A STUDY ON THE GLOBAL REGULARITY FOR TWO-DIMENSIONAL MAGNETOHYDRODYNAMIC AND BOUSSINESQ EQUATIONS

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Whether the classical solutions of two-dimensional incompressible ideal MHD equations or inviscid Boussinesq equations can develop a finite time singularity or globally regular for all time from smooth initial data with finite energy is an outstanding open problem in fluid dynamics. We study these equations to explore how far one can go beyond these two cases and still can prove the global regularity.
First, the global regularity for the 2D MHD equations with horizontal dissipation and horizontal diffusion is studied. We prove that the horizontal components of any solution admit a global bound in any Lebesgue space L^{2r}, $1 \leq r < \infty$ and the bound grows no faster than the order of $\sqrt{r \log r}$ as r increases. Furthermore, we prove that any possible blow-up can be controlled by the L^{∞}-norm of the horizontal components. We establish the global regularity of slightly regularized 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion. The global regularity issue of the MHD equation with horizontal dissipation and horizontal magnetic diffusion is extremely hard. The classical energy method does not work. By using the techniques from the Littlewood- Paley decomposition and logarithmic bound for the horizontal components, we are able to resolve the global regularity issue of the 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion.
Second, the global well-posedness for the 2D Euler-Bousinesq equations with a singular velocity is investigated. We prove the global existence and uniqueness of the solutions to the initial value problem of 2D Euler-Boussinesq equations when the velocity field is double logarithmically more singular than the standard velocity field given by the Biot-Savart law.
Third, the global existence, and uniqueness for the 2D Navier-Stokes-Boussinesq equation with more general dissipation is studied. We prove that the solution is globally regular even the critical dissipation is logarithmically weaker.

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CHAPTER 1

Introduction

This dissertation focuses on the global regularity issue of two well-known nonlinear partial differential equations. The first equation is the two-dimensional Magnetohydrodynamic equations and the second equation is the two-dimensional Boussinesq equations.

1.1 Magnetohydrodynamic Equations

1.1.1 Introduction

Magnetohydrodynamics deals with the dynamics of an electrically conducting fluid under the influence of magnetic field. The word magnetohydrodynamics is made up of three terms *magneto* indicating magnetic field, *hydro* referring liquid, and *dynamics* meaning movement. The magnetic field, which is present everywhere in the universe, generates magnetic force and this force influences the dynamics of moving fluid, potentially changing the geometry or strength of magnetic field itself. The study of magnetohydrodynamics was first initiated by Swedish scientist Hannes Alfvén [4]. Because of his pioneer contribution in magnetohydrodynamics, he received Nobel Prize in Physics in 1970. The Magnetohydrodynamic (MHD) equations govern the dynamics of the velocity and the magnetic field in electrically-conducting fluids and reflect the basic physics laws of conservation. These equations can be implemented to study various problems in plasma, liquid metals, saltwater as well as astrophysics. The MHD equations involve coupling between the Navier-Stokes equations governing the fluid and the Maxwell's equations governing the magnetic field. The standard form of the incompressible MHD equations is

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + b \cdot \nabla b, \qquad x \in \mathbb{R}^n, t > 0 \\ b_t + u \cdot \nabla b = \eta \Delta b + b \cdot \nabla u, \qquad x \in \mathbb{R}^n, t > 0 \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x,0) = u_0(x), b(x,0) = b_0(x) \end{cases} \tag{1.1}$$

where u is the velocity field, b the magnetic field , p the pressure, ν the kinematic viscosity and η the magnetic diffusivity. The term Δu represents dissipation and Δb magnetic diffusion. The MHD equations have been a center of attention to numerous analytical, experimental, and numerical investigations. In [32], Duraut and Lions constructed a class of global weak solution with finite energy and a class of local strong solutions for the system (1.1) in the two dimensions (2D) and three dimensions (3D). The smoothness and uniqueness of such weak solution was demonstrated only in the two dimensional case but this is an outstanding open problem for three dimensions. One of the most fundamental problems in fluid dynamics concerning the 3D MHD equations is whether their classical solutions are globally regular for all time or they develop finite time singularity. This problem is extremely difficult due to the occurrence of nonlinear coupling between the Navier-Stokes equations with a forcing induced by the magnetic field and the induction equation. When $b = 0$, the system (1.1) reduces to the Navier-Stokes equations. Whether the 3D incompressible Navier-Stokes equations can develop a finite time singularity or globally regular for all time from smooth initial data with finite energy is one of the six one millennium dollar prize problems announced by the Clay Mathematical Institute [35]. There are numerous papers related to global regularity criteria published by Mathematicians, Physicists and Engineers ([4, 8, 10, 12, 18, 22, 24, 32, 33, 34, 37, 38, 39, 47, 54, 56, 65, 72, 73, 74, 75, 76, 67, 69, 82, 83, 84, 85, 86]).

Now we will discuss some of the existing results and open problems related to the MHD equations.

1.1.2 Existing Results

The existence and uniqueness of global classical solution for two-dimensional MHD equations with initial data in H^s, $s \geq 2$ is well known ([32], and [67]). However, only the local existence and uniqueness of solution for the the three-dimensional MHD equations (1.1) with initial data in H^s, $s \geq 3$ is known. It is currently unknown that whether solution to the three-dimensional MHD is global in time or it generates finite time singularity. In order to gather some hints of this open problem, there are numerous papers related to regularity criteria under which solution is regular for all time. It would be worthy to mention some known results related to the three-dimensional MHD equations. In [72], Wu proved that if the velocity and magnetic field satisfy $\int_0^T (\|\nabla u(.,t)\|_2^4 + \|\nabla b(.,t)\|_2^4) < \infty$ or $\int_0^T (\|u(.,t)\|_\infty^2 + \|b(.,t)\|_\infty^2) < \infty$ then u and b remain smooth over $[0,T]$. He and Xin realized that the velocity field plays predominant role over the magnetic field for the global regularity [38],[39]. They proved that if u satisfies $\int_0^T \|\nabla u(.,t)\|_\alpha^\beta < \infty$ with $\frac{3}{\alpha} + \frac{2}{\beta} = 2$ and $1 < \beta \leq 2$ then the solution (u, b) is regular on $[0,T]$. Later this assumption was weakened with L^α-norm replaced by norms in Besov Space by Wu ([74]) and was further improved by Chen et al ([24]). In [85], Zhou proved that for initial data in $H^s, s \geq 3$ if the velocity field satisfies $\nabla u \in L^{\alpha,\gamma}$, with $\frac{2}{\alpha} + \frac{3}{\gamma} \leq 2$, $\frac{3}{2} < \gamma \leq \infty$, or $\|\nabla u\|_{L^\infty, \frac{3}{2}}$ is sufficiently small on [0,T], or $u \in L^{\alpha,\gamma}$ with $\frac{2}{\alpha} + \frac{3}{\gamma} \leq 1$, $3 < \gamma \leq \infty$, or $\|u\|_{L^{\infty},3}$ is sufficiently small on [0,T], then the solution remains smooth on $[0,T]$.

In [47], the authors proved the regularity criteria for the 3D MHD equations in terms of partial derivatives of the velocity and the pressure. They proved that if $u_z \in L^\beta(0,T; L^\alpha(\mathbb{R}^3))$, with $\frac{2}{\beta} + \frac{3}{\alpha} \leq \frac{3(\alpha+2)}{4\alpha}$, $\alpha > 2$ or $\nabla_h p \in L^\beta(0,T; L^\alpha(\mathbb{R}^3))$, with $\frac{2}{\beta} + \frac{3}{\alpha} < 3$, $\alpha > \frac{9}{7}$, $\beta \geq 1$ then the weak solution (u, b) is regular. In [14], Cao and Wu investigated regularity criteria in terms of the derivative of the velocity field in one direction. More precisely, they proved that for initial data $(u_0, b_0) \in H^3$, $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. If the velocity field u of the corresponding solution of (1.1) satisfies

$\int_0^T \|u_z(.,t)\|_\alpha^\beta \, dt < \infty$ with $\alpha \geq 3$ and $\frac{3}{\alpha} + \frac{2}{\beta} \leq 1$, then (u, b) remains regular in $[0, T]$. In the same paper, the authors excluded the possibility of finite time blow-up if there is a suitable bound for the derivative of the pressure in a single direction. Their result is for initial data $(u_0, b_0) \in H^1 \cap L^4$, $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$, if the pressure p satisfies $\int_0^T \|p_z(\tau)\|_\alpha^\beta d\tau < \infty$ with $\alpha \geq \frac{12}{7}$ and $\frac{3}{\alpha} + \frac{2}{\beta} \leq \frac{7}{4}$ for some $T > 0$, then (u, b) remains regular on $[0, T]$. There are numerous papers related to the regularity criteria for 3D MHD equations.

When there is no dissipation and magnetic diffusion, the mathematical analysis for the MHD equations becomes more complicated. The equations for ideal MHD equations can be obtained without the dissipation and diffusion, namely

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0. \end{cases} \tag{1.2}$$

Any classical solution of the two-dimensional incompressible Euler equation is global in time [49]. However, it remains a remarkable open problem whether classical solutions of the two-dimensional ideal MHD equation preserve their regularity for all time or finite time blowup. Many attempts have been made but there is no any satisfactory results concerning the regularity of the solution. In [8], Calflish, Klapper and Steele extended the well-known result of Beal, Kato, and Majda [6] to the 3D ideal MHD equations. More precisely, they showed that if the smooth solution (u, b) satisfies $\int_0^T \|\omega\|_{L^\infty} + \|j\|_{L^\infty} dt < \infty$ then the solution can be extended beyond $t = T$. Later this assumption was weakened by norms in Besov space [83].

Since the difficulty arises in two-dimensional ideal MHD equations, many mathematicians and physicists are attracted towards the anisotropic MHD equations. In the case of anisotropic equations, the dissipation and the magnetic diffusion act differently in each direction. In nature, it is possible that the dissipation and the magnetic diffusion coefficient have different roles in different directions. The anisotropic 2D incompressible MHD equations can be written as

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu_1 \, u_{xx} + \nu_2 \, u_{yy} + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = \eta_1 \, b_{xx} + \eta_2 \, b_{yy} + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{1.3}$$

where $(x, y) \in \mathbb{R}^2$, $t \geq 0$, $u = (u_1(x, y, t),\ u_2(x, y, t))$ denotes the 2D velocity field, $p = p(x, y, t)$ the pressure, $b = (b_1(x, y, t), b_2(x, y, t))$ the magnetic field, and ν_1, ν_2, η_1, and η_2 are nonnegative real parameters. When $\nu_1 = \nu_2$ and $\eta_1 = \eta_2$, (1.3) reduces to the standard incompressible MHD equations (1.1).

In [12], Cao and Wu studied the 2D MHD equations with mixed dissipation and diffusion. They established the global regularity issue for the case $\nu_1 > 0$, $\nu_2 = 0$, $\eta_1 = 0$ and $\eta_2 > 0$ or $\nu_1 = 0$, $\nu_2 > 0$, $\eta_1 > 0$, and $\eta_2 = 0$.

When the Reynolds number is very high, which occurs in the turbulent flow, the viscosity of the fluid is very low. So the viscosity parameter is ignored. This leads to the MHD equations with no dissipation but only the magnetic diffusion. The global regularity issue in this case remains open. In [12], the authors established the global existence of H^1 weak solution for (1.3) when $\nu_1 = \nu_2 = 0$ and $\eta_1 = \eta_2 = \eta > 0$ but the uniqueness of such weak solutions is open. They have also shown if the ∇u satisfies $\sup_{p \geq 2} \frac{1}{\sqrt{p}} \int_0^T \|\nabla u(t)\|_{L^p} dt < \infty$, then the solution actually becomes regular.

In the case of extremely high conductive fluids that occurs frequently in the cosmical and geophysical problems, the role of the diffusion phenomenon is insignificant. So the magnetic diffusion is ignored ([22]), yielding the following equations.

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu_1 \, u_{xx} + \nu_2 \, u_{yy} + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0. \end{cases} \tag{1.4}$$

The local existence and uniqueness of the classical solution of the system (1.4) has been proved [65]. In [65], Quansen and Dongjuan proved the regularity condition for two-dimensional MHD equations (1.4) in terms of magnetic field, which is $b \in L^p(0,T;W^{2,p}(\mathbb{R}^2))$, with $\frac{2}{p}+\frac{1}{q} \leq 2$, $1 \leq p \leq \frac{3}{4}$, $2 < q \leq \infty$. Fan and Ozawa proved a regularity criterion in the velocity field as $\nabla u \in L^1(0,T;L^\infty(\mathbb{R}^2))$ [34]. In [84], the authors proved that the solution (u,b) of (1.4) is regular for initial data $(u_0,b_0) \in H^3(\mathbb{R}^2)$ if $\nabla b \in L^1(0,T;BMO(\mathbb{R}^2))$. Lei, Masmoudi and Zhou showed that the classical solution for the system (1.4) is globally regular as long as $b \otimes b \in L^1(0,T;BMO(\mathbb{R}^2))$ [55]. However, whether the solution is globally regular or not under the condition $\int_0^T \|b(.,t)\|^2_{BMO} dt < \infty$ is still open.

The generalized MHD equation can be acquired by replacing the Laplacian in the general MHD equations with a fractional power of $(-\Delta)^\alpha$. This type of dissipation corresponding to a fractional power of Δ arises from modeling real physical phenomena. The generalized MHD equation is

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu \Lambda^{2\alpha} u + b \cdot \nabla b, & x \in \mathbb{R}^n,\ t > 0 \\ b_t + u \cdot \nabla b = \eta \Lambda^{2\beta} b + b \cdot \nabla u, & x \in \mathbb{R}^n,\ t > 0 \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{1.5}$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$ and can be defined by the Fourier transform

$$\widehat{\Lambda^{2\alpha} f}(\xi) = |\xi|^{2\alpha} \hat{f}(\xi)$$

According to the pioneer work of Wu [73] on generalized MHD equation, the best result for n-dimensional MHD equations is, the system (1.5) does not develop finite time singularity as long as

$$\alpha \geq \frac{1}{2} + \frac{n}{4},\ \beta > 0,\ \alpha + \beta \geq 1 + \frac{n}{2}.$$

There are many regularity criteria in the same paper by Wu [73]. In ([74]), the author considered the n-dimensional GMHD equations and proved regularity criterion in Besov space. Very recently, Tran, Yu and Zhai studied the global regularity issue of 2D generalized magnetohydrodynamic equations [69]. They have demonstrated that the generalized 2D MHD equations with initial data $(u_0,b_0) \in H^k$, $k > 2$ is globally regular if α and β satisfy condition $\alpha \geq \frac{1}{2}, \beta \geq 1$ or $0 \leq \alpha < \frac{1}{2}, 2\alpha + \beta > 2$ or $\alpha \geq 2, \beta = 0$. In [82], Yamazaki and in [54] Lin & Du reduced the regularity criteria for the 3D MHD equations which depend only upon two diagonal entries of ∇u.

1.1.3 Statement of the Problems

Despite the numerous work in the MHD equations, the global regularity issue for the two-dimensional MHD equations with horizontal dissipation and horizontal magnetic diffusion or vertical dissipation and vertical magnetic diffusion still needs to be answered. Thus, inspired by the work of Cao and Wu in [12], we study the global regularity issue for 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion for $\mathbb{R}^2$. Our intention to study this type of system is to discover how far one can explore beyond 2D ideal MHD equations.

The first problem of this dissertation is devoted to the global regularity issue in the case when $\nu_1 > 0$, $\nu_2 = 0$, $\eta_1 > 0$ and $\eta_2 = 0$ for MHD equations, namely the two-dimensional MHD equations with horizontal dissipation and horizontal magnetic diffusion

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + u_{xx} + b \cdot \nabla b, \qquad x \in \mathbb{R}^2,\, t > 0 \\ b_t + u \cdot \nabla b = b_{xx} + b \cdot \nabla u, \qquad x \in \mathbb{R}^2,\, t > 0 \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x,0) = u_0(x),\, b(x,0) = b_0(x). \end{cases} \tag{1.6}$$

For simplicity, the positive parameters are assumed to be equal to 1.

Now, a very general question arises: why the possible finite time singularity is ruled out for the two-dimensional MHD equations with mixed dissipation and diffusion ([12]) while the global regularity issue for the 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion (1.6) remains very difficult. We would like to mention the difficulty we encounter when the energy method is applied to answer the question. For any given sufficiently smooth data

$$u(x,0) = u_0(x), \quad b(x,0) = b_0(x),$$

say, $(u_0, b_0) \in H^2(\mathbb{R}^2)$, the corresponding solution obviously obeys global L^2-bound. That is,

$$\begin{aligned} \|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\int_0^t \|u_x(\tau)\|_{L^2}^2 d\tau + 2\int_0^t \|b_x(\tau)\|_{L^2}^2 d\tau \\ = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \end{aligned} \tag{1.7}$$

But the trouble arises when we try to achieve the global H^1-bound. The vorticity $\omega = \nabla \times u$ and the current density $j = \nabla \times b$ satisfy

$$\begin{cases} \omega_t + u \cdot \nabla \omega = \omega_{xx} + b \cdot \nabla j, \\ j_t + u \cdot \nabla j = j_{xx} + b \cdot \nabla \omega + 2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1), \end{cases} \tag{1.8}$$

we then obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\right) + \|\omega_x\|_{L^2}^2 + \|j_x\|_{L^2}^2 \\ = 2\int j\left(\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1)\right) dxdy. \end{aligned} \tag{1.9}$$

In order to obtain suitable bounds for the terms on the right, we need an anisotropic Sobolev inequality stated in the following lemma ([12]).

Lemma 1.1 *If $f, g, h, g_y, h_x \in L^2(\mathbb{R}^2)$, then*

$$\iint_{\mathbb{R}^2} |f\, g\, h|\, dxdy \le C\, \|f\|_2\, \|g\|_2^{\frac{1}{2}} \|g_y\|_2^{\frac{1}{2}}\, \|h\|_2^{\frac{1}{2}} \|h_x\|_2^{\frac{1}{2}}. \tag{1.10}$$

where C is a constant.

If we apply (1.10), two terms on the right of (1.9), $\int j\, \partial_x b_1\, \partial_x u_2$ and $\int j\, \partial_x u_1\, \partial_x b_2$ can be bounded suitably. But if we try to bound remaining two terms, we need to bound either $\int_0^t \|\partial_y b\|_2^2$ or $\int_0^t \|\partial_y u\|_2^2$. Because of the lack of dissipation and diffusion in the vertical direction, we do not know how to bound these terms. As a consequence, we cannot apply the Gronwall's Lemma. This is the place where the direct energy method breaks down and the problem becomes complicated.

Motivated by a recent work of Cao and Wu [13], in this dissertation we explore how the Lebesgue-norm of the horizontal component (u_1, b_1) of a solution would affect the global regularity. We are able to obtain a global *a priori* bound for the norm $\|(u_1, b_1)\|_{2r}$ with $1 \le r < \infty$, where $\|f\|_q$ with $1 \le q \le \infty$ denotes the norm of a function f in the Lebesgue space L^q. The bound depends exponentially on r and we are not sure whether or not $\|(u_1, b_1)\|_\infty$ can be bounded for all time. If we do know that $\int_0^T \|(u_1, b_1)\|_\infty^2 \, dt < \infty$ then we can actually show that the solution is regular on $[0, T]$. Thus we substantiate that the L^∞-norm of the horizontal components controls any possible blowup of classical solution in finite time, which is a conditional global regularity for (3.1). We also prove that the pressure associated with any classical solution obeys the global bound, for any $T > 0$ and $t < T$,

$$\|p(\cdot, t)\|_q \le C(T), \qquad \int_0^T \|p(\cdot, t)\|_{H^s}^2 \, dt < C(T),$$

where $1 < q \le 3$ and $0 < s < 1$. These global bounds together with a decomposition of the pressure into low and high frequency parts , we significantly improve the global bound for $\|(u_1, b_1)\|_{2r}$. We are able to display that $\|(u_1, b_1)\|_{2r}$ does not grow faster than $\sqrt{r \log r}$ for large $r < \infty$.

In addition, we study a slightly regularized version of (1.6), namely

$$\begin{cases} u_t + u \cdot \nabla u + \epsilon(-\Delta)^\delta u = -\nabla p + u_{xx} + b \cdot \nabla b, & x \in \mathbb{R}^2, \ t > 0 \\ b_t + u \cdot \nabla b + \epsilon(-\Delta)^\delta b = b_{xx} + b \cdot \nabla u, & x \in \mathbb{R}^2, \ t > 0 \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0 \end{cases} \tag{1.11}$$

with $\epsilon > 0$ and $\delta > 0$.

We prove that the global regularity of classical solution for the system (1.11)) with the initial data $(u_0, b_0) \in H^2$. These results associated with two-dimensional MHD equations will be available in chapter 3. In chapter 4, we are able to show the global regularity of the 2D MHD equations with the horizontal dissipation and horizontal magnetic diffusion. The global regularity in this case is extremely difficult. We decompose the horizontal components into lower and higher frequency parts by using Littlewood-Paley decomposition. The low frequency part can be bounded by the logarithmic bound for the horizontal components and the high frequency part can be bounded by the horizontal dissipation and horizontal diffusion of the equations (1.6).

1.2 Boussinesq Equations

1.2.1 Introduction

The Boussinesq equations is a system of non-linear differential equations. These equations have been used to study atmospheric and oceanic flow. Mathematically the two-dimensional Boussinesq equations act as a lower dimensional model of the three-dimensional hydrodynamic equations. The Boussinesq equations retain some key features of the three-dimensional Navier-Stokes and the Euler equations such as the vortex stretching term. The inviscid Boussinesq equations can be identified with the three-dimensional Euler equations for axi-symmetric flow away from the z-axis. One of the outstanding open problems in the fluid dynamics is whether classical solutions to the three dimensional Navier-Stokes equations can develop finite time singularities or global regularity. The study of the two-dimensional Boussinesq equations may support some indications to the regularity of the three-dimensional Navier-Stokes equations. Thus these equations have fascinated considerable attention of Mathematicians, Physicists as well as Engineers. The standard two-dimensional

Boussinesq equations can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + \theta e_2 \qquad x \in \mathbb{R}^2,\, t > 0 \\ \partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta \qquad x \in \mathbb{R}^2,\, t > 0 \\ \nabla \cdot u = 0, \end{cases} \tag{1.12}$$

where u denotes velocity field, θ the temperature in the content of thermal convection and the density in the modeling of geophysical fluids, ν the viscosity, κ the thermal diffusivity, and $\mathbf{e}_2$ is the unit vector in the vertical direction.

Now we will be discussing some known pre-existing results and open problems related to the two-dimensional Boussinesq equations.

1.2.2 Existing Results

Cannon and DiBenedett studied the two-dimensional Boussinesq equations (1.12) with full dissipation and thermal diffusion, and achieved the global well-posedness [9]. The local existence and uniqueness of smooth solution of Boussinesq equations and blow-up criterion for the inviscid case, $\nu = 0, \eta = 0$, was proved by Chae and Nam [20]. Moreover, the solution remains smooth if $\int_0^T \|\nabla \theta(.,t)\|_\infty \, dt < \infty$ or $\int_0^t \|\nabla u\|_\infty \, dt < \infty$. Later in [17], Chae proved the global well-posedness for the partial viscosity or partial thermal diffusion, namely $\nu > 0, \kappa = 0$ or $\nu = 0, \kappa > 0$, with initial data $(u_0, \theta_0) \in H^m \times H^m$ with $m > 2$. Hou and Lie [46] provided the similar results for the case $\nu > 0, \kappa = 0$. The results related to the Boussinesq system with zero viscosity was extended in [42] by Hmidi and Keraani to initial data $u_0 \in B_{p,1}^{\frac{2}{p}+1}$ and $\theta_0 \in B_{p,1}^{\frac{2}{p}-1} \cap L^r$ with $r \in (2, \infty)$. In [45], Hmidi and Zerguine considered the Euler-Boussinesq system with a fractional dissipation. Further progress was made by Hmidi, Keraani and Rousset [44] on the Euler-Boussinseq system with critical dissipation or critical diffusion and obtained the global well-posedness. Hmidi considered the Euler-Boussinesq system in which the critical dissipation is reduced by a logarithm of the

Laplacian [40]. The system of such equations is given by

$$
\begin{cases}
\partial_t u + u \cdot \nabla u = -\nabla p + \theta \mathbf{e}_2, \\
\partial_t \theta + u \cdot \nabla \theta + \kappa \mathcal{L} \theta = 0, \\
\nabla \cdot u = 0, \\
u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x),
\end{cases}
\tag{1.13}
$$

where $\mathcal{L} = \dfrac{|D|}{\log^{\alpha}(e^4 + |D|)}$ and $|D| = \sqrt{-\Delta}$, $\alpha \in [0, \frac{1}{2}]$. The global well-posedness has been successfully resolved.

Instead of reducing the dissipation by a logarithmic (1.13), Chae and Wu in [21] considered the generalized Navier-Stokes-Boussinesq equations with a more singular velocity field that was determined by the vorticity through the Biot-Savart law [58]. More precisely they considered the following initial value problem.

$$
\begin{cases}
\partial_t \omega + u \cdot \nabla \omega + \Lambda \omega = \partial_{x_1} \theta, \\
\partial_t \theta + u \cdot \nabla \theta = 0, \\
u = \nabla^{\perp} \psi, \quad \Delta \psi = \Lambda^{\sigma} \log^{\gamma}(I - \Delta) \omega, \\
\omega(x,0) = \omega_0(x), \ \theta(x,0) = \theta_0(x),
\end{cases}
\tag{1.14}
$$

where $\omega = \omega(x,t)$, $\psi = \psi(x,t)$ and $\theta = \theta(x,t)$ are scalar functions of $x = (x_1, x_2) \in \mathbb{R}^2$ and $t \geq 0$, $u = u(x,t) : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field, $\sigma \geq 0$ and $\gamma \geq 0$ are real parameters. The global well-posedness for (1.14) has been resolved [21].

In [31], Danchin and Paicu constructed global strong solution for the Boussinesq system with only horizontal dissipation or horizontal thermal diffusion. Adhikari, Cao and Wu studied the Boussinesq equations with vertical dissipation and vertical diffusion [2, 3], and Cao & Wu proved the global well-posedness [13]. There are many important results on the global well-posedness issue for the two-dimensional Boussinesq equations for partial dissipation ([2, 3, 9, 13, 17, 21, 26, 29, 30, 31, 33, 40, 41, 42, 43, 44, 46, 53, 60, 61, 62]).

1.2.3 Statement of the Problems

The inviscid Boussineq equations is reduced to the widely known two-dimensional Euler equations when $\theta = 0$. The global well-posedness is well-known for the two-dimensional Euler equations, however the global well-posedness for inviscid Boussinesq equations is an outstanding open problem. Many mathematicians are investigating Euler-Boussinesq equations and Navier-Stokes Boussinesq equations to explore how far one can go beyond the critical dissipation or diffusion and still be able to prove the global regularity. Motivated by the recent work of Chae and Wu [21] and Chae, Constantin and Wu [16] on the generalized 2D Euler equations, the second project of this dissertation is to investigate the existence and uniqueness of the solution on the initial-value problem for the generalized Euler-Boussinesq system of equations with a singular velocity.

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \partial_{x_1} \theta, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \Lambda^\sigma P(\Lambda) \omega, \\ \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0, \\ \omega(x,0) = \omega_0(x),\ \theta(x,0) = \theta_0(x), \end{cases} \tag{1.15}$$

where $u = u(x,t)$ is 2D vector field depending on $x = (x_1, x_2) \in \mathbb{R}^2$ and $t \geq 0$, $p = p(x,t)$, $\theta = \theta(x,t)$, and ω are scalar functions, $\mathbf{e}_2$ is the unit vector in the x_2-direction and $\sigma \geq 0$ is a real parameter. The Zygmund operator $\Lambda = (-\Delta)^{1/2}$, Λ^σ and the Fourier multiplier operator $P(\Lambda)$ are defined through the Fourier transform, namely

$$\widehat{\Lambda^\sigma f}(\xi) = |\xi|^\sigma \widehat{f}(\xi) \quad \text{and} \quad \widehat{P(\Lambda) f}(\xi) = P(|\xi|)\, \widehat{f}(\xi).$$

The system (1.15) involves a velocity field that is more singular than the standard velocity which is determined by the vorticity through the Biot-Savart law. The velocity field in the standard case is given by $\nabla u = \nabla \nabla^\perp \Delta^{-1} \omega$ and $\|\nabla u\|_{L^p} \leq C \|\omega\|_{L^p}, p \in$

$(1,\infty)$. Thus $\omega \in L^p$ implies $\nabla u \in L^p$. However, this is not sufficient for (1.15). In fact, $\Lambda^\sigma P(\Lambda)\omega \in L^p$ implies $\nabla u \in L^p$. When $P(\lambda) = I$ and $\sigma = 0$, the system (1.15) is reduced into standard Boussinesq equations. In order to deal with more general dissipation, we assume that P satisfy the following conditions.

Condition 1.1 *The symbol $P(|\xi|)$ is assumed to have the following properties:*

1. *P is continuous on $\mathbb{R}^2$ and $P \in C^\infty(\mathbb{R}^2 \setminus \{0\})$;*
2. *P is radially symmetric;*
3. *$P = P(|\xi|)$ is nondecreasing in $|\xi|$;*
4. *There exist two constants C and C_0 such that*
$$\sup_{2^{-1}\le|\eta|\le 2} \left|(I-\Delta_\eta)^n\, P(2^j|\eta|)\right| \le C\,P(C_0\,2^j)$$
for any integer j and $n = 1, 2$.

The property (4) in Condition 1.1 is a very natural condition on symbols of Fourier multiplier operators and is similar to the main condition in the Mihlin-Hörmander Multiplier Theorem [68]. For notational convenience, we also assume that $P \ge 0$. Some special examples of P are

$$\begin{aligned}
&P(\xi) = \big(\log(1+|\xi|^2)\big)^\gamma \quad \text{with } \gamma \ge 0,\\
&P(\xi) = \big(\log(1+\log(1+|\xi|^2))\big)^\gamma \quad \text{with } \gamma \ge 0,\\
&P(\xi) = |\xi|^\beta \quad \text{with } \beta \ge 0,\\
&P(\xi) = (\log(1+|\xi|^2))^\gamma\,|\xi|^\beta \quad \text{with } \gamma \ge 0 \text{ and } \beta \ge 0.
\end{aligned}$$

We prove in [50] that if the dissipative operator $P(\xi)$ obeys the Condition 1.1 with

$$P(2^k) \le C\,\sqrt{k} \quad \text{for a constant } C \text{ and any large integer } k > 0,$$

and

$$\int_1^\infty \frac{1}{r\log(1+r)\,P(r)}dr = \infty.$$

then the IVP (1.15) with $\sigma = 0$ is globally well-posed in the regularity class

$$\omega \in C([0,T]; B^s_{q,\infty}(\mathbb{R}^2)), \quad \theta \in C([0,T]; B^s_{q,\infty}(\mathbb{R}^2) \cap L^1([0,T]; B^{s+1}_{q,\infty}(\mathbb{R}^2)), \text{for } q > 2 \text{ and } s > 2.$$

As an application, we prove the global well-posedness of classical solutions to (1.15) with double logarithmically singular velocity. More precisely the global regularity of the following system.

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \partial_{x_1} \theta, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \log(1 + \log(1-\Delta))^\gamma \omega, \quad \gamma \in [0,1] \\ \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0, \\ \omega(x,0) = \omega_0(x), \, \theta(x,0) = \theta_0(x), \end{cases} \tag{1.16}$$

As a consequence, when $P(\Lambda) = I$, we recover the classical result of global smooth solutions for the 2D Boussinesq equations with a critical diffusion. On the other hand as $\theta = 0$, our result endorses a result of generalized Euler equations [16].
The global regularity in time can be achieved by showing local existence & uniqueness together with the global bounds. The difficult part is to find appropriate *a priori* global bounds for (ω, θ). The direct energy method cannot be applied due to the presence of the vortex stretching term $\partial_{x_1}\theta$. The vortex stretching term can be concealed by combining ω and $\mathcal{R}$ by $G = \omega + \mathcal{R}\theta$, where $\mathcal{R} \equiv \Lambda^{-1}\partial_{x_1}$. Then G satisfies

$$\partial_t G + u \cdot \ \nabla G = -[\mathcal{R}, u \cdot \nabla]\theta, \tag{1.17}$$

where the commutator $[\mathcal{R}, u \cdot \nabla]\theta = \mathcal{R}(u \cdot \nabla\theta) - u \cdot \nabla(\mathcal{R}\theta)$.

