{1}{2}-\frac{\epsilon}{2}}(|G|^{\frac{q}{2}})\|_{L^2}^2 \\
&\ge C\,\|G\|^q_{L^{\frac{2q}{1+\epsilon}}}.
\end{aligned}$$

For $q \in (2,4)$, we choose $s > 0$ such that

$$s > \epsilon, \qquad s + (1+\epsilon)\left(1 - \frac{2}{q}\right) = \frac{1}{2} - \epsilon.$$

By Hölder's inequality,

$$\left| \int G|G|^{q-2} \nabla \cdot [\mathcal{R}_a, u]\theta \right| \le \|G|G|^{q-2}\|_{\mathring{H}^s} \|[\mathcal{R}_a, u]\theta\|_{\mathring{H}^{1-s}}.$$

By Lemma 4.4 below,

$$\|G|G|^{q-2}\|_{\mathring{H}^s} \le C \|G\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}} \|G\|_{\mathring{H}^{s+(1+\epsilon)(1-\frac{2}{q})}} = C \|G\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}} \|G\|_{\mathring{H}^{\frac{1}{2}-\epsilon}}.$$

In addition, due to the condition in (4.5),

$$\|G\|^2_{\mathring{H}^{\frac{1}{2}-\epsilon}} = \sum_{j \ge -1} 2^{j-2\epsilon j} \|\Delta_j G\|^2_{L^2} \le \sum_{j \ge -1} 2^j a^{-2}(2^j) \|\Delta_j G\|^2_{L^2} \le \|\mathcal{L}^{\frac{1}{2}}(G)\|^2_{L^2}.$$

By Proposition 2.4.2, recalling $s > \epsilon$ and $u = \nabla^\perp \Delta^{-1}\omega$,

$$\begin{aligned}
\|[\mathcal{R}_a, u]\theta\|_{\mathring{H}^{1-s}} &\le C \|u\|_{\mathring{B}^{1-s+\epsilon}_{2,\infty}} \|\theta\|_{B^{-\epsilon,1}_{\infty,2}} + C \|u\|_{L^2} \|\theta\|_{L^2} \\
&\le C \|\omega\|_{B^{0,\frac{1}{a}}_{2,2}} \|\theta\|_{L^\infty} + C \|u\|_{L^2} \|\theta\|_{L^2}.
\end{aligned}$$

Putting the estimates together, we obtain

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\|G\|^q_{L^q} + C \int |\mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})|^2 \, dx + C \|G\|^q_{L^{\frac{2q}{1+\epsilon}}} \\
&\quad \le C \|G\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}} \|\mathcal{L}^{\frac{1}{2}}(G)\|_{L^2} \left(\|\omega\|_{B^{0,\frac{1}{a}}_{2,2}} \|\theta\|_{L^\infty} + C \|u\|_{L^2} \|\theta\|_{L^2} \right).
\end{aligned}$$

Applying Young's inequality to the right-hand side, noticing that $q \in (2,4)$ and resorting to the bounds in Proposition 4.3.1, we obtain (4.19). (4.20) follows from the inequality

$$\|\omega\|_{B^{-\epsilon}_{q,\infty}} \le \|G\|_{B^{-\epsilon}_{q,\infty}} + \|\mathcal{R}_a\theta\|_{B^{-\epsilon}_{q,\infty}} \le \|G\|_{L^q} + \|\theta\|_{L^q}.$$

This completes the proof of Proposition 4.4.1. ■

We have used the following lemma in the proof of Proposition 4.4.1.

Lemma 4.4 *Let $q \in (2,\infty)$, $s \in (0,1)$, $0 < \epsilon(q-2) \le 2$ and $f \in L^{\frac{2q}{1+\epsilon}} \cap \mathring{H}^{s+(1-\frac{2}{q})(1+\epsilon)}$. Then*

$$\||f|^{q-2} f\|_{\mathring{H}^s} \le C\, \|f\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}} \|f\|_{\mathring{B}^s_{\frac{2q}{2-\epsilon(q-2)},2}} \le C\, \|f\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}} \|f\|_{\mathring{H}^{s+(1-\frac{2}{q})(1+\epsilon)}}. \tag{4.21}$$

Proof. This proof modifies that of [35]. Identifying $\mathring{H}^s$ with $\mathring{B}^s_{2,2}$ and by the definition of $\mathring{B}^s_{2,2}$, we have

$$\||f|^{q-2} f\|^2_{\mathring{H}^s} = \int \frac{\||f|^{q-2} f(x+y) - |f|^{q-2} f(x)\|^2_{L^2}}{|y|^{2+2s}}\, dy.$$

Thanks to the inequality

$$\big||f|^{q-2} f(x+y) - |f|^{q-2} f(x)\big| \le C\, \big(|f|^{q-2}(x+y) + |f|^{q-2}(x)\big)\, |f(x+y) - f(x)|,$$

we have, by Hölder's inequality

$$\||f|^{q-2} f(x+y) - |f|^{q-2} f(x)\|^2_{L^2} \le C\, \|f\|^{2(q-2)}_{L^{\frac{2q}{1+\epsilon}}} \|f(x+y) - f(x)\|^2_{L^\rho},$$

where

$$\rho = \frac{2q}{2-\epsilon(q-2)}.$$

Therefore,

$$\||f|^{q-2} f\|^2_{\mathring{H}^s} \le C\, \|f\|^{2(q-2)}_{L^{\frac{2q}{1+\epsilon}}} \|f\|^2_{\mathring{B}^s_{\rho,2}}.$$

Further applying the Besov embedding inequality

$$\|f\|_{\mathring{B}^s_{\rho,2}} \le C\, \|f\|_{\mathring{H}^{s+1-\frac{2}{\rho}}},$$

we obtain (4.21) and this completes the proof of Lemma 4.4. ∎

4.5 Global Bound for $\|G\|_{\widetilde{L}^r_t B^s_{q,1}}$ with $q \in [2,4)$

This section provides a global *a priori* bound for $\|G\|_{\widetilde{L}^r_t B^s_{q,1}}$ with $q \in (2,4)$. This bound serves as an important step towards a global bound for $\|\omega\|_{L^q}$ with general $q \in [2,\infty)$.

Proposition 4.5.1 *Assume that the initial data* (u_0, θ_0) *satisfies the conditions stated in Theorem 4.1. Let*

$$r \in [1, \infty], \quad s \in [0, 1), \quad q \in (2, 4).$$

Then, for any $t > 0$, G *obeys the following global bound*

$$\|G\|_{\tilde{L}^r_t B^s_{q,1}} \leq B(t), \tag{4.22}$$

where B *is integrable on any finite-time interval.*

Proof. Let $j \geq -1$ be an integer. Applying Δ_j to (4.14) yields

$$\partial_t \Delta_j G + \mathcal{L} \Delta_j G = -\Delta_j (u \cdot \nabla G) - \Delta_j [\mathcal{R}_a, u \cdot \nabla] \theta.$$

Taking the inner product with $\Delta_j G |\Delta_j G|^{q-2}$, we have

$$\frac{1}{q} \frac{d}{dt} \|\Delta_j G\|^q_{L^q} + \int \Delta_j G |\Delta_j G|^{q-2} \mathcal{L} \Delta_j G = J_1 + J_2, \tag{4.23}$$

where

$$J_1 = -\int \Delta_j (u \cdot \nabla G)\, \Delta_j G |\Delta_j G|^{q-2}, \tag{4.24}$$

$$J_2 = -\int \Delta_j [\mathcal{R}_a, u \cdot \nabla] \theta \ \Delta_j G |\Delta_j G|^{q-2}.$$

According to Lemma 4.3, for $j \geq 0$, the dissipation part can be bounded below by

$$\int \Delta_j G |\Delta_j G|^{q-2} \mathcal{L} \Delta_j G \geq C P(2^j) \|\Delta_j G\|^q_{L^q}. \tag{4.25}$$

By Lemma 4.5 below, J_1 can be bounded by

$$\begin{aligned} \|J_1\|_{L^q} \ \leq \ & C\, 2^{j(\epsilon + \frac{2}{q})} \, \|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}} \Big[\|\Delta_j G\|_{L^q} + \sum_{m \leq j-2} 2^{(m-j)\frac{2}{q}} \|\Delta_m G\|_{L^q} \\ & + \sum_{k \geq j-1} 2^{(j-k)(1-\frac{2}{q})} \, \|\Delta_k G\|_{L^q} \Big] \, \|\Delta_j G\|^{q-1}_{L^q}, \end{aligned} \tag{4.26}$$

where we have taken ϵ to be small positive number, especially

$$s - 1 + 3\epsilon < 0.$$

To bound J_2, we first apply Hölder's inequality and then employ similar estimates as in the proof of Proposition 2.4.2 to obtain

$$\begin{aligned} |J_2| &\leq \|\Delta_j[\mathcal{R}_a, u\cdot\nabla]\theta\|_{L^q}\|\Delta_j G\|_{L^q}^{q-1} \\ &\leq C\left(2^{j\epsilon}a(2^j)\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\|\theta\|_{L^\infty}+\|u\|_{L^2}\|\theta\|_{L^2}\right)\|\Delta_j G\|_{L^q}^{q-1}. \end{aligned} \tag{4.27}$$

Inserting (4.25), (4.26) and (4.27) in (4.23) and writing the bound for $\|\omega(t)\|_{B^{-\epsilon}_{q,\infty}}$ by $B(t)$, we obtain

$$\begin{aligned} \frac{d}{dt}\|\Delta_j G\|_{L^q} + C2^j\, a^{-1}(2^j)\,\|\Delta_j G\|_{L^q} &\leq C2^{\epsilon j}\, a(2^j)\, B(t) \\ &+ C2^{j(\epsilon+\frac{2}{q})}B(t)\Big[\|\Delta_j G\|_{L^q} + \sum_{m\leq j-2} 2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q} \\ &\qquad + \sum_{k\geq j-1} 2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}\Big]. \end{aligned}$$

Due to (4.5), $a(2^j)\leq 2^{\epsilon j}$. Integrating in time yields

$$\begin{aligned} \|\Delta_j G(t)\|_{L^q} &\leq e^{-C\,2^{(1-\epsilon)j}t}\|\Delta_j G(0)\|_{L^q} + C\,2^{-j(1-3\epsilon)}\,B(t) \\ &+ C\,2^{j(\epsilon+\frac{2}{q})}B(t)\int_0^t e^{-C\,2^{(1-\epsilon)j}(t-\tau)}L(\tau)\,d\tau, \end{aligned}$$

where, for notational convenience, we have written

$$L(t) = \Big[\|\Delta_j G\|_{L^q} + \sum_{m\leq j-2} 2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q} + \sum_{k\geq j-1} 2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}\Big].$$

Taking the L^r norm in time and applying Young's inequality for convolution lead to

$$\begin{aligned} \|\Delta_j G\|_{L^r_t L^q} &\leq C\,2^{-\frac{1}{r}(1-\epsilon)j}\,\|\Delta_j G(0)\|_{L^q} + C\,2^{-j(1-3\epsilon)}\,\widetilde{B}(t) \\ &+ C\,2^{j(-1+2\epsilon+\frac{2}{q})}\widetilde{B}(t)\,\|L\|_{L^r}. \end{aligned}$$

Multiplying by 2^{js}, summing over $j\geq -1$ and noticing $s-1+3\epsilon<0$, we obtain

$$\|G\|_{\widetilde{L}^r_t B^s_{q,1}} \leq C\,\|G(0)\|_{B^{s-1/r(1-\epsilon)}_{q,1}} + C\,\widetilde{B}(t) + K_1 + K_2 + K_3, \tag{4.28}$$

where

$$K_1 = C \sum_{j\geq -1} 2^{j(-1+2\epsilon+\frac{2}{q})}\, \widetilde{B}(t)\, 2^{js} \|\Delta_j G\|_{L^r_t L^q},$$
$$K_2 = C \sum_{j\geq -1} 2^{j(-1+2\epsilon+\frac{2}{q})}\, \widetilde{B}(t)\, 2^{js} \sum_{m\leq j-2} 2^{(m-j)\frac{2}{q}} \|\Delta_m G\|_{L^r_t L^q},$$
$$K_3 = C \sum_{j\geq -1} 2^{j(-1+2\epsilon+\frac{2}{q})}\, \widetilde{B}(t)\, 2^{js} \sum_{k\geq j-1} 2^{(j-k)(1-\frac{2}{q})}\, \|\Delta_k G\|_{L^r_t L^q}.$$

Since $-1+2\epsilon+\frac{2}{q} < 0$, we can choose an integer $N > 0$ such that

$$C\, 2^{N(-1+2\epsilon+\frac{2}{q})} \widetilde{B}(t) \leq \frac{1}{8}.$$

The sums in K_1, K_2 and K_3 can then be split into two parts: $j \leq N$ and $j > N$. Since $\|G\|_{L^q}$ is bounded, the sum for the first part is bounded by $C\, \widetilde{B}(t) 2^{sN}$. The second part of the sum over $j > N$ is bounded by $\frac{1}{8}\|G\|_{\widetilde{L}^r_t B^s_{q,1}}$. Therefore,

$$K_1, K_2, K_3 \leq C\, \widetilde{B}(t) 2^{sN} + \frac{3}{8}\|G\|_{\widetilde{L}^r_t B^s_{q,1}}.$$

Combining these bounds with (4.28) yields the desired estimates. This completes the proof of Proposition 4.5.1. ■

We now provide the details leading to (4.26). They bear some similarities as those in [13], but they are provided here for the sake of completeness.

Lemma 4.5 *Let J_1 be defined as in (4.24). Then we have the following bound*

$$\|J_1\|_{L^q} \leq C\, 2^{j(\epsilon+\frac{2}{q})}\, \|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}} \Big[\|\Delta_j G\|_{L^q} + \sum_{m\leq j-2} 2^{(m-j)\frac{2}{q}} \|\Delta_m G\|_{L^q} + \sum_{k\geq j-1} 2^{(j-k)(1-\frac{2}{q})}\, \|\Delta_k G\|_{L^q}\Big]\, \|\Delta_j G\|^{q-1}_{L^q}.$$

Proof. Using the notion of paraproducts, we write

$$\Delta_j(u\cdot\nabla G) = J_{11} + J_{12} + J_{13} + J_{14} + J_{15},$$

where

$$
\begin{aligned}
J_{11} &= \sum_{|j-k|\le 2} [\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k G,\\
J_{12} &= \sum_{|j-k|\le 2} (S_{k-1}u - S_j u)\cdot\nabla\Delta_j\Delta_k G,\\
J_{13} &= S_j u\cdot\nabla\Delta_j G,\\
J_{14} &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u\cdot\nabla S_{k-1}G),\\
J_{15} &= \sum_{k\ge j-1} \Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k G).
\end{aligned}
$$

Since $\nabla\cdot u = 0$, we have

$$\int J_{13}|\Delta_j G|^{q-2}\Delta_j G\,dx = 0.$$

By Hölder's inequality,

$$\left|\int J_{11}|\Delta_j G|^{q-2}\Delta_j G\right| \le \|J_{11}\|_{L^q}\|\Delta_j G\|_{L^q}^{q-1}.$$

We write the commutator in terms of the integral,

$$J_{11} = \int \Phi_j(x-y)\,(S_{k-1}u(y) - S_{k-1}u(x))\cdot\nabla\Delta_k G(y)\,dy,$$

where Φ_j is the kernel of the operator Δ_j. As in the proof of Lemma 3.3, we have, for any $0<\epsilon<1$,

$$\|J_{11}\|_{L^q} \le \||x|^{1-\epsilon}\Psi_j(x)\|_{L^1}\,\|S_{j-1}u\|_{\mathring{B}^{1-\epsilon}_{q,\infty}}\,\|\nabla\Delta_j G\|_{L^\infty}.$$

By the definition of Φ_j and Bernstein's inequality, we have

$$
\begin{aligned}
\|J_{11}\|_{L^q} &\le C\,2^{j(\epsilon+\frac{2}{q})}\,\||x|^{1-\epsilon}\Psi_0(x)\|_{L^1}\,\|S_{j-1}\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\|\Delta_j G\|_{L^\infty}\\
&\le C\,2^{j(\epsilon+\frac{2}{q})}\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\,\|\Delta_j G\|_{L^q}.
\end{aligned}
$$

Again, by Bernstein's inequality,

$$
\begin{aligned}
\|J_{12}\|_{L^q} &\le C\|\Delta_j u\|_{L^q}\|\nabla\Delta_j G\|_{L^\infty}\\
&\le C2^{j(\epsilon+\frac{2}{q})}\,\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\,\|\Delta_j G\|_{L^q};
\end{aligned}
$$

$$\begin{aligned}\|J_{14}\|_{L^q} &\le C\|\Delta_j u\|_{L^q}\|\nabla S_{j-1}G\|_{L^\infty}\\ &\le C2^{j(\epsilon+\frac{2}{q})}\,\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\sum_{m\le j-2}2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q};\end{aligned}$$

$$\begin{aligned}\|J_{15}\|_{L^q} &\le C2^{j(\epsilon+\frac{2}{q})}\sum_{k\ge j-1}2^{(j-k)(1-\frac{2}{q})}\,\|\Lambda^{1-\epsilon}\Delta_k u\|_{L^q}\|\Delta_k G\|_{L^q}\\ &\le C2^{j(\epsilon+\frac{2}{q})}\,\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\sum_{k\ge j-1}2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}.\end{aligned}$$

Combining the estimates above yields

$$\begin{aligned}\|J_1\|_{L^q} \le\; & C\,2^{j(\epsilon+\frac{2}{q})}\,\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\Big[\|\Delta_j G\|_{L^q}+\sum_{m\le j-2}2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q}\\ &+\sum_{k\ge j-1}2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}\Big]\,\|\Delta_j G\|_{L^q}^{q-1}.\end{aligned}$$

This completes the proof of Lemma 4.5. ■

4.6 Global Bounds for $\|\omega\|_{L^1_tB^{0,a}_{\infty,1}}$ and $\|\omega\|_{L^q}$ for any $q\ge 2$

This section shows that, if the initial data ω_0 is in L^q, then the solution ω is also *a priori* in L^q at any time. This is established by first proving the time integrability $\|\omega\|_{L^1_tB^{0,a}_{\infty,1}}$. More precisely, we have the following proposition.

Proposition 4.6.1 *Assume that the initial data (u_0,θ_0) satisfies the conditions as stated in Theorem 4.1. Then we have the following global a priori bounds. For any $T>0$ and $t\le T$,*

$$\|\omega(t)\|_{L^1_tB^{0,a}_{\infty,1}}\le C(T),\quad \|\theta(t)\|_{B^{0,a^2}_{\infty,1}}\le C(T),\quad \|\omega(t)\|_{L^q}\le C(T),$$

where $C(T)$ are constants depending on T and the initial norms only.

In order to prove this proposition, we need the following fact.

Lemma 4.6 *Let $T>0$ and let u be a divergence-free smooth vector field satisfying*

$$\int_0^T\|\nabla u\|_{L^\infty}\,dt<\infty.$$

Assume that θ solves

$$\partial_t \theta + u \cdot \nabla \theta = f.$$

Let $a : (0, \infty) \to (0, \infty)$ be an nondecreasing and radially symmetric function satisfying (4.5). Let $\rho \in [1, \infty]$. For any $t > 0$,

$$\|\theta\|_{B^{0,a}_{\rho,1}} \le (\|\theta_0\|_{B^{0,a}_{\rho,1}} + \|f\|_{L^1_t B^{0,a}_{\rho,1}}) \left(1 + \int_0^t \|\nabla u\|_{L^\infty} dt\right).$$

This lemma can be proven in a similar fashion as that of Lemma 4.5 in [13]. A crucial assumption is that a satisfies (4.5).

Proof. [Proof of Proposition 4.6.1]

We can write by using Proposition 4.5.1 that, for $t \le T$,

$$\|G\|_{L^1_t B^{0,a}_{\infty,1}} \le C(T).$$

More precisely, if we choose $s \in [0, 1)$ satisfying $s > \frac{2}{q}$ for $q \in (2, 4)$ and set $\epsilon > 0$ satisfying $\epsilon + \frac{2}{q} - s < 0$, then

$$\begin{aligned}
\|G\|_{B^{0,a}_{\infty,1}} &\equiv \sum_{j \ge -1} a(2^j) \|\Delta_j G\|_{L^\infty} \le \sum_{j \ge -1} a(2^j) 2^{\frac{2}{q} j} \|\Delta_j G\|_{L^q} \\
&\le \sum_{j \ge -1} a(2^j) 2^{-\epsilon j}\, 2^{j(\epsilon + \frac{2}{q} - s)} 2^{js} \|\Delta_j G\|_{L^q} \le C \|G\|_{B^s_{q,1}},
\end{aligned}$$

where we have used the fact that $a(2^j) 2^{-\epsilon j} \le C$ for C independent of j. Furthermore,

$$\|\omega\|_{L^1_t B^{0,a}_{\infty,1}} \le \|G\|_{L^1_t B^{0,a}_{\infty,1}} + \|\mathcal{R}_a \theta\|_{L^1_t B^{0,a}_{\infty,1}}.$$

By the definition of the norm in $B^{0,a}_{\infty,1}$ and recalling that $\mathcal{R}_a \theta$ is defined by the multiplier $a(|\xi|) \frac{i\xi_1}{|\xi|}$, we have

$$\begin{aligned}
\|\mathcal{R}_a \theta\|_{B^{0,a}_{\infty,1}} &= a(2^{-1}) \|\Delta_{-1} \mathcal{R}_a \theta\|_{L^\infty} + \sum_{j \ge 0} a(2^j) \|\Delta_j \mathcal{R}_a \theta\|_{L^\infty} \\
&\le C \|\theta_0\|_{L^2} + \sum_{j \ge 0} a^2(2^j) \|\Delta_j \theta\|_{L^\infty} \\
&\le C \|\theta_0\|_{L^2} + \|\theta\|_{B^{0,a^2}_{\infty,1}}.
\end{aligned}$$

By Lemma 4.6,

$$
\begin{aligned}
\|\theta\|_{B^{0,a^2}_{\infty,1}} &\le C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}}\left(1+\int_0^t \|\nabla u\|_{L^\infty}dt\right)\\
&\le C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}}\left(1+\|u\|_{L^1_tL^2}+\|\omega\|_{L^1_tB^0_{\infty,1}}\right)\\
&\le C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}}\left(1+\|u\|_{L^1_tL^2}+\|\omega\|_{L^1_tB^{0,a}_{\infty,1}}\right). \qquad (4.29)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|\omega\|_{L^1_tB^{0,a}_{\infty,1}} &\le \|G\|_{L^1_tB^{0,a}_{\infty,1}} + C\left(\|\theta_0\|_{L^2}+\|\theta_0\|_{B^{0,a^2}_{\infty,1}}\right)t\\
&\quad +C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}}\int_0^t \|u\|_{L^1_\tau L^2}\,d\tau + C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}}\int_0^t \|\omega\|_{L^1_\tau B^{0,a}_{\infty,1}}\,d\tau.
\end{aligned}
$$