Although the vortex stretching term is hidden in the commutator, a compensation of an appropriate estimate for the commutator is needed. First we estimate commutator $[\mathcal{R}, u \cdot \nabla]\theta$ in $B^0_{p,r}$ for $p, r \in [1,\infty]$. Then the bounds for $\|\omega\|_{L^\infty_t L^q}, \|\theta\|_{L^1_t B^{0,P}_{\infty,2}}$ and $\|\omega\|_{L^\infty_t L^\infty}$ are achieved by using equations for G and $\mathcal{R}\theta$ together with the commutator estimates and the estimates of θ in $B^\epsilon_{\infty,1}$. Finally the desired bounds for ω

and θ in $B^s_{q,\infty}$, for $q > 2, s > 2$ can be obtained in two steps. In the first step we bound ω and θ in $B^\beta_{q,\infty}$ for $\frac{2}{q} < \beta < 1$ by using the regularity on G and a logarithmic bound for $\|\nabla u\|_{L^\infty}$ in terms of $\|\omega\|_{L^q \cap L^\infty}$. The second step is to bound ω and θ in $B^{\beta_1}_{q,\infty}$ for $1 < \beta_1 \leq 2 - \frac{2}{q}$ by implementing the bound obtained in the first step. After repetition of this process, we achieve the desired bound for ω and θ in $B^s_{q,\infty}$ for $s > 2$ and $q > 2$.

In order to achieve uniqueness, we estimate the difference of two solutions in the regularity class (1.16). It is easy to estimate directly because the solution is in the high regularity class. The details will be provided later in the chapter 5.

Inspired by the work of Chae and Wu ([21]), and Hmidi ([40], the third problem we study in this dissertation is the global regularity issue for the Boussinesq-Navier-Stokes equations with Logarithmic supercritical dissipation.

$$\begin{cases} \partial_t u + u \cdot \nabla u + \mathcal{L}u = -\nabla p + \theta \mathbf{e}_2, \qquad x \in \mathbb{R}^2,\, t > 0 \\ \partial_t \theta + u \cdot \nabla \theta = 0, \qquad x \in \mathbb{R}^2,\, t > 0 \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.18}$$

where u is a vector field denoting the velocity, θ is a scalar function, $\mathbf{e}_2$ is the unit vector in the x_2 direction, and $\mathcal{L}$ is a nonlocal dissipation operator defined by

$$\mathcal{L}f(x) = \text{ p.v. } \int_{\mathbb{R}^2} \frac{f(x) - f(y)}{|x-y|^2} m(|x-y|)dy \tag{1.19}$$

and $m\colon (0,\infty) \to (0,\infty)$ is a smooth, positive, and non-increasing function with the following three properties.

(i) there exists $C_1 > 0$ such that $\quad rm(r) \leq C_1 \qquad$ for all $r \leq 1$;

(ii) there exists $C_2 > 0$ such that $\quad r|m'(r)| \leq C_2 m(r) \qquad$ for all $r > 0$;

(iii) there exists $\beta > 0$ such that $r^\beta m(r)$ is non-increasing.

In [51], we prove the global well-posedness of the system (1.18). As an application, we prove the global well-posedness of the 2D Boussinesq-Navier-Stokes equations with logarithmically supercritical dissipation. More precisely, we have manifested that when the initial data is in the following functional setting;

$$u_0 \in H^1(\mathbb{R}^2), \quad \omega_0 \in L^q(\mathbb{R}^2) \cap B^0_{\infty,1}(\mathbb{R}^2)\ , \quad \theta_0 \in L^2(\mathbb{R}^2) \cap B^{0,a^2}_{\infty,1}(\mathbb{R}^2),$$

where $\omega_0 = \nabla \times u_0$ is the initial vorticity. Then (1.18) has a unique global solution (u, θ) satisfying, for all $t > 0$,

$$u \in L^\infty_t H^1, \quad \omega \in L^\infty_t L^q \cap L^1_t B^0_{\infty,1}\ , \quad \theta \in L^\infty_t L^2 \cap L^\infty_t B^{0,a^2}_{\infty,1} \cap L^1_t B^{0,a}_{\infty,1}.$$

The local well-posedness for (1.18) is well-known. The global well-possedness can be achieved by obtaining the global bound. The global bound for $\|\omega\|_{L^2}$ cannot be obtained from the vorticity equation, $\partial_t \omega + u \cdot \nabla \omega + \mathcal{L}\omega = \partial_{x_1}\theta$, because of the vortex stretching term $\partial_{x_1}\theta$ unless $\mathcal{L}$ is very dissipative. In order to tackle such difficulty, we combine two equations. The combined quantity $G = \omega - \mathcal{R}_a\theta$ with $\mathcal{R}_a = \mathcal{L}^{-1}\partial_{x_1}$ satisfies

$$\partial_t G + u \cdot \nabla G + \mathcal{L}G = [\mathcal{R}_a, u \cdot \nabla]\theta. \tag{1.20}$$

By using the appropriate bound for commutator together with the lower bound for dissipative operator, we prove the global *a priori* bound for $\|G\|_{L^q}, q \in [2, 4)$. The global bound for $\|G\|_{L^q}$ and $\|\omega\|_{L^q}, q \geq 4$ is acquired by implementing bound of $\|G\|_{\widetilde{L}^r_t B^s_{q,1}}$ and $\|G\|_{L^1_t B^{0,a}_{\infty,1}}$. Finally, we establish the global bound for $\|\omega\|_{L^1_t B^{0,a}_{\infty,1}}$ and for $\|\theta\|_{L^1_t B^{0,a^2}_{\infty,1}}$ by using the algebraic bound of $\|\theta\|_{L^1_t B^{0,a}_{\infty,1}}$ in terms of $\|\nabla u\|_{L^1_t L^\infty}$. To complement uniqueness for the system, we consider the difference of two solutions in the above regularity class and show that the difference must vanish. Then, the uniqueness is achieved by using two inequalities, the velocity difference $u^{(2)} - u^{(1)}$ in $B^0_{2,\infty}$ and the difference $\theta^{(2)} - \theta^{(1)}$ in $B^{-1,a}_{2,\infty}$. The details will be provided in the chapter 6.

1.3 Organization of the Dissertation

This dissertation is organized as follows. The second chapter deals with some basic notations, definitions, Littlewood-Paley theory and some basic lemmas. We present our main work in the Chapter three, Chapter four and Chapter five.

In chapter three we account the two-dimensional MHD equations with horizontal dissipation and horizontal magnetic diffusion. This chapter is further divided into five sections. In section 3.1, we present the exponential global L^r, $2 \leq r < \infty$ bound for the horizontal components of the velocity field and magnetic field. The global bound for pressure is illustrated in section 3.2. The section 3.3 deals with the improved global bound for the horizontal components. More precisely, we present the L^r-norms of the horizontal components cannot grow faster than $\sqrt{r \log r}$, for large r. The conditional global regularity for the two-dimensional MHD equations with the horizontal dissipation and horizontal magnetic diffusion is elucidated in section 3.4 which is followed by the global regularity for the slightly improved two-dimensional MHD equations with horizontal dissipation and horizontal magnetic diffusion in section 3.5.

The fourth chapter establishes the global regularity of the 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion. This chapter is divided into two sections. The first section deals with the H^1 bound and the second section deals with the H^2 bound.

The fifth chapter is devoted to the study of the 2D Euler-Boussinesq equations with a more singular velocity. The chapter five is further divided into four sections. In section 5.1 we put forward interpolation inequality and commutator estimate in the Besov space which are essential to our study. In section 5.2, we discuss global *a priori* bound for the vorticity ω and θ in the L^q and Besov space $B^{0,P}_{\infty,2}$. The global bounds for ω and θ are presented in section 5.3, which is followed by the proof of the main theorem presented in section 5.4.

The sixth chapter is dedicated to the study of Boussinesq-Navier stokes equations with a logarithmically supercritical dissipation. This chapter is further divided into six sections. Section 6.1 provides some preliminary estimates including commutator. The global bounds for $\|\omega\|_{B^{0,a-1}_{2,2}}$, $\|G\|_{L^q}$ for $q \in (2,4)$ are presented in sections 6.2 and 6.3. The sections 6.4 and 6.5 present global bound for $\|G\|_{\tilde{L}^r_t B^s_{q,1}}$ for $q \in [2,4)$, $\|\omega\|_{L^1_t B^{0,a}_{\infty,1}}$ and $\|\omega\|_{L^q}$ for $q \geq 2$. The last section deals with the the proof of main theorem.

CHAPTER 2

Preliminary

This chapter is devoted to some basic notations, definitions, and facts, which are relevant to this dissertation. Harmonic analysis, specially Besov space techniques, have been very effective tools to study the well-posedness of nonlinear partial differential equations. We present Littlewood-Paley decomposition, Bony's para-products and properties of the Besov space followed by notations and some definitions.

Notation:

Throughout this dissertation, the following notations will be used.

- C is a harmless constant which may have different values in different steps.
- For every $p \in [1, \infty]$, $\|.\|_{L^p}$ or $\|.\|_p$ denotes the norm in the Lebesgue Space L^p.
- The norm in the space $L^p([0,T]; L^q(\mathbb{R}^2)$ is denoted by $\|.\|_{L^p_t L^q}$ or $\|.\|_{p,q}$.
- $\|(f,g)\|_p = \|(f,g)\|_{L^p} = \|f\|^p_{L^p} + \|g\|^p_{L^p}$.
- $\partial_x^2 f = f_{xx} = \frac{\partial^2 f}{\partial x^2}$.
- For any pair of operators P and Q , the commutator $[P,Q] = PQ - QP$.

Definition 2.1 **(L^p space)** *For $1 \leq p \leq \infty$, $L^p(\mathbb{R}^n)$ is the space of functions such that*

$$\|u\|_{L^p} = \|u\|_p = \begin{cases} \left(\int_{\mathbb{R}^n} |u(x,\tau)|^p \, dx\right)^{\frac{1}{p}}, & \textit{if } 1 \leq p < \infty, \\ ess\sup_{x \in \mathbb{R}^n} |u(x,\tau)|, & \textit{if } p = \infty, \end{cases}$$

is finite.

Definition 2.2 *$L^{p,q} = L^p([0,T]; L^q(\mathbb{R}^d))$ is the space of functions such that*

$$\|u\|_{L^{p,q}} = \|u\|_{p,q} = \begin{cases} \left(\int_0^t \|u(.,\tau)\|_{L^q}^p d\tau\right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ ess\sup_{0<\tau<t} \|u(.,\tau)\|_{L^q}, & \text{if } p = \infty, \end{cases}$$

is finite.

Definition 2.3 *The space l^p consists of sequence $\{x_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$*

Definition 2.4 (Convolution) *The convolution of two functions f and $g \in L^1(\mathbb{R}^d)$ is defined as*

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y)g(x-y)\, dy$$

Definition 2.5 (Schwartz Space) *The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ consists of smooth function $f \in C^{\infty}(\mathbb{R}^d)$ such that for every k and m*

$$\sup_{x \in \mathbb{R}^d,\, |\alpha| \leq m} (1+|x|)^k |D^{\alpha} f| < \infty$$

Definition 2.6 (Riesz Transform) *For $1 \leq j \leq d$, the jth Riesz transform of f is given by convolution*

$$\mathcal{R}_j(f) = C_d\ p.v \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\, dy$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$

In terms of Fourier multiplier

$$\mathcal{R}_j(f)(x) = (-\frac{i\xi_j}{|\xi|}\hat{f}(\xi))^{\vee}(x)$$

The Riesz transform satisfies $\sum_{j=1}^{d} \mathcal{R}_j^2 = -I$, where I is the identity operator.

Definition 2.7 (Sobolev Space) *Let $\Omega \subset \mathbb{R}^d$ be an open set, for integer $k \geq 0$ and $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}(\Omega)$ consists of the functions $f \in L^p(\Omega)$ that have weak derivatives $D^{\alpha} f \in L^p(\Omega)$ of all orders $|\alpha| \leq k$. The norm is defined as*

$$\|f\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^{\alpha} f\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}}, & \text{if} \quad 1 \leq p < \infty, \\ \max_{|\alpha| \leq k} \|D^{\alpha} f\|_{L^{\infty}(\Omega)}, & \text{if} \quad p = \infty. \end{cases}$$

When $p = 2$ we write $W^{k,2} = H^k$.

Definition 2.8 (H^k-norm) *For any $k \in \mathbb{R}$, the H^k-norm is equivalently defined as*

$$\|f\|_{H^k} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 \, d\xi$$

2.1 Besov Space

Let $\mathcal{S}$ be the Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. $\mathcal{S}_0$ denotes a subspace of $\mathcal{S}$ defined by

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S} : \int_{\mathbb{R}^d} \phi(x)\, x^\gamma \, dx = 0,\ |\gamma| = 0, 1, 2, \cdots \right\}$$

and $\mathcal{S}_0'$ denotes its dual. $\mathcal{S}_0'$ can be identified as

$$\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P}$$

where $\mathcal{P}$ denotes the space of multinomials.

To introduce the Littlewood-Paley decomposition we write for each $j \in \mathbb{Z}$,

$$A_j = \left\{ \xi \in \mathbb{R}^d : \ 2^{j-1} \le |\xi| < 2^{j+1} \right\}. \tag{2.1}$$

The Littlewood-Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j \in \mathbb{Z}} \in \mathcal{S}$ such that

$$\operatorname{supp} \widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1 \ , & \text{if } \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0 \ , & \text{if } \xi = 0. \end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) \widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

In addition, if $\psi \in \mathcal{S}_0$, then

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for any } \xi \in \mathbb{R}^d.$$

That is, for $\psi \in \mathcal{S}_0$,

$$\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f, \qquad f \in \mathcal{S}_0'$$

in the sense of weak-$*$ topology of $\mathcal{S}_0'$. For notational convenience, we define

$$\mathring{\Delta}_j f = \Phi_j * f, \qquad j \in \mathbb{Z}. \tag{2.2}$$

Definition 2.9 (Homogeneous Besov Space) *For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the homogeneous Besov space $\mathring{B}^s_{p,q}$ consists of $f \in \mathcal{S}_0'$ satisfying*

$$\|f\|_{\mathring{B}^s_{p,q}} \equiv \|2^{js}\|\mathring{\Delta}_j f\|_{L^p}\|_{l^q} < \infty.$$

In order to define inhomogeneous Besove space, choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{2.3}$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \leq -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \cdots. \end{cases} \tag{2.4}$$

Definition 2.10 (Inhomogeneous Besov Space) *The inhomogeneous Besov space $B^s_{p,q}$ with $1 \le p, q \le \infty$ and $s \in \mathbb{R}$ consists of functions $f \in \mathcal{S}'$ satisfying*

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js}\|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

More precisely the inhomogeneous Besov norm is

$$\|f\|_{B^s_{p,q}} = \begin{cases} \|\Delta_{-1} f\|_{L^p} + \left(\sum_{j=0}^{\infty}(2^{js}\|\Delta_j f\|_{L^p})^q\right)^{\frac{1}{q}}, & \text{if} \quad q < \infty, \\ \|\Delta_{-1} f\|_{L^p} + \sup_{0 \le j < \infty} 2^{js}\|\Delta_j f\|_{L^p}, & \text{if} \quad q = \infty. \end{cases}$$

The Besov spaces $\mathring{B}^s_{p,q}$ and $B^s_{p,q}$ with $s \in (0,1)$ and $1 \le p, q \le \infty$ can be equivalently defined by the norms

$$\|f\|_{\mathring{B}^s_{p,q}} = \left(\int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} dt\right)^{1/q},$$

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \left(\int_{\mathbb{R}^d} \frac{(\|f(x+t) - f(x)\|_{L^p})^q}{|t|^{d+sq}} dt\right)^{1/q}.$$

When $q = \infty$, the expressions are interpreted in the normal way.

Definition 2.11 (Littlewood-Paley Decomposition) *We define non-homogeneous Littlewood-Paley decomposition as*

$$f = \sum_{j=-1}^{\infty} \Delta_j f$$

for any $f \in \mathcal{S}'$ and Δ_j is called the the Fourier localization operators.

We also frequently use the notation for partial sum or low frequency cut-off S_j: for an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j. For notational convenience, we write Δ_j for $\mathring{\Delta}_j$. We would like to mention some facts related to Δ_j:

$$\Delta_j \Delta_k \equiv 0, \quad \text{if} \quad |j-k| \geq 2$$

$$S_j \equiv \sum_{k=-\infty}^{j-1} \Delta_k \to I, \quad \text{as} \quad j \to \infty$$

$$\Delta_k (S_{j-1} f \Delta_j f) = 0, \quad if \quad |j-k| \geq 4$$

We can define classical spaces, L^p, H^s and Hölder space by using the Littlewood-Paley decomposition. The equivalent L^p norm in terms of Besov norm is

$$\|f\|_{L^p} \simeq \|S_0 f\|_{L^p} + \| \left(\sum_{j=0}^{\infty} (\Delta_j f)^2 \right)^{\frac{1}{2}} \|_{L^p}, \quad \text{if} \quad 1 \leq p < \infty.$$

The equivalent H^s for $s \in \mathbb{R}$, $\|f\|_{H^s} = \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}$, is equivalent to

$$\|f\|_{H^s} \simeq \|S_0 f\|_{L^2} + \left(\sum_{j=0}^{\infty} 2^{2js} \|\Delta_j f\|_{L^2}^2 \right)^{\frac{1}{2}}$$

Thus $L^p(\mathbb{R}^d) = B^0_{2,2}$, for $1 \leq p < \infty$ and $H^s(\mathbb{R}^d) = B^s_{2,2}(\mathbb{R}^d)$

2.2 Generalized Besov Space

We will use the following generalized Besov spaces to include an algebraic part of the modes.

Definition 2.12 *For $s, \gamma \in \mathbb{R}$ and $p, q \in [1, \infty]$, the generalized Besov spaces $B^{s,\gamma}_{p,q}$ are defined by*

$$\|f\|_{B^{s,\gamma}_{p,q}} \equiv \|2^{js}(1+|j|)^\gamma \|\Delta_j f\|_{L^p}\|_{l^q} < \infty.$$

Definition 2.13 *Let $a(x) = a(|x|) : (0, \infty) \to (0, \infty)$ be a non-decreasing function satisfying (6.6), namely*

$$\lim_{|x| \to \infty} \frac{a(x)}{|x|^\sigma} = 0\,, \quad \forall \sigma > 0.$$

For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the generalized Besov spaces $\mathring{B}^{s,a}_{p,q}$ and $B^{s,a}_{p,q}$ are defined through the norms

$$\|f\|_{\mathring{B}^{s,a}_{p,q}} \equiv \|2^{js} a(2^j)\, \|\mathring{\Delta}_j f\|_{L^p}\|_{l^q} < \infty,$$
$$\|f\|_{B^{s,a}_{p,q}} \equiv \|2^{js} a(2^j)\, \|\Delta_j f\|_{L^p}\|_{l^q} < \infty. \tag{2.5}$$

We have also used the space-time spaces defined below.

Definition 2.14 *For $t > 0$, $s \in \mathbb{R}$ and $1 \le p, q, r \le \infty$, the space-time spaces $\widetilde{L}^r_t \mathring{B}^s_{p,q}$ and $\widetilde{L}^r_t B^s_{p,q}$ are defined through the norms*

$$\|f\|_{\widetilde{L}^r_t \mathring{B}^s_{p,q}} \equiv \|2^{js} \|\mathring{\Delta}_j f\|_{L^r_t L^p}\|_{l^q},$$

$$\|f\|_{\widetilde{L}^r_t B^s_{p,q}} \equiv \|2^{js} \|\Delta_j f\|_{L^r_t L^p}\|_{l^q}.$$

$\widetilde{L}^r_t \mathring{B}^{s,a}_{p,q}$ and $\widetilde{L}^r_t B^{s,a}_{p,q}$ are similarly defined.

These spaces are related to the classical space-time spaces $L^r_t \mathring{B}^s_{p,q}$, $L^r_t B^{s,\gamma}_{p,q}$, $L^r_t \mathring{B}^{s,a}_{p,q}$ and $L^r_t B^{s,a}_{p,q}$ via the Minkowski inequality.

Most frequently used function spaces and Besov spaces are related by the following embedding relations.

Proposition 2.2.1 *For any $s \in \mathbb{R}$,*

$$\mathring{H}^s \sim \mathring{B}^s_{2,2}, \quad H^s \sim B^s_{2,2}.$$

For any $s \in \mathbb{R}$ and $1 < q < \infty$,

$$\mathring{B}^s_{q,\min\{q,2\}} \hookrightarrow \mathring{W}^s_q \hookrightarrow \mathring{B}^s_{q,\max\{q,2\}}.$$

In particular, $\mathring{B}^0_{q,\min\{q,2\}} \hookrightarrow L^q \hookrightarrow \mathring{B}^0_{q,\max\{q,2\}}$.

Bernstein's inequalities are useful tools in dealing with Fourier localized functions and these inequalities trade integrability for derivatives. Proposition (2.2.2) provides Bernstein type inequalities for fractional derivatives.

Proposition 2.2.2 *Let $\alpha \geq 0$. Let $1 \leq p \leq q \leq \infty$.*

1) If f satisfies

$$supp\,\widehat{f} \subset \{\xi \in \mathbb{R}^d : \ |\xi| \leq K2^j\},$$

for some integer j and a constant $K > 0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_1\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}.$$

2) If f satisfies

$$supp\,\widehat{f} \subset \{\xi \in \mathbb{R}^d : \ K_1 2^j \leq |\xi| \leq K_2 2^j\}$$

for some integer j and constants $0 < K_1 \leq K_2$, then

$$C_1\, 2^{2\alpha j}\|f\|_{L^q(\mathbb{R}^d)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_2\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p and q only.

2.3 Bony's Para-products

The main concept of para-product is the decomposition of a product of two functions u and v into three parts. The first part is denoted by $T_u v$ called para-product of v by u. The second part is $T_v u$ is para-product of u and v. The last part is the remainder $R(u, v)$. The para-product operator $T_u v$ is defined as

$$T_u v = \sum_j \Delta_j v S_{j-1} u = \sum_j \sum_{k \leq j-2} \Delta_j v \Delta_k u, \qquad \text{where partial sum} \quad S_j = \sum_{k \leq j-1} \Delta_k$$

The regularity of $T_u v$ is mainly determined by v. The remainder $R(u, v)$ is defined as

$$R(u, v) = \sum_{|i-j| \leq 1} \Delta_i u \Delta_j v = \sum_i \Delta_i u \tilde{\Delta}_j v, \quad \text{where} \quad \tilde{\Delta}_i = \Delta_{i-1} + \Delta_i + \Delta_{i+1}$$

Since $\sum_j \Delta_j = I$, we can write

$$uv = \sum_j \sum_k \Delta_j u \Delta_k v$$

Definition 2.15 (Para-products) *The Bony's para-products of two tempered distribution u and u is defined as*

$$uv = T_u v + T_v u + R(u, v).$$

2.4 Some Useful Lemmas and Inequalities

Lemma 2.1 (Hölder Inequality) *For any Lebesgue measurable functions f and g*

$$\|fg\|_{L^p} \leq \|f\|_q \|g\|_r, \quad \textit{where} \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{r}.$$

Lemma 2.2 (Minkowski's Inequality)

$$\left(\int \left| \int f(x,y) dy \right|^p dx \right)^{\frac{1}{p}} \leq \int \left(\int |f(x,y)|^p \, dx \right)^{\frac{1}{p}} dy.$$

Lemma 2.3 (Young's Inequality)

1. *(Product) If a and b are non-negative numbers then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \textit{where} \quad \frac{1}{p} + \frac{1}{q} = 1$$

2. *(Convolutions) If $f \in L^p$, $g \in L^q$, $1 \leq p, q \leq \infty$ then $f * g \in L^r$, where $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. More precisely,*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Lemma 2.4 (Gagliardo-Nirenberg Inequality) *For $1 \leq q, r \leq \infty$ and α and β are integer satisfying $0 \leq \alpha < \beta$. Then for any smooth function f,*

$$\|\Lambda^\alpha f\|_{L^p} \leq C \|\Lambda^\beta f\|_{L^q}^a \|f\|_{L^r}^{1-a}, \quad \textit{where} \quad (\frac{1}{p} - \frac{\alpha}{d}) = a(\frac{1}{q} - \frac{\beta}{d}) + \frac{(1-a)}{r}$$

Lemma 2.5 (Grönwall Inequality) *If f, g and $h \geq 0$ are continuous on $[0, t]$, h is differential, and*

$$g(t) \leq h(t) + \int_0^t f(s) g(s) ds,$$

then

$$g(t) \leq h(0) \exp\left(\int_0^t f(s) ds \right) + \int_0^t h'(s) \left[\exp \int_s^t f(\tau) \, d\tau \right] ds$$

Lemma 2.6 (Osgood Inequality) *Let $\alpha(t) > 0$ be a locally integrable function. Assume $\omega(t) \geq 0$ satisfies*

$$\int_0^\infty \frac{1}{\omega(r)} dr = \infty.$$

Suppose that $\rho(t) > 0$ satisfies

$$\rho(t) \leq a + \int_{t_0}^t \alpha(s)\omega(\rho(s))ds$$

for some constant $a \geq 0$. Then if $a = 0$, then $\rho \equiv 0$; if $a > 0$, then

$$-\Omega(\rho(t)) + \Omega(a) \leq \int_{t_0}^t \alpha(\tau)d\tau,$$

where

$$\Omega(x) = \int_x^1 \frac{dr}{\omega(r)}.$$

CHAPTER 3

The 2D Magnetohydrodynamic Equations with Horizontal Dissipation and Horizontal Magnetic Diffusion

In this chapter, we consider the initial value problem for the 2D magnetohydrodynamic equations with horizontal dissipation and horizontal magnetic diffusion.

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + u_{xx} + b \cdot \nabla b, \qquad x \in \mathbb{R}^2,\, t > 0 \\ b_t + u \cdot \nabla b = b_{xx} + b \cdot \nabla u, \qquad x \in \mathbb{R}^2,\, t > 0 \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \end{cases} \tag{3.1}$$

where u is the velocity field, b the magnetic field, and p the pressure.

First we prove that the horizontal component of any solution admits a global (in time) bound in any Lebesgue space L^{2r} with $1 \leq r < \infty$ and the bound is exponential. Our main efforts are devoted to improve the global bound for $\|(u_1, b_1)\|_{2r}$. We prove that the pressure associated with any classical solution obeys the global bound, for any $T > 0$ and $t < T$,

$$\|p(\cdot,t)\|_q \leq C(T), \qquad \int_0^T \|p(\cdot,t)\|_{H^s}^2\, dt < C(T),$$

where $1 < q \leq 3$ and $0 < s < 1$. These global bounds together with a decomposition of the pressure into low and high frequency parts , we significantly improve the global bound for $\|(u_1, b_1)\|_{2r}$. We are able to show that $\|(u_1, b_1)\|_{2r}$ does not grow faster than $\sqrt{r \log r}$ for large $r < \infty$.

In addition, we establish a conditional global regularity in terms of the $L_t^2 L_x^\infty$-norm of the horizontal components.

Furthermore, we study the global regularity of a slightly regularized version of the MHD equations, namely

$$\begin{cases} u_t + u \cdot \nabla u + \epsilon(-\Delta)^\delta u = -\nabla p + u_{xx} + b \cdot \nabla b, \qquad x \in \mathbb{R}^2,\ t > 0 \\ b_t + u \cdot \nabla b + \epsilon(-\Delta)^\delta b = b_{xx} + b \cdot \nabla u, \qquad x \in \mathbb{R}^2,\ t > 0 \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{3.2}$$

with $\epsilon > 0$ and $\delta > 0$.

We establish the global well-posedness for the system (3.2) with initial data $(u_0, b_0) \in H^s, s \geq 2$.

The proofs of our results take advantage of the symmetric structure of (3.1). That is,

$$w^\pm = u \pm b$$

satisfies

$$\begin{cases} \partial_t w^+ + (w^- \cdot \nabla) w^+ = -\nabla p + \partial_x^2 w^+, \\ \partial_t w^- + (w^+ \cdot \nabla) w^- = -\nabla p + \partial_x^2 w^-, \\ \nabla \cdot w^+ = 0, \quad \nabla \cdot w^- = 0. \end{cases} \tag{3.3}$$

We remark that this symmetric formulation is still more complex than the 2D Boussinesq equations dealt with in [13]. (3.3) consists of a system of two vector equations and the interaction between w^+ and w^- makes it mathematically more difficult.

In later section, we will need the following anisotropic triple product lemma (see [12]).

Lemma 3.1 *If $f, g, h, g_y, h_x \in L^2(\mathbb{R}^2)$, then*

$$\iint_{\mathbb{R}^2} |f\, g\, h|\, dxdy \leq C\, \|f\|_2\, \|g\|_2^{\frac{1}{2}} \|g_y\|_2^{\frac{1}{2}}\, \|h\|_2^{\frac{1}{2}} \|h_x\|_2^{\frac{1}{2}}. \tag{3.4}$$

Where C is a constant.

The vorticity formulation of (3.1) is very useful to study global well-posedness. The vorticity $\omega = \nabla \times u$ and the current density $j = \nabla \times b$ satisfy

$$\begin{cases} \omega_t + u \cdot \nabla \omega = \omega_{xx} + b \cdot \nabla j, \\ j_t + u \cdot \nabla j = j_{xx} + b \cdot \nabla \omega + 2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1), \end{cases} \tag{3.5}$$

We will present the theorems and their proofs in the later sections.

3.1 A Global Bound in the Lebesgue Spaces

Assume that (u,b) is a classical solution of (3.1). This section shows that its component in the $x-$direction (u_1,b_1) admits a global (in time) bound in $L^{2r}(\mathbb{R}^2)$ for any $1\le r<\infty$. The bound obtained here depends exponentially on r. More precisely, we develop the following theorem.

Theorem 3.1 *Assume that $(u_0,b_0)\in H^2(\mathbb{R}^2)$ and let (u,b) be the corresponding solution of (3.1). Then, for any $1\le r<\infty$, (u_1,b_1) obeys the global bound*

$$\|(u_1,b_1)\|_{2r}\le C_1 e^{C_2\,r^3}, \tag{3.6}$$

where C_1 and C_2 are constants depending on $\|(u_0,b_0)\|_{2r}$ only.

In order to prove this theorem, we need the global L^2-bound.

Lemma 3.2 *Let $(u_0,b_0)\in H^2(\mathbb{R}^2)$ and let (u,b) be the corresponding solution of (3.1). Then, (u,b) obeys the following global L^2-bound,*

$$\|u(t)\|_2^2+\|b(t)\|_2^2+2\int_0^t\|\partial_x u(\tau)\|_2^2\,d\tau+2\int_0^t\|\partial_x b(\tau)\|_2^2\,d\tau\ \le \|u_0\|_2^2+\|b_0\|_2^2 \tag{3.7}$$

for any $t\ge 0$.