By Gronwall's inequality, $\|\omega\|_{L^1_tB^{0,a}_{\infty,1}} \le C(T)$, which, in turn, implies that, by (4.29),

$$
\|\theta(t)\|_{B^{0,a^2}_{\infty,1}} \le C(T).
$$

Now we prove the bound for $\|\omega\|_{L^q}$. From the equations of G and $\mathcal{R}_a\theta$,

$$
\begin{aligned}
\|\omega\|_{L^q} &\le \|G\|_{L^q}+\|\mathcal{R}_a\theta\|_{L^q}\\
&\le \|G_0\|_{L^q}+\|\mathcal{R}_a\theta_0\|_{L^q}+2\int_0^t \|[\mathcal{R}_a, u\cdot\nabla]\theta\|_{L^q}\,d\tau\\
&\le \|G_0\|_{L^q}+\|\mathcal{R}_a\theta_0\|_{L^q}+2\int_0^t \|[\mathcal{R}_a, u\cdot\nabla]\theta\|_{B^0_{q,1}}\,d\tau.
\end{aligned}
$$

Following the steps as in the proof of Proposition 2.4.2, we can show that

$$
\|[\mathcal{R}_a, u\cdot\nabla]\theta\|_{B^0_{q,1}} \le C\|\omega\|_{L^q}\,\|\theta\|_{B^{0,a}_{\infty,1}} + C\,\|\theta_0\|_{L^2}\,\|u\|_{L^2}.
$$

Gronwall's inequality and the bound $\|\theta\|_{L^1_tB^{0,a}_{\infty,1}} \le C(T)$ then imply the bound for $\|\omega\|_{L^q}$. This completes the proof of Proposition 4.6.1. ■

4.7 Uniqueness and Proof of Theorem 4.1

This section proves the uniqueness of solutions in the class stated in Theorem 4.1 and sketches the proof of Theorem 4.1. First we state and prove the uniqueness theorem.

Theorem 4.2 *Assume that the initial data (u_0, θ_0) satisfies the conditions stated in Theorem 4.1. Then, the solutions (u, θ) in the class*

$$u \in L^\infty([0,T]; H^1), \quad \omega \in L^\infty([0,T]; L^q) \cap L^1_T B^{0,a}_{\infty,1}, \quad \theta \in L^\infty([0,T], L^2 \cap B^{0,a}_{\infty,1}) \tag{4.30}$$

must be unique.

Proof. Assume that $(u^{(1)}, \theta^{(1)})$ and $(u^{(2)}, \theta^{(2)})$ are two solutions in the class (4.30). Let $p^{(1)}$ and $p^{(2)}$ be the associated pressure. The differences

$$u = u^{(2)} - u^{(1)}, \qquad p = p^{(2)} - p^{(1)}, \qquad \theta = \theta^{(2)} - \theta^{(1)}$$

satisfy

$$\begin{cases} \partial_t u + u^{(1)} \cdot \nabla u + u \cdot \nabla u^{(2)} + \mathcal{L} u = -\nabla p + \theta \mathbf{e}_2, \\ \partial_t \theta + u^{(1)} \cdot \nabla \theta + u \cdot \nabla \theta^{(2)} = 0. \end{cases}$$

By Lemmas 4.7 and 4.8 below, we have the following estimates

$$\begin{aligned} \|u(t)\|_{B^0_{2,\infty}} &\le \|u(0)\|_{B^0_{2,\infty}} + C\,\|\theta\|_{L^\infty_t B^{-1,a}_{2,\infty}} \\ &+ C \int_0^t \|u(\tau)\|_{L^2} \left(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}\right) d\tau \end{aligned}$$

and

$$\begin{aligned} \|\theta(t)\|_{B^{-1,a}_{2,\infty}} &\le \|\theta(0)\|_{B^{-1,a}_{2,\infty}} + C \int_0^t \|\theta(\tau)\|_{B^{-1,a}_{2,\infty}} (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}})\, d\tau \\ &\quad + C \int_0^t \|u(\tau)\|_{L^2} \|\theta^{(2)}\|_{B^{0,a}_{\infty,1}}\, d\tau. \end{aligned}$$

In addition, we bound $\|u\|_{L^2}$ by the following interpolation inequality

$$\|u\|_{L^2} \le C\,\|u\|_{B^0_{2,\infty}} \log\left(1 + \frac{\|u\|_{H^1}}{\|u\|_{B^0_{2,\infty}}}\right)$$

together with $\|u\|_{H^1} \le \|u^{(1)}\|_{H^1} + \|u^{(2)}\|_{H^1}$. These inequalities allow us to conclude that

$$Y(t) \equiv \|u(t)\|_{B^0_{2,\infty}} + \|\theta(t)\|_{B^{-1,a}_{2,\infty}}$$

obeys

$$Y(t) \le 2\,Y(0) + C \int_0^t D_1(\tau) Y(\tau)\, \log\left(1 + D_2(\tau)/Y(\tau)\right)\, d\tau, \tag{4.31}$$

where

$$\begin{aligned} D_1 &= \|\theta^{(2)}\|_{B^{0,a}_{\infty,1}} + \|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}, \\ D_2 &= \|u^{(1)}\|_{H^1} + \|u^{(2)}\|_{H^1}. \end{aligned}$$

Applying Osgood's inequality to (4.31) and noticing that $Y(0) = 0$, we conclude that $Y(t) = 0$. This completes the proof of Theorem 4.2. ■

We now state and prove two estimates used in the proof of Theorem 4.2.

Lemma 4.7 *Assume that $u^{(1)}$, $u^{(2)}$, u, p and θ are defined as in the proof of Theorem 4.2 and satisfy*

$$\partial_t u + u^{(1)} \cdot \nabla u + u \cdot \nabla u^{(2)} + \mathcal{L} u = -\nabla p + \theta \mathbf{e}_2. \tag{4.32}$$

Then we have the a priori bound

$$\begin{aligned} \|u(t)\|_{B^0_{2,\infty}} &\le \|u(0)\|_{B^0_{2,\infty}} + C\,\|\theta\|_{L^\infty_t B^{-1,a}_{2,\infty}} \\ &\quad + C \int_0^t \|u(\tau)\|_{L^2} \left(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}\right) d\tau. \end{aligned} \tag{4.33}$$

Proof. [Proof of Lemma 4.7] Let $j \ge -1$ be an integer. Applying Δ_j to (4.32) and taking the inner product with $\Delta_j u$, we obtain, after integration by parts,

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u\|_{L^2}^2 + \|\mathcal{L}^{\frac{1}{2}} \Delta_j u\|_{L^2}^2 = J_1 + J_2 + J_3, \tag{4.34}$$

where

$$\begin{aligned} J_1 &= -\int \Delta_j u\, \Delta_j(u^{(1)} \cdot \nabla u)\, dx, \\ J_2 &= -\int \Delta_j u\, \Delta_j(u \cdot \nabla u^{(2)})\, dx, \\ J_3 &= \int \Delta_j u\, \Delta_j(\theta \mathbf{e}_2)\, dx. \end{aligned}$$

By Plancherel's theorem,

$$\|\mathcal{L}^{\frac{1}{2}}\Delta_j u\|_{L^2}^2 \geq C\, 2^j a^{-1}(2^j)\, \|\Delta_j u\|_{L^2}^2,$$

where $C = 0$ in the case of $j = -1$ and $C > 0$ for $j \geq 0$. The estimate for J_3 is easy and we have, by Hölder's inequality,

$$|J_3| \leq \|\Delta_j u\|_{L^2}\, \|\Delta_j \theta\|_{L^2} \leq 2^j a^{-1}(2^j)\, \|\Delta_j u\|_{L^2}\, \|\theta\|_{B^{-1,a}_{2,\infty}}.$$

To estimate J_1, we need to use a commutator structure to shift one derivative to $u^{(1)}$. For this purpose, we write

$$\Delta_j(u^{(1)} \cdot \nabla u) = J_{11} + J_{12} + J_{13} + J_{14} + J_{15}, \tag{4.35}$$

where

$$\begin{aligned}
J_{11} &= \sum_{|j-k|\leq 2} [\Delta_j, S_{k-1}u^{(1)} \cdot \nabla]\Delta_k u, \\
J_{12} &= \sum_{|j-k|\leq 2} (S_{k-1}u^{(1)} - S_j u^{(1)}) \cdot \nabla \Delta_j \Delta_k u, \\
J_{13} &= S_j u^{(1)} \cdot \nabla \Delta_j u, \\
J_{14} &= \sum_{|j-k|\leq 2} \Delta_j(\Delta_k u^{(1)} \cdot \nabla S_{k-1} u), \\
J_{15} &= \sum_{k\geq j-1} \Delta_j(\Delta_k u^{(1)} \cdot \nabla \widetilde{\Delta}_k u).
\end{aligned}$$

Since $\nabla \cdot u^{(1)} = 0$, we have

$$\int J_{13}\, \Delta_j u\, dx = 0.$$

J_{11}, J_{12}, J_{14} and J_{15} can be bounded in a similar fashion as in the proof of Lemma 4.5 and we have

$$\begin{aligned}
&\|J_{11}\|_{L^2},\, \|J_{12}\|_{L^2} \leq C\, (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}})\|\Delta_j u\|_{L^2}, \\
&\|J_{14}\|_{L^2} \leq C\, (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \sum_{m\leq j-1} 2^{m-j}\|\Delta_m u\|_{L^2}, \\
&\|J_{15}\|_{L^2} \leq C\, (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \sum_{k\geq j-1} 2^{j-k}\|\Delta_k u\|_{L^2}.
\end{aligned}$$

To estimate J_2, we write

$$\Delta_j(u \cdot \nabla u^{(2)}) = J_{21} + J_{22} + J_{23}, \tag{4.36}$$

where

$$\begin{aligned}
J_{21} &= \sum_{|j-k|\le 2} \Delta_j(S_{k-1}u \cdot \nabla \Delta_k u^{(2)}), \\
J_{22} &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u \cdot \nabla S_{k-1} u^{(2)}), \\
J_{23} &= \sum_{k\ge j-1} \Delta_j(\Delta_k u \cdot \nabla \widetilde{\Delta}_k u^{(2)}).
\end{aligned}$$

Therefore, by Hölder's inequality,

$$\begin{aligned}
&\|J_{21}\|_{L^2} \le C\, \|u\|_{L^2}\, \|\nabla \Delta_j u^{(2)}\|_{L^\infty}, \\
&\|J_{22}\|_{L^2} \le C\, \|\Delta_j u\|_{L^2} (\|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}), \\
&\|J_{23}\|_{L^2} \le C\, (\|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}) \sum_{k\ge j-1} 2^{j-k} \|\Delta_k u\|_{L^2}.
\end{aligned}$$

Inserting the estimates above in (4.34), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u\|_{L^2} + C\, 2^j a^{-1}(2^j)\, \|\Delta_j u\|_{L^2} \le C\, 2^j a^{-1}(2^j)\, \|\theta\|_{B^{-1,a}_{2,\infty}} + K(t), \tag{4.37}$$

where

$$\begin{aligned}
K(t) \;\; = \;\; & C\, (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}) \|\Delta_j u\|_{L^2} \\
& + C\, \|u\|_{L^2}\, \|\nabla \Delta_j u^{(2)}\|_{L^\infty} + (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \sum_{m\le j-1} 2^{m-j} \|\Delta_m u\|_{L^2} \\
& + C\, (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}) \sum_{k\ge j-1} 2^{j-k} \|\Delta_k u\|_{L^2}.
\end{aligned}$$

Integrating (4.37) in time and taking $\sup_{j\ge -1}$, we obtain (4.33). This completes the proof of Lemma 4.7. ■

Lemma 4.8 *Assume that θ, $u^{(1)}$, u and $\theta^{(2)}$ are defined as in the proof of Theorem 4.2 and satisfy*

$$\partial_t \theta + u^{(1)} \cdot \nabla \theta + u \cdot \nabla \theta^{(2)} = 0. \tag{4.38}$$

Then we have the a priori bound

$$\begin{aligned}\|\theta(t)\|_{B^{-1,a}_{2,\infty}} &\le \|\theta(0)\|_{B^{-1,a}_{2,\infty}} + C\int_0^t \|\theta(\tau)\|_{B^{-1,a}_{2,\infty}}(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}})\,d\tau \\ &\quad + C\int_0^t \|u(\tau)\|_{L^2}\|\theta^{(2)}\|_{B^{0,a}_{\infty,1}}\,d\tau. \qquad (4.39)\end{aligned}$$

Proof. [Proof of Lemma 4.8] Let $j \ge -1$ be an integer. Applying Δ_j to (4.38) and taking the inner product with $\Delta_j\theta$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j\theta\|^2_{L^2} = K_1 + K_2, \qquad (4.40)$$

where

$$\begin{aligned}K_1 &= -\int \Delta_j\theta\,\Delta_j(u^{(1)}\cdot\nabla\theta)\,dx, \\ K_2 &= -\int \Delta_j\theta\,\Delta_j(u\cdot\nabla\theta^{(2)})\,dx.\end{aligned}$$

To estimate K_1, we decompose $\Delta_j(u^{(1)}\cdot\nabla\theta)$ as in (4.35) and estimate each component in a similar fashion to obtain

$$\begin{aligned}|K_1| &\le C\,\|\Delta_j\theta\|^2_{L^2}\,(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \\ &\quad + C\,\|\Delta_j\theta\|_{L^2}\,2^j a^{-1}(2^j)\,\|\theta\|_{B^{-1,a}_{2,\infty}}(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}).\end{aligned}$$

To estimate K_2, we decompose $\Delta_j(u\cdot\nabla\theta^{(2)})$ as in (4.36) and bound the components in a similar fashion to have

$$|K_2| \le C\,\|\Delta_j\theta\|_{L^2}\,\|u\|_{L^2}2^j a^{-1}(2^j)\,\|\theta^{(2)}\|_{B^{0,a}_{\infty,1}}.$$

Combining these estimates, we find

$$\begin{aligned}\frac{d}{dt}\|\Delta_j\theta\|_{L^2} &\le C\,2^j a^{-1}(2^j)\,\|\theta\|_{B^{-1,a}_{2,\infty}}(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \\ &\quad + C\,\|u\|_{L^2}2^j a^{-1}(2^j)\,\|\theta^{(2)}\|_{B^{0,a}_{\infty,1}}.\end{aligned}$$

Integrating in time, multiplying by $2^{-j}a(2^j)$ and taking $\sup_{j\ge -1}$, we obtain (4.39). This completes the proof of Lemma 4.8. ∎

We now sketch the proof of Theorem 4.1.

Proof. [Proof of Theorem 4.1] Thanks to Theorem 4.2, it suffices to establish the existence of solutions. The first step is to obtain a local (in time) solution and then extend it into a global solution through the global *a priori* bounds obtained in the previous section. The local solution can be constructed through the method of successive approximation. That is, we consider a successive approximation sequence $\{(\omega^{(n)},\theta^{(n)})\}$ solving

$$\begin{cases}\omega^{(1)} = S_2\omega_0, \quad \theta^{(1)} = S_2\theta_0,\\ \partial_t\omega^{(n+1)} + u^{(n)}\cdot\nabla\omega^{(n+1)} + \mathcal{L}\omega^{(n+1)} = \partial_{x_1}\theta^{(n+1)},\\ \partial_t\theta^{(n+1)} + u^{(n)}\cdot\nabla\theta^{(n+1)} = 0,\\ \omega^{(n+1)}(x,0) = S_{n+2}\omega_0(x), \quad \theta^{(n+1)}(x,0) = S_{n+2}\theta_0(x).\end{cases} \tag{4.41}$$

To show that $\{(\omega^{(n)},\theta^{(n)})\}$ converges to a solution of (4.4), it suffices to prove that $\{(\omega^{(n)},\theta^{(n)})\}$ obeys the following properties:

(1) There exists a time interval $[0,T_1]$ over which $\{(\omega^{(n)},\theta^{(n)})\}$ are bounded uniformly in terms of n. More precisely, we show that

$$\|\omega^{(n)}\|_{L_t^\infty(L^2\cap L^q)\cap L_t^1 B^{0,a}_{\infty,1}} \le C(T_1), \quad \|\theta^{(n)}\|_{L_t^\infty(L^2\cap B^{0,a^2}_{\infty,1})\cap L_t^1 B^{0,a}_{\infty,1}} \le C(T_1),$$

where $C(T_1)$ is a constant independent of n.

(2) There exists $T_2 > 0$ such that $\omega^{(n+1)} - \omega^{(n)}$ is a Cauchy sequence in $L_t^\infty B^{-1}_{\infty,1}$ and $\theta^{(n+1)} - \theta^{(n)}$ is Cauchy in $L_t^1 B^{-1,a}_{\infty,1}$, namely

$$\|\omega^{(n+1)} - \omega^{(n)}\|_{L_t^\infty B^{-1}_{\infty,1}} \le C(T_2)\,2^{-n}, \quad \|\theta^{(n+1)} - \theta^{(n)}\|_{L_t^1 B^{-1,a}_{\infty,1}} \le C(T_2)\,2^{-n}$$

for any $t\in[0,T_2]$, where $C(T_2)$ is independent of n.

If the properties stated in (1) and (2) hold, then there exists (ω,θ) satisfying

$$\omega \in L_t^\infty(L^2\cap L^q)\cap L_t^1 B^{0,a}_{\infty,1}, \quad \theta \in L_t^\infty(L^2\cap B^{0,a^2}_{\infty,1})\cap L_t^1 B^{0,a}_{\infty,1},$$

$$\omega^{(n)} \to \omega \quad \text{in} \quad L^\infty_t B^{-1}_{\infty,1}, \qquad \theta^{(n)} \to \theta \quad \text{in} \quad L^1_t B^{-1,a}_{\infty,1}$$

for any $t \le \min\{T_1, T_2\}$. It is then easy to show that (ω, θ) solves (4.4) and we thus obtain a local solution and the global bounds in the previous sections allow us to extend it into a global solution. It then remains to verify the properties stated in (1) and (2). Property (1) can be shown as in the previous sections (Section 4.3 through Section 4.6) while Property (2) can be checked as in the proof of Theorem 4.2. We thus omit further detail. This completes the proof of Theorem 4.1. ■

CHAPTER 5

Lans-α Magnetohydrodynamics System

5.1 Introduction

As a numerical regularization of MHD equations, various magnetohydrodynamics-α models have been proposed in [48]. One of the commonly used one is Lans-α magnetohydrodynamics (MHD) system. This chapter studies the global regularity issue of Lans-α MHD system when the dissipation or the diffusion is weakened by a logarithmic factor [44]. Attention will be focused on the system of the form

$$\begin{cases} \partial_t v + (u\cdot\nabla)v + \sum_{k=1}^{2} v_k \nabla u_k + \nabla(\pi + \frac{1}{2}|b|^2) + \nu\mathcal{L}^2 v = (b\cdot\nabla)b, \\ \partial_t b + (u\cdot\nabla)b - (b\cdot\nabla)u + \eta\mathcal{L}^2 b = 0. \end{cases} \tag{5.1}$$

together with

$$v = (1-\alpha^2\Delta)u, \;\; \nabla\cdot u = \nabla\cdot b = 0, \;\; (v,b)(x,0) = (v_0,b_0)(x). \tag{5.2}$$

Where $v(x,t)$ denotes the two-dimensional velocity vector field, $u(x,t)$ the filtered velocity, $b(x,t)$ the two-dimensional magnetic vector field and π the pressure scalar field. Here ν, $\eta \geq 0$ are the kinematic viscosity and magnetic diffusivity constants respectively and the Fourier operator $\mathcal{L}$ is defined through the Fourier transform by

$$\widehat{\mathcal{L}f}(\xi) = m(\xi)\hat{f}(\xi), \quad m(\xi) \geq \frac{|\xi|}{g(|\xi|)}. \tag{5.3}$$

and α denotes the length-scale parameter representing the width of the filters.

The author in [29] obtained the global regularity results of the system (5.1) in case $\nu > 0,\ \eta = 0,\ g \equiv 1$ and $\nu = 0,\ \eta > 0,\ g \equiv 1$. Our main purpose of this chapter is to extend the global regularity results of the system (5.1) logarithmically. More precisely, we prove the following theorems.

Theorem 5.1 *Suppose $\nu > 0$, $\eta = 0$ and $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a radially symmetric, non-decreasing function such that $g \geq 1$ and satisfies*

$$\int_e^\infty \frac{d\tau}{g(\tau)\sqrt{\ln(\tau)}\,\tau} = \infty. \tag{5.4}$$

Then for any v_0, $b_0 \in H^4(\mathbb{R}^2)$, there exists a unique classical solution pair to the system (5.1), (5.2).

Theorem 5.2 *Suppose $\nu = 0$, $\eta > 0$ and $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a radially symmetric, non-decreasing function such that $g \geq 1$ and satisfies*

$$\int_e^\infty \frac{d\tau}{g^4(\tau)\ln(\tau)\tau} = \infty. \tag{5.5}$$

Then for any v_0, $b_0 \in H^4(\mathbb{R}^2)$, there exists a unique classical solution pair to the system (5.1), (5.2).

Throughout this chapter, we use the notation $A \lesssim_{a,b} B$, $A \approx_{a,b} B$ to imply that there exists a non-negative constant c that depends on a, b such that $A \leq cB$, $A = cB$ respectively. We write fractional Laplacians $\Lambda := (-\Delta)^{\frac{1}{2}}$, vorticity and current density by

$$\omega := \nabla \times v, \quad j := \nabla \times b.$$

For the simplicity, we set $\nu = \eta = \alpha = 1$.

5.2 *A priori* Estimates for Theorem 5.1

We first obtain the basic energy conservation at $\nu = 1$, $\eta = 0$. Taking L^2-inner products on $(5.1)_1$ with u and $(5.1)_2$ with b and integrating in time, we obtain

$$\begin{aligned} \sup_{t\in[0,T]} (\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|b\|_{L^2}^2)(t) \quad &+ \quad \int_0^T (\|\mathcal{L}u\|_{L^2}^2 + \|\mathcal{L}\nabla u\|_{L^2}^2) d\tau \\ &\leq \quad (\|u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2), \end{aligned} \tag{5.6}$$

where we have used (5.2).