Proof. (Proof of Theorem 3.1) It is more convenient to use the symmetric form of (3.1), namely (3.3). Multiplying the first component of the first equation of (3.3) by $w_1^+|w_1^+|^{2r-2}$ and integrating it with respect to space variable, we obtain, after integration by parts,

$$\frac{1}{2r}\frac{d}{dt}\|w_1^+\|_{2r}^{2r}+(2r-1)\int|\partial_x w_1^+|^2|w_1^+|^{2r-2}=(2r-1)\int p\,\partial_x w_1^+|w_1^+|^{2r-2}. \tag{3.8}$$

By Hölder's and Sobolev's inequalities,

$$\begin{aligned}\int p\,\partial_x w_1^+|w_1^+|^{2r-2}\ &\le\ \|p\|_{2r}\,\|\partial_x w_1^+|w_1^+|^{r-1}\|_2\,\||w_1^+|^{r-1}\|_{\frac{2r}{r-1}}\\ &\le\ Cr\|\nabla p\|_{\frac{2r}{r+1}}\,\|\partial_x w_1^+|w_1^+|^{r-1}\|_2\,\|w_1^+\|_{2r}^{r-1},\end{aligned}$$

where C is a constant independent of r. Therefore, by Young's inequality,

$$(2r-1)\int p\,\partial_x w_1^+|w_1^+|^{2r-2} \le \frac{2r-1}{4}\|\partial_x w_1^+|w_1^+|^{r-1}\|_2^2 + Cr^3\|\nabla p\|_{\frac{2r}{r+1}}^2\|w_1^+\|_{2r}^{2(r-1)}.$$

To bound the pressure, we take the divergence of (3.3) to get

$$-\Delta p = \partial_x(w_1^-\partial_x w_1^+ + w_1^+\partial_x w_1^-) + \partial_y(w_1^+\partial_x w_2^- + w_1^-\partial_x w_2^+). \tag{3.9}$$

Due to the boundedness of Riesz transforms on L^q for any $1<q<\infty$, we have

$$\begin{aligned}\|\nabla p\|_{\frac{2r}{r+1}} &\le \|w_1^-\partial_x w_1^+\|_{\frac{2r}{r+1}} + \|w_1^+\partial_x w_1^-\|_{\frac{2r}{r+1}} + \|w_1^+\partial_x w_2^-\|_{\frac{2r}{r+1}} + ||w_1^-\partial_x w_2^+||_{\frac{2r}{r+1}}\\ &\le \|w_1^-\|_{2r}(\|\partial_x w_1^+\|_2 + \|\partial_x w_2^+\|_2) + \|w_1^+\|_{2r}(\|\partial_x w_1^-\|_2 + \|\partial_x w_2^-\|_2).\end{aligned}$$

Consequently,

$$\begin{aligned}&Cr^3\|\nabla p\|_{\frac{2r}{r+1}}^2\|w_1^+\|_{2r}^{2(r-1)}\\ &\le Cr^3(\|\partial_x w_1^+\|_2^2 + \|\partial_x w_2^+\|_2^2 + \|\partial_x w_1^-\|_2^2 + \|\partial_x w_2^-\|_2^2)(\|w_1^-\|_{2r} + \|w_1^+\|_{2r})^2\|w_1^+\|_{2r}^{2(r-1)}\\ &\le Cr^3(\|\partial_x w_1^+\|_2^2 + \|\partial_x w_2^+\|_2^2 + \|\partial_x w_1^-\|_2^2 + \|\partial_x w_2^-\|_2^2)(\|w_1^+\|_{2r}^{2r} + \|w_1^-\|_{2r}^{2r}).\end{aligned}$$

Combining the estimates above, we obtain

$$\begin{aligned}&\frac{1}{r}\frac{d}{dt}\|w_1^+\|_{2r}^{2r} + \frac{(2r-1)}{2}\int|\partial_x w_1^+|^2|w_1^+|^{2r-2}\\ &\le Cr^3(\|\partial_x w_1^+\|_2^2 + \|\partial_x w_2^+\|_2^2 + \|\partial_x w_1^-\|_2^2 + \|\partial_x w_2^-\|_2^2)(\|w_1^+\|_{2r}^{2r} + \|w_1^-\|_{2r}^{2r}).\end{aligned}$$

Similarly,

$$\begin{aligned}&\frac{1}{r}\frac{d}{dt}\|w_1^-\|_{2r}^{2r} + \frac{(2r-1)}{2}\int|\partial_x w_1^-|^2|w_1^-|^{2r-2}\\ &\le Cr^3(\|\partial_x w_1^+\|_2^2 + \|\partial_x w_2^+\|_2^2 + \|\partial_x w_1^-\|_2^2 + \|\partial_x w_2^-\|_2^2)(\|w_1^+\|_{2r}^{2r} + \|w_1^-\|_{2r}^{2r}).\end{aligned}$$

Adding these two inequalities yields

$$\begin{aligned}&\frac{1}{r}\frac{d}{dt}\left(\|w_1^+\|_{2r}^{2r} + \|w_1^-\|_{2r}^{2r}\right) + \frac{(2r-1)}{2}\int(|\partial_x w_1^+|^2|w_1^+|^{2r-2} + |\partial_x w_1^-|^2|w_1^-|^{2r-2})\\ &\le Cr^3\left(\|\partial_x w_1^+\|_2^2 + \|\partial_x w_2^+\|_2^2 + \|\partial_x w_1^-\|_2^2 + \|\partial_x w_2^-\|_2^2\right)\left(\|w_1^+\|_{2r}^{2r} + \|w_1^-\|_{2r}^{2r}\right).\end{aligned}$$

It then follows from Gronwall's inequality that

$$\begin{aligned}\|w_1^+\|_{2r}^{2r} + \|w_1^-\|_{2r}^{2r} &\le \left(\|w_1^+(0)\|_{2r}^{2r} + \|w_1^-(0)\|_{2r}^{2r}\right) \\ &\quad \times \exp\left(Cr^4 \int_0^t \left(\|\partial_x w_1^+\|_2^2 + \|\partial_x w_2^+\|_2^2 + \|\partial_x w_1^-\|_2^2 + \|\partial_x w_2^-\|_2^2\right) d\tau\right).\end{aligned}$$

This inequality together with (6.14) yields (3.6). ■

3.2 Global Bounds for the Pressure

In this section we provide the L^{2r}-bounds for the second components (u_2, b_2) when $r = 2$ and 3. Then we establish two global bounds for the pressure: one for $\|p\|_q$ with $1 < q \le 3$ and the other for $\int_0^t \|p(\tau)\|_{H^s}^2 d\tau$ with $s \in [0, 1)$. The precise results can be stated as follows.

Theorem 3.2 *Assume that $(u_0, b_0) \in H^2(\mathbb{R}^2)$ and let (u, b) be the corresponding solution of (3.1). Let p be the corresponding pressure. Let $s \in (0, 1)$. Then, for any $T > 0$ and $t \le T$,*

$$\|(u_2, b_2)(t)\|_{L^{2r}} \le C, \quad r = 2, 3, \tag{3.10}$$

and, for any $1 < q \le 3$,

$$\|p(t)\|_q \le C, \qquad \int_0^T \|p(\tau)\|_{H^s}^2 \, d\tau < C, \tag{3.11}$$

where C is a constant depending on T and the initial data.

Proof. There seems to be no uniform approach to prove the bounds in (3.10) simultaneously for $r = 2$ and $r = 3$. We prove them separately and start with the L^4-bound. It is more convenient to use the symmetric form (3.3). Multiplying the second component of the first equation of (3.3) by $w_2^+|w_2^+|^2$ and integrating by parts yield

$$\frac{1}{4}\frac{d}{dt}\|w_2^+\|_4^4 + 3\int |\partial_x w_2^+|^2 |w_2^+|^2 = 3\int p\, \partial_y w_2^+ |w_2^+|^2.$$

To bound the term on the right, we use $\nabla \cdot w^+ = 0$ and integrate by parts to get

$$\begin{aligned}
\int p\,\partial_y w_2^+ |w_2^+|^2 &= -\int p\,\partial_x w_1^+ |w_2^+|^2 \\
&= \int \partial_x p\, w_1^+ |w_2^+|^2 + 2\int p\, w_1^+ \partial_x w_2^+ w_2^+ \\
&= J_1 + 2J_2. \qquad (3.12)
\end{aligned}$$

By Hölder's and Sobolev's inequalities,

$$\begin{aligned}
|J_2| &\leq \|p\|_4\, \|w_1^+\|_4\, \|w_2^+ \partial_x w_2^+\|_2 \\
&\leq C\, \|\nabla p\|_{\frac{4}{3}}\, \|w_1^+\|_4\, \|w_2^+ \partial_x w_2^+\|_2.
\end{aligned}$$

According to (3.10),

$$\|\nabla p\|_{\frac{4}{3}} \leq \|w_1^-\|_4 (\|\partial_x w_1^+\|_2 + \|\partial_x w_2^+\|_2) + \|w_1^+\|_4 (\|\partial_x w_1^-\|_2 + \|\partial_x w_2^-\|_2).$$

Therefore, by Young's inequality,

$$\begin{aligned}
|J_2| &\leq \frac{1}{16} \|w_2^+ \partial_x w_2^+\|_2^2 \\
&\quad + C\,(\|w_1^-\|_4^4 + \|w_1^+\|_4^4)(\|\partial_x w_1^+\|_2^2 + \|\partial_x w_2^+\|_2^2 + \|\partial_x w_1^-\|_2^2 + \|\partial_x w_2^-\|_2^2).
\end{aligned}$$

To bound J_1, we first apply Hölder's inequality,

$$|J_1| \leq \|p_x\|_{\frac{8}{5}}\, \|w_1^+\|_8\, \|(w_2^+)^2\|_4.$$

By Lemma 3.3 below and $\nabla \cdot w^+ = 0$,

$$\|(w_2^+)^2\|_4 \leq C \|\partial_x (w_2^+)^2\|_2^{\frac{1}{2}}\, \|\partial_y (w_2^+)^2\|_1^{\frac{1}{2}} \leq C \|w_2^+ \partial_x w_2^+\|_2^{\frac{1}{2}}\, \|w_2^+ \partial_x w_1^+\|_1^{\frac{1}{2}}.$$

According to (3.10),

$$\begin{aligned}
\|\nabla p\|_{\frac{8}{5}} &\leq C\, \|w_1^-\|_8 (\|\partial_x w_1^+\|_2 + \|\partial_x w_2^+\|_2) + \|w_1^+\|_8 (\|\partial_x w_1^-\|_2 + \|\partial_x w_2^-\|_2) \\
&\leq C\, (\|w_1^-\|_8 + \|w_1^+\|_8)(\|\partial_x w^-\|_2 + \|\partial_x w^+\|_2).
\end{aligned}$$

Therefore,

$$\begin{aligned}|J_1| &\le C\|w_1^+\|_8(\|w_1^-\|_8+\|w_1^+\|_8)(\|\partial_x w^-\|_2+\|\partial_x w^+\|_2)\|w_2^+\partial_x w_2^+\|_2^{\frac12}\|w_2^+\partial_x w_1^+\|_1^{\frac12}\\ &\le \frac{1}{16}\|w_2^+\partial_x w_2^+\|_2^2\\ &\quad +C(\|\partial_x w^-\|_2^2+\|\partial_x w^+\|_2^2)+\|w_1^+\|_8^4(\|w_1^-\|_8+\|w_1^+\|_8)^4\|w_2^+\|_2^2\|\partial_x w_1^+\|_2^2.\end{aligned}$$

Inserting the estimates for J_1 and J_2 in (3.12) and recalling Theorem 3.1, we obtain a global bound for $\|w_2^+\|_4$. The bound for $\|w_2{}^-\|_4 < C$ can be similarly established. To prove the L^6-bound in (3.10), we obtain from (3.3) that

$$\begin{aligned}&\frac{1}{6}\frac{d}{dt}\left(\|w_2^+\|_6^6+\|w_2^-\|_6^6\right)+5\left\||w_2^+|^2\,|\partial_x w_2^+|\right\|_2^2+5\left\||w_2^-|^2\,|\partial_x w_2^-|\right\|_2^2\\ &=5\int p(|w_2^+|^4\partial_y w_2^+ + |w_2^-|^4\partial_y w_2^-)\\ &=-5\int p(|w_2^+|^4\partial_x w_1^+ + |w_2^-|^4\partial_x w_1^-)\\ &=5\int \partial_x p(|w_2^+|^4 w_1^+ + |w_2^-|^4 w_1^-)+20\int p(|w_2^+|^3\,\partial_x w_2^+ w_1^+ + |w_2^-|^3\,\partial_x w_2^- w_1^-).\end{aligned}$$

Applying Hölder's inequality, (3.9) and Lemma 3.3, we have

$$\begin{aligned}&\int \partial_x p(|w_2^+|^4 w_1^+ + |w_2^-|^4 w_1^-)\\ &\le \|\partial_x p\|_{\frac{36}{19}}(\||w_2^+|^3\|_3^{4/3}\|w_1^+\|_{36}+\||w_2^-|^3\|_3^{4/3}\|w_1^-\|_{36})\\ &\le C\,(\|w_1^+\|_{36}+\|w_1^-\|_{36})^2(\|\partial_x w^+\|_2+\|\partial_x w^-\|_2)\\ &\quad\times(\|\partial_y|w_2^+|^3\|_1^{\frac49}\,\|\partial_x|w_2^+|^3\|_2^{\frac49}\,\||w_2^+|^3\|_2^{\frac49}+\|\partial_y|w_2^-|^3\|_1^{\frac49}\,\|\partial_x|w_2^-|^3\|_2^{\frac49}\,\||w_2^-|^3\|_2^{\frac49})\\ &\le C\,(\|w_1^+\|_{36}+\|w_1^-\|_{36})^2\,(\|\partial_x w^+\|_2+\|\partial_x w^-\|_2)\left(\||w_2^+|^3\|_2^{\frac49}+\||w_2^-|^3\|_2^{\frac49}\right)\\ &\quad\times\left(\|w_2^+\|_4^{\frac89}\,\|\partial_y w_2^+\|_2^{\frac49}+\|w_2^-\|_4^{\frac89}\,\|\partial_y w_2^-\|_2^{\frac49}\right)\left(\|\partial_x|w_2^+|^3\|_2^{\frac49}+\|\partial_x|w_2^-|^3\|_2^{\frac49}\right).\end{aligned}$$

Also, by Hölder's inequality and (3.9),

$$\begin{aligned}&\int p(|w_2^+|^3\,\partial_x w_2^+ w_1^+ + |w_2^-|^3\,\partial_x w_2^- w_1^-)\\ &\le \|p\|_6(\|w_2^+\|_6\,\||w_2^+|^2\,\partial_x w_2^+\|_2\|w_1^+\|_6+\|w_2^-\|_6\,\||w_2^-|^2\,\partial_x w_2^-\|_2\|w_1^-\|_6)\\ &\le C\,(\|w_1^+\|_6+\|w_1^-\|_6)(\|\partial_x w^+\|_2+\|\partial_x w^-\|_2)\\ &\quad\times(\|w_1^+\|_6\||w_2^+|^2\,\partial_x w_2^+\|_2\|w_2^+\|_6+\|w_1^-\|_6\||w_2^-|^2\,\partial_x w_2^-\|_2\|w_2^-\|_6).\end{aligned}$$

Therefore, by Young's and Gronwall's inequalities,

$$\|w_2^+\|_6^6 + \|w_2^-\|_6^6 + \int_0^t \left(\left\| |w_2^+|^2 \, |\partial_x w_2^+| \right\|_2^2 + \left\| |w_2^-|^2 \, |\partial_x w_2^-| \right\|_2^2 \right) \leq C.$$

We now prove the first inequality in (3.11). Taking the divergence of the first two equations in (3.3), we have

$$-\Delta p = \nabla \cdot (w^- \cdot \nabla w^+).$$

By the boundedness of Riesz transforms on L^q,

$$\|p\|_q \leq C \|w^-\|_{2q} \, \|w^+\|_{2q}.$$

For $1 < q \leq 3$, $\|w^-\|_{2q}$ and $\|w^+\|_{2q}$ are bounded according to Theorem 3.1 and (3.10) and thus $\|p\|_q < C$.

Now we prove the second inequality in (3.11). Recall that Λ^s is defined through its Fourier transform

$$\widehat{\Lambda^s f}(\xi) = |\xi|^s \hat{f}(\xi).$$

Combining (3.9), the boundedness of Riesz transforms on L^2 and the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
\|\Lambda^s p\|_2 &\leq \|\Lambda^s(-\Delta)^{-1}\partial_x(w_1^-\partial_x w_1^+ + w_1^+\partial_x w_1^-)\|_2 + \|\Lambda^s(-\Delta)^{-1}\partial_y(w_1^+\partial_x w_2^- + w_1^-\partial_x w_2^+)\|_2 \\
&\leq \|\Lambda^{-(1-s)}(w_1^-\partial_x w_1^+ + w_1^+\partial_x w_1^-)\|_2 + \|\Lambda^{-(1-s)}(w_1^+\partial_x w_2^- + w_1^-\partial_x w_2^+)\|_2 \\
&\leq C \, \|w_1^-\partial_x w_1^+ + w_1^+\partial_x w_1^-\|_q + \|w_1^+\partial_x w_2^- + w_1^-\partial_x w_2^+\|_q \\
&\leq C \, \left(\|\partial_x w^+\|_2 + \|\partial_x w^-\|_2\right) \left(\|w_1^+\|_{\frac{2}{1-s}} + \|w_1^-\|_{\frac{2}{1-s}}\right),
\end{aligned}$$

where q satisfies $\frac{1}{q} = \frac{1}{2} + \frac{1-s}{2}$ and C is a constant independent of s. This completes the proof of Theorem 3.2. ∎

We have used the following two calculus inequalities in the proof of above theorem.

Lemma 3.3 *Assume that* $f \in L^2(\mathbb{R}^2)$, $\partial_x f \in L^1(\mathbb{R}^2)$ *and* $\partial_y f \in L^2(\mathbb{R}^2)$. *Then*

$$\|f\|_4 \le \sqrt{3}\,\|\partial_x f\|_1^{\frac{1}{2}}\,\|\partial_y f\|_2^{\frac{1}{2}}, \tag{3.13}$$

$$\|f\|_3 \le \sqrt[3]{2}\,\|f\|_2^{\frac{1}{3}}\,\|\partial_x f\|_1^{\frac{1}{3}}\,\|\partial_y f\|_2^{\frac{1}{3}}, \tag{3.14}$$

Proof. Write

$$\begin{aligned}
f^4(x,y) &= \int_{-\infty}^{y} \partial_z(f^3(x,z))\,dz \int_{-\infty}^{x} \partial_z f(z,y)\,dz \\
&\le 3\left(\int_{-\infty}^{\infty} f^2(x,z)\partial_z f\,dz\right)\left(\int_{-\infty}^{\infty} \partial_z f(z,y)\,dz\right) \\
\int_{\mathbb{R}^2} |f|^4 dxdy &\le 3\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f|^2|\partial_z f(x,z)|\,dz\,dx\right)\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_z f(z,y)|\,dz\,dy\right) \\
\|f\|_4^4 &\le 3\left(\int_{\mathbb{R}^2} |f|^4\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2} |\partial_z f(x,z)|^2\right)^{\frac{1}{2}}\|\partial_x f\|_1 \\
&\le 3\|f\|_4^2\,\|\partial_y f\|_2\,\|\partial_x f\|_1.
\end{aligned}$$

This yields (3.13).

For (3.14) write

$$f^3 = f^2\,f = \left(\int_{-\infty}^{y} \partial_y(f^2)\,dy\right)\left(\int_{-\infty}^{x} \partial_x f\,dx\right).$$

then we obtain (3.14) by integrating and using Hölder inequality. ■

3.3 An Improved Global Lebesgue-bound

This section establishes the improved global bound for $\|(u_1,b_1)\|_{L^{2r}}$, which states that for $r > 2$, $\|(u_1,b_1)\|_{L^{2r}}$ does not grow faster than the order of $\sqrt{r\log r}$. More precisely we prove the following theorem.

Theorem 3.3 *Assume that* $(u_0,b_0) \in H^2(\mathbb{R}^2)$ *and let* (u,b) *be the corresponding solution of (3.1). Let* $2 < r < \infty$. *Then,*

$$\|(u_1,b_1)(t)\|_{L^{2r}(\mathbb{R}^2)} \le B_0(t)\,\sqrt{r\log r} + B_1, \tag{3.15}$$

where B_0 *is a smooth function of* t *and* B_1 *depends only on* $\|(u_0,b_0)\|_{2r}$.

In order to prove this theorem, we need several facts that we now state and prove.

Lemma 3.4 *Let $q \in [2, \infty)$ and $s \in (\frac{1}{2}, 1]$. Assume that $f, g, \partial_y g \in L^2(\mathbb{R}^2)$, $h \in L^{2(q-1)}(\mathbb{R}^2)$ and $\Lambda_x^s h \in L^2(\mathbb{R}^2)$. Then,*

$$\left| \iint_{R^2} f\, g\, h\, dxdy \right| \le C\, \|f\|_2\, \|g\|_2^{\rho}\, \|\partial_y g\|_2^{1-\rho}\, \|h\|_{2(q-1)}^{\vartheta}\, \|\Lambda_x^s h\|_2^{1-\vartheta}. \tag{3.16}$$

where ρ and ϑ are given by

$$\rho = \frac{1}{2} + \frac{(2s-1)(q-2)}{2(2s-1)(q-1)+2}, \quad \vartheta = \frac{(2s-1)(q-1)}{(2s-1)(q-1)+1},$$

and Λ_x^s denotes a fractional derivative with respect to x and is defined by

$$\Lambda_x^s h(x) = \int e^{ix\cdot\xi} |\xi_1|^s \widehat{h}(\xi)\, d\xi.$$

Proof. To prove this inequality, we recall the one-dimensional Sobolev inequality

$$\|h\|_{L_x^\infty(\mathbb{R})} \le C\, \|h\|_{L_x^{2(q-1)}(\mathbb{R})}^{\vartheta} \|\Lambda_x^s h\|_{L_x^2(\mathbb{R})}^{1-\vartheta}, \tag{3.17}$$

where we have used the sub-index x with the Lebesgue spaces to emphasize that the norms are taken in one-dimensional Lebesgue spaces with respect to x. By Hölder's inequality and (3.17),

$$\begin{aligned} \left| \iint f\, g\, h\, dxdy \right| &\le C \int \|f\|_{L_x^2} \|g\|_{L_x^2} \|h\|_{L_x^{2(q-1)}}^{\vartheta} \|\Lambda_x^s h\|_{L_x^2}^{1-\vartheta}\, dy \\ &\le C \left(\int \|f\|_{L_x^2}^2\, dy \right)^{\frac{1}{2}} \left(\int \|g\|_{L_x^2}^{\mu}\, dy \right)^{\frac{1}{\mu}} \\ &\qquad \times \left(\int \|h\|_{L_x^{2(q-1)}}^{2(q-1)}\, dy \right)^{\frac{\vartheta}{2(q-1)}} \left(\int \|\Lambda_x^s h\|_{L_x^2}^2\, dy \right)^{\frac{(1-\vartheta)}{2}} \\ &= C\, \|f\|_2\, \|g\|_{L_x^2 L_y^{\mu}}\, \|h\|_{2(q-1)}^{\vartheta} \|\Lambda_x^s h\|_2^{1-\vartheta}, \end{aligned} \tag{3.18}$$

where $\mu = 2(q-1)/(\vartheta(q-2))$. Clearly, $\mu \ge 2$. By Minkowski's inequality followed by a Sobolev inequality,

$$\|g\|_{L_x^2 L_y^{\mu}} \le \|g\|_{L_y^{\mu} L_x^2} \le C\, \|g\|_2^{\rho}\, \|\partial_y g\|_2^{1-\rho}. \tag{3.19}$$

Inserting (3.19) in (3.18) yields the desired inequality in (3.16). ∎

The following lemma allows us to bound the high frequency and low frequency parts of a function in H^s $(0 < s < 1)$ separately.

Lemma 3.5 *Let $f \in H^s(\mathbb{R}^2)$ with $s \in (0,1)$. Let $R \in (0,\infty)$. Denote by $B(0,R)$ the ball centered at zero with radius R and by $\chi_{B(0,R)}$ the characteristic function on $B(0,R)$. Write*

$$f = \overline{f} + \widetilde{f} \quad \text{with} \quad \overline{f} = \mathcal{F}^{-1}(\chi_{B(0,R)}\mathcal{F}f) \quad \text{and} \quad \widetilde{f} = \mathcal{F}^{-1}((1-\chi_{B(0,R)})\mathcal{F}f), \tag{3.20}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse fourier transform, respectively. Then we have the following estimates for $\overline{f}$ and $\widetilde{f}$.

(1) For a pure constant C_0 (independent of s),

$$\|\overline{f}\|_{L^\infty(\mathbb{R}^2)} \le \frac{C_0}{\sqrt{1-s}} R^{1-s} \|f\|_{H^s(\mathbb{R}^2)}, \tag{3.21}$$

(2) For any $2 \le q < \infty$ satisfying $1 - s - \frac{2}{q} < 0$, there is a constant C_1 independent of s, q, R and f such that

$$\|\widetilde{f}\|_{L^q(\mathbb{R}^2)} \le C_1\, q\, R^{1-s-\frac{2}{q}} \|f\|_{H^s(\mathbb{R}^2)}. \tag{3.22}$$

A similar type of lemma is proved in [13]. The lemma in [13] involves H^1 functions while this lemma 3.5 allows to deal with H^s-functions with $s \in (0,1)$. This lemma can be similarly proven as Lemma 2.3 of [13].

Proof. For any $f \in H^s(\mathbf{R}^2)$, $0 < s < 1$, we can write

$$f = \overline{f} + \widetilde{f} \quad \text{with} \quad \overline{f} = \mathcal{F}^{-1}(\chi_{B(0,R)}\mathcal{F}f) \quad \text{and} \quad \widetilde{f} = \mathcal{F}^{-1}((1-\chi_{B(0,R)})\mathcal{F}f),$$

where

$$\chi_{(0,R)} = \begin{cases} 1 \ \text{ if } \ |\xi| \le R \\ 0 \ \text{ if } \ |\xi| > R. \end{cases}$$

$$
\begin{aligned}
\|\overline{f}\|_\infty \quad &\le \int_{|\xi|\le R} |\widehat{\overline{f}}(\xi)|\, d\xi \\
&= \int_{|\xi|\le R} (1+|\xi|)^{-s}(1+|\xi|)^s\, |\widehat{\overline{f}}(\xi)|\, d\xi \\
&\le C\|f\|_{H^s}(\int_0^R (1+r)^{-2s} r dr)^{\frac{1}{2}} \\
&\le CR^{1-s}\|f\|_{H^s}
\end{aligned}
$$

To prove (3.22), we use the embedding relation: for any $1 \le q < \infty$,

$$
\dot{B}^0_{q,\min\{q,2\}} \hookrightarrow L^q \hookrightarrow \dot{B}^0_{q,\max\{q,2\}},
$$

where $\dot{B}^0_{q,r}$ denotes the homogeneous Besov space. So for $2 \le q < \infty$

$$
\|\tilde{f}\|_q \le Cq\|\tilde{f}\|_{\dot{B}^0_{q,2}} = Cq \left[\sum_{j=-\infty}^{\infty} \|\Delta_j \tilde{f}\|_q^2 \right]^{\frac{1}{2}}
$$

Since $\tilde{f}(\xi)$ has support $|\xi| > R$ so choose $j \le j_0 = [\log_2 R] - 1$ and use Bernsteins inequality,

$$
\begin{aligned}
\|\tilde{f}\|_q \quad &\le Cq \left[\sum_{j=j_0}^{\infty} 2^{4j(\frac{1}{2}-\frac{1}{q})} \|\Delta_j \tilde{f}\|_2^2 \right]^{\frac{1}{2}} = Cq \left[\sum_{j=j_0}^{\infty} 2^{4j(\frac{1}{2}-\frac{1}{q}-\frac{s}{2})} 2^{2js} \|\Delta_j \tilde{f}\|_2^2 \right]^{\frac{1}{2}} \\
&\le Cq 2^{j_0(1-s-\frac{2}{q})} \|\tilde{f}\|_{H^s} \quad \le CqR^{1-s-\frac{2}{q}} \|f\|_{H^s}
\end{aligned}
$$

This completes the proof of the lemma. ■

In [13], the authors had proved the following lemma.

Lemma 3.6 *Let $1 < q < \infty$. Let $f \in L^q(\mathbb{R}^d)$ and let $\widetilde{f}$ be defined as in (3.20). Then, for a constant C depending on q only such that*

$$
\|\widetilde{f}\|_{L^q(\mathbb{R}^d)} \le C\, \|f\|_{L^q(\mathbb{R}^d)}.
$$

After collecting all lemmas we need, we are now ready to prove the main theorem of this section.

Proof. (Proof of Theorem 3.3) As in the proof of Theorem 3.1, we use the symmetric form (3.3) and start with (3.8) with $r > 2$

$$\frac{1}{2r}\frac{d}{dt}\|w_1^+\|_{2r}^{2r} + (2r-1)\int |\partial_x w_1^+|^2 |w_1^+|^{2r-2} = (2r-1)\int p\,\partial_x w_1^+ |w_1^+|^{2r-2}. \qquad (3.23)$$

The term on the right will be treated differently. To start, we fix $R > 0$ (to be specified later) and write

$$(2r-1)\int p\,\partial_x w_1^+ |w_1^+|^{2r-2} = J_1 + J_2,$$

where

$$J_1 = (2r-1)\int \overline{p}\,\partial_x w_1^+\,|w_1^+|^{2r-2}, \qquad J_2 = (2r-1)\int \widetilde{p}\,\partial_x w_1^+\,|w_1^+|^{2r-2}$$

with $\overline{p}$ and $\widetilde{p}$ as defined in (3.20). To estimate J_1 and J_2, we choose two parameters s and q satisfying

$$\frac{\sqrt{5}-1}{2} < s < 1, \quad 2 < q \le \frac{5}{2}, \quad \frac{3}{2} + \frac{1}{2(2s-1)} < q < 1 + \frac{1}{1-s}. \qquad (3.24)$$

The technical constraints in (3.24) will become clear later. By Hölder's and Young's inequalities, we find

$$\begin{aligned} |J_1| &\le (2r-1)\|\overline{p}\|_\infty \|(w_1^+)^{r-1}\|_2 \|\partial_x w_1^+ (w_1^+)^{r-1}\|_2 \\ &\le (2r-1)\|\overline{p}\|_\infty^2 \|(w_1^+)^{r-1}\|_2^2 + \frac{2r-1}{4}\|\partial_x w_1^+ (w_1^+)^{r-1}\|_2^2. \end{aligned}$$

Assuming s and q satisfying (3.24) and applying Lemma 3.5, we have

$$\|\overline{p}\|_\infty \le \frac{C_0}{\sqrt{1-s}}\,R^{1-s}\,\|p\|_{H^s}, \qquad (3.25)$$

where C_0 is a constant independent of s. In the rest of the proof, we pay special attention to whether a constant is bounded uniformly as $s \to 1^-$. By (3.25) and the interpolation inequality

$$\int (w_1^+)^{2r-2} \le \|w_1^+\|_2^{\frac{2}{r-1}} \|w_1^+\|_{2r}^{\frac{2r^2-4r}{r-1}}, \qquad (3.26)$$

we have

$$
\begin{aligned}
|J_1| &\le \frac{2r-1}{4}\|\partial_x w_1^+ (w_1^+)^{r-1}\|_2^2 \\
&\quad + \frac{C_0^2}{1-s}\,(2r-1)\,R^{2(1-s)}\,\|p\|_{H^s}^2\|w_1^+\|_2^{\frac{2}{r-1}}\|w_1^+\|_{2r}^{\frac{2r^2-4r}{r-1}}, \qquad (3.27)
\end{aligned}
$$

where C_0 is independent of s. To bound J_2, we first apply Lemma 3.4 to obtain

$$
|J_2| \le C(2r-1)\,\|\partial_x w_1^+|w_1^+|^{r-1}\|_2\,\|\widetilde{p}\|_{2(q-1)}^{\vartheta}\,\|\Lambda_y^s\widetilde{p}\|_2^{1-\vartheta}\,\|(w_1^+)^{r-1}\|_2^{\rho}\,\|\partial_x(w_1^+)^{r-1}\|_2^{1-\rho}
$$

where s and q satisfy (3.24), ϑ and ρ are given explicitly in terms of s and q,

$$
\vartheta = \frac{(2s-1)(q-1)}{(2s-1)(q-1)+1}, \qquad \rho = \frac{1}{2} + \frac{(2s-1)(q-2)}{2[(2s-1)(q-1)+1]}, \qquad (3.28)
$$

and C is bounded uniformly as $s \to 1^-$. According to (3.26), we have

$$
\|(w_1^+)^{r-1}\|_2^{\rho} \le \|w_1^+\|_2^{\frac{\rho}{r-1}}\|w_1^+\|_{2r}^{\frac{\rho(r^2-2r)}{(r-1)}}.
$$

By Hölder's inequality,

$$
\begin{aligned}
\|\partial_x(w_1^+)^{r-1}\|_2^{1-\rho} &= (r-1)^{1-\rho}\left(\int(\partial_x w_1^+)^2(w_1^+)^{2(r-2)}\right)^{\frac{1}{2}(1-\rho)} \\
&= (r-1)^{1-\rho}\left(\int(\partial_x w_1^+)^{\frac{2}{r-1}}(\partial_x w_1^+)^{\frac{2(r-2)}{r-1}}(w_1^+)^{2(r-2)}\right)^{\frac{1}{2}(1-\rho)} \\
&= (r-1)^{1-\rho}\|\partial_x w_1^+\|_2^{\frac{1-\rho}{r-1}}\left(\int(w_1^+)^{2(r-1)}(\partial_x w_1^+)^2\right)^{\frac{(r-2)(1-\rho)}{2(r-1)}}
\end{aligned}
$$

Therefore, by Young's inequality,

$$
\begin{aligned}
|J_2| &\le C\,(2r-1)(r-1)^{1-\rho}\|\partial_x w_1^+\|_2^{\frac{1-\rho}{r-1}}\|w_1^+\|_2^{\frac{\rho}{r-1}}\|w_1^+\|_{2r}^{\frac{\rho(r^2-2r)}{r-1}} \\
&\qquad \times\|\tilde{p}\|_{2(q-1)}^{\vartheta}\|\Lambda^s\tilde{p}\|_2^{1-\vartheta}\left(\int(\partial_x w_1^+)^2(w_1^+)^{2r-2}\right)^{\frac{1}{2}+\frac{(r-2)(1-\rho)}{2(r-1)}} \\
&\le \frac{2r-1}{4}\int(\partial_x w_1^+)^2(w_1^+)^{2r-2} + C(2r-1)(r-1)^{\frac{2(1-\rho)(r-1)}{\sigma}}\|w_1^+\|_2^{\frac{2\rho}{\sigma}} \\
&\qquad \times\|\partial_x w_1^+\|_2^{\frac{2(1-\rho)}{\sigma}}\|w_1^+\|_{2r}^{\frac{2\rho(r^2-2r)}{\sigma}}\|\tilde{p}\|_{2(q-1)}^{\frac{2\vartheta(r-1)}{\sigma}}\|\Lambda^s\tilde{p}\|_2^{\frac{2(1-\vartheta)(r-1)}{\sigma}}. \qquad (3.29)
\end{aligned}
$$

where C is again bounded uniformly as $s \to 1^-$, and, for notational convenience, we have written

$$\sigma = (r-1) - (1-\rho)(r-2) = 1 + \rho r - 2\rho. \tag{3.30}$$

To further estimate, we split $\|\tilde{p}\|_{2(q-1)}$ into two parts and bound one of them via Lemma 3.5. More precisely, we have, for any $0 \le \beta \le 1$,

$$\begin{aligned} \|\tilde{p}\|_{2(q-1)} &= \|\tilde{p}\|_{2(q-1)}^{1-\beta} \, \|\tilde{p}\|_{2(q-1)}^{\beta} \\ &\le C \, \|\tilde{p}\|_{2(q-1)}^{1-\beta} \, R^{(1-s-\frac{1}{q-1})\beta} \, \|p\|_{H^s}^{\beta} \\ &\le C \, \|p\|_{2(q-1)}^{1-\beta} \, R^{(1-s-\frac{1}{q-1})\beta} \, \|p\|_{H^s}^{\beta}. \end{aligned} \tag{3.31}$$