5.2.1 $\|v\|_{L^2}^2 + \|\nabla b\|_{L^2}^2$-estimate

Proposition 5.2.1 *Suppose $\nu = 1$, $\eta = 0$ and $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a radially symmetric, non-decreasing function such that $g \geq 1$ and satisfies (5.4). Then the solution pair to the system (5.1), (5.2) in $[0, T]$ satisfies*

$$\sup_{t\in[0,T]} (\|v\|_{L^2}^2 + \|\nabla b\|_{L^2}^2)(t) + \int_0^T \|\mathcal{L}v\|_{L^2}^2 d\tau \lesssim 1.$$

Proof. We denote by $A(t) = e + (\|v\|_{L^2}^2 + \|\nabla b\|_{L^2}^2)(t)$ and take L^2-inner products on (5.1) with $(v, -\Delta b)$, sum, integrate by parts and use (5.2), (5.6) and Hölder's inequality to obtain

$$\begin{aligned}
& \frac{1}{2}\partial_t(\|v\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\mathcal{L}v\|_{L^2}^2 \qquad (5.7)\\
= & \int -\sum_{k=1}^{2} v_k \nabla u_k \cdot v + (b\cdot\nabla)b\cdot v - (b\cdot\nabla)u\cdot\Delta b + (u\cdot\nabla)b\cdot\Delta b\\
= & -\int \sum_{k=1}^{2} v_k \nabla u_k \cdot v + (b\cdot\nabla)u\cdot b + \int \nabla b\cdot\nabla b\cdot\nabla u + \nabla b\cdot\nabla u\cdot\nabla b - \nabla u\cdot\nabla b\cdot\nabla b\\
\lesssim & \|\nabla u\|_{L^\infty} A(t).
\end{aligned}$$

Using Littlewood-Paley decomposition, Bernstein's inequality, Plancherel theorem, (5.3) and the fact that g is non-decreasing: we obtain for some $M > 1$ to be determined subsequently

$$\begin{aligned}
\|\nabla u\|_{L^\infty} &\leq \sum_{k\geq -1} \|\Delta_k \nabla u\|_{L^\infty}\\
&\lesssim \sum_{2^k\leq M} \frac{2^k}{g(2^k)} \|\Delta_k \nabla u\|_{L^2} g(2^k) + \sum_{2^k\geq M} 2^{-k} \frac{2^{2k}}{g(2^k)} \|\Delta_k \nabla u\|_{L^2} g(2^k)\\
&\lesssim g(M) \sum_{2^k\leq M} \|\Delta_k \mathcal{L}\nabla u\|_{L^2} + \sum_{2^k\geq M} 2^{-k} g(2^k) \|\Delta_k \mathcal{L}v\|_{L^2}.
\end{aligned}$$

We also used the fact that $\|\Delta u\|_{L^2} \lesssim \|v\|_{L^2}$. Finally, by applying Hölder's inequality, we obtain

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\lesssim g(M)\left(\sum_{2^k\le M}1^2\right)^{\frac12}\left(\sum_{2^k\le M}\|\Delta_k\mathcal{L}\nabla u\|_{L^2}^2\right)^{\frac12} \\ &+ \left(\left(\sum_{2^k\ge M}2^{-2k}\right)^{\frac12}\left(\sum_{2^k\ge M}2^{-2k}g^4(2^k)\right)^{\frac12}\right)^{\frac12}\left(\sum_{2^k\ge M}\|\Delta_k\mathcal{L}v\|_{L^2}^2\right)^{\frac12} \\ &\lesssim g(M)\sqrt{\ln(M)}\|\mathcal{L}\nabla u\|_{L^2}+M^{-\frac12}\|\mathcal{L}v\|_{L^2}.\end{aligned} \tag{5.8}$$

Using (5.7) and (5.8), applying Young's inequality and choosing $M = cA(t)$ for sufficiently large c, we have

$$\begin{aligned}\frac12\partial_t A(t)+\|\mathcal{L}v\|_{L^2}^2 &\lesssim \left(g(cA(t))\sqrt{\ln(cA(t))}\|\mathcal{L}\nabla u\|_{L^2}+(cA(t))^{-\frac12}\|\mathcal{L}v\|_{L^2}\right)A(t) \\ &\lesssim \frac{\|\mathcal{L}v\|_{L^2}^2}{2}+c\left(g(cA(t))\sqrt{\ln(cA(t))}(1+\|\mathcal{L}\nabla u\|_{L^2}^2)A(t)\right).\end{aligned}$$

Thus, by absorbing the dissipative term, we obtain

$$\partial_t A(t)+\|\mathcal{L}v\|_{L^2}^2 \lesssim g(cA(t)\sqrt{\ln(cA(t))}(1+\|\mathcal{L}\nabla u\|_{L^2}^2)A(t) \tag{5.9}$$

which implies for any $t\in[0,T]$,

$$\int_{cA(0)}^{cA(t)}\frac{d\tau}{g(\tau)\sqrt{\ln(\tau)}\tau}\lesssim\int_0^t(1+\|\mathcal{L}\nabla u\|_{L^2}^2)d\tau\lesssim\int_0^T(1+\|\mathcal{L}\nabla u\|_{L^2}^2)d\tau\lesssim 1$$

by (5.6). This implies that

$$\sup_{t\in[0,T]}A(t)\lesssim 1 \tag{5.10}$$

by (5.4). Now, integrating (5.9) in time over $[0,T]$, using (5.6) and (5.10) completes the proof of Proposition 5.2.1. ■

5.2.2 $\|\omega\|_{L^2}^2+\|\Delta b\|_{L^2}^2$-estimate

Proposition 5.2.2 *Suppose $\nu = 1$, $\eta = 0$ and $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a radially symmetric, non-decreasing function such that $g \ge 1$ and satisfies (5.4). Then the solution pair to the system (5.1), (5.2) in $[0,T]$ satisfies*

$$\sup_{t\in[0,T]}(\|\omega\|_{L^2}^2+\|\Delta b\|_{L^2}^2)(t)+\int_0^T\|\mathcal{L}\omega\|_{L^2}^2d\tau\lesssim 1.$$

Proof. Taking a curl on $(5.1)_1$, we get

$$\partial_t \omega + (u \cdot \nabla)\omega + \mathcal{L}^2 \omega = (b \cdot \nabla) j. \tag{5.11}$$

Taking L^2-inner products of (5.11) with ω, we have

$$\frac{1}{2}\partial_t \|\omega\|_{L^2}^2 + \|\mathcal{L}\omega\|_{L^2}^2 = \int (b \cdot \nabla) j \omega := I_1. \tag{5.12}$$

Applying Λ^2 on $(5.2)_2$ and taking L^2-inner products with $\Lambda^2 b$ leads to

$$\frac{1}{2}\partial_t \|\Delta b\|_{L^2}^2 = -\int \Lambda^2((u \cdot \nabla) b) \cdot \Lambda^2 b + \int \Lambda^2((b \cdot \nabla) u) \cdot \Lambda^2 b := II_1 + II_2. \tag{5.13}$$

First, we estimate

$$I_1 \leq \|b\|_{L^8} \|\nabla j\|_{L^2} \|\omega\|_{L^{\frac{8}{3}}} \tag{5.14}$$

by Hӧlder's inequalities. Further we estimate

$$\|\omega\|_{L^{\frac{8}{3}}} \lesssim \sum_{k \geq -1} 2^{-k(\frac{3}{4})} g(2^k) \|\Delta_k \mathcal{L}\omega\|_{L^2} \lesssim \|\mathcal{L}\omega\|_{L^2} \tag{5.15}$$

due to Littlewood Paley decomposition, Bernstein's inequality, (5.3) and Hӧlder's inequalitiy, Thus, we deduce by (5.14) and (5.15)

$$I_1 \leq \|b\|_{H^1} \|\Delta b\|_{L^2} \|\mathcal{L}\omega\|_{L^2} \leq \frac{\|\mathcal{L}\omega\|_{L^2}^2}{2} + c\|\Delta b\|_{L^2}^2 \tag{5.16}$$

due to Sobolev embedding of $H^1(\mathbb{R}^2) \hookrightarrow L^8(\mathbb{R}^2)$, (5.6), Proposition 5.2.1 and Young's inequality.

Next, we estimate

$$\begin{aligned} II_1 &= -\int [\Lambda^2((u \cdot \nabla) b) - u \cdot \nabla \Lambda^2 b] \cdot \Lambda^2 b \\ &\lesssim (\|\nabla u\|_{L^\infty} \|\Lambda \nabla b\|_{L^2} + \|\Lambda^2 u\|_{L^4} \|\nabla b\|_{L^4}) \|\Lambda^2 b\|_{L^2} \\ &\lesssim \|\nabla u\|_{L^\infty} \|\Delta b\|_{L^2}^2 + \|\Lambda^2 u\|_{L^2}^{\frac{1}{2}} \|\omega\|_{L^2}^{\frac{1}{2}} \|\nabla b\|_{L^2}^{\frac{1}{2}} \|\Delta b\|_{L^2}^{\frac{3}{2}} \\ &\lesssim (1 + \|\nabla u\|_{L^\infty})(\|\omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) \end{aligned} \tag{5.17}$$

by Hölder's inequality, Lemma 2.10, Gagliardo-Nirenberg inequality, Proposition 5.2.1 and Young's inequality. Now we use Littlewod Paley decomposition to estimate

$$\begin{aligned} \|\nabla u\|_{L^\infty} &\le \|\Delta_{-1}\nabla u\|_{L^\infty} + \sum_{k\ge 0}\|\Delta_k \nabla u\|_{L^\infty} \\ &\lesssim 1 + \sum_{k\ge 0} 2^{-k} g(2^k)\|\Delta_k \mathcal{L}\Delta u\|_{L^2} \\ &\lesssim 1 + \|\mathcal{L}v\|_{L^2} \end{aligned} \tag{5.18}$$

by Bernstein's inequality, Plancherel theorem and Hölder's inequality. Thus, by (5.17) and (5.18), we get

$$II_1 \lesssim (1 + \|\mathcal{L}v\|_{L^2}^2)(\|\omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) \tag{5.19}$$

Finallly,

$$\begin{aligned} II_2 &\lesssim (\|\nabla u\|_{L^\infty}\|\Lambda^2 b\|_{L^2} + \|b\|_{L^\infty}\|\Lambda^2\nabla u\|_{L^2})\|\Lambda^2 b\|_{L^2} \\ &\lesssim (1 + \|\mathcal{L}v\|_{L^2})\|\Lambda^2 b\|_{L^2}^2 \\ &\quad +(\|b\|_{L^2} + \|b\|_{H^1}\log_2(2 + \|b\|_{H^2}) + 1)\|\omega\|_{L^2}\|\Delta b\|_{L^2} \\ &\lesssim (\|\omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2)(1 + \|\mathcal{L}v\|_{L^2}^2)\log_2(2 + \|\omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) \end{aligned} \tag{5.20}$$

by Hölder's inequality, Lemma 2.9, (5.18), Lemma 2.13, (5.6) and Young's inequality. Applying (5.16), (5.19) and (5.20) in the sum of (5.12) and (5.13), we obtain after absorbing,

$$\begin{aligned} &\partial_t(\|\omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + \|\mathcal{L}\omega\|_{L^2}^2 \\ \lesssim\ &(\|\omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2)(1 + \|\mathcal{L}v\|_{L^2}^2)\log_2(2 + \|\omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) \end{aligned} \tag{5.21}$$

By Proposition 5.2.1, this implies $\sup_{t\in[0,T]}(\|\omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2)$ is bounded and further taking the L^1-norm in time on (5.21) completes the proof of Proposition 5.2.2. ■

5.3 Higher Regularity for Theorem 5.1

To obtain the higher regularity, we apply Λ^4 on $(5.1)_1$, Λ^5 on $(5.1)_2$, take L^2-inner products with $(\Lambda^4 v, \Lambda^5 b)$ respectively to get

$$\begin{aligned}
&\frac{1}{2}\partial_t(\|\Lambda^4 v\|_{L^2}^2 + \|\Lambda^5 b\|_{L^2}^2) + \|\mathcal{L}\Lambda^4 v\|_{L^2}^2 \qquad (5.22)\\
= \;& -\int [\Lambda^4((u\cdot\nabla)v) - u\cdot\nabla\Lambda^4 v]\cdot\Lambda^4 v - \sum_{k=1}^{2}\int \Lambda^4(v_k\nabla u_k)\cdot\Lambda^4 v\\
& -\int [\Lambda^5((u\cdot\nabla)b) - u\cdot\nabla\Lambda^5 b]\cdot\Lambda^5 b\\
& +\int \Lambda^4((b\cdot\nabla)b)\cdot\Lambda^4 v + \int \Lambda^5((b\cdot\nabla)u)\cdot\Lambda^5 b = \sum_{i=1}^{5} III_i
\end{aligned}$$

First we estimate

$$\begin{aligned}
III_1 &\lesssim (\|\nabla u\|_{L^\infty}\|\Lambda^4 v\|_{L^2} + \|\Lambda^4 u\|_{L^4}\|\nabla v\|_{L^4})\|\Lambda^4 v\|_{L^2} \qquad (5.23)\\
&\lesssim ((\|\nabla u\|_{L^2} + \|\omega\|_{L^2})\|\Lambda^4 v\|_{L^2} + \|\Lambda^3 u\|_{L^2}^{\frac{1}{2}}\|\Lambda^6 u\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^2}^{\frac{5}{6}}\|\Lambda^4 v\|_{L^2}^{\frac{1}{6}})\|\Lambda^4 v\|_{L^2}\\
&\lesssim 1 + \|\Lambda^4 v\|_{L^2}^2
\end{aligned}$$

where we have used Hölder's inequality, Lemma 2.10, Sobolev embedding of $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, Gagliardo-Nirenberg inequality, (5.6), Proposition 5.2.2 and Young's inequality.

Next, by Hölder's inequality, Lemma 2.9, Sobolev embedding of $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, Gagliardo-Nirenberg and Young's inequalities, we obtain

$$\begin{aligned}
III_2 &\lesssim (\|\nabla u\|_{L^\infty}\|\Lambda^4 v\|_{L^2} + \|v\|_{L^4}\|\Lambda^5 u\|_{L^4})\|\Lambda^4 v\|_{L^2} \qquad (5.24)\\
&\lesssim (\|\Lambda^4 v\|_{L^2}(\|\nabla u\|_{L^2} + \|\omega\|_{L^2}) + \|v\|_{L^2}^{\frac{7}{8}}\|\Lambda^4 v\|_{L^2}^{\frac{1}{8}}\|\Lambda^3 u\|_{L^2}^{\frac{1}{6}}\|\Lambda^6 u\|_{L^2}^{\frac{5}{6}})\|\Lambda^4 v\|_{L^2}\\
&\lesssim 1 + \|\Lambda^4 v\|_{L^2}^2.
\end{aligned}$$

Next, by Hölder's inequality, Lemma 2.10, Sobolev embedding of $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, Propositions 5.2.1 and 5.2.2, Gagliardo-Nirenberg and Young's inequalities, we have

$$\begin{aligned} III_3 &\lesssim (\|\nabla u\|_{L^\infty}\|\Lambda^5 b\|_{L^2} + \|\nabla b\|_{L^4}\|\Lambda^5 u\|_{L^4})\|\Lambda^5 b\|_{L^2} \\ &\lesssim (\|\Lambda^5 b\|_{L^2}(\|\nabla u\|_{L^2} + \|\omega\|_{L^2}) + \|\Lambda^3 u\|_{L^2}^{\frac{1}{6}}\|\Lambda^6 u\|_{L^2}^{\frac{5}{6}}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|\Delta b\|_{L^2}^{\frac{1}{2}})\|\Lambda^5 b\|_{L^2} \\ &\lesssim 1 + \|\Lambda^4 v\|_{L^2}^2 + \|\Lambda^5 b\|_{L^2}^2. \end{aligned} \tag{5.25}$$

Next, by Hölder's inequality, Lemma 2.9, Sobolev embedding of $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, Propositions 5.2.1 and 5.2.2, (5.6), Gagliardo-Nirenberg and Young's inequalities, we get

$$\begin{aligned} III_4 &\lesssim (\|\Lambda^4 b\|_{L^4}\|\nabla b\|_{L^4} + \|b\|_{L^\infty}\|\Lambda^5 b\|_{L^2})\|\Lambda^4 v\|_{L^2} \\ &\lesssim (\|\Lambda^2 b\|_{L^2}^{\frac{1}{6}}\|\Lambda^5 b\|_{L^2}^{\frac{5}{6}}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|\Delta b\|_{L^2}^{\frac{1}{2}} + (\|b\|_{L^2} + \|\Delta b\|_{L^2})\|\Lambda^5 b\|_{L^2})\|\Lambda^4 v\|_{L^2} \\ &\lesssim 1 + \|\Lambda^4 v\|_{L^2}^2 + \|\Lambda^5 b\|_{L^2}^2. \end{aligned} \tag{5.26}$$

Finally, by Hölder's inequality, Lemma 2.9, Sobolev embedding of $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, Propositions 5.2.1 and 5.2.2, Gagliardo-Nirenberg and Young's inequalities, we obtain

$$\begin{aligned} III_5 &\lesssim (\|\Lambda^5 b\|_{L^2}\|\nabla u\|_{L^\infty} + \|b\|_{L^\infty}\|\Lambda^6 u\|_{L^2})\|\Lambda^5 b\|_{L^2} \\ &\lesssim (\|\Lambda^5 b\|_{L^2}(\|\nabla u\|_{L^2} + \|\omega\|_{L^2}) + (\|b\|_{L^2} + \|\Delta b\|_{L^2})\|\Lambda^4 v\|_{L^2})\|\Lambda^5 b\|_{L^2} \\ &\lesssim \|\Lambda^4 v\|_{L^2}^2 + \|\Lambda^5 b\|_{L^2}^2. \end{aligned} \tag{5.27}$$

Inserting the estimates (5.23), (5.24), (5.25), (5.26) and (5.27) in (5.22), Gronwall's inequality implies

$$\sup_{t\in[0,T]} (\|\Lambda^4 v\|_{L^2}^2 + \|\Lambda^5 b\|_{L^2}^2)(t) \lesssim 1.$$

5.4 A priori Estimates for Theorem 5.2

This section establishes a global a priori bounds for the solution pair (v, b). In contrast to the classical MHD system, $\|v\|_{L^2}$ is not conserved for the solutions to (5.1). Therefore, we first establish the basic energy conservation for the pair (u, b). Taking

L^2-inner products on (5.1) at $\nu = 0$, $\eta = 1$ with (u, b) and integrating in time we obtain

$$\begin{aligned}\sup_{t\in[0,T]} (\|u\|_{L^2}^2 \ + \ \|\nabla u\|_{L^2}^2 \ &+ \ \|b\|_{L^2}^2)(t) + \int_0^T \|\mathcal{L}b\|_{L^2}^2 d\tau \\ &\leq \ (\|u_0\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2)\end{aligned} \tag{5.28}$$

For the rest of this section, we set $B(t) := e + \|\mathcal{L}b\|_{L^2}^2(t)$ and $E(t) := e + (\|\omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2)(t)$.

5.4.1 $\|\mathcal{L}b\|_{L^2}^2$-estimate

The $\|v\|_{L^2}^2$-estimate established for Theorem 5.1 will not go through in this case without dissipation because of the term $\sum_{k=1}^2 v_k \nabla u_k \cdot v$. This is precisely the problematic term remarked by the authors in [78]. To overcome this difficulty, we take advantage of the vorticity formulation of (5.1). However, we cannot estimate $\|\omega\|_{L^2}$ without improving our estimate on b due to the term $(b \cdot \nabla)j$. We prove the following proposition:

Proposition 5.4.1 *Suppose $\nu = 0$, $\eta = 1$ and $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a radially symmetric, non-decreasing function such that $g \geq 1$ and satisfies (5.5). Then the solution pair to the system (5.1), (5.2) in $[0, T]$ satisfies*

$$\sup_{t\in[0,T]} \|\mathcal{L}b\|_{L^2}^2(t) + \int_0^T \|\mathcal{L}^2 b\|_{L^2}^2 d\tau \lesssim 1.$$

Proof. Taking L^2-inner product on $(5.1)_2$ with $\mathcal{L}^2 b$ and applying Hölder's inequality and Lemma 2.9, we obtain

$$\begin{aligned}\frac{1}{2}\partial_t \|\mathcal{L}b\|_{L^2}^2 + \|\mathcal{L}^2 b\|_{L^2}^2 \ &\leq \ (\|\text{div}(u \otimes b)\|_{L^2} + \|\text{div}(b \otimes u)\|_{L^2})\|\mathcal{L}^2 b\|_{L^2} \\ &\lesssim \ (\|u\|_{L^6}\|\Lambda b\|_{L^3} + \|\Lambda u\|_{L^2}\|b\|_{L^\infty})\|\mathcal{L}^2 b\|_{L^2}\end{aligned} \tag{5.29}$$

We first note that by (5.28) and by the Sobolev embedding of $H^1(\mathbb{R}^2) \hookrightarrow L^6(\mathbb{R}^2)$,

$$\|u\|_{L^6} \lesssim \|u\|_{H^1} \lesssim 1.$$

Now by Bernstein's inequality, Plancherel theorem, Hölder's inequality and (5.3), we have for some $M > 1$ to be determined subsequently.

$$\begin{aligned}\|\Lambda b\|_{L^3} &\lesssim \sum_{2^k \le M} \frac{2^{\frac{k}{3}}}{g(2^k)} \|\Delta_k \Lambda b\|_{L^2} g(2^k) + \sum_{2^k > M} \frac{2^{-\frac{2k}{3}} 2^k}{g^2(2^k)} \|\Delta_k \Lambda b\|_{L^2} g^2(2^k) \quad (5.30)\\ &\lesssim g(M) \sum_{2^k \le M} 2^{\frac{k}{3}} \|\Delta_k \mathcal{L} b\|_{L^2} + \sum_{2^k > M} 2^{-\frac{2k}{3}} \|\Delta_k \mathcal{L}^2 b\|_{L^2} g^2(2^k)\\ &\lesssim g(M) M^{\frac{1}{3}} \sqrt{\ln(M)} \|\mathcal{L} b\|_{L^2} + M^{-\frac{1}{3}} \|\mathcal{L}^2 b\|_{L^2}.\end{aligned}$$

On the other hand, with the same argument, we can bound

$$\begin{aligned}\|b\|_{L^\infty} &\lesssim \sum_{2^k \le M} \frac{2^k}{g(2^k)} \|\Delta_k b\|_{L^2} g(2^k) + \sum_{2^k > M} 2^{-k} \frac{2^{2k}}{g^2(2^k)} \|\Delta_k b\|_{L^2} g^2(2^k) \quad (5.31)\\ &\lesssim g(M) \sqrt{\ln(M)} \|\mathcal{L} b\|_{L^2} + M^{-\frac{1}{2}} \|\mathcal{L}^2 b\|_{L^2}\end{aligned}$$

Inserting the estimates (5.30) and (5.31) in (5.29), there exists a constant c_0 such that

$$\begin{aligned}&\frac{1}{2}\partial_t \|\mathcal{L} b\|_{L^2}^2 + \|\mathcal{L}^2 b\|_{L^2}^2 \quad (5.32)\\ \le\ & c_0 (g(M) M^{\frac{1}{3}} \sqrt{\ln(M)} \|\mathcal{L} b\|_{L^2} \|\mathcal{L}^2 b\|_{L^2} + M^{-\frac{1}{3}} \|\mathcal{L}^2 b\|_{L^2}^2\\ &+ g(M) \sqrt{\ln(M)} \|\mathcal{L} b\|_{L^2} \|\mathcal{L}^2 b\|_{L^2} + M^{-\frac{1}{2}} \|\mathcal{L}^2 b\|_{L^2}^2).\end{aligned}$$