Where the last inequality follows from Lemma 3.6 and C is a constant independent of s. Due to the conditions on s and q in (3.24), this bound allows us to generate $R^{(1-s-\frac{1}{q-1})\beta}$ with $(1-s-\frac{1}{q-1})\beta \le 0$. Inserting (3.31) in (3.29) yields

$$\begin{aligned} |J_2| \;\le\; & \frac{2r-1}{4} \int (\partial_x w_1^+)^2 (w_1^+)^{2r-2} \\ & + C(2r-1)(r-1)^{\frac{2(1-\rho)(r-1)}{\sigma}} \, R^{(1-s-\frac{1}{q-1})\beta \frac{2\vartheta(r-1)}{\sigma}} \, \|w_1^+\|_2^{\frac{2\rho}{\sigma}} \\ & \times \|\partial_x w_1^+\|_2^{\frac{2(1-\rho)}{\sigma}} \, \|w_1^+\|_{2r}^{\frac{2\rho(r^2-2r)}{\sigma}} \, \|p\|_{2(q-1)}^{(1-\beta)\frac{2\vartheta(r-1)}{\sigma}} \, \|p\|_{H^s}^{\beta\frac{2\vartheta(r-1)}{\sigma} + \frac{2(1-\vartheta)(r-1)}{\sigma}}, \end{aligned}$$

where, again, C is bounded uniformly as $s \to 1^-$. We choose β so that the sum of the powers of $\|\partial_x w_1^+\|_2$ and of $\|p\|_{H^s}$ is equal to 2, namely

$$\frac{2(1-\rho)}{\sigma} + \beta \frac{2\vartheta(r-1)}{\sigma} + \frac{2(1-\vartheta)(r-1)}{\sigma} = 2.$$

Recalling (3.28) and (3.30), we find that

$$\beta = \frac{(2s-1)(2q-3)-1}{(2q-2)(2s-1)}. \tag{3.32}$$

The condition in (3.24) guarantees that $0 < \beta \le 1$. Then

$$\|\partial_x w_1^+\|_2^{\frac{2(1-\rho)}{\sigma}} \, \|p\|_{H^s}^{\beta\frac{2\vartheta(r-1)}{\sigma} + \frac{2(1-\vartheta)(r-1)}{\sigma}} \le C \left(\|\partial_x w_1^+\|_2^2 + \|p\|_{H^s}^2 \right).$$

Therefore, for β given by (3.32), we have

$$\begin{aligned}
|J_2| \;\leq\; & \frac{2r-1}{4}\int(\partial_x w_1^+)^2(w_1^+)^{2r-2} \\
& + C(2r-1)(r-1)^{\frac{2(1-\rho)(r-1)}{\sigma}}\, R^{(1-s-\frac{1}{q-1})\beta\frac{2\vartheta(r-1)}{\sigma}}\, \|w_1^+\|_2^{\frac{2\rho}{\sigma}} \\
& \times \|p\|_{2(q-1)}^{(1-\beta)\frac{2\vartheta(r-1)}{\sigma}} \left(\|\partial_x w_1^+\|_2^2 + \|p\|_{H^s}^2\right) \|w_1^+\|_{2r}^{\frac{2\rho(r^2-2r)}{\sigma}}.
\end{aligned} \tag{3.33}$$

Combining (3.23), (6.27) and (6.28), we obtain

$$\begin{aligned}
& \frac{1}{2r}\frac{d}{dt}\|w_1^+\|_{2r}^{2r} + \frac{2r-1}{4}\int|\partial_x w_1^+|^2|w_1^+|^{2r-2} \\
& \leq \frac{C_0^2}{1-s}\,(2r-1)\,R^{2(1-s)}\,\|p\|_{H^s}^2\|w_1^+\|_2^{\frac{2}{r-1}}\|w_1^+\|_{2r}^{\frac{2r^2-4r}{r-1}} \\
& \quad + C(2r-1)(r-1)^{\frac{2(1-\rho)(r-1)}{\sigma}}\, R^{(1-s-\frac{1}{q-1})\beta\frac{2\vartheta(r-1)}{\sigma}}\, \|w_1^+\|_2^{\frac{2\rho}{\sigma}} \\
& \quad \times \|p\|_{2(q-1)}^{(1-\beta)\frac{2\vartheta(r-1)}{\sigma}} \left(\|\partial_x w_1^+\|_2^2 + \|p\|_{H^s}^2\right) \|w_1^+\|_{2r}^{\frac{2\rho(r^2-2r)}{\sigma}}
\end{aligned} \tag{3.34}$$

where C_0 is independent of s and C is bounded uniformly as $s \to 1^-$. We now choose R such that

$$R^{2(1-s)} = (r-1)^{\frac{2(1-\rho)(r-1)}{\sigma}}\, R^{(1-s-\frac{1}{q-1})\beta\frac{2\vartheta(r-1)}{\sigma}}.$$

Solving this equation for R, we find

$$R^{2(1-s)} = (r-1)^{\frac{2(1-s)(1-\rho)(r-1)}{(1-s)\sigma+\beta\vartheta\left(s-1+\frac{1}{q-1}\right)(r-1)}}.$$

We then use (3.28),(3.30) and (3.32) to simplify this index and obtain

$$\frac{2(1-s)(1-\rho)(r-1)}{(1-s)\sigma+\beta\vartheta\left(s-1+\frac{1}{q-1}\right)(r-1)} = \frac{2(1-s)(q-1)}{q-2+(r-1)^{-1}(1-s)(q-1)}.$$

We denote this index by δ,

$$\delta \equiv \frac{2(1-s)(q-1)}{q-2+(r-1)^{-1}(1-s)(q-1)} \tag{3.35}$$

and therefore $R^{2(1-s)} = (r-1)^{\delta}$. Clearly, $\delta \to 0$ as $s \to 1$, and

$$\frac{1}{1-s} = \frac{q-1}{q-2}\left(2 - \frac{\delta}{r-1}\right)\frac{1}{\delta} \leq \frac{2q-2}{q-2}\,\frac{1}{\delta}.$$

In addition, we notice that

$$\frac{2r^2-4r}{r-1} \le 2r-2, \qquad \frac{2\rho(r^2-2r)}{\sigma} \le 2r-2.$$

Without loss of generality, we assume $\|w_1^+\|_{2r} \ge 1$. It then follows from (3.34) that

$$\frac{d}{dt}\|w_1^+\|_{2r}^2 \le \frac{C}{\delta}\, B(t)\,(2r-1)\,(r-1)^{\delta}, \tag{3.36}$$

where C is bounded uniformly as $\delta \to 0^+$, and

$$B(t) = \|p\|_{H^s}^2 \|w_1^+\|_2^{\frac{2}{r-1}} + \|w_1^+\|_2^{\frac{2\rho}{\sigma}}\ \|p\|_{2(q-1)}^{(1-\beta)\frac{2\vartheta(r-1)}{\sigma}} \left(\|\partial_x w_1^+\|_2^2 + \|p\|_{H^s}^2\right).$$

Since (6.10) holds for any $\delta > 0$, we set

$$\delta = \frac{1}{\log(r-1)}$$

to obtain the optimal upper bound

$$\frac{d}{dt}\|w_1^+\|_{2r}^2 \le C\, B(t)\,(2r-1)\,\log(r-1) \tag{3.37}$$

After δ is selected, we choose s and q satisfying (**??**) to fulfill (3.35) Since we have chosen $2 < q \le \frac{5}{2}$, $2 < 2(q-1) \le 3$. According to Theorem 3.2, $B(t)$ is integrable on any time interval. We obtain (3.15) after integrating (3.37) in time. This completes the proof of Theorem 3.3. ■

3.4 Conditional Global Regularity

This section establishes the global bounds for $\|(u,b)\|_{H^2}$ in terms of the norms of the horizontal components u_1 and b_1 in $L_t^2 L_x^\infty$. We prove that any possible blowup of the classical solution to the system (3.1) can be controlled by the L^∞-norm of the horizontal components of the velocity field and the magnetic field. More precisely, we have the following theorem.

Theorem 3.4 *Assume $(u_0, b_0) \in H^2(\mathbb{R}^2)$ and let (u, b) be the corresponding solution of (3.1). If*

$$\int_0^T \|(u_1, b_1)(t)\|_\infty^2 \, dt < \infty$$

for some $T > 0$, then $\|(u, b)\|_{H^2}$ is finite on $[0, T]$.

The proof of this theorem is divided into two major parts. The first part bounds the H^1-norm while the second bounds the H^2-norm.

3.4.1 H^1 Bound in terms of $\|(u_1, b_1)\|_{L_t^2 L_x^\infty}$

This subsection proves the following proposition.

Proposition 3.4.1 *Assume $(u_0, b_0) \in H^2(\mathbb{R}^2)$ and let (u, b) be the corresponding solution of (3.1). Then, for any $T > 0$ and $t \le T$,*

$$\|(u, b)(t)\|_{H^1} \le C_1 \, e^{C_2 \int_0^t \left(\|u_1(\tau)\|_\infty^2 + \|b_1(\tau)\|_\infty^2 \right) d\tau}, \tag{3.38}$$

where C_1 depends on T and the initial data only, and C_2 is a pure constant.

Proof. Taking the inner product of the first equation of (3.3) with Δw^+ and integrating by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla w^+\|_2^2 + \|\partial_x \nabla w^+\|_2^2 = I_1 + I_2 + I_3 + I_4 + I_5 + I_6,$$

where

$$\begin{aligned} I_1 &= \int \partial_x w_1^- \partial_x w_2^+ \partial_x w_2^+, \quad I_2 = \int \partial_x w_2^- \partial_y w_1^+ \partial_x w_1^+, \quad I_3 = \int \partial_x w_2^- \partial_y w_2^+ \partial_x w_2^+, \\ I_4 &= \int \partial_y w_1^- \partial_x w_1^+ \partial_y w_1^+, \quad I_5 = \int \partial_y w_1^- \partial_x w_2^+ \partial_y w_2^+, \quad I_6 = \int \partial_y w_2^- \partial_y w_1^+ \partial_y w_1^+. \end{aligned}$$

The terms can be bounded as follows. By Lemma 3.1,

$$\begin{aligned} |I_1| &\le \|\partial_x w_1^-\|_2 \, \|\partial_x w_2^+\|_2^{\frac{1}{2}} \, \|\partial_x^2 w_2\|_2^{\frac{1}{2}} \, \|\partial_x w_2^+\|_2^{\frac{1}{2}} \, \|\partial_{xy}^2 w_2^+\|_2^{\frac{1}{2}} \\ &\le \|\partial_x w_1^-\|_2 \, \|\partial_x w_2^+\|_2 \, \|\nabla \partial_x w_2^+\|_2 \\ &\le \frac{1}{16} \|\nabla \partial_x w_2^+\|_2^2 + C \|\partial_x w_1^-\|_2^2 \, \|\nabla w_2^+\|_2^2. \end{aligned}$$

Similarly,

$$|I_2| \leq \frac{1}{16}\|\nabla\partial_x w_1^+\|_2^2 + C\|\partial_x w_2^-\|_2^2\,\|\nabla w_1^+\|_2^2.$$

$$|I_3| \leq \frac{1}{16}\|\nabla\partial_x w_2^+\|_2^2 + C\|\partial_x w_2^-\|_2^2\,\|\nabla w_2^+\|_2^2.$$

Integrating by parts, we have

$$I_4 = -\int \partial_{xy}^2 w_1^- w_1^+ \partial_y w_1^+ - \int \partial_y w_1^- w_1^+ \partial_{xy}^2 w_1^+.$$

By Hölder's inequality,

$$\begin{aligned} I_4 &\leq 2\|w_1^+\|_\infty\,\|\nabla\partial_x w_1^-\|_2\,\|\nabla w_1^-\|_2 \\ &\leq \frac{1}{16}\|\nabla\partial_x w_1^-\|_2^2 + C\|w_1^+\|_\infty^2\,\|\nabla w_1^-\|_2^2. \end{aligned}$$

I_5 and I_6 admit similar bounds as I_4,

$$|I_5| \leq \frac{1}{16}\|\nabla w_x^+\|_2^2 + C\|w_1^-\|_\infty^2\|\nabla w_2^+\|_2^2.$$

$$|I_6| \leq \frac{1}{16}\|\nabla\partial_x w_1^+\|^2 + C\|w_1^-\|_\infty^2\|\nabla w_1^+\|_2^2.$$

Similar estimates can be obtained for ∇w^-. Their combination yields

$$\begin{aligned} &\frac{d}{dt}\left(\|\nabla w^+\|_2^2 + \|\nabla w^-\|_2^2\right) + \left(\|\partial_x \nabla w^+\|_2^2 + \|\partial_x \nabla w^-\|_2^2\right) \\ &\quad \leq \left(\|\partial_x w^+\|_2^2 + \|\partial_x w^-\|_2^2 + \|w_1^-\|_\infty^2 + ||w_1^+\|_\infty^2\right)\left(\|\nabla w^+||^2 + ||\nabla w^-\|^2\right). \end{aligned}$$

Gronwall's lemma then yields the desired L^2-bound for $(\nabla u, \nabla b)$. Combining with the global L^2-bound in Lemma 1.7 leads to (3.38). ■

3.4.2 H^2 Bound in terms of $\|(u_1, b_1)\|_{L_t^2 L_x^\infty}$

This subsection is dedicated to prove the theorem 3.4.

Proof. Taking the inner product of the first equation in (3.3) with $\Delta^2 w^+$ and integrating by parts, we find

$$\frac{1}{2}\frac{d}{dt}\|\Delta w^+\|_2^2 + \|\partial_x \Delta w^+\|_2^2 = -\int \Delta(w^- \cdot \nabla w^+) \cdot \Delta w^+. \tag{3.39}$$

In order to make use of the anisotropic dissipation, we need to decompose the nonlinear term into different parts which show explicit dependence on the horizontal and vertical derivatives. We write

$$\int \Delta(w^- \cdot \nabla w^+) \cdot \Delta w^+ = J_1 + J_2 + J_3,$$

where

$$J_1 = \int (\Delta w^- \cdot \nabla w^+) \cdot \Delta w^+, \quad J_2 = 2\int (\partial_x w^- \cdot \nabla \partial_x w^+) \cdot \Delta w^+,$$
$$J_3 = 2\int (\partial_y w^- \cdot \nabla \partial_y w^+) \cdot \Delta w^+.$$

We further decompose J_1 into four terms, $J_1 = J_{11} + J_{12} + J_{13} + J_{14}$, where

$$J_{11} = \int (\Delta w_1^- \partial_x w_1^+)\, \Delta w_1^+, \quad J_{12} = \int (\Delta w_1^- \partial_x w_2^+)\, \Delta w_2^+,$$
$$J_{13} = \int (\Delta w_2^- \partial_y w_1^+)\, \Delta w_1^+, \quad J_{14} = \int (\Delta w_2^- \partial_y w_2^+)\, \Delta w_2^+.$$

It is clear that, after integration by parts and applying Hölder's inequality,

$$|J_{11}| \le \frac{1}{16}\left(\|\Delta\partial_x w_1^+\|_2^2 + \|\Delta\partial_x w_1^-\|_2^2\right) + 4\|w_1^+\|_\infty^2 \left(\|\Delta w_1^+\|_2^2 + \|\Delta w_1^-\|_2^2\right).$$

Similarly, after invoking $\partial_x w_1^+ + \partial_y w_2^+ = 0$,

$$|J_{14}| \le \frac{1}{16}\left(\|\Delta\partial_x w_2^+\|_2^2 + \|\Delta\partial_x w_2^-\|_2^2\right) + 4\|w_1^+\|_\infty^2 \left(\|\Delta w_2^+\|_2^2 + \|\Delta w_2^-\|_2^2\right).$$

To bound J_{12}, we apply Lemma 3.1 to obtain

$$\begin{aligned}
|J_{12}| &\le \|\partial_x w_2^+\|_2 \,\|\Delta w_1^-\|_2^{\frac{1}{2}} \,\|\Delta\partial_x w_1^-\|_2^{\frac{1}{2}} \,\|\Delta w_2^+\|_2^{\frac{1}{2}} \,\|\Delta\partial_y w_2^+\|_2^{\frac{1}{2}} \\
&= \|\partial_x w_2^+\|_2 \,\|\Delta w_1^-\|_2^{\frac{1}{2}} \,\|\Delta\partial_x w_1^-\|_2^{\frac{1}{2}} \,\|\Delta w_2^+\|_2^{\frac{1}{2}} \,\|\Delta\partial_x w_1^+\|_2^{\frac{1}{2}} \\
&\le \frac{1}{16}\left(\|\Delta\partial_x w_1^+\|_2^2 + \|\Delta\partial_x w_1^-\|_2^2\right) + 4\|\partial_x w_2^+\|_2^2 \left(\|\Delta w_2^+\|_2^2 + \|\Delta w_1^-\|_2^2\right).
\end{aligned}$$

To bound J_{13}, we need the H^1-bound from Proposition 3.4.1. By Lemma 3.1,

$$\begin{aligned}
|J_{13}| &\le \|\Delta w_1^+\|_2\, \|\Delta w_2^-\|_2^{\frac12}\, \|\Delta\partial_y w_2^-\|_2^{\frac12}\, \|\partial_y w_1^+\|_2^{\frac12}\, \|\partial_x\partial_y w_1^+\|_2^{\frac12} \\
&\le \|\Delta w_1^+\|_2\, \|\Delta w_2^-\|_2^{\frac12}\, \|\Delta\partial_x w_1^-\|_2^{\frac12}\, \|\nabla w_1^+\|_2^{\frac12}\, \|\nabla\partial_x w_1^+\|_2^{\frac12} \\
&\le \frac{1}{16}\|\Delta\partial_x w_1^-\|_2^2 + C\|\nabla w_1^+\|_2\|\Delta w_1^+\|_2^2 + C\|\partial_x\nabla w_1^+\|_2^2\|\Delta w_2^-\|_2^2.
\end{aligned}$$

Collecting the estimates for J_1, we have found that four terms

$$\begin{aligned}
|J_1| \le{}& \frac14\left(\|\Delta\partial_x w^+\|_2^2 + \|\Delta\partial_x w^-\|_2^2\right) \\
&+\left(\|w_1^+\|_\infty^2 + \|\partial_x w^+\|_2^2 + \|\nabla w_1^+\|_2 + \|\partial_x\nabla w_1^+\|_2^2\right)\left(\|\Delta w^+\|_2^2 + \|\Delta w^-\|_2^2\right).
\end{aligned}$$

J_2 and J_3 can be estimated in a similar fashion and we omit further details. Combining the estimates for all of them and applying Gronwall's inequality yields the desired global result. This completes the proof of the theorem 3.4. ■

3.5 Global Regularity for a Slightly Regularized System

This section establishes that (3.2) possesses global regular solutions if the initial data are sufficiently smooth. More precisely, we have the following theorem.

Theorem 3.5 *Let $\epsilon > 0$ and $\delta > 0$ be real parameters. Consider (3.2) with an initial data $(u_0, b_0) \in H^2(\mathbb{R}^2)$. Then the corresponding solution (u, b) obeys the following global a priori bounds, for any $T > 0$ and $t \le T$,*

$$\|(u,b)\|_{H^2}^2 + \int_0^t \left(\|(\partial_x u, \partial_x b)\|_{H^2}^2 + \epsilon\|(\Lambda^\delta u, \Lambda^\delta b)\|_{H^2}^2\right)\, d\tau \le C,$$

where C is a constant depending on T and $\|(u_0, b_0)\|_{H^2}$ only.

Proof. We show that (u, b) admits a global H^2 bound. Clearly,

$$\begin{aligned}
&\|(u,b)(t)\|_2^2 + 2\int_0^t \left(\|\partial_x u(\tau)\|_2^2 + \|\partial_x b(\tau)\|_2^2\right)\, d\tau \\
&\qquad +2\epsilon\int_0^t \left(\|\Lambda^\delta u(\tau)\|_2^2 + \|\Lambda^\delta b(\tau)\|_2^2\right)\, d\tau = \|(u_0, b_0)\|_2^2.
\end{aligned}$$

To obtain the global bound for the H^1-norm, we take advantage of the vorticity formulation. Taking the curl of (3.2), we find that $\omega = \nabla \times u$ and $j = \nabla \times b$ satisfy

$$\begin{cases} \omega_t + u \cdot \nabla\omega + \epsilon(-\Delta)^\delta \omega = b \cdot \nabla j + \omega_{xx}, \\ j_t + u \cdot \nabla j + \epsilon(-\Delta)^\delta j = b \cdot \nabla\omega + j_{xx} \\ \qquad\qquad + 2\partial_x b_1(\partial_y u_1 + \partial_x u_2) - 2\partial_x u_1(\partial_y b_1 + \partial_x b_2). \end{cases} \tag{3.40}$$

Taking the inner product of (3.40) with (ω, j) and integrating by parts, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|\omega\|_2^2 + \|j\|_2^2\right) &+ \|\partial_x\omega\|_2^2 + \|\partial_x j\|_2^2 + \epsilon\|\Lambda^\delta\omega\|_2^2 + \epsilon\|\Lambda^\delta j\|_2^2 \\ &= J_1 + J_2 + J_3 + J_4, \end{aligned} \tag{3.41}$$

where

$$\begin{aligned} J_1 &= 2\int \partial_x b_1\, \partial_y u_1\, j\, dxdy, \quad J_2 = 2\int \partial_x b_1\, \partial_x u_2\, j\, dxdy, \\ J_3 &= 2\int \partial_x u_1\, \partial_y b_1\, j\, dxdy, \quad J_4 = 2\int \partial_x u_1\, \partial_x b_2\, j\, dxdy. \end{aligned}$$

The terms above can be bounded as follows. Integrating by parts, we have

$$J_1 = -2\int b_1\, \partial_{xy} u_1\, j - 2\int b_1\, \partial_y u_1\, \partial_x j.$$

Choose q large enough such that $q\delta > 2$. By Hölder's inequality,

$$|J_1| \le 2\|b_1\|_q\, \|\partial_{xy}u_1\|_2\, \|j\|_{\frac{2q}{q-2}} + 2\,\|b_1\|_q\, \|\partial_x j\|_2\, \|\partial_y u_1\|_{\frac{2q}{q-2}}. \tag{3.42}$$

By the boundedness of singular integral operators,

$$\|\partial_{xy}u_1\|_2 \le C\,\|\partial_x\omega\|_2, \quad \|\partial_y u_1\|_{\frac{2q}{q-2}} \le C\,\|\omega\|_{\frac{2q}{q-2}}.$$

Applying the Sobolev inequality, for $q > 2$ and $q\delta > 2$

$$\|f\|_{\frac{2q}{q-2}} \le C\,\|f\|_2^{1-\frac{2}{q\delta}}\, \|\Lambda^\delta f\|_2^{\frac{2}{q\delta}},$$

and Young's inequality, we obtain

$$\begin{aligned} |J_1| \le\ & \frac{1}{8}\|\partial_x\omega\|_2^2 + \frac{\epsilon}{4}\|\Lambda^\delta j\|_2^2 + C\,\|b_1\|_q^{\frac{2q\delta}{q\delta-2}}\, \|j\|_2^2 \\ & + \frac{1}{8}\|\partial_x j\|_2^2 + \frac{\epsilon}{4}\|\Lambda^\delta\omega\|_2^2 + C\,\|b_1\|_q^{\frac{2q\delta}{q\delta-2}}\, \|\omega\|_2^2. \end{aligned}$$

J_2 can be bounded through Lemma 3.1,

$$\begin{aligned}|J_2| &\le C\,\|\partial_x b_1\|_2\,\|\partial_x u_2\|_2^{\frac{1}{2}}\,\|\partial_{xy} u_2\|_2^{\frac{1}{2}}\,\|j\|_2^{\frac{1}{2}}\,\|\partial_x j\|_2^{\frac{1}{2}}\\ &\le \frac{1}{8}\|\partial_x\omega\|_2^2+\frac{1}{8}\|\partial_x j\|_2^2+C\,\|\partial_x b_1\|_2^2\,\left(\|\omega\|_2^2+\|j\|_2^2\right).\end{aligned}$$

To bound J_3, we first integrate by parts to obtain

$$J_3=-2\int u_1\,\partial_{xy}b_1\,j-2\int u_1\,\partial_y b_1\,\partial_x j.$$

The terms on the right can then be estimated in a similar fashion as in (3.42) and the bound is

$$|J_3|\le\frac{1}{8}\|\partial_x\omega\|_2^2+\frac{1}{8}\|\partial_x j\|_2^2+\frac{\epsilon}{4}\|\Lambda^\delta j\|_2^2+C\,\|u_1\|_q^{\frac{2q\delta}{q\delta-2}}\,\|j\|_2^2.$$

J_4 can be bounded in a a similar fashion as J_2 and

$$|J_4|\le\frac{1}{8}\|\partial_x j\|_2^2+C\,\|\partial_x u_1\|_2^2\,\|j\|_2^2.$$

Inserting the estimates for J_1, J_2, J_3 and J_4 in (3.41) yields the desired global H^1-bound.

To establish the global H^2-bound, we take the inner product of (3.40) with $(\Delta\omega,\Delta j)$ to obtain, after integration by parts,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla\omega\|_2^2+\|\nabla j\|_2^2)+\|\nabla\partial_x\omega\|_2^2+\|\nabla\partial_x j\|_2^2+\epsilon\|\Lambda^{\delta+1}\omega\|_2^2+\epsilon\|\Lambda^{\delta+1}j\|_2^2\\ &\quad=L_1+L_2+L_3+L_4+L_5,\end{aligned}\tag{3.43}$$

where

$$\begin{aligned}L_1&=-\int\nabla\omega\cdot\nabla u\cdot\nabla\omega\;dxdy,\\ L_2&=-\int\nabla j\cdot\nabla u\cdot\nabla j\;dxdy,\\ L_3&=\int\nabla\omega\cdot(\nabla b+(\nabla b)^t)\cdot\nabla j\;dxdy,\\ L_4&=2\int\nabla[\partial_x b_1(\partial_x u_2+\partial_y u_1)]\cdot\nabla j\;dxdy,\\ L_5&=-2\int\nabla[\partial_x u_1(\partial_x b_2+\partial_y b_1)]\cdot\nabla j\;dxdy.\end{aligned}$$

To estimate L_1, we write the integrand explicitly

$$L_1 \;=\; \int (\partial_x u_1 (\partial_x \omega)^2 + (\partial_x u_2 + \partial_y u_1)\partial_x \omega \partial_y \omega + \partial_y u_2 (\partial_y \omega)^2)\, dxdy.$$

Each one of them can be bounded by Lemma 3.1 and then by Young's inequality. For example,

$$\begin{aligned}
\int \partial_x u_1 (\partial_x \omega)^2\, dxdy \;&\leq\; C\|\partial_x u_1\|_2 \|\partial_x \omega\|_2^{\frac{1}{2}} \|\partial_x^2 \omega\|_2^{\frac{1}{2}} \|\partial_x \omega\|_2^{\frac{1}{2}} \|\partial_{xy}^2 \omega\|_2^{\frac{1}{2}} \\
&\leq\; \frac{1}{32}\|\nabla \partial_x \omega\|_2^2 + C\|\omega\|_2^2 \|\nabla \omega\|_2^2
\end{aligned}$$

Since the estimates for other terms are similar, we obtain

$$|L_1| \leq \frac{1}{8}\|\nabla \partial_x \omega\|_2^2 + C\,\|\omega\|_2^2 \|\nabla \omega\|_2^2 + C\,\|\nabla \omega\|_2^2 \|\omega\|_2^{\frac{2}{3}} \|\partial_x \omega\|_2^{\frac{2}{3}}.$$

Similarly, L_2, L_3, L_4 and L_5 are bounded by

$$\begin{aligned}
|L_2| \;&\leq\; \frac{1}{8}\|\nabla \partial_x j\|_2^2 + C\,\left(\|\omega\|_2^2 + \|\omega\|_2^{\frac{2}{3}} \|\partial_x \omega\|_2^{\frac{2}{3}}\right) \|\nabla j\|_2^2, \\
|L_3| \;&\leq\; \frac{1}{8}\|\nabla \partial_x \omega\|_2^2 + \frac{1}{8}\|\nabla \partial_x j\|_2^2 \\
&\quad + C\,\|j\|_2^2 (\|\nabla \omega\|_2^2 + \|\nabla j\|_2^2) + C\,\|\partial_x j\|_2^2 \|\nabla j\|_2^2, \\
|L_4| \;&\leq\; \frac{1}{8}\|\nabla \partial_x \omega\|_2^2 + \frac{1}{8}\|\nabla \partial_x j\|_2^2 \\
&\quad + C\,\left(\|j\|_2^2 + \|\omega\|_2^2 + \|\partial_x j\|_2^2 + \|\omega\|_2^{\frac{2}{3}} \|\partial_x \omega\|_2^{\frac{2}{3}}\right) (\|\nabla \omega\|_2^2 + \|\nabla j\|_2^2), \\
|L_5| \;&\leq\; \frac{1}{8}\|\nabla \partial_x \omega\|_2^2 + \frac{1}{8}\|\nabla \partial_x j\|_2^2 \\
&\quad + C\,\left(\|j\|_2^2 + \|\omega\|_2^2 + \|\partial_x j\|_2^2 + \|\partial_x \omega\|_2^2\right) (\|\nabla \omega\|_2^2 + \|\nabla j\|_2^2).
\end{aligned}$$

Inserting these estimates in (3.43), applying Gronwall's inequality and invoking the global H^1-bound, we achieved the desired global H^2-bound for the solution. This concludes the proof of Theorem 3.5. ■

CHAPTER 4

Global Regularity for the 2D Magnetohydrodynamic Equations with Horizontal Dissipation and Horizontal Magnetic Diffusion

In this chapter, we consider 2D anisotropic MHD equations.

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \partial_x^2 u + b \cdot \nabla b, \qquad (x,y) \in \mathbb{R}^2,\ t>0, \\ \partial_t b + u \cdot \nabla b = \partial_x^2 b + b \cdot \nabla u, \qquad (x,y) \in \mathbb{R}^2,\ t>0, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \qquad (x,y) \in \mathbb{R}^2,\ t>0, \\ u(x,y,0) = u_0(x,y),\ b(x,y,0) = b_0(x,y), \qquad (x,y) \in \mathbb{R}^2. \end{cases} \tag{4.1}$$

We establish the global regularity issue of the system (4.1).

The local well-posedness of (4.1) can be obtained by the standard classical method. The difficult part is the global bounds for the velocity and magnetic field. The global regularity issue of (4.1) is reduced to finding the global H^1 and H^2-bound for the velocity and magnetic field. More precisely, we prove the following theorem in this chapter.

Theorem 4.1 *Assume that $(u_0, b_0) \in H^2(\mathbb{R}^2)$, $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then, (4.1) has a unique global solution (u, b) satisfying, for any $T > 0$ and $t \leq T$,*

$$u,\ b,\ \partial_x u, \partial_x b \in L^\infty([0,T]; H^2(\mathbb{R}^2)).$$

The global regularity issue on the MHD equations with this type of partial dissipation is extremely difficult. The direct energy method fails because of the dissipation and diffusion in only one direction.

In the chapter 3 we realized that that the time square integrability of the L^∞-norm of (u_1, b_1) controls the regularity. However, it is extremely difficult to achieve the global bound for $\|(u_1, b_1)\|_{L^\infty}$. Because of this difficulty we would like to use the bound for L^q, $q > 2$ for the global regularity. We proved the following bound in the previous chapter.

$$\|(u_1, b_1)\|_{L^q} \leq C\sqrt{q\log q} \text{ for any } q \in (2, \infty).$$

This bound is very helpful to bound the lower part of the horizontal components. In order to achieve the global H^2 bound for the velocity field u and the magnetic field b, we first decompose the horizontal components u_1 and b_1 into lower and higher frequency part by using Littlewood-Paley decomposition. The low frequency part of (u_1, b_1) can be bounded by the logarithmic bound of the horizontal components and the higher frequency part of (u_1, b_1) can be bounded by anisotropic dissipation and diffusion.