We choose $M = cB(t)$ for sufficiently large c so that

$$c_0 \left(M^{-\frac{1}{3}} \|\mathcal{L}^2 b\|_{L^2}^2 + M^{-\frac{1}{2}} \|\mathcal{L}^2 b\|_{L^2}^2 \right) < \frac{\|\mathcal{L}^2 b\|_{L^2}^2}{4} \quad (5.33)$$

Moreover, by Young's inequality, we get

$$c_0 g(M) M^{\frac{1}{3}} \sqrt{\ln(M)} \|\mathcal{L} b\|_{L^2} \|\mathcal{L}^2 b\|_{L^2} \le \frac{\|\mathcal{L}^2 b\|_{L^2}^2}{8} + c g^2(M) M^{\frac{2}{3}} \ln(M) \|\mathcal{L} b\|_{L^2}^2, \quad (5.34)$$

$$c_0 g(M) \sqrt{\ln(M)} \|\mathcal{L} b\|_{L^2} \|\mathcal{L}^2 b\|_{L^2} \le \frac{\|\mathcal{L}^2 b\|_{L^2}^2}{8} + c g^2(M) \ln(M) \|\mathcal{L} b\|_{L^2}^2. \quad (5.35)$$

Inserting (5.33), (5.34) and (5.35) in (5.32) and absorbing the diffusive term, we obtain

$$\partial_t B(t) + \|\mathcal{L}^2 b\|_{L^2}^2 \lesssim g^2(cB(t) \ln(cB(t)) B(t) B^{\frac{2}{3}}(t). \quad (5.36)$$

Thus, by integrating and applying Hölder's inequality and (5.28), we have

$$\int_{cB(0)}^{cB(t)} \frac{d\tau}{g^2(\tau)\ln(\tau)\tau} \leq \int_0^t (e + \|\mathcal{L}b\|_{L^2}^2)^{\frac{2}{3}} d\tau \lesssim T^{\frac{1}{3}} \left(\int_0^T (e + \|\mathcal{L}b\|_{L^2}^2) d\tau \right)^{\frac{2}{3}} \lesssim 1$$

By (5.5), we obtain the bound on $\sup_{t\in[0,T]} \|\mathcal{L}b\|_{L^2}^2(t)$ and hence integrating (5.36) in time completes the proof of Propsition 5.4.1. ∎

5.4.2 $\|\omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2$-estimate

The regularity obtained in Proposition 5.4.1 for b is logarithmically worse than H^1-estimate. However, it is enough to obtain higher regularity due to the nice structure of the vorticity formulation of $(5.1)_1$. We now prove the following proposition:

Proposition 5.4.2 *Suppose $\nu = 0$, $\eta = 1$ and $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a radially symmetric, non-decreasing function such that $g \geq 1$ and satisfies (5.5). Then the solution pair to the system (5.1), (5.2) in $[0, T]$ satisfies*

$$\sup_{t\in[0,T]} (\|\omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2)(t) + \int_0^T \|\mathcal{L}\Delta b\|_{L^2}^2 d\tau \lesssim 1.$$

Proof. The vorticity formulation of $(5.1)_1$ is given by

$$\partial_t \omega + (u \cdot \nabla)\omega = (b \cdot \nabla) j. \tag{5.37}$$

Taking L^2-inner product on (5.37) with ω leads to

$$\frac{1}{2}\partial_t \|\omega\|_{L^2}^2 = \int (b \cdot \nabla) j\omega. \tag{5.38}$$

We apply Δ on $(5.1)_2$, take L^2-inner product with Δb and sum to (5.38) to obtain

$$\begin{aligned} &\frac{1}{2}\partial_t(\|\omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + \|\mathcal{L}\Delta b\|_{L^2}^2 \\ &= \int (b \cdot \nabla) j\omega - \int \Delta((u \cdot \nabla)b) \cdot \Delta b + \int \Delta((b \cdot \nabla)u) \cdot \Delta b \\ &= \sum_{i=1}^3 IV_i \end{aligned} \tag{5.39}$$

By Hölder's and Young's inequalities, IV_1 can be bounded by

$$IV_1 \le \|b\|_{L^\infty}\|\nabla j\|_{L^2}\|\omega\|_{L^2} \lesssim \|b\|_{L^\infty}(\|\nabla j\|_{L^2}^2 + \|\omega\|_{L^2}^2)$$

Now, we estimate

$$\begin{aligned}
\|b\|_{L^\infty} &\lesssim \sum_{2^k\le M} \frac{2^k}{g(2^k)}\|\Delta_k b\|_{L^2} g(2^k) + \sum_{2^k>M} \frac{2^{-2k}2^{3k}}{g(2^k)}\|\Delta_k b\|_{L^2} g(2^k) \\
&\lesssim g(M)\sum_{2^k\le M}\|\Delta_k \mathcal{L} b\|_{L^2} + \left(\left(\sum_{2^k>M} 2^{-2k}\right)^{\frac{1}{2}}\left(\sum_{2^k>M} 2^{-2k} g^4(2^k)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\|\mathcal{L}\Delta b\|_{L^2} \\
&\lesssim g(M)\sqrt{\ln(M)} + M^{-1}\|\mathcal{L}\Delta b\|_{L^2}
\end{aligned}$$

by Bernstein's inequality, Plancherel theorem, (5.3), Hölder's inequality and Proposition 5.4.1. Therefore,

$$IV_1 \le g(M)\sqrt{\ln(M)}E(t) + M^{-1}\|\mathcal{L}\Delta b\|_{L^2}E(t). \tag{5.40}$$

We use Hölder's inequality, Gagliardo-Nirenberg inequality, Leema 2.10 and (5.28) to estimate

$$\begin{aligned}
IV_2 &= \int [\Delta((u\cdot\nabla)b - u\cdot\nabla\Delta b]\cdot\Delta b \\
&\lesssim (\|\nabla u\|_{L^3}\|\Delta b\|_{L^3} + \|\Delta u\|_{L^2}\|\nabla b\|_{L^6})\|\Delta b\|_{L^3} \\
&\lesssim \|\nabla u\|_{L^2}^{\frac{5}{6}}\|\omega\|_{L^2}^{\frac{1}{6}}\|\Delta b\|_{L^3}^2 + \|\nabla u\|_{L^2}^{\frac{1}{2}}\|\omega\|_{L^2}^{\frac{1}{2}}\|b\|_{L^2}^{\frac{2}{7}}\|\Delta b\|_{L^3}^{\frac{12}{7}} \\
&\lesssim \|\omega\|_{L^2}^{\frac{1}{6}}\|\Delta b\|_{L^3}^2 + \|\omega\|_{L^2}^{\frac{1}{2}}\|\Delta b\|_{L^3}^{\frac{12}{7}},
\end{aligned}$$

where we estimate

$$\begin{aligned}
\|\Delta b\|_{L^3} &\lesssim \sum_{2^k\le M} \frac{2^{\frac{k}{3}}}{g^2(2^k)}\|\Delta_k\Delta b\|_{L^2} g^2(2^k) + \sum_{2^k>M}\frac{2^{-\frac{2k}{3}}2^k}{g(2^k)}\|\Delta_k\Delta b\|_{L^2} g(2^k) \\
&\lesssim g^2(M)M^{\frac{1}{3}}\sum_{2^k\le M}\|\Delta_k\mathcal{L}^2 b\|_{L^2} + \left(\left(\sum_{2^k>M} 2^{-\frac{k}{6}}\right)^{\frac{1}{2}}\left(\sum_{2^k>M} 2^{-\frac{5k}{2}} g^4(2^k)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\|\mathcal{L}\Delta b\|_{L^2} \\
&\lesssim g^2(M)M^{\frac{1}{3}}\sqrt{\ln(M)}\|\mathcal{L}^2 b\|_{L^2} + M^{-\frac{1}{24}}\|\mathcal{L}\Delta b\|_{L^2}
\end{aligned}$$

by Bernstein's inequality, Plancherel theorem, (5.3) and Hölder's inequality. Thus, by using Lemma 2.7, we have

$$\|\omega\|_{L^2}^{\frac{1}{6}}\|\Delta b\|_{L^3}^2 \lesssim \|\omega\|_{L^2}^{\frac{1}{6}} g^4(M)M^{\frac{2}{3}}\ln(M)\|\mathcal{L}^2 b\|_{L^2}^2 + \|\omega\|_{L^2}^{\frac{1}{6}} M^{-\frac{1}{12}}\|\mathcal{L}\Delta b\|_{L^2}^2.$$

On the other hand, by Bernstein's inequality, Plancherel theorem, (5.3) and Hölder's inequality, we obtain

$$\begin{aligned}
\|\Delta b\|_{L^3} &\lesssim \sum_{2^k\le M}\frac{2^{\frac{k}{3}}}{g^2(2^k)}\|\Delta_k\Delta b\|_{L^2}g^2(2^k) + \sum_{2^k>M}\frac{2^{-\frac{2k}{3}}2^k}{g(2^k)}\|\Delta_k\Delta b\|_{L^2}g(2^k)\\
&\lesssim g^2(M)M^{\frac{1}{3}}\sum_{2^k\le M}\|\Delta_k\mathcal{L}^2 b\|_{L^2} + \left(\left(\sum_{2^k>M}2^{-\frac{7k}{12}}\right)^{\frac{1}{2}}\left(\sum_{2^k>M}2^{-\frac{25k}{12}}g^4(2^k)\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\|\mathcal{L}\Delta b\|_{L^2}\\
&\lesssim g^2(M)M^{\frac{1}{3}}\sqrt{\ln(M)}\|\mathcal{L}^2 b\|_{L^2} + M^{-\frac{7}{48}}\|\mathcal{L}\Delta b\|_{L^2}.
\end{aligned}$$

Thus, by using Lemma 2.7 again, we get

$$\|\omega\|_{L^2}^{\frac{1}{2}}\|\Delta b\|_{L^3}^{\frac{12}{7}} \lesssim \|\omega\|_{L^2}^{\frac{1}{2}} g^{\frac{24}{7}}(M)M^{\frac{4}{7}}(\ln(M))^{\frac{6}{7}}\|\mathcal{L}^2 b\|_{L^2}^{\frac{12}{7}} + \|\omega\|_{L^2}^{\frac{1}{2}} M^{-\frac{1}{4}}\|\mathcal{L}\Delta b\|_{L^2}^{\frac{12}{7}}.$$

Collecting the estimates above, we obtain

$$\begin{aligned}
IV_2 &\lesssim \|\omega\|_{L^2}^{\frac{1}{6}} g^4(M)M^{\frac{2}{3}}\ln(M)\|\mathcal{L}^2 b\|_{L^2}^2 + \|\omega\|_{L^2}^{\frac{1}{6}} M^{-\frac{1}{12}}\|\mathcal{L}\Delta b\|_{L^2}^2\\
&\quad+\|\omega\|_{L^2}^{\frac{1}{2}} g^{\frac{24}{7}}(M)M^{\frac{4}{7}}(\ln(M))^{\frac{6}{7}}\|\mathcal{L}^2 b\|_{L^2}^{\frac{12}{7}} + \|\omega\|_{L^2}^{\frac{1}{2}} M^{-\frac{1}{4}}\|\mathcal{L}\Delta b\|_{L^2}^{\frac{12}{7}}.
\end{aligned} \tag{5.41}$$