4.1 H^1 Bound for u and b

The global H^2-bound for u and b can be obtained by using the global L^2 and H^1-bound. The global L^2-bound for u and b is obvious. In this section we establish the global H^1-bound. More precisely, we prove the following proposition.

Proposition 4.1.1 *Assume that $(u_0, b_0) \in H^2(\mathbb{R}^2)$, $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Let (u, b) be the corresponding solution of (4.1). Then, for any $T > 0$ and $t \leq T$,*

$$\|(u(t),\, b(t))\|_{H^1} \leq C(T, u_0, b_0).$$

In order to prove this theorem, first we prove the following lemma [77].

Lemma 4.1 *Let $\alpha > \frac{1}{2}$. Then, there exists a constant $C = C(\alpha)$ such that*

$$\|f\|_{L^\infty(\mathbb{R}^2)} \leq C\,\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\,\|\partial_y f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\,\|\partial_x f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}-\frac{1}{4\alpha}}\,\|\Lambda_y^\alpha \partial_x f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4\alpha}}, \tag{4.2}$$

where the one-dimensional fractional operator Λ_y^α is defined as a Fourier multiplier operator, namely

$$\Lambda_y^\alpha f(x, y) = \int_{\mathbb{R}^2} |\xi_2|^\alpha e^{i(x\xi_1 + y\xi_2)}\widehat{f}(\xi_1, \xi_2)\, d\xi_1 d\xi_2.$$

Proof. (Proof of Lemma 4.1) We have one-dimensional Sobolev inequalities;

$$\|f\|_{L^\infty_x} \le C\,\|f\|_{L^2_x}^{\frac{1}{2}}\,\|\partial_x f\|_{L^2_x}^{\frac{1}{2}}, \qquad \|f\|_{L^\infty_y} \le C(\alpha)\,\|f\|_{L^2_y}^{1-\frac{1}{2\alpha}}\,\|\Lambda_y^\alpha f\|_{L^2_y}^{\frac{1}{2\alpha}}.$$

Therefore, by Minkowski's inequality,

$$\begin{aligned}
\|f\|_{L^\infty(\mathbb{R}^2)} &= \|\|f\|_{L^\infty_x}\|_{L^\infty_y} \\
&\le C\,\|\|f\|_{L^\infty_y}\|_{L^2_x}^{\frac{1}{2}}\,\|\|\partial_x f\|_{L^\infty_y}\|_{L^2_x}^{\frac{1}{2}} \\
&\le C\,\|\left(\|f\|_{L^2_y}^{\frac{1}{2}}\,\|\partial_y f\|_{L^2_y}^{\frac{1}{2}}\right)\|_{L^2_x}^{\frac{1}{2}}\,\|\left(\|\partial_x f\|_{L^2_x}^{1-\frac{1}{2\alpha}}\,\|\Lambda_y^\alpha \partial_x f\|_{L^2_x}^{\frac{1}{2\alpha}}\right)\|_{L^2_x}^{\frac{1}{2}}
\end{aligned}$$

(4.2) then follows from Young's inequality. This completes the proof of Lemma 4.1.

■

Proof. (Proof of Proposition 4.1.1) Multiplying the vorticity equation by ω and current density equation by j, integrating with respect to space variable, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\right) + \|\partial_x \omega\|_{L^2}^2 + \|\partial_x j\|_{L^2}^2 \\
&\qquad = 2\int \partial_x b_1(\partial_x u_2 + \partial_y u_1)\, j\, dxdy - 2\int \partial_x u_1(\partial_x b_2 + \partial_y b_1)\, j\, dxdy. \quad (4.3)
\end{aligned}$$

For notational convenience, we will omit $dxdy$ from the spatial integral and set

$$Y(t) = \|\omega(\cdot,t)\|_{L^2}^2 + \|j(\cdot,t)\|_{L^2}^2.$$

The first term can be bounded by Lemma 3.1

$$J_1 \equiv \left|2\int \partial_x b_1\,\partial_x u_2\, j\right| \le C\,\|\partial_x u_2\|_{L^2}\,\|\partial_x b_1\|_{L^2}^{\frac{1}{2}}\,\|\partial_x\partial_y b_1\|_{L^2}^{\frac{1}{2}}\,\|j\|_{L^2}^{\frac{1}{2}}\,\|\partial_x j\|_{L^2}^{\frac{1}{2}}.$$

Applying Young's inequality and the fact that

$$\|\partial_x b_1\|_{L^2} \le \|j\|_{L^2}, \qquad \|\partial_x\partial_y b_1\|_{L^2} \le \|\partial_x j\|_{L^2},$$

we have

$$J_1 \le \frac{1}{8}\|\partial_x j\|_{L^2}^2 + C\,\|\partial_x u_2\|_{L^2}^2\,\|j\|_{L^2}^2.$$

Similarly,

$$J_3 \equiv \left| 2\int \partial_x u_1\, \partial_x b_2\, j \right| \le \frac{1}{8}\|\partial_x j\|_{L^2}^2 + C\,\|\partial_x u_1\|_{L^2}^2\,\|j\|_{L^2}^2.$$

The second term on the right of (4.3) has to be handled differently. By integration by parts,

$$J_2 \equiv 2\int \partial_x b_1\, \partial_y u_1\, j = -2\int b_1 \partial_x \partial_y u_1\, j - 2\int b_1 \partial_y u_1\, \partial_x j \equiv J_{21} + J_{22}. \tag{4.4}$$

By the Littlewood-Paley decomposition, for a positive integer N to be specified later,

$$J_{21} = -2\int b_1 \partial_x \partial_y u_1\, j = J_{211} + J_{212}, \tag{4.5}$$

where

$$\begin{aligned} J_{211} &= -2\int S_N b_1\, \partial_x \partial_y u_1\, j, \\ J_{212} &= -2\int \sum_{k\ge N} \Delta_k b_1\, \partial_x \partial_y u_1\, j. \end{aligned}$$

By Hölder's inequality and then Bernstein's inequality, for any $q \in [2,\infty)$,

$$\begin{aligned} J_{211} &\le 2\|S_N b_1\|_{L^\infty}\,\|\partial_x\partial_y u_1\|_{L^2}\,\|j\|_{L^2} \\ &\le C\,2^{N\frac{2}{q}}\,\|S_N b_1\|_{L^q}\,\|\partial_x\partial_y u_1\|_{L^2}\,\|j\|_{L^2} \\ &\le C\,2^{N\frac{2}{q}}\,\sqrt{q\log(q)}\,\sup_{q\in[2,\infty)} \frac{\|b_1\|_{L^q}}{\sqrt{q\log(q)}}\,\|\partial_x\partial_y u_1\|_{L^2}\,\|j\|_{L^2}. \end{aligned}$$

By taking $q = N$ and applying Young's inequality, we have

$$J_{211} \le \frac{1}{8}\|\partial_x\partial_y u_1\|_{L^2}^2 + C\,N\log(N)\,\left[\sup_{q\in[2,\infty)} \frac{\|b_1\|_{L^q}}{\sqrt{q\log(q)}}\right]^2 \|j\|_{L^2}^2. \tag{4.6}$$

By Young's inequality,

$$J_{212} \le \frac{1}{8}\|\partial_x\partial_y u_1\|_{L^2}^2 + C\,\left[\sum_{k\ge N} \|\Delta_k b_1\|_{L^\infty}\right]^2 \|j\|_{L^2}^2. \tag{4.7}$$

To bound $\sum_{k\ge N} \|\Delta_k b_1\|_{L^\infty}$, we apply Lemma 4.1 with $\alpha \in (\frac{1}{2}, 1]$ to obtain

$$\|\Delta_k b_1\|_{L^\infty} \le C\,\|\Delta_k b_1\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\,\|\partial_y \Delta_k b_1\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\,\|\partial_x \Delta_k b_1\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}-\frac{1}{4\alpha}}\,\|\Lambda_y^\alpha \partial_x \Delta_k b_1\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4\alpha}}.$$

Therefore, invoking the simple fact that

$$\|\partial_y \Delta_k b_1\|_{L^2} \le \|\Delta_k j\|_{L^2}, \quad \|\partial_x \Delta_k b_1\|_{L^2} \le \|\Delta_k j\|_{L^2},$$

and applying Hölder's inequality for series, we have

$$\sum_{k\ge N} \|\Delta_k b_1\|_{L^\infty} \le C\, \|b_1\|_{L^2}^{\frac14}\, \|j\|_{L^2}^{\frac34-\frac{1}{4\alpha}} \left[\sum_{k\ge N} \|\Delta_k \Lambda_y^\alpha \partial_x b_1\|_{L^2}^{\frac{1}{\alpha+1}}\right]^{\frac14+\frac{1}{4\alpha}}.$$

By Bernstein's inequality and Young's inequality,

$$\begin{aligned}
\sum_{k\ge N} \|\Delta_k b_1\|_{L^\infty} &\le C\, \|b_1\|_{L^2}^{\frac14}\, \|j\|_{L^2}^{\frac34-\frac{1}{4\alpha}} \left[\sum_{k\ge N} 2^{-k\frac{1-\alpha}{\alpha+1}} \|\Delta_k \partial_y \partial_x b_1\|_{L^2}^{\frac{1}{\alpha+1}}\right]^{\frac14+\frac{1}{4\alpha}} \\
&\le C\, 2^{-N\frac{1-\alpha}{4\alpha}}\, \|b_1\|_{L^2}^{\frac14}\, \|j\|_{L^2}^{\frac34-\frac{1}{4\alpha}}\, \|\partial_x j\|_{L^2}^{\frac{1}{4\alpha}} \\
&\le C\, \|b_1\|_{L^2} + 2^{-N\frac{(1-\alpha)}{3\alpha}} \left(\|j\|_{L^2} + \|\partial_x j\|_{L^2}\right).
\end{aligned}$$

Inserting this inequality in (4.7) leads to

$$J_{212} \le \frac18 \|\partial_x\partial_y u_1\|_{L^2}^2 + C\, \|b_1\|_{L^2}^2\, \|j\|_{L^2}^2 + 2^{-N\frac{2(1-\alpha)}{3\alpha}} \left(\|j\|_{L^2}^2 + \|\partial_x j\|_{L^2}^2\right)\, \|j\|_{L^2}^2$$

Now we take N to be the smallest integer such that

$$2^{-N\frac{2(1-\alpha)}{3\alpha}} \left(\|j\|_{L^2} + \|\omega\|_{L^2}\right) \le \frac18$$

Then

$$J_{212} \le \frac18 \|\partial_x\partial_y u_1\|_{L^2}^2 + C\,(1 + \|b_1\|_{L^2}^2)\, \|j\|_{L^2}^2 + \frac18 \|\partial_x j\|_{L^2}^2. \tag{4.8}$$

Combining (4.6) and (4.8) into (4.5), we have

$$\begin{aligned}
J_{21} &\le \frac14 \|\partial_x\partial_y u_1\|_{L^2}^2 + \frac18 \|\partial_x j\|_{L^2}^2 + C\,(1 + \|b_1\|_{L^2}^2)\, Y(t) \\
&\quad + C \left[\sup_{q\in[2,\infty)} \frac{\|b_1\|_{L^q}}{\sqrt{q\log(q)}}\right]^2 Y(t)\, \log(e+Y(t))\, \log(e+\log(e+Y(t))).
\end{aligned}$$

The term J_{22} in (4.4) can be estimated in a similar fashion and we have

$$\begin{aligned}
J_{22} &\le \frac18 \|\partial_x \omega\|_{L^2}^2 + \frac14 \|\partial_x j\|_{L^2}^2 + C\,(1 + \|b_1\|_{L^2}^2)\, Y(t) \\
&\quad + C \left[\sup_{q\in[2,\infty)} \frac{\|b_1\|_{L^q}}{\sqrt{q\log(q)}}\right]^2 Y(t)\, \log(e+Y(t))\, \log(e+\log(e+Y(t))).
\end{aligned}$$

This settles the estimate for J_2. The estimate for the last term

$$J_4 = -2 \int \partial_x u_1 \, \partial_y b_1 \, j$$

is very similar to that for J_2 and the bound is

$$\begin{aligned} J_4 \;\le\; & \frac{1}{4}\|\partial_x \omega\|_{L^2}^2 + \frac{1}{4}\|\partial_x j\|_{L^2}^2 + C\,(1+\|u_1\|_{L^2}^2)\,Y(t) \\ & + C\left[\sup_{q\in[2,\infty)} \frac{\|u_1\|_{L^q}}{\sqrt{q\log(q)}}\right]^2 Y(t)\,\log(e+Y(t))\,\log(e+\log(e+Y(t))). \end{aligned}$$

Inserting the bounds for J_1, J_2, J_3 and J_4 in (4.3), we have

$$\begin{aligned} & \frac{d}{dt} Y(t) + \frac{1}{2}(\|\partial_x \omega\|_{L^2}^2 + \|\partial_x j\|_{L^2}^2) \\ & \quad \le C\,(1+\|(u_1,b_1)\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2 + \|\partial_x b\|_{L^2}^2)\,Y(t) \\ & \quad + C\left[\sup_{q\in[2,\infty)} \frac{\|(u_1,b_1)\|_{L^q}}{\sqrt{q\log(q)}}\right]^2 Y(t)\,\log(e+Y(t))\,\log(e+\log(e+Y(t))). \end{aligned}$$

By the Osgood inequality, we obtain, for any $T>0$ and $t \le T$

$$Y(t) = \|\omega(\cdot,t)\|_{L^2}^2 + \|j(\cdot,t)\|_{L^2}^2 \le C(T, u_0, b_0).$$

This completes the proof for Proposition 4.1.1. ■

4.2 Global H^2 Bound

The section provides a global bound for the H^2-norm of (u,b). More precisely, we prove the following proposition.

Proposition 4.2.1 *Assume that $(u_0,b_0) \in H^2(\mathbb{R}^2)$, $\nabla\cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Let (u,b) be the corresponding solution of (**??**). Then, for any $T>0$ and $t\le T$,*

$$\|(u(t),\, b(t))\|_{H^2} \le C(T, u_0, b_0).$$

Proof. It follows from the equations of (ω, j) in (3.5) that

$$Z(t) \equiv \|\nabla \omega(\cdot,t)\|_{L^2}^2 + \|\nabla j(\cdot,t)\|_{L^2}^2$$

obeys

$$\frac{d}{dt}Z + \|\nabla\partial_x\omega\|_{L^2}^2 + \|\nabla\partial_x j\|_{L^2}^2 = K_1 + \cdots + K_6,$$

where

$$\begin{aligned}
K_1 &= -\int \nabla\omega \cdot \nabla u \cdot \nabla\omega,\\
K_2 &= \int \nabla\omega \cdot \nabla b \cdot \nabla j,\\
K_3 &= -\int \nabla j \cdot \nabla u \cdot \nabla j,\\
K_4 &= \int \nabla j \cdot \nabla b \cdot \nabla\omega,\\
K_5 &= 2\int \nabla(\partial_x b_1(\partial_x u_2 + \partial_y u_1)) \cdot \nabla j,\\
K_6 &= -2\int \nabla(\partial_x u_1(\partial_x b_2 + \partial_y b_1)) \cdot \nabla j.
\end{aligned}$$

To bound K_1, we further write K_1 into four terms

$$K_1 = K_{11} + K_{12} + K_{13} + K_{14},$$

where

$$\begin{aligned}
K_{11} &= -\int \partial_x u_1\,(\partial_x\omega)^2,\\
K_{12} &= -\int \partial_x u_2\,\partial_x\omega\,\partial_y\omega,\\
K_{13} &= -\int \partial_y u_1\,\partial_x\omega\,\partial_y\omega,\\
K_{14} &= -\int \partial_y u_2\,(\partial_y\omega)^2.
\end{aligned}$$

By Lemma 3.1,

$$\begin{aligned}
|K_{11}| &\le C\,\|\partial_x u_1\|_{L^2}\,\|\partial_x\omega\|_{L^2}\,\|\partial_x\partial_y\omega\|_{L^2}^{\frac{1}{2}}\,\|\partial_x^2\omega\|_{L^2}^{\frac{1}{2}}\\
&\le \frac{1}{64}\|\nabla\partial_x\omega\|_{L^2}^2 + C\,\|\partial_x u_1\|_{L^2}^2\,\|\partial_x\omega\|_{L^2}^2.
\end{aligned}$$

Similarly,

$$|K_{12}| \le \frac{1}{64}\|\nabla\partial_x\omega\|_{L^2}^2 + C\,\|\partial_x u_2\|_{L^2}^2\,\|\partial_x\omega\|_{L^2}\,\|\partial_y\omega\|_{L^2}.$$

Realizing that $\omega = \partial_x u_2 - \partial_y u_1$, we have

$$\begin{aligned}
|K_{13}| &\leq \int |\omega\, \partial_x \omega\, \partial_y \omega| + |K_{12}| \\
&\leq C\, \|\partial_x \omega\|_{L^2}\, \|\omega\|_{L^2}^{\frac{1}{2}}\, \|\partial_y \omega\|_{L^2}\, \|\partial_x \partial_y \omega\|_{L^2}^{\frac{1}{2}} + |K_{12}| \\
&\leq \frac{1}{32}\|\nabla \partial_x \omega\|_{L^2}^2 + C\, \|\partial_x u_2\|_{L^2}^2\, \|\partial_x \omega\|_{L^2}\, \|\partial_y \omega\|_{L^2} \\
&\quad + C\, (\|\omega\|_{L^2}^2 + \|\partial_x \omega\|_{L^2}^2)\, \|\nabla \omega\|_{L^2}^2.
\end{aligned}$$

By the divergence-free condition and integration by parts,

$$K_{14} = 2 \int u_1\, \partial_y \omega\, \partial_x \partial_y \omega.$$

K_{14} can be further estimated as in J_{21} in the previous subsection. That is, we decompose u_1 into two parts,

$$u_1 = S_N u_1 + \sum_{k \geq N} \Delta_k u_1$$

and then bound each corresponding part separately as in the previous subsection to obtain the following bound

$$\begin{aligned}
|K_{14}| &\leq \frac{1}{64}\|\nabla \partial_x \omega\|_{L^2}^2 + C\, (1 + \|u_1\|_{L^2}^2)\, Z(t) \\
&\quad + C \left[\sup_{q \in [2,\infty)} \frac{\|u_1\|_{L^q}}{\sqrt{q \log(q)}} \right]^2 Z(t)\, \log(e + Z(t))\, \log(e + \log(e + Z(t))).
\end{aligned}$$

Therefore, K_1 is bounded by

$$\begin{aligned}
|K_1| &\leq \frac{1}{16}\|\nabla \partial_x \omega\|_{L^2}^2 + C\, (1 + \|u_1\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \|\partial_x \omega\|_{L^2}^2)\, Z(t) \\
&\quad + C \left[\sup_{q \in [2,\infty)} \frac{\|u_1\|_{L^q}}{\sqrt{q \log(q)}} \right]^2 Z(t)\, \log(e + Z(t))\, \log(e + \log(e + Z(t))).
\end{aligned}$$

K_2, K_3 and K_4 can be estimates similarly and obeys the bound, for $i = 2, 3, 4$,

$$\begin{aligned}
|K_i| &\leq \frac{1}{16}\|\nabla \partial_x \omega\|_{L^2}^2 + \frac{1}{16}\|\nabla \partial_x j\|_{L^2}^2 \\
&\quad + C\, (1 + \|(u_1, b_1)\|_{L^2}^2 + \|(\omega, j)\|_{L^2}^2 + \|\partial_x (\omega, j)\|_{L^2}^2)\, Z(t) \\
&\quad + C \left[\sup_{q \in [2,\infty)} \frac{\|(u_1, b_1)\|_{L^q}}{\sqrt{q \log(q)}} \right]^2 Z(t)\, \log(e + Z(t))\, \log(e + \log(e + Z(t))).
\end{aligned}$$

It can be checked that K_5 and K_6 also admit the same bound. For example, K_5 can be written as

$$K_5 = 2\int (\partial_x u_2 + \partial_y u_1)\partial_x \nabla b_1 \cdot \nabla j + 2\int \partial_x \nabla b_1 (\partial_x \nabla u_2 \cdot \nabla j + \partial_y \nabla u_1 \cdot \nabla j)$$

and all the terms can be bounded as in the first four terms. We omit the details for the sake of conciseness. Putting all the estimates together, we find

$$\begin{aligned}
&\frac{d}{dt} Z + \|\nabla \partial_x \omega\|_{L^2}^2 + \|\nabla \partial_x j\|_{L^2}^2 \\
&\quad \leq C\,(1 + \|(u_1, b_1)\|_{L^2}^2 + \|(\omega, j)\|_{L^2}^2 + \|\partial_x(\omega, j)\|_{L^2}^2)\, Z(t) \\
&\quad + C \left[\sup_{q\in[2,\infty)} \frac{\|(u_1, b_1)\|_{L^q}}{\sqrt{q\log(q)}} \right]^2 Z(t)\, \log(e + Z(t))\, \log(e + \log(e + Z(t))).
\end{aligned}$$

Osgood's inequality then yields the desired bound. This completes the proof of Proposition 4.2.1. ■

CHAPTER 5

The 2D Euler-Boussinesq Equations with a Singular Velocity

In this chapter, we consider the initial-value problem for the 2D Euler-Boussinesq equations with a singular velocity

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \partial_{x_1} \theta, \\ \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0, \\ u = \nabla^{\perp} \psi, \quad \Delta \psi = \Lambda^{\sigma} P(\Lambda) \omega, \\ \omega(x,0) = \omega_0(x), \ \theta(x,0) = \theta_0(x), \end{cases} \tag{5.1}$$

where $u = u(x,t)$ is 2D vector field depending on $x = (x_1, x_2) \in \mathbb{R}^2$ and $t \geq 0$, $p = p(x,t)$, $\theta = \theta(x,t)$ $\omega = \omega(x,t)$, and $\psi = \psi(x,t)$ are scalar functions of $x = (x_1, x_2) \in \mathbb{R}^2$ and $t \geq 0$, $\mathbf{e}_2$ is the unit vector in the x_2-direction and $\sigma \geq 0$ is a real parameter. Here the Zygmund operator $\Lambda = (-\Delta)^{1/2}$, Λ^{σ} and the Fourier multiplier operator $P(\Lambda)$ are defined through the Fourier transform, namely

$$\widehat{\Lambda^{\sigma} f}(\xi) = |\xi|^{\sigma} \widehat{f}(\xi) \quad \text{and} \quad \widehat{P(\Lambda) f}(\xi) = P(|\xi|)\, \widehat{f}(\xi).$$

The objective of this chapter is to establish the global (in time) existence and uniqueness of solutions to (5.1) when the initial data is in a suitable functional setting.

The Euler-Boussinesq system in (5.1) is a generalization of the 2D Boussinesq equations. The standard velocity formulation of the 2D Boussinesq equations with fractional dissipation and fractional thermal diffusion is given by

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nu(-\Delta)^\alpha u = -\nabla p + \theta \mathbf{e}_2, \\ \nabla \cdot u = 0, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\beta \theta = 0 \end{cases} \tag{5.2}$$

with the corresponding vorticity $\omega = \nabla \times u$ satisfying

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \nu(-\Delta)^\alpha \omega = \partial_{x_1} \theta, \\ \partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\beta \theta = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \omega, \end{cases} \tag{5.3}$$

where $\nu \geq 0$, $\kappa \geq 0$, $\alpha \in (0,1]$ and $\beta \in (0,1]$ are real parameters, and $\mathbf{e}_2$ is the unit vector in the x_2-direction. When $\sigma = 0$ and $P(\Lambda) = I$ then the equations (5.1) converted to (5.3) with $\nu = 0$ and $\beta = \frac{1}{2}$. It is worth mentioning here that the vorticity in our model (5.1) is more singular than the vorticity in (5.3). In addition, (5.1) can be reformulated in terms of the quasi-velocity v given by $\omega = \nabla \times v$,

$$\begin{cases} \partial_t v + u \cdot \nabla v - \sum_{j=1}^{2} u_j \nabla v_j = -\nabla p + \theta \mathbf{e}_2, \\ u = \Lambda^\sigma P(\Lambda) v, \quad \nabla \cdot v = 0, \\ \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0. \end{cases} \tag{5.4}$$

When $P(\Lambda) = I, \sigma = 0$ then $u = v$ and the system () reduces to standard Boussinesq equations with the pressure $p - \frac{1}{2}|u|^2$.

In order to work with more general operator, P is assumed to satisfy the following condition.

Condition 5.1 *The symbol $P(|\xi|)$ assumes the following properties:*

1. *P is continuous on $\mathbb{R}^2$ and $P \in C^\infty(\mathbb{R}^2 \setminus \{0\})$;*

2. P is radially symmetric;

3. $P = P(|\xi|)$ is nondecreasing in $|\xi|$;

4. There exist two constants C and C_0 such that

$$\sup_{2^{-1}\le|\eta|\le 2} \left|(I-\Delta_\eta)^n\, P(2^j|\eta|)\right| \le C\, P(C_0\, 2^j)$$

for any integer j and $n = 1, 2$.

The last assumption in Condition 5.1 is a very natural condition on symbols of Fourier multiplier operators and is similar to the main condition in the Mihlin-Hörmander Multiplier Theorem ([68]). For notational convenience, we also assume that $P \ge 0$. This type of symbols $P(|\xi|)$ for the operator $P(\Lambda)$ is already used to study generalized 2D Euler equations in the work of Chae, Constantin and Wu [19]. Some special examples of P are

$$\begin{aligned}
&P(\xi) = \left(\log(1+|\xi|^2)\right)^\gamma \quad \text{with } \gamma \ge 0,\\
&P(\xi) = \left(\log(1+\log(1+|\xi|^2))\right)^\gamma \quad \text{with } \gamma \ge 0,\\
&P(\xi) = |\xi|^\beta \quad \text{with } \beta \ge 0,\\
&P(\xi) = (\log(1+|\xi|^2))^\gamma\, |\xi|^\beta \quad \text{with } \gamma \ge 0 \text{ and } \beta \ge 0.
\end{aligned}$$

The main theorem of this chapter is the following.

Theorem 5.2 *Let $\sigma = 0$. Assume the symbol $P(|\xi|)$ obeys Condition 5.1 and*

$$P(2^k) \le C\,\sqrt{k} \quad \textit{for a constant } C \textit{ and any large integer } k > 0, \tag{5.5}$$

$$\int_1^\infty \frac{1}{r\log(1+r)\,P(r)}\,dr = \infty. \tag{5.6}$$

Let $q > 2$ and $s > 2$. Consider the IVP (5.1) with $\omega_0 \in B^s_{q,\infty}(\mathbb{R}^2)$ and $\theta_0 \in B^s_{q,\infty}(\mathbb{R}^2)$. Then the IVP (5.1) has a unique global solution (ω, θ) satisfying, for any $T > 0$ and $t \le T$,

$$\omega \in C([0,T]; B^s_{q,\infty}(\mathbb{R}^2)), \quad \theta \in C([0,T]; B^s_{q,\infty}(\mathbb{R}^2) \cap L^1([0,T]; B^{s+1}_{q,\infty}(\mathbb{R}^2)). \tag{5.7}$$

As an application of this theorem, we prove the global regularity of generalized Euler-Boussinesq equations (5.1) with the double logarithmically supercritical velocity. More precisely we have the following.

Corollary 5.1 *Let $q > 2$ and $s > 2$ and $\sigma = 0$. Consider*

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \partial_{x_1} \theta, \\ u = \nabla^{\perp} \psi, \quad \Delta \psi = \Lambda^{\sigma} (\log(1 + \log(1 - \Delta)))^{\gamma} \omega, \ \gamma \in [0, 1] \\ \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0, \\ \omega(x, 0) = \omega_0(x), \ \theta(x, 0) = \theta_0(x). \end{cases} \tag{5.8}$$

Let $\omega_0 \in B^s_{q,\infty}(\mathbb{R}^2)$ and $\theta_0 \in B^s_{q,\infty}(\mathbb{R}^2)$. Then the IVP (5.8) has a unique global solution.

When $\theta \equiv 0$, the result in Corollary 5.1 includes to Theorem 1.3 for the generalized 2D Euler in [19]. On the other hand if P is the identity operator, we reproduce the global well-posedness for one of the critical Boussinesq equations ([44]). It is worth to mention that the Theorem 5.2 does not allow logarithmically supercritical velocity in equations (5.1).

The global *a priori* bound for ω and θ cannot be achieved through the direct energy estimates due to the vortex stretching term $\partial_{x_1} \theta$. We combine the equation for Riesz transform $\mathcal{R} \equiv \Lambda^{-1} \partial_{x_1}$ for θ and ω. Applying the Riesz transform $\mathcal{R} \equiv \Lambda^{-1} \partial_{x_1}$ to the θ equations, we have

$$\partial_t \mathcal{R}\theta + u \cdot \nabla \mathcal{R}\theta + \Lambda \mathcal{R}\theta = -[\mathcal{R}, u \cdot \nabla]\theta$$

Then the combined quantity $G = \omega + \mathcal{R}\theta$ satisfies

$$\partial_t G + u \cdot \nabla G = -[\mathcal{R}, u \cdot \nabla]\theta, \tag{5.9}$$

where the commutator $[\mathcal{R}, u \cdot \nabla]\theta = \mathcal{R}(u \cdot \nabla \theta) - u \cdot \nabla(\mathcal{R}\theta)$. Although the vortex stretching term is hidden in the commutator, a compensation of an appropriate estimate for the commutator is needed.

This chapter is further divided into four sections. The second section provides logarithmically interpolation inequality of $\|\nabla u\|_{L^\infty}$ and estimates the commutator $[\mathcal{R}, u\cdot\nabla]\theta$ in the Besov spaces $B^0_{p,r}$. The third section establishes global *a priori* bounds for $\|\omega\|_{L^q}$ and $\|\theta\|_{B^{0,P}_{\infty,2}}$ and $\|\omega\|_{L^\infty}$. The global *a priori* bounds for ω and θ in $B^s_{q,\infty}$ will be provided in the fourth section.

5.1 Preliminary Estimates

This section is mainly dedicated to two types of estimates. The first estimate provides the interpolation inequality to bound $\|\nabla u\|_{L^\infty}$ in terms of $\|\omega\|_{L^q\cap L^\infty}$. The second estimate deals with the commutator $[\mathcal{R}, u\cdot\nabla]\theta$ in the Besov space $B^0_{p,r}$ and $B^0_{\infty,r}$. Some facts regarding $\|\Delta_j\nabla u\|_{L^p}$ and $\|S_N\nabla u\|_{L^p}$ are needed for the commutator estimates.

The velocity field u in (5.1) is determined by the vorticity ω through a Fourier multiplier operator, namely

$$u = \nabla^\perp\Delta^{-1}P(\Lambda)\omega.$$

In order to estimate the solutions of (5.1) in Besov type spaces, we often need to bound ∇u in terms of ω and the basic ingredients involved are $\|\Delta_j\nabla u\|_{L^p}$ and $\|S_N\nabla u\|_{L^p}$. In [19], Chae, Constantin and Wu proved the following lemma for a very general Fourier multiplier operator $Q(\Lambda)$.

Lemma 5.1 *Assume that the symbol Q satisfies Condition 5.1 and that u and ω are related through*

$$u = \nabla^\perp\Delta^{-1}Q(\Lambda)\omega.$$

Then, for any integer $j\ge 0$ and $N\ge 0$,

$$\begin{aligned}\|S_N\nabla u\|_{L^p} &\le C_p\,Q(C_0 2^N)\,\|S_N\omega\|_{L^p}, \quad 1<p<\infty,\\ \|\Delta_j\nabla u\|_{L^q} &\le C\,Q(C_0 2^j)\,\|\Delta_j\omega\|_{L^q}, \quad 1\le q\le\infty,\end{aligned}$$

where C_p is a constant depending on p only, C_0 and C are pure constants.

The L^∞-norm of ∇u in general is not bounded by $\|\omega\|_{L^\infty}$. In fact, we need a small correction, which is given by the following logarithmic type interpolation inequality.

Proposition 5.1.1 *Assume that the symbol Q satisfies Condition 5.1 and (5.5). Let u and ω be related through*

$$u = \nabla^\perp \Delta^{-1} Q(\Lambda)\omega.$$

Then, for any $1 \le q \le \infty$, $\beta > 2/q$, and $1 < p < \infty$,

$$\|\nabla u\|_{L^\infty} \le C\,(1 + \|\omega\|_{L^p}) + C\,\|\omega\|_{L^\infty}\,\log(1 + \|\omega\|_{B^\beta_{q,\infty}})\,Q\left(\|\omega\|^{\frac{2q}{q\beta-2}}_{B^\beta_{q,\infty}}\right),$$

where C's are constants that depend on p, q and β only.

Proof. (Proof of Proposition 5.1.1) For any integer $N \ge 0$, we have

$$\|\nabla u\|_{L^\infty} \le \|\Delta_{-1}\nabla u\|_{L^\infty} + \sum_{k=0}^{N-1} \|\Delta_k \nabla u\|_{L^\infty} + \sum_{k=N}^{\infty} \|\Delta_k \nabla u\|_{L^\infty}.$$

By Bernstein's inequality and Lemma 5.1, we have

$$\|\nabla u\|_{L^\infty} \;\; \le \;\; C\,\|\omega\|_{L^p} + C\,N\,Q(2^N)\,\|\omega\|_{L^\infty} + C\,\sum_{k=N}^{\infty} (2^k)^{\frac{2}{q}}\,\|\nabla \Delta_k u\|_{L^q}.$$

By Lemma 5.1,

$$\|\nabla u\|_{L^\infty} \;\; \le \;\; C\,\|\omega\|_{L^p} + C\,N\,Q(2^N)\,\|\omega\|_{L^\infty} + \,C_d\,\sum_{k=N}^{\infty} (2^k)^{\frac{2}{q}}\,Q(2^k)\|\Delta_k \omega\|_{L^q}.$$

By the definition of Besov space $B^\beta_{q,\infty}$,

$$\|\Delta_k \omega\|_{L^q} \le 2^{-\beta\,k}\|\omega\|_{B^\beta_{q,\infty}}.$$

Therefore,

$$\|\nabla u\|_{L^\infty} \;\; \le \;\; C\,\|\omega\|_{L^p} + C\,N\,Q(2^N)\,\|\omega\|_{L^\infty} + \,C\,\|\omega\|_{B^\beta_{q,\infty}}\sum_{k=N}^{\infty} (2^k)^{(\frac{2}{q}-\beta)}\;Q(2^k).$$

Due to $\frac{2}{q} - \beta < 0$ and (5.5), we can choose $\epsilon > 0$ such that

$$\epsilon + \frac{2}{q} - \beta < 0 \quad \text{and} \quad Q(2^N) \le 2^{\epsilon N}.$$

Especially, we take $\epsilon = \frac{1}{2}(\beta - \frac{2}{q})$ to get

$$\|\nabla u\|_{L^\infty} \leq C\,\|\omega\|_{L^p} + C\,N\,Q(2^N)\,\|\omega\|_{L^\infty} + C\,\|\omega\|_{B^\beta_{q,\infty}}\,(2^N)^{(\frac{1}{q}-\frac{\beta}{2})}.$$

If we choose N to be the largest integer satisfying

$$N \leq \frac{1}{\frac{\beta}{2} - \frac{1}{q}} \log_2\left(1 + \|\omega\|_{B^\beta_{q,\infty}}\right),$$

we then obtain the desired result in Proposition 5.1.1. ∎

The following lemma will be used when we present the commutator $[\mathcal{R}, u \cdot \nabla]\theta$ estimates in this chapter as well as in chapter 5.

Lemma 5.2 *Consider two different cases: $\delta \in (0,1)$ and $\delta = 1$.*

1. *Let $\delta \in (0,1)$ and $q \in [1,\infty]$. If $|x|^\delta h \in L^1$, $f \in \mathring{B}^\delta_{q,\infty}$ and $g \in L^\infty$, then*

$$\|h * (fg) - f(h * g)\|_{L^q} \leq C\,\||x|^\delta h\|_{L^1}\,\|f\|_{\mathring{B}^\delta_{q,\infty}}\|g\|_{L^\infty}, \tag{5.10}$$

   where C is a constant independent of f, g and h.

2. *Let $\delta = 1$. Let $q \in [1,\infty]$. Let $r_1 \in [1,q]$ and $r_2 \in [1,\infty]$ satisfying $\frac{1}{r_1} + \frac{1}{r_2} = 1$. Then*

$$\|h * (fg) - f(h * g)\|_{L^q} \leq C\,\||x|h\|_{L^{r_1}}\,\|\nabla f\|_{L^q}\,\|g\|_{L^{r_2}}, \tag{5.11}$$

Where $\mathring{B}^\delta_{q,\infty}$ denotes a homogeneous Besov space. The proof of (5.10) is in [21] while (5.11) is in [43].

Proposition 5.1.2 (Commutator Estimates) *Let $\mathcal{R} = \Lambda^{-1}\partial_{x_1}$ denote the Riesz transform. Assume that the symbol P satisfies Condition 5.1 and*

$$\text{for any } \epsilon > 0, \quad \lim_{|\xi|\to\infty} \frac{P(|\xi|)}{|\xi|^\epsilon} = 0. \tag{5.12}$$

Assume that u and ω are related by

$$u = \nabla^\perp \Delta^{-1} \Lambda^\sigma P(\Lambda)\omega$$

with $\sigma \in [0,1)$. Then, for any $p \in (1,\infty)$ and $r \in [1,\infty]$,

$$\|[\mathcal{R}, u\cdot\nabla]\theta\|_{B^0_{p,r}} \le C\|\omega\|_{L^p}\|\theta\|_{B^{\sigma,P}_{\infty,r}} + C\,\|\omega\|_{L^p}\,\|\theta\|_{L^p} \tag{5.13}$$

and, for any $r \in [1,\infty]$, $q \in (1,\infty)$ and any $\epsilon > 0$,

$$\|[\mathcal{R}, u\cdot\nabla]\theta\|_{B^0_{\infty,r}} \le C(\|\omega\|_{L^q} + \|\omega\|_{L^\infty})\|\theta\|_{B^{\sigma+\epsilon}_{\infty,r}} + C\,\|\omega\|_{L^q}\,\|\theta\|_{L^q} \tag{5.14}$$

for some constant C, where the generalized Besov space $B^{\sigma,P}_{\infty,r}$ with P being the symbol of the operator P , and $B^{\sigma+\epsilon}_{\infty,r}$ is a standard Besov space.

Proof. (Proof of Proposition 6.6) By the definition of $B^0_{p,r}$,

$$\|[\mathcal{R}, u\cdot\nabla]\theta\|_{B^0_{p,r}} = \left[\sum_{j=-1}^{\infty} \|\Delta_j[\mathcal{R}, u\cdot\nabla]\theta\|^r_{L^p}\right]^{\frac{1}{r}}.$$

Using the notion of paraproducts, we decompose $\Delta_j[\mathcal{R}, u\cdot\nabla]\theta$ into three parts,

$$\Delta_j[\mathcal{R}, u\cdot\nabla]\theta \;\;=\;\; J_1 + J_2 + J_3,$$

where

$$\begin{aligned}
J_1 &= \sum_{|k-j|\le 2} \Delta_j(\mathcal{R}(S_{k-1}u\cdot\nabla\Delta_k\theta) - S_{k-1}u\cdot\nabla\mathcal{R}\Delta_k\theta),\\
J_2 &= \sum_{|k-j|\le 2} \Delta_j(\mathcal{R}(\Delta_k u\cdot\nabla S_{k-1}\theta) - \Delta_k u\cdot\nabla\mathcal{R}S_{k-1}\theta),\\
J_3 &= \sum_{k\ge j-1} \Delta_j(\mathcal{R}(\Delta_k u\cdot\nabla\tilde{\Delta}_k\theta) - \Delta_k u\cdot\nabla\mathcal{R}\tilde{\Delta}_k\theta)
\end{aligned}$$

with $\tilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. The Fourier transform of $S_{k-1}u\cdot\nabla\Delta_k\theta$ is supported in the annulus 2^kA, where A denotes a fixed annulus. $\mathcal{R}$ acting on this term can be represented as a convolution with the kernel $h_k(x) = 2^{dk}h(2^kx)$ with $d=2$, where h is a smooth function with compact support. That is,

$$\begin{aligned}
&\mathcal{R}(S_{k-1}u\cdot\nabla\Delta_k\theta) - S_{k-1}u\cdot\nabla\mathcal{R}\Delta_k\theta\\
&\qquad = h_k * (S_{k-1}u\cdot\nabla\Delta_k\theta) - S_{k-1}u\cdot\nabla(h_k * \Delta_k\theta).
\end{aligned}$$

Therefore, according to Lemma 5.2,

$$\|J_1\|_{L^p} \le C\, \||x|h_j\|_{L^1}\, \|\nabla S_{j-1}u\|_{L^p}\, \|\nabla \Delta_j \theta\|_{L^\infty}. \tag{5.15}$$

Applying Lemma 5.1, Bernstein's inequality and the equality

$$\||x|h_j\|_{L^1} = 2^{-j}\, \||x|h(x)\|_{L^1} = C\, 2^{-j},$$

we have

$$\begin{aligned} \|J_1\|_{L^p} &\le C\, 2^{\sigma j}\, P(2^j) \|S_{j-1}\omega\|_{L^p}\, \|\Delta_j \theta\|_{L^\infty} \\ &\le C\, 2^{\sigma j}\, P(2^j)\, \|\omega\|_{L^p}\, \|\Delta_j \theta\|_{L^\infty}. \end{aligned}$$

Similarly,

$$\begin{aligned} \|J_2\|_{L^p} &\le C\, 2^{-j}\, 2^{\sigma j}\, P(2^j) \|\Delta_j \omega\|_{L^p}\, \|\nabla S_{j-1}\theta\|_{L^\infty} \\ &\le C\, 2^{-(1-\sigma)j}\, P(2^j)\, \|\Delta_j\omega\|_{L^p} \sum_{m\le j-1} 2^m \|\Delta_m \theta\|_{L^\infty} \\ &\le C\, \|\Delta_j \omega\|_{L^p} \sum_{m \le j-1} \frac{2^{(1-\sigma)m} P(2^j)}{2^{(1-\sigma)j} P(2^m)}\, 2^{\sigma m} P(2^m)\, \|\Delta_m \theta\|_{L^\infty}. \end{aligned}$$

But the estimate of $\|J_3\|_{L^p}$ is different. We need to distinguish between low frequency and high frequency terms. For the high frequency terms, the commutator structure is not essential. For $j = 0, 1$, the terms in J_3 with $k = -1, 0, 1$ have Fourier transforms containing the origin in their support and the lower bound part of Bernstein's inequality does not apply. To deal with these low frequency terms, we take advantage of the commutator structure and bound them by Lemma 5.2. More precisely, for $j = 0, 1$ and $k = -1, 0, 1$,

$$\begin{aligned} &\|\Delta_j(\mathcal{R}(\Delta_k u \cdot \nabla \tilde{\Delta}_k \theta) - \Delta_k u \cdot \nabla \mathcal{R} \tilde{\Delta}_k \theta)\|_{L^p} \\ &\qquad \le C\, \|\nabla \Delta_k u\|_{L^p}\, \|\Delta_k \theta\|_{L^p} \le C\, \|\omega\|_{L^p}\, \|\theta\|_{L^p}. \end{aligned}$$

For higher frequency terms, we first apply Bernstein's inequality to obtain

$$\begin{aligned}
\|J_3\|_{L^p} &\leq C \sum_{k\geq j-1} 2^j \|\mathcal{R}(\Delta_k u \cdot \tilde{\Delta}_k \theta)\|_{L^p} + C \sum_{k\geq j-1} 2^j \|\Delta_k u \cdot \mathcal{R}\tilde{\Delta}_k \theta\|_{L^p} \\
&\leq C \sum_{k\geq j-1} 2^{j-k} \, \|\nabla \Delta_k u\|_{L^p} \, \|\Delta_k \theta\|_{L^\infty} \\
&\leq C \sum_{k\geq j-1} 2^{j-k} \, \|\Delta_k \omega\|_{L^p} 2^{\sigma k} P(2^k) \|\Delta_k \theta\|_{L^\infty}.
\end{aligned}$$

Since $\sigma \in [0,1)$ and the assumption on P in (5.12), we obtain, by Young's inequality for series convolution,

$$\begin{aligned}
\|[\mathcal{R}, u\cdot\nabla]\theta\|_{B^0_{p,r}} &= C \left[\sum_{j=-1}^{\infty} \|J_1\|^r_{L^p} + \|J_2\|^r_{L^p} + \|J_3\|^r_{L^p} \right]^{\frac{1}{r}} \\
&= C \, \|\omega\|_{L^p} \|\theta\|_{B^{\sigma,P}_{\infty,r}} + C \, \|\omega\|_{L^p} \, \|\theta\|_{L^p}.
\end{aligned}$$

This completes the proof of (5.13). In order to prove (5.14), only the inequality we estimate differently is J_1.

$$\|J_1\|_{L^\infty} \leq C \, \||x| h_j\|_{L^1} \, \|\nabla S_{j-1} u\|_{L^\infty} \, \|\nabla \Delta_j \theta\|_{L^\infty}.$$

But we bound $\|\nabla S_{j-1} u\|_{L^\infty}$ here in a different way. By Lemma 5.2 and the assumption in (5.12), we obtain, for $\sigma \in [0,1)$ and for any $\epsilon > 0$,

$$\begin{aligned}
\|\nabla S_{j-1} u\|_{L^\infty} &\leq \|\nabla \Delta_{-1} u\|_{L^\infty} + \sum_{0\leq m\leq j-2} \|\Delta_m \nabla u\|_{L^\infty} \\
&\leq C\|w\|_{L^q} + \sum_{0\leq m\leq j-2} 2^{\sigma m} \, P(2^m) \, \|\Delta_m \omega\|_{L^\infty} \\
&\leq C\|w\|_{L^q} + C \, 2^{(\sigma+\epsilon)j} \|\omega\|_{L^\infty}.
\end{aligned}$$

Therefore,

$$\|J_1\|_{L^\infty} \leq C \, (\|w\|_{L^q} + \|\omega\|_{L^\infty}) 2^{(\sigma+\epsilon)j} \|\Delta_j \theta\|_{L^\infty}.$$

The bounds for J_2 and J_3 can be obtained by simply setting $p = \infty$ in the corresponding bounds for $\|J_2\|_{L^p}$ and $\|J_3\|_{L^p}$ above. This completes the proof of Proposition 6.6. ■

5.2 Global Bounds for $\|\omega\|_{L_t^\infty L^q}$, $\|\theta\|_{L_t^1 B^{0,P}_{\infty,2}}$ and $\|\omega\|_{L_t^\infty L^\infty}$

This section establishes global *a priori* bounds for $\|\omega\|_{L_t^\infty L^q}$ and $\|\theta\|_{L_t^1 B^{0,P}_{\infty,2}}$ simultaneously when $\sigma = 0$.

Proposition 5.2.1 *Let $\sigma = 0$ and $q > 2$. Assume the symbol P satisfies Condition 5.1 and (5.5). Let (ω, θ) be a smooth solution of (5.1) with $\omega_0 \in B^s_{q,\infty}$ and $\theta_0 \in B^s_{q,\infty}$. Then, for any $T > 0$ and $0 < t \le T$,*

$$\|\omega(t)\|_{L^q} \le C(T), \qquad \|\theta\|_{L_t^1 B^{0,P}_{\infty,2}} \le C(T), \qquad \|\omega(t)\|_{L^\infty} \le C(T)$$

for some constant C depending T and the initial norms of ω_0 and θ_0.

Note that (5.5) implies (5.12). In order to prove this proposition, first we prove the following two lemmas.

Lemma 5.3 *Let $\sigma \in [0, 1)$. Assume that the symbol P satisfies Condition 5.1 and (5.12). Let (ω, θ) be a smooth solution of (5.1). Then, for any $q \in [2, \infty)$ and for any $t > 0$,*

$$\|\omega(t)\|_{L^q} \le C\left(\|\omega_0\|_{L^q} + \|\theta_0\|_{L^q}\right) e^{C\,t\|\theta_0\|_{L^q}}\, e^{C\int_0^t \|\theta(\tau)\|_{B^{\sigma,P}_{\infty,2}}\, d\tau}, \tag{5.16}$$

where C's are pure constants.

Proof. We start with the equations satisfied by G and $\mathcal{R}\theta$,

$$\begin{aligned} &\partial_t G + u\cdot\ \nabla G = -[\mathcal{R}, u\cdot\nabla]\theta, \\ &\partial_t \mathcal{R}\theta + u\cdot\nabla\mathcal{R}\theta + \Lambda\mathcal{R}\theta = -[\mathcal{R}, u\cdot\nabla]\theta. \end{aligned} \tag{5.17}$$

By the embedding $B^0_{q,2} \hookrightarrow L^q$ for $q \ge 2$ and Lemma 6.6,

$$\begin{aligned} \|\omega(t)\|_{L^q} &\le \|G_0\|_{L^q} + \|\mathcal{R}\theta_0\|_{L^q} + 2\int_0^t \|[\mathcal{R}, u\cdot\nabla]\theta\|_{L^q} d\tau \\ &\le \|G_0\|_{L^q} + \|\mathcal{R}\theta_0\|_{L^q} + 2\int_0^t \|[\mathcal{R}, u\cdot\nabla]\theta\|_{B^0_{q,2}} d\tau \\ &\le \|G_0\|_{L^q} + \|\theta_0\|_{L^q} + C\int_0^t \left[\|\omega(\tau)\|_{L^q}(\|\theta(\tau)\|_{B^{\sigma,P}_{\infty,2}} + \|\theta_0\|_{L^q})\right] d\tau, \end{aligned}$$

which implies (5.16), by Gronwall's inequality. ∎

The second lemma makes use of the dissipation in the θ-equation,

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \Lambda\theta = 0, \\ u = \nabla^\perp\psi, \quad \Delta\psi = \Lambda^\sigma P(\Lambda)\omega, \\ \theta(x,0) = \theta_0(x). \end{cases} \tag{5.18}$$

Lemma 5.4 *Let $\sigma\in[0,1)$. Assume that the symbol P satisfies Condition 5.1 and (5.12). Let $q\in(1,\infty)$. Then, any smooth solution (ω,θ) solving (5.18) satisfies, for each integer $j\ge 0$,*

$$2^{j(1-\sigma)}\|\Delta_j\theta\|_{L^1_tL^q} \le 2^{-j\sigma}\|\Delta_j\theta_0\|_{L^q} + C\,P(2^j)\,\|\theta_0\|_{L^\infty}\int_0^t \|\omega(\tau)\|_{L^q}d\tau, \tag{5.19}$$

where C is a pure constant.

Proof. Letting $j\ge 0$ and applying Δ_j to (5.18), multiplying by $\Delta_j\theta|\Delta_j\theta|^{q-2}$ and integrating over $\mathbb{R}^2$, we obtain, after integrating by parts,

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j\theta\|_{L^q}^q + \int \Delta_j\theta|\Delta_j\theta|^{q-2}\Lambda\Delta_j\theta\,dx = -\int \Delta_j\theta|\Delta_j\theta|^{q-2}\Delta_j(u\cdot\nabla\theta)\,dx.$$

Due to the lower bound ([24, 76])

$$\int \Delta_j\theta|\Delta_j\theta|^{q-2}\Lambda\Delta_j\theta\,dx \ge C2^j\|\Delta_j\theta\|_{L^q}^q$$

and the decomposition of $[\Delta_j,\,u\cdot\nabla]\theta$ into five parts,

$$\Delta_j(u\cdot\nabla\theta) = J_1+J_2+J_3+J_4+J_5$$

with

$$\begin{aligned} J_1 &= \sum_{|j-k|\le 2}[\Delta_j,\,S_{k-1}u\cdot\nabla]\Delta_k\theta, \\ J_2 &= \sum_{|j-k|\le 2}(S_{k-1}u - S_ju)\cdot\nabla\Delta_j\Delta_k\theta, \\ J_3 &= S_ju\cdot\nabla\Delta_j\theta, \\ J_4 &= \sum_{|j-k|\le 2}\Delta_j(\Delta_ku\cdot\nabla S_{k-1}\theta), \\ J_5 &= \sum_{k\ge j-1}\Delta_j(\Delta_ku\cdot\nabla\widetilde{\Delta}_k\theta), \end{aligned}$$

we obtain, by Hölder's inequality,

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j\theta\|_{L^q}^q + C2^j\|\Delta_j\theta\|_{L^q}^q \le \|\Delta_j\theta\|_{L^q}^{q-1}\left(\|J_1\|_{L^q} + \|J_2\|_{L^q} + \|J_4\|_{L^q} + \|J_5\|_{L^q}\right).$$

The integral involving J_3 becomes zero due to the divergence-free condition $\nabla\cdot S_ju = 0$. The terms on the right can be bounded as follows. To bound $\|J_1\|_{L^q}$, we write $[\Delta_j,\, S_{k-1}u\cdot\nabla]\Delta_k\theta$ as an integral,

$$[\Delta_j,\, S_{k-1}u\cdot\nabla]\Delta_k\theta = \int \Phi_j(x-y)(S_{k-1}u(y) - S_{k-1}u(x))\cdot\nabla\Delta_k\theta(y)dy,$$

where Φ_j is the kernel associated with the operator Δ_j. By Lemma 5.2 and the inequality

$$\|\Phi_j(x)|x|^{1-\sigma}\|_{L^1} \le 2^{-j(1-\sigma)}\,\|\Phi_0(x)|x|^{1-\sigma}\|_{L^1} \le C\,2^{-j(1-\sigma)},$$

we have

$$\begin{aligned}
\|J_1\|_{L^q} &\le \sum_{|j-k|\le 2}\|\Phi_j(x)|x|^{1-\sigma}\|_{L^1}\,\|S_{k-1}u\|_{B^{1-\sigma}_{q,\infty}}\,\|\nabla\Delta_k\theta\|_{L^\infty}\\
&\le C\sum_{|j-k|\le 2} 2^{-j(1-\sigma)}\,\|S_{k-1}u\|_{\mathring{B}^{1-\sigma}_{q,\infty}}\,2^k\|\Delta_k\theta\|_{L^\infty}.
\end{aligned}$$

Recalling that $\Lambda^{1-\sigma}u = \nabla^\perp\Delta^{-1}\Lambda P(\Lambda)\omega$ and applying Lemma 5.1, we obtain

$$\|S_{k-1}u\|_{\mathring{B}^{1-\sigma}_{q,\infty}} \le C\,\|\Lambda^{1-\sigma}S_{k-1}u\|_{L^q} \le C\,P(2^j)\,\|S_{k-1}\omega\|_{L^q} \le C\,P(2^j)\,\|\omega\|_{L^q}.$$

Therefore,

$$\|J_1\|_{L^q} \le C2^{j\sigma}\,P(2^j)\,\|\omega\|_{L^q}\|\Delta_j\theta\|_{L^\infty}.$$

By Bernstein's inequality,

$$\begin{aligned}
\|J_2\|_{L^q} &\le \sum_{|j-k|\le 2}\|S_ju - S_{k-1}u\|_{L^q}\,\|\nabla\Delta_j\theta\|_{L^\infty} \le C\|\Delta_ju\|_{L^q}2^j\|\Delta_j\theta\|_{L^\infty}\\
&\le C\,\|\nabla\Delta_ju\|_{L^q}\|\Delta_j\theta\|_{L^\infty}\\
&\le C2^{j\sigma}\,P(2^j)\,\|\Delta_j\omega\|_{L^q}\|\Delta_j\theta\|_{L^\infty}.
\end{aligned}$$

We have applied the lower bound part of Bernstein's inequality in the second inequality above, which is valid for $j \geq 0$. Similarly,

$$
\begin{aligned}
\|J_4\|_{L^q} &\leq C\,\|\Delta_j u\|_{L^q}\|\nabla S_{j-1}\theta\|_{L^\infty} \leq C\,\|\Delta_j u\|_{L^q} 2^j \|S_j\theta\|_{L^\infty} \\
&\leq C\,\|\nabla\Delta_j u\|_{L^q}\|\theta\|_{L^\infty} \leq C\,2^{j\sigma}\,P(2^j)\,\|\Delta_j\omega\|_{L^q}\|\theta\|_{L^\infty}.
\end{aligned}
$$

Since $\sigma \in [0,1)$ and the condition on P in (5.12),

$$
\begin{aligned}
\|J_5\|_{L^q} &\leq C \sum_{k\geq j-1} 2^j \|\Delta_k u\|_{L^q}\|\widetilde{\Delta}_k\theta\|_{L^\infty} \\
&\leq C \sum_{k\geq j-1} 2^{j-k}\|\nabla\Delta_k u\|_{L^q}\,\|\Delta_k\theta\|_{L^\infty} \\
&\leq 2^{j\sigma} \sum_{k\geq j-1} 2^{(j-k)(1-\sigma)}\,P(2^k)\,\|\Delta_k\omega\|_{L^q}\|\Delta_k\theta\|_{L^\infty} \\
&\leq C 2^{j\sigma}\,P(2^j)\,\|\omega\|_{L^q}\|\theta\|_{L^\infty}.
\end{aligned}
$$

Collecting the estimates above, we obtain

$$
\frac{d}{dt}\|\Delta_j\theta\|_{L^q} + C\,2^j\|\Delta_j\theta\|_{L^q} \leq C 2^{j\sigma}\,P(2^j)\,\|\omega\|_{L^q}\|\theta_0\|_{L^\infty}.
$$

Integrating with respect to time yields

$$
\|\Delta_j\theta(t)\|_{L^q} \leq e^{-C2^j t}\,\|\Delta_j\theta_0\|_{L^q} + C 2^{j\sigma}\,P(2^j)\,\|\theta_0\|_{L^\infty}\int_0^t e^{-C2^j(t-\tau)}\,\|\omega(\tau)\|_{L^q}\,d\tau.
$$

Hence

$$
2^j\|\Delta_j\theta\|_{L^1_t L^q} \leq \|\Delta_j\theta_0\|_{L^q} + C 2^{j\sigma}\,P(2^j)\,\|\theta_0\|_{L^\infty}\int_0^t \|\omega(\tau)\|_{L^q} d\tau,
$$

This completes the proof of Lemma 5.4. ■

Now we are ready to prove main proposition of this section.

Proof. (Proof of Proposition 5.2.1) Using the definition of $B^{0,P}_{\infty,2}$ and the embedding $B^{0,P}_{\infty,1} \hookrightarrow B^{0,P}_{\infty,2}$,

$$
\|\theta\|_{L^1_t B^{0,P}_{\infty,2}} \leq \int_0^t \left[\sum_{j=-1}^{N-1} (P(2^j))^2 \|\Delta_j\theta\|^2_{L^\infty}\right]^{\frac{1}{2}} d\tau + \int_0^t \sum_{j=N}^{\infty} P(2^j)\|\Delta_j\theta\|_{L^\infty}\,d\tau.
$$

Applying the condition on P in (5.5),

$$\|\theta\|_{L^1_t B^{0,P}_{\infty,2}} \;\le\; t\|\theta_0\|_{L^\infty} N + \sum_{j\ge N} P(2^j)\|\Delta_j\theta\|_{L^1_t L^\infty}. \tag{5.20}$$

Since $q\in(2,\infty)$ and P satisfies (5.5), we choose $\epsilon>0$ such that

$$-1+\epsilon+\frac{2}{q}<0, \qquad (P(2^j))^2\, 2^{-j\epsilon}\le 1.$$

By Bernstein's inequality and Lemma 5.4 with $\sigma=0$,

$$\begin{aligned}
\sum_{j\ge N} P(2^j)\,\|\Delta_j\theta\|_{L^1_t L^\infty} &\le \sum_{j\ge N} P(2^j)\,2^{j\frac{2}{q}}\|\Delta_j\theta\|_{L^1_t L^q}\\
&\le C\sum_{j\ge N}(P(2^j))^2\,2^{j(\frac{2}{q}-1)}(\|\theta_0\|_{L^q}+\|\theta_0\|_{L^\infty}\|\omega\|_{L^1_t L^q})\\
&\le C\sum_{j\ge N}2^{j(\frac{2}{q}+\epsilon-1)}(\|\theta_0\|_{L^q}+\|\theta_0\|_{L^\infty}\|\omega\|_{L^1_t L^q})\\
&\le C\,\|\theta_0\|_{L^q}+C\,2^{N(-1+\epsilon+\frac{2}{q})}\|\theta_0\|_{L^\infty}\|\omega\|_{L^1_t L^q}.
\end{aligned}$$

Inserting the estimates above in (5.20) and choosing N to be the largest integer satisfying

$$N\le \frac{\log(1+\|\omega\|_{L^1_t L^q})}{(1-\epsilon-\frac{2}{q})}+1$$

yields

$$\|\theta\|_{L^1_t B^{0,P}_{\infty,2}}\le C\,\|\theta_0\|_{L^\infty\cap L^q}+C\,\|\theta_0\|_{L^\infty} t\;\log\left(1+\int_0^t\|\omega(\tau)\|_{L^q}\,d\tau\right).$$

Applying (5.16) with $\sigma=0$ yields

$$\|\theta\|_{L^1_t B^{0,P}_{\infty,2}}\le C\,t\log(1+C\,t)+C\,t\|\theta\|_{L^1_t B^{0,P}_{\infty,2}}, \tag{5.21}$$

where C's are constants depending on $\|\theta_0\|_{L^q}$ and $\|\theta_0\|_{L^\infty}$. This inequality allows us to conclude that, for any $T>0$ and $t\le T$,

$$\|\theta\|_{L^1_t B^{0,P}_{\infty,2}}\le C(T,\|\omega_0\|_{L^q},\|\theta_0\|_{L^q\cap L^\infty}). \tag{5.22}$$

In fact, (5.22) is first obtained on a finite-time interval and the global bound is then obtained through an iterative process. Finally we prove the global bound for $\|\omega\|_{L^\infty}$.

By (5.19) with $\sigma = 0$ and (5.5), we have, for any integer $j \geq 0$ and any $\epsilon > 0$,

$$2^{j(1-\epsilon)}\|\Delta_j\theta\|_{L^1_t L^q} \leq \|\theta_0\|_{L^q} + C\|\theta_0\|_{L^\infty}\int_0^t \|\omega(\tau)\|_{L^q}d\tau \leq C(T). \tag{5.23}$$

Since $q \in (2,\infty)$, we can choose $\epsilon > 0$ such that

$$2\epsilon + \frac{2}{q} - 1 < 0.$$

By Bernstein's inequality,

$$\|\theta\|_{B^\epsilon_{\infty,1}} \leq \sum_{j\geq -1} 2^{(2\epsilon+\frac{2}{q}-1)j}2^{(1-\epsilon)j}\|\Delta_j\theta\|_{L^q} \leq C\sup_{j\geq -1} 2^{j(1-\epsilon)}\|\Delta_j\theta\|_{L^q}.$$

It then follows from (5.23) that, for any $t \leq T$,

$$\|\theta\|_{L^1_t B^\epsilon_{\infty,1}} \leq C(T). \tag{5.24}$$

Starting with the equations of G and $\mathcal{R}\theta$, namely (5.17), and applying Lemma 5.1.2, we have, for any $\epsilon > 0$, From the equations of G and $\mathcal{R}\theta$

$$\begin{aligned}
\|G\|_{L^\infty} + \|\mathcal{R}\theta\|_{L^\infty} &\leq \|G_0\|_{L^\infty} + \|\mathcal{R}\theta_0\|_{L^\infty} + 2\int_0^t \|[\mathcal{R},\ \ u\cdot\nabla]\theta\|_{B^0_{\infty,1}}d\tau \\
&\leq \|G_0\|_{L^\infty} + \|\mathcal{R}\theta_0\|_{L^\infty} \\
&\qquad + \int_0^t ((\|\omega\|_{L^q} + \|\omega\|_{L^\infty})\|\theta\|_{B^\epsilon_{\infty,1}} + \|\omega\|_{L^q}\|\theta\|_{L^q})d\tau \\
&\leq \|G_0\|_{L^\infty} + \|\mathcal{R}\theta_0\|_{L^\infty} + \int_0^t (\|G\|_{L^\infty} + \|\mathcal{R}\theta\|_{L^\infty})\|\theta\|_{B^\epsilon_{\infty,1}}d\tau \\
&\qquad + \int_0^t (\|\omega\|_{L^q}\|\theta\|_{B^\epsilon_{\infty,1}} + \|\omega\|_{L^q}\|\theta\|_{L^q})\,d\tau.
\end{aligned}$$

By Gronwall's inequality, (5.24) and the global bound for $\|\omega\|_{L^q}$, we have

$$\|\omega\|_{L^\infty} \leq \|G\|_{L^\infty} + \|\mathcal{R}\theta\|_{L^\infty} \leq C(T).$$

This completes the proof of Proposition 5.2.1. ∎

5.3 Global Bound for $\|\omega\|_{B^s_{q,\infty}}$ and $\|\theta\|_{B^s_{q,\infty}}$

This section is dedicated to obtain a global bound for $\|\omega\|_{B^s_{q,\infty}}$ and $\|\theta\|_{B^s_{q,\infty}}$. This can be done in two main steps. The first step provides bounds for $\|\omega\|_{B^\beta_{q,\infty}}$ and $\|\theta\|\|_{B^\beta_{q,\infty}}$ for β in the range $\frac{2}{q} < \beta < 1$ while the second step proves the global bounds for $\|\omega\|_{B^{\beta_1}_{q,\infty}}$ and $\|\theta\|\|_{B^{\beta_1}_{q,\infty}}$ for $1 \leq \beta_1 < 2 - \frac{2}{q}$. The desired bounds in $B^s_{q,\infty}$ with $s > 2$ can be obtained by a repetition of the second step. The detail is provided below.