Finally, we estimate

$$IV_3 \lesssim (\|\Delta b\|_{L^3}\|\nabla u\|_{L^3} + \|b\|_{L^6}\|\Lambda^3 u\|_{L^2})\|\Delta b\|_{L^3}$$

by Hölder's inequality and Lemma 2.10. As above, we have the bound of

$$\|\nabla u\|_{L^3}\|\Delta b\|_{L^3}^2 \lesssim \|\omega\|_{L^2}^{\frac{1}{6}} g^4(M)M^{\frac{2}{3}}\ln(M)\|\mathcal{L}^2 b\|_{L^2}^2 + \|\omega\|_{L^2}^{\frac{1}{6}} M^{-\frac{1}{12}}\|\mathcal{L}\Delta b\|_{L^2}^2.$$

We now estimate by Gagliardo-Nirenberg inequality and (5.28)

$$\|b\|_{L^6}\|\Lambda^3 u\|_{L^2}\|\Delta b\|_{L^3} \lesssim \|b\|_{L^2}^{\frac{5}{7}}\|\omega\|_{L^2}\|\Delta b\|_{L^3}^{\frac{9}{7}} \lesssim \|\omega\|_{L^2}\|\Delta b\|_{L^3}^{\frac{9}{7}}$$

where we estimate

$$
\begin{aligned}
\|\Delta b\|_{L^3} &\lesssim \sum_{2^k\le M}\frac{2^{\frac{k}{3}}}{g^2(2^k)}\|\Delta_k\Delta b\|_{L^2}g^2(2^k)+\sum_{2^k>M}\frac{2^{-\frac{2k}{3}}2^k}{g(2^k)}\|\Delta_k\Delta b\|_{L^2}g(2^k)\\
&\lesssim g^2(M)M^{\frac{1}{3}}\sum_{2^k\le M}\|\Delta_k\mathcal{L}^2 b\|_{L^2}+\left(\left(\sum_{2^k>M}2^{-\frac{14k}{9}}\right)^{\frac12}\left(\sum_{2^k>M}2^{-\frac{10k}{9}}g^4(2^k)\right)^{\frac12}\right)^{\frac12}\|\mathcal{L}\Delta b\|_{L^2}\\
&\lesssim g^2(M)M^{\frac13}\sqrt{\ln(M)}\|\mathcal{L}^2 b\|_{L^2}+M^{-\frac{7}{18}}\|\mathcal{L}\Delta b\|_{L^2}
\end{aligned}
$$

by Bernstein's inequality, Plancherel theorem, (5.3) and Hölder's inequality. Thus, by Lemma 2.7

$$
\begin{aligned}
&\|b\|_{L^6}\|\Lambda^3 u\|_{L^2}\|\Delta b\|_{L^3}\\
\lesssim\ &\|\omega\|_{L^2}g^{\frac{18}{7}}(M)M^{\frac37}(\ln(M)^{\frac{9}{14}}\|\mathcal{L}^2 b\|_{L^2}^{\frac97}+\|\omega\|_{L^2}M^{-\frac12}\|\mathcal{L}\Delta b\|_{L^2}^{\frac97}.
\end{aligned}
$$

Collecting the estimates above, we obtain

$$
\begin{aligned}
IV_3 &\lesssim \|\omega\|_{L^2}^{\frac16}g^4(M)M^{\frac23}\ln(M)\|\mathcal{L}^2 b\|_{L^2}^2+\|\omega\|_{L^2}^{\frac16}M^{-\frac{1}{12}}\|\mathcal{L}\Delta b\|_{L^2}^2 \qquad (5.42)\\
&\quad +\|\omega\|_{L^2}g^{\frac{18}{7}}(M)M^{\frac37}(\ln(M))^{\frac{9}{14}}\|\mathcal{L}^2 b\|_{L^2}^{\frac97}+\|\omega\|_{L^2}M^{-\frac12}\|\mathcal{L}\Delta b\|_{L^2}^{\frac97}.
\end{aligned}
$$

Inserting the estimates (5.40), (5.41), (5.42) into (5.39), we have shown that there exists $c_0>0$ such that

$$
\begin{aligned}
&\partial_t E(t)+2\|\mathcal{L}\Delta b\|_{L^2}^2\\
&\lesssim c_0[g(M)\sqrt{\ln(M)}E(t)+M^{-1}\|\mathcal{L}\Delta b\|_{L^2}E(t)\\
&+E^{\frac{1}{12}}(t)g^4(M)M^{\frac23}\ln(M)\|\mathcal{L}^2 b\|_{L^2}^2+E^{\frac{1}{12}}(t)M^{-\frac{1}{12}}\|\mathcal{L}\Delta b\|_{L^2}^2\\
&+E^{\frac14}(t)g^{\frac{24}{7}}(M)M^{\frac47}(\ln(M))^{\frac67}\|\mathcal{L}^2 b\|_{L^2}^{\frac{12}{7}}+E^{\frac14}(t)M^{-\frac14}\|\mathcal{L}\Delta b\|_{L^2}^{\frac{12}{7}}\\
&+\sqrt{E(t)}g^{\frac{18}{7}}(M)M^{\frac37}(\ln(M))^{\frac{9}{14}}\|\mathcal{L}^2 b\|_{L^2}^{\frac97}+\sqrt{E(t)})M^{-\frac12}\|\mathcal{L}\Delta b\|_{L^2}^{\frac97}].
\end{aligned}
$$

Taking $M=cE(t)$ for sufficiently large c so that after absorbing, we have

$$
\begin{aligned}
&\partial_t E(t) + \|\mathcal{L}\Delta b\|_{L^2}^2 &(5.43)\\
&\lesssim g(cE(t))\sqrt{\ln(cE(t))}E(t) + E^{\frac{1}{12}}(t)g^4(cE(t))(cE(t))^{\frac{2}{3}}\ln(cE(t))\|\mathcal{L}^2 b\|_{L^2}^2\\
&+E^{\frac{1}{4}}(t)g^{\frac{24}{7}}(cE(t))(cE(t))^{\frac{4}{7}}(\ln(cE(t)))^{\frac{6}{7}}\|\mathcal{L}^2 b\|_{L^2}^{\frac{12}{7}}\\
&+\sqrt{E(t)}g^{\frac{18}{7}}(cE(t))(cE(t))^{\frac{3}{7}}(\ln(cE(t)))^{\frac{9}{14}}\|\mathcal{L}^2 b\|_{L^2}^{\frac{9}{7}} + 1\\
&\lesssim g^4(cE(t))\ln(cE(t))(1+\|\mathcal{L}^2 b\|_{L^2}^2)E(t).
\end{aligned}
$$

The bound on $\sup_{t\in[0,T]}(\|\omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2)$ then follows from (5.5) and Proposition 5.4.1. Taking L^1-norm in time on (5.43) completes the proof of Proposition 5.4.2. ■

5.5 Higher Regularity of Theorem 5.2

While obtaining the higher regularity estimate for Theorem 5.1, we haven't used the dissipation. We only used the uniform bound of $\sup_{t\in[0,T]}(\|\omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2)$. Thus, in the similar way, we can obtain the higher regularity estimates for Theorem 5.2.

5.6 Proofs of Theorem 5.1 and Theorem 5.2

The local existence and niqueness of the solution pair (v, b) to (5.1) if $(v_0, b_0) \in H^4(\mathbb{R}^2)$ may be shown following the work in [50] on the Navier-Stoke's equation. Due to the high regularity of the solution pair (v, b), the uniqueness of the solution is obvious. We will focus on establishing the existence of solutions.

Let us first define the mollification $\mathcal{J}_\epsilon f$ of $f \in L^p(\mathbb{R}^2)$, $1 \le p \le \infty$ by

$$
(\mathcal{J}_\epsilon f)(x) = \epsilon^{-2}\int_{\mathbb{R}^2}\rho\left(\frac{x-y}{\epsilon}\right)f(y)dy, \quad \epsilon > 0,
$$

where $\rho(x)$ is a radial function satisfying

$$
\rho(|x|) \in C_0^\infty, \quad \rho \ge 0, \quad \int_{\mathbb{R}^2}\rho dx = 1.
$$

It is standard technique to establish the global existence of the unique solution by using the properties of mollifiers and Picard theorem to the following regularized system of (5.1):

$$
\begin{cases}
\partial_t v^\epsilon + \mathcal{J}_\epsilon((\mathcal{J}_\epsilon u^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon v^\epsilon)) + \sum_{k=1}^{2}\mathcal{J}_\epsilon((\mathcal{J}_\epsilon v_k^\epsilon \nabla(\mathcal{J}_\epsilon u_k^\epsilon)) + \nabla\pi^\epsilon + \mathcal{J}_\epsilon^2\mathcal{L}^2 v^\epsilon = \mathcal{J}_\epsilon((\mathcal{J}_\epsilon b^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon b^\epsilon)),\\
\partial_t b^\epsilon + \mathcal{J}_\epsilon((\mathcal{J}_\epsilon u^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon b^\epsilon)) + \mathcal{J}_\epsilon^2\mathcal{L}^2 b^\epsilon = +\mathcal{J}_\epsilon((\mathcal{J}_\epsilon b^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon u^\epsilon)),\\
\nabla\cdot u^\epsilon = \nabla\cdot b^\epsilon = 0, \quad v^\epsilon = (1-\Delta)u^\epsilon.
\end{cases}
$$

We further write

$$
y^\epsilon := \begin{pmatrix} v^\epsilon \\ b^\epsilon \end{pmatrix}, \quad \|y^\epsilon\|_{H^4}^2 := \|v^\epsilon\|_{H^4}^2 + \|b^\epsilon\|_{H^4}^2, \quad \theta^\epsilon := \begin{pmatrix} u^\epsilon \\ \nabla u^\epsilon \\ b^\epsilon \end{pmatrix}
$$

and

$$
\begin{cases}
\partial_t y^\epsilon = F_\epsilon(y^\epsilon),\\
y^\epsilon(x,0) = y_0^\epsilon(x) = (v_0, b_0)(x)
\end{cases}
$$

where

$$
F_\epsilon(y^\epsilon) = \begin{cases}
-P\mathcal{J}_\epsilon[(\mathcal{J}_\epsilon u^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon v^\epsilon)] + \sum_{k=1}^{2} P\mathcal{J}_\epsilon((\mathcal{J}_\epsilon v_k^\epsilon\nabla(\mathcal{J}_\epsilon u_k^\epsilon)) + P\mathcal{J}_\epsilon[(\mathcal{J}_\epsilon b^\epsilon)\cdot\nabla\mathcal{J}_\epsilon b^\epsilon] - \mathcal{J}_\epsilon^2\mathcal{L}^2 v^\epsilon,\\
-\mathcal{J}_\epsilon[(\mathcal{J}_\epsilon u^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon b^\epsilon)] + \mathcal{J}_\epsilon[(\mathcal{J}_\epsilon b^\epsilon)\cdot\nabla(\mathcal{J}_\epsilon u^\epsilon)] - \mathcal{J}_\epsilon^2\mathcal{L}^2 b^\epsilon
\end{cases}
$$

in which P is the Leray-projection onto the divergence-free vector fields. By the a priority estimates we have already obtained, one can show that $\{\theta^\epsilon\}$ is cauchy in $C([0,T];L^2)$. Finally, by using the Alaoglu's theorem, we obtain the existence of the solution to the system (5.1). This completes the proof of Theorem 5.1 and Theorem 5.2.

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