Proposition 5.3.1 *Assume that $\sigma = 0$ and the symbol $P(|\xi|)$ obeys Condition 5.1, (5.5) and (5.6). Let $q > 2$ and let $s > 2$. Consider the IVP (5.1) and (**??**) with $\omega_0 \in B^s_{q,\infty}(\mathbb{R}^2)$ and $\theta_0 \in B^s_{q,\infty}(\mathbb{R}^2)$. Let (ω, θ) be a smooth solution of (5.1). Then (ω, θ) admits a global a priori bound. More precisely, for any $T > 0$ and $t \leq T$,*

$$\|(\omega(t), \theta(t))\|_{B^s_{q,\infty}} \leq C(s, q, T, \|(\omega_0, \theta_0)\|_{B^s_{q,\infty}}),$$

where C is a constant depending on s, q, T and the initial norm.

Proof. Let $j \geq -1$ be an integer. Applying Δ_j to the equation of G & multiplying by $\Delta_j G|\Delta_j G|^{q-2}$ and integrating over $\mathbb{R}^2$, we obtain, after integrating by parts,

$$\begin{aligned} \frac{1}{q}\frac{d}{dt}\|\Delta_j G\|^q_{L^q} &= -\int \Delta_j G|\Delta_j G|^{q-2}\Delta_j(u\cdot\nabla G)\,dx \\ &\quad -\int \Delta_j[\mathcal{R}, u\cdot\nabla]\theta\,\Delta_j G|\Delta_j G|^{q-2}\,dx. \end{aligned}$$

Following the notion of paraproducts, we decompose $\Delta_j(u\cdot\nabla G)$ into five parts,

$$\Delta_j(u\cdot\nabla G) = J_1 + J_2 + J_3 + J_4 + J_5$$

with

$$\begin{aligned}
J_1 &= \sum_{|j-k|\le 2} [\Delta_j,\, S_{k-1}u\cdot\nabla]\Delta_k G,\\
J_2 &= \sum_{|j-k|\le 2} (S_{k-1}u - S_j u)\cdot\nabla\Delta_j\Delta_k G,\\
J_3 &= S_j u\cdot\nabla\Delta_j G,\\
J_4 &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u\cdot\nabla S_{k-1}G),\\
J_5 &= \sum_{k\ge j-1} \Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k G).
\end{aligned}$$

By Hölder's inequality,

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j G\|_{L^q}^q \le \|\Delta_j G\|_{L^q}^{q-1}\left(\|J_1\|_{L^q} + \|J_2\|_{L^q} + \|J_4\|_{L^q} + \|J_5\|_{L^q} + \|J_6\|_{L^q}\right),$$

where $J_6 = \Delta_j[\mathcal{R}, u\cdot\nabla]\theta$. The integral involving J_3 becomes zero due to the divergence-free condition $\nabla\cdot S_j u = 0$. The terms on the right can be bounded as follows. To bound $\|J_1\|_{L^q}$, we write $[\Delta_j,\, S_{k-1}u\cdot\nabla]\Delta_k G$ as an integral,

$$[\Delta_j,\, S_{k-1}u\cdot\nabla]\Delta_k G = \int \Phi_j(x-y)(S_{k-1}u(y) - S_{k-1}u(x))\cdot\nabla\Delta_k G(y)dy,$$

where Φ_j is the kernel associated with the operator Δ_j. By a standard commutator estimate ([**?**], [76]),

$$\|J_1\|_{L^q} \le C\sum_{|j-k|\le 2}\|\nabla S_{k-1}u\|_{L^\infty}\|\Delta_k G\|_{L^q}.$$

By Hölder's and Bernstein's inequalities,

$$\|J_2\|_{L^q} \le C\,\|\nabla\widetilde{\Delta}_j u\|_{L^\infty}\,\|\Delta_j G\|_{L^q}.$$

We have especially applied the lower bound part in Bernstein's inequalities, which is given in Proposition 2.2.2. The purpose is to shift the derivative ∇ from G to u. It is worth pointing out that the lower bound does not apply when $j=-1$. In the case when $j=-1$, J_2 involves only low modes and there is no need to shift the derivative from G to u. J_2 is bounded differently. When $j=-1$, J_2 becomes

$$J_2 = -S_0(u)\cdot\nabla\Delta_1\Delta_{-1}G = -\Delta_{-1}u\cdot\nabla\Delta_1\Delta_{-1}G,$$

whose L^q-norm can be bounded by

$$\|J_2\|_{L^q} \le C\|\Delta_{-1}u\|_{L^\infty}\,\|\Delta_{-1}G\|_{L^q} \le C\|\omega\|_{L^q}\,\|G\|_{L^q}.$$

For J_4 and J_5, we have, by Bernstein's inequality,

$$\begin{aligned}
\|J_4\|_{L^q} &\le C\sum_{|j-k|\le 2}\|\Delta_k u\|_{L^\infty}\,\|\nabla S_{k-1}G\|_{L^q}\\
&\le C\sum_{|j-k|\le 2}\|\nabla\Delta_k u\|_{L^\infty}\sum_{m\le k-1}2^{m-k}\|\Delta_m G\|_{L^q},\\
\|J_5\|_{L^q} &\le C\sum_{k\ge j-1}2^j\,\|\Delta_k u\|_{L^\infty}\|\widetilde{\Delta}_k G\|_{L^q}\\
&\le C\sum_{k\ge j-1}2^{j-k}\,\|\nabla\Delta_k u\|_{L^\infty}\,\|\widetilde{\Delta}_k G\|_{L^q}.
\end{aligned}$$

Furthermore, for any $\beta\in\mathbb{R}$,

$$\begin{aligned}
\|J_1\|_{L^q} &\le C\sum_{|j-k|\le 2}\|\nabla u\|_{L^\infty}2^{-\beta(k+1)}\,2^{\beta(k+1)}\|\Delta_k G\|_{L^q} &(5.25)\\
&\le C\,2^{-\beta(j+1)}\,\|G\|_{B^\beta_{q,\infty}}\,\|\nabla u\|_{L^\infty}\sum_{|j-k|\le 2}2^{\beta(j-k)} &(5.26)\\
&\le C\,2^{-\beta(j+1)}\,\|G\|_{B^\beta_{q,\infty}}\,\|\nabla u\|_{L^\infty}, &(5.27)
\end{aligned}$$

where C is a constant depending on β only. It is clear that $\|J_2\|_{L^q}$ admits the same bound. For any $\beta<1$, we have

$$\begin{aligned}
\|J_4\|_{L^q} &\le C\,\|\nabla u\|_{L^\infty}\sum_{|j-k|\le 2}\sum_{m<k-1}2^{m-k}\,2^{-\beta(m+1)}\,2^{\beta(m+1)}\,\|\Delta_m G\|_{L^q}\\
&\le C\,\|\nabla u\|_{L^\infty}\,\|G\|_{B^\beta_{q,\infty}}\sum_{|j-k|\le 2}\sum_{m<k-1}2^{m-k}\,2^{-\beta(m+1)}\\
&= C\,2^{-\beta(j+1)}\,\|G\|_{B^\beta_{q,\infty}}\,\|\nabla u\|_{L^\infty}\sum_{|j-k|\le 2}2^{\beta(j-k)}\sum_{m<k-1}2^{(m-k)(1-\beta)}\\
&\le C\,2^{-\beta(j+1)}\,\|G\|_{B^\beta_{q,\infty}}\,\|\nabla u\|_{L^\infty}.
\end{aligned}$$

where C is a constant depending on β only and the condition $\beta<1$ is used to guarantee that $(m-k)(1-\beta)<0$. For any $\beta>-1$,

$$\begin{aligned}
\|J_5\|_{L^q} &\le C\,\|\nabla u\|_{L^\infty}\,2^{-\beta(j+1)}\sum_{k\ge j-1}2^{(\beta+1)(j-k)}\,2^{\beta(k+1)}\,\|\widetilde{\Delta}_k G\|_{L^q}\\
&\le C\,2^{-\beta(j+1)}\,\|G\|_{B^\beta_{q,\infty}}\,\|\nabla u\|_{L^\infty}.
\end{aligned}$$

$\|J_6\|_{L^q} = \|\Delta_j[\mathcal{R}, u\cdot\nabla]\theta\|_{L^q}$ can be estimated as in the proof of Proposition 6.6,

$$\|J_6\|_{L^q} \le C\,(\|\omega\|_{L^q} + \|\omega\|_{L^\infty}) 2^{\epsilon j} \|\Delta_j \theta\|_{L^q}$$

for any fixed $\epsilon > 0$, where C is a constant depending on ϵ. For the purpose to be specified later, we choose

$$\epsilon > 0, \quad \beta + \epsilon < 1.$$

Collecting these estimates and invoking the global bounds for $\|\omega\|_{L^q \cap L^\infty}$, we obtain, for any $-1 < \beta < 1$,

$$\frac{d}{dt}\|\Delta_j G\|_{L^q} \le C\, 2^{-\beta(j+1)}\, \|G\|_{B^\beta_{q,\infty}}\, \|\nabla u\|_{L^\infty} + C\, 2^{\epsilon j} \|\Delta_j \theta\|_{L^q} + C.$$

Let $\widetilde{\beta} = \beta + \epsilon < 1$. By applying the process above to the equation for θ and making use of the fact that

$$\int \Delta_j \theta |\Delta_j \theta|^{q-2} \Lambda \Delta_j \theta \, dx \ge 0,$$

we obtain

$$\frac{d}{dt}\|\Delta_j \theta\|_{L^q} \le C\, 2^{-\widetilde{\beta}(j+1)}\, \|\theta\|_{B^{\widetilde{\beta}}_{q,\infty}}\, \|\nabla u\|_{L^\infty}.$$

Integrating the inequalities in time and adding them up, we obtain

$$X(t) \le C + X(0) + C \int_0^t (1 + \|\nabla u(\tau)\|_{L^\infty}) X(\tau)\, d\tau, \tag{5.28}$$

where

$$X(t) \equiv \|G(t)\|_{B^\beta_{q,\infty}} + \|\theta(t)\|_{B^{\widetilde{\beta}}_{q,\infty}}.$$

By Proposition 5.1.1, for any $\frac{2}{q} < \beta$,

$$\begin{aligned} \|\nabla u\|_{L^\infty} &\le C\,(1 + \|\omega\|_{L^p}) + C\,\|\omega\|_{L^\infty}\, P\left(\|\omega\|^{\frac{2q}{q\beta-2}}_{B^\beta_{q,\infty}}\right) \log(1 + \|\omega\|_{B^\beta_{q,\infty}}) \\ &\le C\,(1 + \|\omega\|_{L^p}) + C\,\|\omega\|_{L^\infty}\, P(X(t)^{\frac{2q}{q\beta-2}}) \log(1 + X(t)). \end{aligned}$$

Inserting this inequality in (5.28) and applying Osgood's inequality, we obtain desired bound, for $t \le T$,

$$\|\omega(t)\|_{B^\beta_{q,\infty}} \le \|G(t)\|_{B^\beta_{q,\infty}} + \|\theta(t)\|_{B^{\widetilde{\beta}}_{q,\infty}} = X(t) \le C(T).$$

We now proceed to show that, for any $t \le T$,

$$\|\omega(t)\|_{B^{\beta_1}_{q,\infty}} \le C(T) \quad \text{for any } \beta_1 \text{ satisfying } 1 < \beta_1 < 2 - \tfrac{2}{q}.$$

The idea is first to get the global bound for $\|\theta(t)\|_{B^{\beta_1}_{q,\infty}}$ from the equation for θ and then get the global bound for $\|G\|_{B^{\beta_1}_{q,\infty}}$. As we have seen from the previous part, J_4 is the only term that requires $\beta < 1$. In the process of estimating $\|\theta(t)\|_{B^{\beta_1}_{q,\infty}}$, the corresponding terms $\widetilde{J}_1$, $\widetilde{J}_2$, $\widetilde{J}_5$ can be bounded the same way as before, namely

$$\|\widetilde{J}_1\|_{L^q},\, \|\widetilde{J}_2\|_{L^q},\, \|\widetilde{J}_5\|_{L^q} \le C\, 2^{-\beta_1(j+1)}\, \|\theta\|_{B^{\beta_1}_{q,\infty}}\, \|\nabla u\|_{L^\infty}. \tag{5.29}$$

$\|\widetilde{J}_4\|_{L^q}$ is estimated differently. We start with the basic bound

$$\|\widetilde{J}_4\|_{L^q} \le C \sum_{|j-k|\le 2} \|\nabla \Delta_k u\|_{L^\infty} \sum_{m<k-1} 2^{m-k} \|\Delta_m \theta\|_{L^q}.$$

Since $\beta_1 + \frac{2}{q} < 2$, we can choose $\frac{2}{q} < \beta < 1$ and $\epsilon > 0$ such that

$$\beta_1 + \frac{2}{q} + \epsilon < 2\beta. \tag{5.30}$$

By Berntsein's inequality and Lemma 5.1,

$$\begin{aligned} \|\nabla \Delta_k u\|_{L^\infty} &\le C\, 2^{\frac{2k}{q}}\, \|\nabla \Delta_k u\|_{L^q} \le C\, 2^{\frac{2k}{q}}\, P(2^k) \|\Delta_k \omega\|_{L^q} \\ &\le C\, 2^{k(\frac{2}{q}+\epsilon)} \|\Delta_k \omega\|_{L^q} \le C\, 2^{k(\frac{2}{q}+\epsilon-\beta)}\, \|\omega\|_{B^{\beta}_{q,\infty}}. \end{aligned}$$

Clearly, for any $\beta < 1$,

$$\begin{aligned} \sum_{m<k-1} 2^{m-k}\, \|\Delta_m \theta\|_{L^q} &= 2^{-\beta k} \sum_{m<k-1} 2^{(m-k)(1-\beta)} 2^{\beta m}\, \|\Delta_m \theta\|_{L^q} \\ &\le C\, 2^{-\beta k} \|\theta\|_{B^{\beta}_{q,\infty}}. \end{aligned}$$

Therefore, according to (5.30) and the global bound in the first step,

$$\|\widetilde{J}_4\|_{L^q} \le C\, 2^{-\beta_1(j+1)}\, \|\omega\|_{B^{\beta}_{q,\infty}}\, \|\theta\|_{B^{\beta}_{q,\infty}}\, 2^{(\beta_1+\frac{2}{q}+\epsilon-2\beta)j} \le C\, 2^{-\beta_1(j+1)}. \tag{5.31}$$

Collecting the estimates in (5.29) and (5.31), we have

$$\frac{d}{dt}\|\Delta_j \theta\|_{L^q} \le C\, 2^{-\beta_1(j+1)}\, \|\theta\|_{B^{\beta_1}_{q,\infty}}\, \|\nabla u\|_{L^\infty} + C\, 2^{-\beta_1(j+1)}.$$

Bounding $\|\nabla u\|_{L^\infty}$ by the interpolation inequality in Proposition 5.1.1 and applying Osgood inequality lead to the desired global bound for $\|\theta\|_{B^{\beta_1}_{q,\infty}}$. With this bound at our disposal, we then obtain a global bound for $\|G\|_{B^{\beta_1}_{q,\infty}}$ by going through a similar process on the equation of G. Therefore, for any $t \le T$,

$$\|\omega\|_{B^{\beta_1}_{q,\infty}} \le \|\theta\|_{B^{\beta_1}_{q,\infty}} + \|G\|_{B^{\beta_1}_{q,\infty}} \le C(T).$$

If necessary, we can repeat the second step a few times to achieve the global bound for ω and θ in $B^s_{q,\infty}$ for any $s > 2$. This completes the proof of Proposition 5.3.1. ■

5.4 Proof of the Main Theorem

This section establishes the existence and uniqueness of the global (in time) solution. The uniqueness part will be followed directly due to the high regularity in the class (5.7) of solutions. We will be focused on the existence part. Once we have local existence and uniqueness, the global bounds in previous sections allow us to extend the solution for all time.

Proof. (Proof of Theorem 5.2) To show existence, wet start with the construction of a local solution through the method of successive approximation. That is, we consider a successive approximation sequence $\{(\omega^{(n)}, \theta^{(n)})\}$ solving

$$\begin{cases} \omega^{(1)} = S_2\omega_0, \quad \theta^{(1)} = S_2\theta_0, \\ u^{(n)} = \nabla^\perp \Delta^{-1} P(\Lambda)\, \omega^{(n)}, \\ \partial_t \omega^{(n+1)} + u^{(n)} \cdot \nabla \omega^{(n+1)} = \partial_{x_1} \theta^{(n+1)}, \\ \partial_t \theta^{(n+1)} + u^{(n)} \cdot \nabla \theta^{(n+1)} + \Lambda \theta^{(n+1)} = 0, \\ \omega^{(n+1)}(x,0) = S_{n+2}\omega_0(x), \quad \theta^{(n+1)}(x,0) = S_{n+2}\theta_0(x). \end{cases} \tag{5.32}$$

In order to show that $\{(\omega^{(n)}, \theta^{(n)})\}$ converges to a solution of (5.1), it suffices to prove that $\{(\omega^{(n)}, \theta^{(n)})\}$ obeys the following properties:

(1) There exists a time interval $[0, T_1]$ over which $\{(\omega^{(n)}, \theta^{(n)})\}$ are bounded uniformly in terms of n. More precisely, we show that

$$\|(\omega^{(n)}, \theta^{(n)})\|_{B^s_{q,\infty}} \leq C(T_1, \|(\omega_0, \theta_0)\|_{B^s_{q,\infty}}),$$

for a constant depending on T_1 and the initial norm only.

(2) There exists $T_2 > 0$ such that $\omega^{(n+1)} - \omega^{(n)}$ and $\theta^{(n+1)} - \theta^{(n)}$ are Cauchy in $B^{s-1}_{q,\infty}$, namely

$$\|\omega^{(n+1)} - \omega^{(n)}\|_{B^{s-1}_{q,\infty}} \leq C(T_2)\, 2^{-n}, \quad \|\theta^{(n+1)} - \theta^{(n)}\|_{B^{s-1}_{q,\infty}} \leq C(T_2)\, 2^{-n}$$

for any $t \in [0, T_2]$, where $C(T_2)$ is independent of n.

If the properties stated in (1) and (2) hold, then there exists (ω, θ) satisfying, for $T = \min\{T_1, T_2\}$,

$$\omega(\cdot, t) \in B^s_{q,\infty}, \quad \theta(\cdot, t) \in B^s_{q,\infty} \quad \text{for} \quad 0 \leq t \leq T,$$

$$\omega^{(n)}(\cdot, t) \to \omega(\cdot, t) \quad \text{in} \quad B^{s-1}_{q,\infty}, \qquad \theta^{(n)}(\cdot, t) \to \theta(\cdot, t) \quad \text{in} \quad B^{s-1}_{q,\infty}.$$

It is then easy to show that (ω, θ) solves (5.1) and we thus obtain a local solution and the global bounds in Sections 5.2 and 5.3 allow us to extend it into a global solution. It then remains to verify the properties stated in (1) and (2). Property (1) can be shown as in Sections 5.2 and 5.3. To verify Property (2), we consider the equations for the differences $\omega^{(n+1)} - \omega^{(n)}$ and $\theta^{(n+1)} - \theta^{(n)}$ and prove Property (2) inductively in n. The bounds can be achieved in a similar fashion in Sections 5.2 and 5.3. This completes the proof of Theorem 5.2. ■

CHAPTER 6

The 2D Boussinesq-Navier-Stokes Equations with Logarithmically Supercritical Dissipation

This chapter is dedicated to the global well-posedness of the following Cauchy problem for the Boussinseq-Navier-Stokes equations with dissipation given by a general integral operator.

$$\begin{cases} \partial_t u + u \cdot \nabla u + \mathcal{L}u = -\nabla p + \theta \mathbf{e}_2, \\ \partial_t \theta + u \cdot \nabla \theta = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{6.1}$$

where u is a velocity vector field, θ is a scalar function, and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. A nonlocal operator $\mathcal{L}$ is defined by

$$\mathcal{L}f(x) = \text{ p.v. } \int_{\mathbb{R}^2} \frac{f(x) - f(y)}{|x-y|^2} m(|x-y|) dy \tag{6.2}$$

and $m\colon (0,\infty) \to (0,\infty)$ is a smooth, positive, non-increasing function with the following three properties.

(i) there exists $C_1 > 0$ such that $rm(r) \leq C_1 \qquad$ for all $r \leq 1$.

(ii) there exists $C_2 > 0$ such that $r|m'(r)| \leq C_2 m(r) \qquad$ for all $r > 0$.

(iii) there exists $\beta > 0$ such that $r^\beta m(r)$ is non-increasing.

In [28], Dabkowski, Kiselev, Silvestre and Vicol introduced this type of dissipative operator to study the well-posedness of slightly supercritical active scalar equations. Some special examples examples of $m(r)$ are

1. $m(r) = \dfrac{1}{r^\alpha} \qquad$ for $r > 0$ and $\alpha \in (0,1],$ which gives $\mathcal{L} = \Lambda^\alpha$;
2. $m(r) = \dfrac{1}{r \log^\gamma (e + 1/r)} \qquad$ for $r > 0, \gamma \geq 0$;
3. $m(r) = \dfrac{1}{r \log\log(e^2 + 1/r)} \qquad$ for $r > 0$.

The first example represents the supercritical dissipation $\mathcal{L}u$, the second example yields the logarithmically supercritical dissipation $\mathcal{L}u$ and the third includes the log-log supercritical dissipation. Thus, the dissipative operator mention above is a more general operator.

This dissipative operator $\mathcal{L}$ can be equivalently defined by Fourier multiplier [28].

$$\widehat{\mathcal{L}f}(\xi) = P(|\xi|)\widehat{f}(\xi) \tag{6.3}$$

for $P(|\xi|) = m(\frac{1}{|\xi|})$ and $P(\xi)$ satisfies the following four conditions.

1. (The doubling condition): For any $\xi \in \mathbb{R}^2$, $P(2|\xi|) \leq c_D P(|\xi|),$ with constant $c_D \geq 1$.

2. (The Hörmander-Mikhlin condition [68]): for any $\xi \in \mathbb{R}^2$, $|\xi|^{|k|}\,|\partial_\xi^k P(|\xi|)| \leq c_H P(|\xi|)$ for some constant $c_H \geq 1$, and for all multi-indices $k \in \mathbb{Z}^d$ with $|k| \leq N$, with N only depending on c_D.

3. (Sub-quadratic growth at ∞): $\int_0^1 P(|\xi|^{-1})|\xi| d|\xi| < \infty$.

4. $(-\Delta)^2 P(|\xi|) \geq c_H^{-1} P(\xi)|\xi|^{-4}$ for all sufficiently large $|\xi|$.

We assume that $\mathcal{L}$ satisfies both (6.2) and (6.3) with $P(|\xi|) = m(\frac{1}{|\xi|})$ satisfying the aforementioned properties. Taking the curl in the u equation and $\mathcal{R}_\alpha \equiv \mathcal{L}^{-1}\partial_{x_1}$ to θ

equation in (6.1), we obtain

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \mathcal{L}\omega = \partial_{x_1}\theta, \\ \partial_t \mathcal{R}_\alpha \theta + u \cdot \nabla \mathcal{R}_\alpha \theta = -[\mathcal{R}_\alpha, u \cdot \nabla]\theta \\ u = \nabla^\perp \psi, \quad \Delta \psi = \omega, \end{cases} \tag{6.4}$$

where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ and ψ denotes the stream function. In order to show global regularity, we need to show existence, uniqueness and global bounds of solutions. The most difficult part is to find appropriate global bounds. The global regularity can be achieved by getting appropriate bounds from the vorticity equations. However due to the vortex stretching term, the direct energy method does not work. The vortex stretching term can be concealed by combining ω equations and $\mathcal{R}_\alpha\theta$ equations by $G = \omega - \mathcal{R}_\alpha\theta$. Then G satisfies

$$\partial_t G + u \cdot \nabla G + \mathcal{L}G = [\mathcal{R}_a, u \cdot \nabla]\theta. \tag{6.5}$$

Now the vortex stretching term is hiding in the commutator, however a compensation of an appropriate estimate for the commutator is needed.

Our main result is a global well-posedness theorem for the IVP (6.1) or (6.4) when $\mathcal{L}$ is slightly supercritical. More precisely, we prove the following theorem.

Statement of the main theorem

Theorem 6.1 *Consider the IVP (6.1) and assume that $\mathcal{L}$ satisfies (6.2) and (6.3) with $P(|\xi|) = m(\frac{1}{|\xi|})$ obeying the aforementioned conditions. We further assume that $a(\xi) = a(|\xi|) \equiv |\xi|/P(|\xi|)$ is positive, non-decreasing and satisfies*

$$\lim_{|\xi|\to\infty} \frac{a(|\xi|)}{|\xi|^\sigma} = 0\,, \quad \forall\, \sigma > 0. \tag{6.6}$$

Let $q > 2$ and let the initial data (u_0, θ_0) be in the class

$$u_0 \in H^1(\mathbb{R}^2), \quad \omega_0 \in L^q(\mathbb{R}^2) \cap B^0_{\infty,1}(\mathbb{R}^2)\,, \quad \theta_0 \in L^2(\mathbb{R}^2) \cap B^{0,a^2}_{\infty,1}(\mathbb{R}^2),$$

where $\omega_0 = \nabla \times u_0$ *is the initial vorticity. Then (6.1) has a unique global solution* (u, θ) *satisfying, for all* $t > 0$,

$$u \in L_t^\infty H^1, \quad \omega \in L_t^\infty L^q \cap L_t^1 B_{\infty,1}^0\ , \quad \theta \in L_t^\infty L^2 \cap L_t^\infty B_{\infty,1}^{0,a^2} \cap L_t^1 B_{\infty,1}^{0,a}.$$

Here $B_{\infty,1}^0$ denotes an inhomogeneous Besov space and $B_{q,r}^{s,a}$ with $a \geq 0$ being a non-decreasing function is defined through the norm

$$\|f\|_{B_{q,r}^{s,a}} = \|2^{js} a(2^j) \|\Delta_j f\|_{L^q}\|_{l^r} < \infty, \tag{6.7}$$

where Δ_j denotes the Fourier localization operator. A special consequence of Theorem 6.1 is the global existence and uniqueness of classical solutions of (6.1) with logarithmically supercritical dissipation,

$$\widehat{\mathcal{L}u}(\xi) = P(|\xi|)\widehat{u}(\xi) \equiv \frac{|\xi|}{\log^\gamma(e + |\xi|)}\, \widehat{u}(\xi) \quad \text{for any } \gamma \geq 0. \tag{6.8}$$

More precisely, we have the following.

Corollary 6.1 *Consider the IVP*

$$\begin{cases} \partial_t u + u \cdot \nabla u + \frac{\Lambda}{\log^\gamma(e+|\Lambda|)} u = -\nabla p + \theta \mathbf{e}_2, \\ \partial_t \theta + u \cdot \nabla \theta = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{6.9}$$

Assume that $(u_0, \theta_0) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ *with* $s > 1$. *Then IVP (6.9) has a unique global solution* $(u, \theta) \in L^\infty([0,T]; H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2))$ *for any* $T > 0$.

6.1 Preliminary Estimates

This section is dedicated to provide some useful estimates for our purpose.

It is worth here to recall that $\mathcal{L}$ denotes the operator defined by both (6.2) and (6.3), and $a(|\xi|) \equiv \frac{|\xi|}{P(|\xi|)}, \qquad \mathcal{R}_a = \mathcal{L}^{-1}\partial_{x_1}$.

The following two lemmas provide the lower bound for the dissipative operator $\mathcal{L}$. These bounds are very useful when we estimate the L^p-norms of the solution. Similar type of estimate can be found in [27].

Lemma 6.1 *Let $\mathcal{L}$ be the operator defined by (6.2). Then, for $p > 1$,*

$$|f(x)|^{p-2}f(x)(\mathcal{L}f(x)) \geq \frac{1}{p}\mathcal{L}(|f|^p).$$

Proof.

$$\begin{aligned}
\mathcal{L}f(x) &= \text{p.v.} \int \frac{f(x)-f(y)}{|x-y|^d} m(|x-y|)dy \\
|f(x)|^{p-2}f(x)\mathcal{L}f(x) &= \text{p.v.} \int \frac{|f(x)|^p - |f(x)|^{p-2}f(x)f(y)}{|x-y|^d} m(|x-y|)dy.
\end{aligned}$$

By Young's inequality,

$$|f(x)|^{p-2}f(x)f(y) \leq |f(x)|^{p-1}|f(y)| \leq \frac{p-1}{p}|f(x)|^p + \frac{1}{p}|f(y)|^p$$

Therefore,

$$\begin{aligned}
&|f(x)|^{p-2}f(x)\mathcal{L}f(x) \\
&\geq \frac{1}{p}\ \text{p.v.} \int \frac{p|f(x)|^p - (p-1)|f(x)|^p - |f(y)|^p}{|x-y|^d} m(|x-y|)dy \\
&\geq \frac{1}{p}\mathcal{L}(|f|^p).
\end{aligned}$$

This completes the proof of Lemma 6.1. ■

Lemma 6.2 *Let $\mathcal{L}$ be the operator defined by (6.2). Then, for $p \geq 2$,*

$$\int |f|^{p-2}f(\mathcal{L}f)\,dx \geq \frac{2}{p}\int \left|\mathcal{L}^{\frac{1}{2}}(|f|^{\frac{p}{2}})\right|^2 dx.$$

Proof. The $p = 2$ case is trivial. For $p > 2$, let $\beta = \frac{p}{2} - 2$. By Lemma 6.1,

$$\begin{aligned}
\int |f|^{p-2}f(\mathcal{L}f)dx &= \int |f|^{\frac{p}{2}}|f|^{\beta}f(\mathcal{L}f)dx \\
&\geq \int |f|^{\frac{p}{2}}\frac{2}{p}(\mathcal{L}(|f|^{\frac{p}{2}})dx \\
&= \frac{2}{p}\int \left|\mathcal{L}^{\frac{1}{2}}(|f|^{\frac{p}{2}})\right|^2 dx.
\end{aligned}$$

This completes the proof of Lemma 6.2. ■

We also need the generalized version of the Bernstein type inequality for the dissipative operator.

Lemma 6.3 *Let $j \geq 0$ be an integer and $p \in [2, \infty)$. Let $\mathcal{L}$ be defined by (6.2) and (6.3). Then, for any $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$P(2^j)\|\Delta_j f\|^p_{L^p(\mathbb{R}^d)} \leq C \int_{\mathbb{R}^d} |\Delta_j f|^{p-2} \Delta_j f \mathcal{L} \Delta_j f \, dx, \tag{6.10}$$

where C is a constant depending on p and d only.

In order to prove this lemma, let us recall two Lemmas from [**?**].

Lemma 6.4 *Let $2 < p < \infty$. Then there exist two positive constants c_p and C_p such that for every $f \in \mathcal{S}'$ and every $j \in \mathbb{Z}$, we have*

$$c_p 2^{\frac{2j}{p}} \|\Delta_j f\|_{L^p} \leq \|\nabla(|\Delta_j f|^{\frac{p}{2}})\|^{\frac{2}{p}}_{L^2} \leq C_p 2^{\frac{2j}{p}} \|\Delta_j f\|_{L^p}.$$

Lemma 6.5 *Let $p \in [1, \infty)$, $s \in [0, p)$. Suppose that l, r, m satisfying $1 < l \leq r < \infty$, $1 < m < \infty$, $\frac{1}{l} = \frac{1}{r} + \frac{p-1}{m}$.*
Then for $f(u) = |u|^p$, the following estimate holds

$$\|f(z)\|_{\dot{B}^s_{l,2}} \leq C_p \|z\|^{p-1}_{\dot{B}^0_{m,2}} \|z\|_{\dot{B}^s_{r,2}}.$$

Proof. (Proof of Lemma 6.3)

When $p = 2$ the above inequality follows Plancherel's theorem. For $p > 2$, we modify the proof from [24, 40]. Let $N > 0$ be an integer to be specified later. After applying triangle inequality,

$$\|\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2} \leq \|S_N \Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2} + \|(Id - S_N)\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2} \equiv I_1 + I_2.$$

By the standard Bernstein inequality, for $s > 0$,

$$I_2 \leq C 2^{-Ns} \||\Delta_j f|^{\frac{p}{2}}\|_{B^{1+s}_{2,2}}.$$

Applying Lemma 6.5 for $s \in (0, \min(\frac{p}{2}-1, 2))$,

$$\||\Delta_j f|^{\frac{p}{2}}\|_{B^{1+s}_{2,2}} \le C\|\Delta_j f\|^{\frac{p}{2}-1}_{B^0_{p,2}} \|\Delta_j f\|_{B^{1+s}_{p,2}} \le C2^{j(1+s)}\|\Delta_j f\|^{\frac{p}{2}}_{L^p}.$$

Therefore,

$$I_2 \le C2^{-Ns}2^{j(1+s)}\|\Delta_j f\|^{\frac{p}{2}}_{L^p}.$$

By Lemma 5.1,

$$I_1 = \|S_N \Lambda \mathcal{L}^{-\frac{1}{2}}\mathcal{L}^{\frac{1}{2}}(|\Delta_j f|^{\frac{p}{2}})\|_{L^2} \le C\,2^N\,(P(2^N))^{-\frac{1}{2}}\,\|\mathcal{L}^{\frac{1}{2}}(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}.$$

Combining the estimates leads to

$$\|\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2} \le C2^{-Ns}2^{j(1+s)}\|\Delta_j f\|^{\frac{p}{2}}_{L^p} + C\,2^N(P(2^N))^{-\frac{1}{2}}\|\mathcal{L}^{\frac{1}{2}}(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}.$$

Applying lemma 6.4 for Λ.

$$2^j\|\Delta_j f\|^{\frac{p}{2}}_{L^p} \le C\,\|\Lambda(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}.$$

Therefore,

$$2^j\|\Delta_j f\|^{\frac{p}{2}}_{L^p} \le C2^{-Ns}2^{j(1+s)}\|\Delta_j f\|^{\frac{p}{2}}_{L^p} + C\,2^N(P(2^N))^{-\frac{1}{2}}\|\mathcal{L}^{\frac{1}{2}}(|\Delta_j f|^{\frac{p}{2}})\|_{L^2}. \tag{6.11}$$

We now choose $j < N \le j + N_0$ with N_0 independent of j such that

$$C\,2^{-(N-j)s} \le \frac{1}{2}.$$

From (6.11), we can write

$$(P(2^N))^{\frac{1}{2}}2^j\|\Delta_j f\|^{\frac{p}{2}}_{L^p} \le C2^N\|\mathcal{L}^{\frac{1}{2}}(|\Delta_j f|^{\frac{p}{2}})\|_{L^2} \tag{6.12}$$

(6.10) then follows from (6.12). This completes the proof of Lemma 6.3. ■

The following lemma provides the commutator estimates.

Lemma 6.6 *Let a and $\mathcal{R}_a$ be defined as in (6.1). Assume*

$$p \in [2, \infty), \quad q \in [1, \infty], \quad 0 < s < \delta.$$

Let $[\mathcal{R}_a, u]F = \mathcal{R}_a(uF) - u\mathcal{R}_a F$ be a standard commutator. Then

$$\|[\mathcal{R}_a, u]F\|_{B^{s,a}_{p,q}} \le C\,(\|u\|_{\mathring{B}^{\delta}_{p,\infty}}\|F\|_{B^{s-\delta,\,a^2}_{\infty,q}} + \|u\|_{L^2}\,\|F\|_{L^2}),$$

where C denotes a constant independent of a and $\mathcal{R}_a$.

Proof. (Proof of Proposition 6.6) Let $j \ge -1$ be an integer. Decompose $\Delta_j[\mathcal{R}_a, u]F$ into three parts,

$$\Delta_j[\mathcal{R}_a, u]F \;=\; I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &= \sum_{|k-j|\le 2} \Delta_j(\mathcal{R}_a(S_{k-1}u\cdot\Delta_k F) - S_{k-1}u\cdot\mathcal{R}_a\Delta_k F),\\
I_2 &= \sum_{|k-j|\le 2} \Delta_j(\mathcal{R}_a(\Delta_k u\cdot S_{k-1}F) - \Delta_k u\cdot\mathcal{R}_a S_{k-1}F),\\
I_3 &= \sum_{k\ge j-1} \Delta_j(\mathcal{R}_a(\Delta_k u\cdot\tilde{\Delta}_k F) - \Delta_k u\mathcal{R}_a\cdot\tilde{\Delta}_k F).
\end{aligned}$$

When the operator $\mathcal{R}_a$ acts on a function whose Fourier transform is supported on an annulus, it can be represented as a convolution kernel. Since the Fourier transform of $S_{k-1}u\cdot\Delta_k F$ is supported on an annulus around the radius of 2^k, we can write

$$h_k \star (S_{k-1}u\cdot\Delta_k F) - S_{k-1}u\cdot(h_k\star\Delta_k F),$$

where h_k is given by the inverse Fourier transform of $i\xi_1 P^{-1}(|\xi|)\,\widetilde{\Phi}_k(\xi)$, namely

$$h_k(x) = \Big(i\xi_1 P^{-1}(|\xi|)\,\widetilde{\Phi}_k(\xi)\Big)^{\vee}(x).$$

Here $\widetilde{\Phi}_k(\xi) \in C_0^\infty(\mathbb{R}^2)$, $\widetilde{\Phi}_k(\xi)$ is also supported on an annulus around the radius of 2^k and is identically equal to 1 on the support of $S_{k-1}u\cdot\Delta_k F$. Therefore, recalling (6.1), we can write

$$i\xi_1 P^{-1}(|\xi|)\,\widetilde{\Phi}_k(\xi) = i\frac{\xi_1}{|\xi|}\tilde{\Phi}_0(2^{-k}\xi)\,a(|\xi|).$$

Therefore,

$$h_k(x) = 2^{2k}\, h_0(2^k x) * a^\vee(x), \quad h_0(x) = \left(\frac{\xi_1}{|\xi|}\tilde{\Phi}_0(\xi)\right)^\vee.$$

By Lemma 5.2,

$$\begin{aligned}
\|I_1\|_{L^p} &\le C\, \||x|^\delta h_j\|_{L^1} \|S_{j-1}u\|_{\mathring{B}^\delta_{p,\infty}} \|\Delta_j F\|_{L^\infty} \\
&\le C\, 2^{-\delta j}\, a(2^j)\, \|S_{j-1}u\|_{\mathring{B}^\delta_{p,\infty}} \|\Delta_j F\|_{L^\infty}.
\end{aligned}$$

I_2 in L^p can be estimated as follows.

$$\begin{aligned}
\|I_2\|_{L^p} &\le C\, 2^{-\delta j}\, a(2^j)\, \|S_{j-1}F\|_{L^\infty} \|\Delta_j u\|_{\mathring{B}^\delta_{p,\infty}} \\
&\le C\, 2^{-\delta j}\, a(2^j) \sum_{m\le j-1} \|\Delta_m F\|_{L^\infty}\, \|\Delta_j u\|_{\mathring{B}^\delta_{p,\infty}} \\
&= C\, 2^{-sj} a^{-1}(2^j) \sum_{m\le j-1} 2^{(s-\delta)(j-m)} \frac{a^2(2^j)}{a^2(2^m)}\; 2^{(s-\delta)m} a^2(2^m)\, \|\Delta_m F\|_{L^\infty} \|\Delta_j u\|_{\mathring{B}^\delta_{p,\infty}}.
\end{aligned}$$

On the other hand, we estimate $\|I_3\|_{L^p}$ differently. We need to distinguish between low frequency and high frequency terms. For $j = 0, 1$, the terms in I_3 with $k = -1, 0, 1$ have Fourier transforms containing the origin in their support and the lower bound part of Bernstein's inequality does not apply. To deal with these low frequency terms, we take advantage of the commutator structure and bound them by Lemma 5.2. The kernel h corresponding to $\mathcal{R}_a$ still satisfies, for any $r_1 \in (1, \infty)$,

$$\||x|\, h\|_{L^{r_1}} \le C.$$

Therefore, by Lemma 5.2 and Bernstein's inequality, for $j = 0, 1$ and $k = -1, 0, 1$,

$$\begin{aligned}
\|\Delta_j(\mathcal{R}_a(\Delta_k u \cdot \tilde{\Delta}_k F) - \Delta_k u \cdot \mathcal{R}_a \tilde{\Delta}_k F)\|_{L^p} &\le C\, \||x|h\|_{L^{r_1}}\, \|\nabla \Delta_k u\|_{L^p}\, \|\Delta_k F\|_{L^{r_2}} \\
&\le C\, \|u\|_{L^2}\, \|F\|_{L^2}.
\end{aligned}$$

where $\frac{1}{r_1} + \frac{1}{r_2} = 1$. For the high frequency terms, we do not need the commutator structure. By Lemma 5.1 and Hölder's inequality,

$$\begin{aligned}\|I_{31}\|_{L^p} \equiv \left\| \sum_{k\geq j-1} \Delta_j(\mathcal{R}_a(\Delta_k u \cdot \tilde{\Delta}_k F)) \right\|_{L^p} &\leq \sum_{k\geq j-1} C\, a(2^j)\, \|\Delta_k u\|_{L^p}\, \|\Delta_k F\|_{L^\infty} \\ \leq C\ a(2^j) \sum_{k\geq j-1} 2^{-\delta k}\, 2^{\delta k} \|\Delta_k u\|_{L^p}\, \|\Delta_k F\|_{L^\infty} \\ \leq C\, 2^{-sj}\, a^{-1}(2^j)\, \|u\|_{\mathring{B}^\delta_{p,\infty}} \sum_{k\geq j-1} 2^{s(j-k)} \frac{a^2(2^j)}{a^2(2^k)}\, 2^{(s-\delta)k} a^2(2^k)\ \|\Delta_k F\|_{L^\infty}.\end{aligned}$$

$I_{32} \equiv \sum_{k\geq j-1} \Delta_k u \cdot \mathcal{R}_a \tilde{\Delta}_k F$ admits the same bound. By the definition

$$\begin{aligned}\|[\mathcal{R}_a, u]F\|_{B^{s,a}_{p,q}} \ \leq\ & \left[\sum_{j\geq -1} 2^{qsj} a^q(2^j) \|I_1\|^q_{L^p}\right]^{\frac{1}{q}} + \left[\sum_{j\geq -1} 2^{qsj} a^q(2^j) \|I_2\|^q_{L^p}\right]^{\frac{1}{q}} \\ & + \left[\sum_{j\geq -1} 2^{qsj} a^q(2^j) (\|I_{31}\|^q_{L^p} + \|I_{32}\|^q_{L^p})\right]^{\frac{1}{q}} + C\, \|u\|_{L^2}\, \|F\|_{L^2}.\end{aligned}$$

The first term on the right is clearly bounded by

$$C\, \|u\|_{\mathring{B}^\delta_{p,\infty}} \left[\sum_{j\geq -1} 2^{q(s-\delta)j} a^{2q}(2^j) \|\Delta_j F\|^q_{L^\infty}\right]^{\frac{1}{q}} = C\, \|u\|_{\mathring{B}^\delta_{p,\infty}}\, \|F\|_{B^{s-\delta,\, a^2}_{\infty,q}}.$$

Since $s < \delta$, (6.6) and a convolution inequality for series,

$$\left[\sum_{j\geq -1} 2^{qsj} a^q(2^j) \|I_2\|^q_{L^p}\right]^{\frac{1}{q}} \leq C\, \|u\|_{\mathring{B}^\delta_{p,\infty}}\, \|F\|_{B^{s-\delta,\, a^2}_{\infty,q}}.$$

Since $0 < s$, (6.6) and a convolution inequality for series,

$$\left[\sum_{j\geq -1} 2^{qsj} a^q(2^j) \|I_{31}\|^q_{L^p}\right]^{\frac{1}{q}} \leq C\, \|u\|_{\mathring{B}^\delta_{p,\infty}}\, \|F\|_{B^{s-\delta,\, a^2}_{\infty,q}}.$$

This completes the proof of Proposition 6.6. ■

6.2 Global Bound for ω in $B^{0,a^{-1}}_{2,2}$

In this section we discuss a global *a priori* estimates for $\|G\|_{L^2}$ and $\|\omega\|_{B^{0,a^{-1}}_{2,2}}$.

Lemma 6.7 *Assume that the initial data (u_0, θ_0) satisfies the conditions in Theorem 6.1. Let (u, θ) be the corresponding solution and let $\omega = \nabla \times u$ be the vorticity. Let*

$$G = \omega - \mathcal{R}_a \theta, \qquad \mathcal{R}_a = \mathcal{L}^{-1} \partial_{x_1}. \tag{6.13}$$

Then, for any $t \geq 0$,

$$\|G\|_{L^2}^2 + \int_0^t \|\mathcal{L}^{\frac{1}{2}} G(\tau)\|_{L^2}^2 \, d\tau \leq B(t)$$

and consequently

$$\|\omega(t)\|_{B^{0,\, a-1}_{2,2}} \leq B(t),$$

where $B(t)$ is integrable on any finite-time interval $[0, T]$.

Proof. Trivially u and θ obey the following global *a priori* bounds

$$\|\theta(t)\|_{L^2 \cap L^\infty} \leq \|\theta_0\|_{L^2 \cap L^\infty}, \quad \|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + t\|\theta_0\|_{L^2}. \tag{6.14}$$

It is easy to check that G satisfies

$$\partial_t G + u \cdot \nabla G + \mathcal{L} G = [\mathcal{R}_a, u \cdot \nabla]\theta. \tag{6.15}$$

Taking the inner product with G leads to

$$\frac{1}{2} \frac{d}{dt} \|G\|_{L^2}^2 + \int G \, \mathcal{L} G \, dx = \int G \nabla \cdot [\mathcal{R}_a, u]\theta \, dx. \tag{6.16}$$

By the Hölder inequality and the boundedness of Riesz transforms on L^2,

$$\left| \int G \nabla \cdot [\mathcal{R}_a, u]\theta dx \right| \leq \|\mathcal{L}^{\frac{1}{2}} G\|_{L^2} \, \|\mathcal{L}^{-\frac{1}{2}} \Lambda [\mathcal{R}_a, u]\theta\|_{L^2}.$$

Inserting this estimate in (6.16) and applying Young's inequality, we obtain

$$\frac{d}{dt} \|G\|_{L^2}^2 + \|\mathcal{L}^{\frac{1}{2}} G\|_{L^2}^2 \leq \|\mathcal{L}^{-\frac{1}{2}} \Lambda [\mathcal{R}_a, u]\theta\|_{L^2}^2. \tag{6.17}$$

By the definition of the norm, $\|\mathcal{L}^{-\frac{1}{2}} \Lambda f\|_2 \leq \|f\|_{B^{\frac{1}{2},\, \frac{a}{2}}_{2,2}}$. Applying Proposition 6.6 with $\delta > \frac{1}{2}$ and $p = q = 2$, we obtain

$$\|[\mathcal{R}_a, u]\theta\|_{B^{\frac{1}{2},\frac{a}{2}}_{2,2}} \leq C\|u\|_{B^\delta_{2,\infty}} \, \|\theta\|_{B^{\frac{1}{2}-\delta,\, \frac{a2}{4}}_{\infty,2}} + C \, \|u\|_{L^2} \, \|\theta\|_{L^2}.$$

Since $u = \nabla^\perp \Delta^{-1} \omega$,

$$\begin{aligned} \|u\|_{B^\delta_{2,\infty}} &= \sup_{j \geq -1} 2^{\delta j} \, \|\Delta_j u\|_{L^2} \leq \|\Delta_{-1} u\|_{L^2} + \sup_{j \geq 0} 2^{\delta j} \, \|\Delta_j \nabla^\perp \Delta^{-1} \omega\|_{L^2} \\ &\leq \|u\|_{L^2} + \sup_{j \geq 0} 2^{(\delta-1)j} \|\Delta_j \omega\|_{L^2} \leq \|u\|_{L^2} + \|\omega\|_{B^{0,a-1}_{2,2}}. \end{aligned}$$

For $\delta > \frac{1}{2}$, $\|\theta\|_{B^{\frac{1}{2}-\delta,\frac{a^2}{4}}_{\infty,2}} \le \|\theta\|_{L^\infty}$. Therefore,

$$\|\mathcal{L}^{-\frac{1}{2}}\Lambda[\mathcal{R}_a,u]\theta\|_{L^2} \le \|[\mathcal{R}_a,u]\theta\|_{B^{\frac{1}{2},\frac{a}{2}}_{2,2}} \le C\,\|u\|_{L^2}\,\|\theta\|_{L^2\cap L^\infty} + \|\omega\|_{B^{0,a-1}_{2,2}}\|\theta\|_{L^\infty}. \tag{6.18}$$

We can bound the $\|\omega\|_{B^{0,a-1}_{2,2}}$ by

$$\|\omega\|_{B^{0,a-1}_{2,2}} \le \|G\|_{B^{0,a-1}_{2,2}} + \|\mathcal{R}_a\theta\|_{B^{0,a-1}_{2,2}} \le \|G\|_2 + \|\theta\|_2. \tag{6.19}$$

Since $\|u\|_{L^2}$ and $\|\theta\|_{L^2\cap L^\infty}$ are bounded by (6.14), we combine (6.17), (6.18) and (6.19) to achieve the desired result. ■

6.3 Global Bound for G in L^q with $q \in (2,4)$

This section establishes a global *a priori* bounds for $\|\omega\|_{L^q}$ with $q \in (2,4)$.

Proposition 6.3.1 *Assume that the initial data (u_0,θ_0) satisfies the conditions stated in Theorem 6.1. Let (u,θ) be the corresponding solution and G be defined as in (6.13). Then, for any $q \in (2,4)$, G obeys the global bound, for any $T > 0$ and $t \le T$,*

$$\|G(t)\|_{L^q}^q + C\int_0^t \int \left|\mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})\right|^2 dxdt + C\int_0^t \|G\|^q_{L^{\frac{2q}{1+\epsilon}}}\,d\tau \le B(t), \tag{6.20}$$

where C is a constant depending on q only and $B(t)$ is integrable on any finite time interval. A special consequence is that, for any small $\epsilon > 0$,

$$\|\omega(t)\|_{B^{-\epsilon}_{q,\infty}} \le B(t). \tag{6.21}$$

Proof. Multiplying (6.15) by $G|G|^{q-2}$ and integrating with respect to x, we get

$$\frac{1}{q}\frac{d}{dt}\|G\|_{L^q}^q + \int G|G|^{q-2}\mathcal{L}G dx = -\int G|G|^{q-2}\nabla\cdot[\mathcal{R}_a,u]\theta\,dx.$$

By Lemma 6.2,

$$\int G|G|^{q-2}\mathcal{L}G dx \ge C\int |\mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})|^2\,dx.$$

Set $\epsilon > 0$ to be small, say, for $q \in (2,4)$,

$$(1+\epsilon)\left(1-\frac{2}{q}\right) < \frac{1}{2}.$$

Applying the condition (6.6) and by a Sobolev embedding,

$$
\begin{aligned}
\|\mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})\|_{L^2}^2 &= \sum_{j\geq -1} \|\Delta_j \mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})\|_{L^2}^2 \\
&= \sum_{j\geq -1} 2^j a^{-1}(2^j) \|\Delta_j(|G|^{\frac{q}{2}})\|_{L^2}^2 \\
&\geq C \sum_{j\geq -1} 2^{(1-\epsilon)j} \|\Delta_j(|G|^{\frac{q}{2}})\|_{L^2}^2 \\
&= C\, \|\Lambda^{\frac{1}{2}-\frac{\epsilon}{2}}(|G|^{\frac{q}{2}})\|_{L^2}^2 \\
&\geq C\, \|G\|^q_{L^{\frac{2q}{1+\epsilon}}}.
\end{aligned}
$$

For $q \in (2,4)$, we choose $s > 0$ such that

$$
s > \epsilon, \qquad s + (1+\epsilon)\left(1 - \frac{2}{q}\right) = \frac{1}{2} - \epsilon.
$$

By Hölder's inequality,

$$
\left| \int G|G|^{q-2} \nabla \cdot [\mathcal{R}_a, u]\theta \right| \leq \|G|G|^{q-2}\|_{\mathring{H}^s} \|[\mathcal{R}_a, u]\theta\|_{\mathring{H}^{1-s}}.
$$

By Lemma 6.8 below,

$$
\|G|G|^{q-2}\|_{\mathring{H}^s} \leq C\, \|G\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}} \|G\|_{\mathring{H}^{s+(1+\epsilon)(1-\frac{2}{q})}} = C\, \|G\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}} \|G\|_{\mathring{H}^{\frac{1}{2}-\epsilon}}.
$$

In addition, due to the condition in (6.6),

$$
\|G\|^2_{\mathring{H}^{\frac{1}{2}-\epsilon}} = \sum_{j\geq -1} 2^{j-2\epsilon j} \|\Delta_j G\|_{L^2}^2 \leq \sum_{j\geq -1} 2^j a^{-2}(2^j) \|\Delta_j G\|_{L^2}^2 \leq \|\mathcal{L}^{\frac{1}{2}}(G)\|_{L^2}^2.
$$

By Proposition 6.6, recalling $s > \epsilon$ and $u = \nabla^\perp \Delta^{-1}\omega$,

$$
\begin{aligned}
\|[\mathcal{R}_a, u]\theta\|_{\mathring{H}^{1-s}} &\leq C\, \|u\|_{\mathring{B}^{1-s+\epsilon}_{2,\infty}} \|\theta\|_{B^{-\epsilon,1}_{\infty,2}} + C\, \|u\|_{L^2} \|\theta\|_{L^2} \\
&\leq C\, \|\omega\|_{B^{0,\frac{1}{a}}_{2,2}} \|\theta\|_{L^\infty} + C\, \|u\|_{L^2} \|\theta\|_{L^2}.
\end{aligned}
$$

Putting the estimates together, we get

$$
\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\|G\|^q_{L^q} + C \int |\mathcal{L}^{\frac{1}{2}}(|G|^{\frac{q}{2}})|^2\, dx + C\, \|G\|^q_{L^{\frac{2q}{1+\epsilon}}} \\
&\qquad \leq C\, \|G\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}} \|\mathcal{L}^{\frac{1}{2}}(G)\|_{L^2} \left(\|\omega\|_{B^{0,\frac{1}{a}}_{2,2}} \|\theta\|_{L^\infty} + C\, \|u\|_{L^2} \|\theta\|_{L^2} \right).
\end{aligned}
$$

Applying Young's inequality to the right-hand side, noticing that $q \in (2,4)$ and resorting to the bounds in Proposition 6.7, we obtain (6.20) and (6.21) follows from the inequality

$$\|\omega\|_{B^{-\epsilon}_{q,\infty}} \le \|G\|_{B^{-\epsilon}_{q,\infty}} + \|\mathcal{R}_a\theta\|_{B^{-\epsilon}_{q,\infty}} \le \|G\|_{L^q} + \|\theta\|_{L^q}.$$

This completes the proof of Proposition 6.3.1. ■

We have applied the following lemma in the proof of Proposition 6.3.1.

Lemma 6.8 *Let $q \in (2,\infty)$, $s \in (0,1)$, $0 < \epsilon(q-2) \le 2$ and $f \in L^{\frac{2q}{1+\epsilon}} \cap \mathring{H}^{s+(1-\frac{2}{q})(1+\epsilon)}$. Then*

$$\||f|^{q-2} f\|_{\mathring{H}^s} \le C\,\|f\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}} \|f\|_{\mathring{B}^s_{\frac{2q}{2-\epsilon(q-2)},2}} \le C\,\|f\|^{q-2}_{L^{\frac{2q}{1+\epsilon}}} \|f\|_{\mathring{H}^{s+(1-\frac{2}{q})(1+\epsilon)}}. \tag{6.22}$$

Proof. This proof modifies the proof given by [43]. Identifying $\mathring{H}^s$ with $\mathring{B}^s_{2,2}$ and by the definition of $\mathring{B}^s_{2,2}$, we have

$$\||f|^{q-2} f\|^2_{\mathring{H}^s} = \int \frac{\||f|^{q-2} f(x+y) - |f|^{q-2} f(x)\|^2_{L^2}}{|y|^{2+2s}}\, dy.$$

Applying the inequality

$$\big||f|^{q-2} f(x+y) - |f|^{q-2} f(x)\big| \le C\,\left(|f|^{q-2}(x+y) + |f|^{q-2}(x)\right) |f(x+y) - f(x)|,$$

we have, by Hölder's inequality

$$\||f|^{q-2} f(x+y) - |f|^{q-2} f(x)\|^2_{L^2} \le C\,\|f\|^{2(q-2)}_{L^{\frac{2q}{1+\epsilon}}} \|f(x+y) - f(x)\|^2_{L^\rho},$$

where

$$\rho = \frac{2q}{2-\epsilon(q-2)}.$$

Therefore,

$$\||f|^{q-2} f\|^2_{\mathring{H}^s} \le C\,\|f\|^{2(q-2)}_{L^{\frac{2q}{1+\epsilon}}} \|f\|^2_{\mathring{B}^s_{\rho,2}}.$$

Further applying the Besov embedding inequality

$$\|f\|_{\mathring{B}^s_{\rho,2}} \le C\,\|f\|_{\mathring{H}^{s+1-\frac{2}{\rho}}},$$

we achieve (6.22) and this accomplishes the proof of Lemma 6.8. ■

6.4 Global Bound for $\|G\|_{\widetilde{L}^r_t B^s_{q,1}}$ with $q \in [2,4)$

This section establishes a global *a priori* bound for $\|G\|_{\widetilde{L}^r_t B^s_{q,1}}$ with $q \in (2,4)$. By using this bound, we will show the global bound $\|\omega\|_{L^q}$ with general $q \in [2,\infty)$.

Proposition 6.4.1 *Assume that the initial data (u_0, θ_0) satisfies the conditions stated in Theorem 6.1. Let*

$$r \in [1,\infty], \quad s \in [0,1), \quad q \in (2,4).$$

Then, for any $t > 0$, G obeys the following global bound

$$\|G\|_{\tilde{L}^r_t B^s_{q,1}} \le B(t), \tag{6.23}$$

where B is integrable on any finite-time interval.

Proof. Let $j \ge -1$ be an integer. Applying Δ_j to (6.15) yields

$$\partial_t \Delta_j G + \mathcal{L}\Delta_j G = -\Delta_j(u\cdot\nabla G) - \Delta_j[\mathcal{R}_a, u\cdot\nabla]\theta.$$

Taking the inner product with $\Delta_j G|\Delta_j G|^{q-2}$, we have

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j G\|_{L^q}^q + \int \Delta_j G|\Delta_j G|^{q-2}\mathcal{L}\Delta_j G = J_1 + J_2, \tag{6.24}$$

where

$$J_1 = -\int \Delta_j(u\cdot\nabla G)\,\Delta_j G|\Delta_j G|^{q-2}, \tag{6.25}$$

$$J_2 = -\int \Delta_j[\mathcal{R}_a, u\cdot\nabla]\theta \;\Delta_j G|\Delta_j G|^{q-2}.$$

According to Lemma 6.3, for $j \ge 0$, the dissipation part can be bounded below by

$$\int \Delta_j G|\Delta_j G|^{q-2}\mathcal{L}\Delta_j G \ge C P(2^j)\|\Delta_j G\|_{L^q}^q. \tag{6.26}$$

By Lemma 6.9 below, J_1 can be bounded by

$$\begin{aligned}\|J_1\|_{L^q} \le\; & C\,2^{j(\epsilon+\frac{2}{q})}\,\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\Big[\|\Delta_j G\|_{L^q} + \sum_{m\le j-2} 2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q} \\ & + \sum_{k\ge j-1} 2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}\Big]\,\|\Delta_j G\|_{L^q}^{q-1},\end{aligned} \tag{6.27}$$

where we have taken ϵ to be small positive number, especially

$$s-1+3\epsilon<0.$$

To bound J_2, we first apply Hölder's inequality and then employ similar estimates as in the proof of Proposition 6.6 to obtain

$$\begin{aligned} |J_2| &\le \|\Delta_j[\mathcal{R}_a, u\cdot\nabla]\theta\|_{L^q}\|\Delta_j G\|_{L^q}^{q-1} \\ &\le C\left(2^{j\epsilon}a(2^j)\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\|\theta\|_{L^\infty}+\|u\|_{L^2}\|\theta\|_{L^2}\right)\|\Delta_j G\|_{L^q}^{q-1}. \end{aligned} \tag{6.28}$$

Inserting (6.26), (6.27) and (6.28) in (6.24) and writing the bound for $\|\omega(t)\|_{B^{-\epsilon}_{q,\infty}}$ by $B(t)$, we obtain

$$\begin{aligned} \frac{d}{dt}\|\Delta_j G\|_{L^q}+C2^j\,a^{-1}(2^j)\,\|\Delta_j G\|_{L^q} &\le C2^{\epsilon j}\,a(2^j)\,B(t) \\ &+C2^{j(\epsilon+\frac{2}{q})}B(t)\Big[\|\Delta_j G\|_{L^q}+\sum_{m\le j-2}2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q} \\ &\qquad+\sum_{k\ge j-1}2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}\Big]. \end{aligned}$$

Due to (6.6), $a(2^j)\le 2^{\epsilon j}$. Integrating in time yields

$$\begin{aligned} \|\Delta_j G(t)\|_{L^q} &\le e^{-C\,2^{(1-\epsilon)j}t}\|\Delta_j G(0)\|_{L^q}+C\,2^{-j(1-3\epsilon)}\,B(t) \\ &+C\,2^{j(\epsilon+\frac{2}{q})}B(t)\int_0^t e^{-C\,2^{(1-\epsilon)j}(t-\tau)}L(\tau)\,d\tau, \end{aligned}$$

where, for notational convenience, we have written

$$L(t)=\Big[\|\Delta_j G\|_{L^q}+\sum_{m\le j-2}2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q}+\sum_{k\ge j-1}2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}\Big].$$

Taking the L^r norm in time and applying Young's inequality for convolution lead to

$$\begin{aligned} \|\Delta_j G\|_{L^r_t L^q} &\le C\,2^{-\frac{1}{r}(1-\epsilon)j}\,\|\Delta_j G(0)\|_{L^q}+C\,2^{-j(1-3\epsilon)}\,\widetilde{B}(t) \\ &\quad+C\,2^{j(-1+2\epsilon+\frac{2}{q})}\,\widetilde{B}(t)\,\|L\|_{L^r}. \end{aligned}$$

Multiplying by 2^{js}, summing over $j\ge -1$ and noticing $s-1+3\epsilon<0$, we obtain

$$\|G\|_{\widetilde{L}^r_t B^s_{q,1}} \le C\,\|G(0)\|_{B^{s-1/r(1-\epsilon)}_{q,1}}+C\,\widetilde{B}(t)+K_1+K_2+K_3, \tag{6.29}$$

where

$$K_1 = C \sum_{j \geq -1} 2^{j(-1+2\epsilon+\frac{2}{q})}\, \widetilde{B}(t)\, 2^{js} \|\Delta_j G\|_{L^r_t L^q},$$
$$K_2 = C \sum_{j \geq -1} 2^{j(-1+2\epsilon+\frac{2}{q})}\, \widetilde{B}(t)\, 2^{js} \sum_{m \leq j-2} 2^{(m-j)\frac{2}{q}} \|\Delta_m G\|_{L^r_t L^q},$$
$$K_3 = C \sum_{j \geq -1} 2^{j(-1+2\epsilon+\frac{2}{q})}\, \widetilde{B}(t)\, 2^{js} \sum_{k \geq j-1} 2^{(j-k)(1-\frac{2}{q})} \|\Delta_k G\|_{L^r_t L^q}.$$

Since $-1 + 2\epsilon + \frac{2}{q} < 0$, we can choose an integer $N > 0$ such that

$$C\, 2^{N(-1+2\epsilon+\frac{2}{q})} \widetilde{B}(t) \leq \frac{1}{8}.$$

The sums in K_1, K_2 and K_3 can then be split into two parts: $j \leq N$ and $j > N$. Since $\|G\|_{L^q}$ is bounded, the sum for the first part is bounded by $C\, \widetilde{B}(t) 2^{sN}$. The second part of the sum over $j > N$ is bounded by $\frac{1}{8}\|G\|_{\widetilde{L}^r_t B^s_{q,1}}$. Therefore,

$$K_1, K_2, K_3 \leq C\, \widetilde{B}(t) 2^{sN} + \frac{3}{8}\|G\|_{\widetilde{L}^r_t B^s_{q,1}}.$$

Combining these bounds with (6.29) yields the desired estimates. This completes the proof of Proposition 6.4.1. ■

We now provide the details leading to (6.27). They bear some similarities as those in [21], but they are provided here for the sake of completeness.

Lemma 6.9 *Let J_1 be defined as in (6.25). Then we have the following bound*

$$\begin{aligned} \|J_1\|_{L^q} \;\leq\; & C\, 2^{j(\epsilon+\frac{2}{q})}\, \|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}} \Big[\|\Delta_j G\|_{L^q} + \sum_{m \leq j-2} 2^{(m-j)\frac{2}{q}} \|\Delta_m G\|_{L^q} \\ & + \sum_{k \geq j-1} 2^{(j-k)(1-\frac{2}{q})} \|\Delta_k G\|_{L^q} \Big] \|\Delta_j G\|^{q-1}_{L^q}. \end{aligned}$$

Proof. Using the notion of paraproducts, we write

$$\Delta_j (u \cdot \nabla G) = J_{11} + J_{12} + J_{13} + J_{14} + J_{15},$$

where

$$
\begin{aligned}
J_{11} &= \sum_{|j-k|\le 2} [\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k G,\\
J_{12} &= \sum_{|j-k|\le 2} (S_{k-1}u - S_j u)\cdot\nabla\Delta_j\Delta_k G,\\
J_{13} &= S_j u\cdot\nabla\Delta_j G,\\
J_{14} &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u\cdot\nabla S_{k-1}G),\\
J_{15} &= \sum_{k\ge j-1} \Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k G).
\end{aligned}
$$

Since $\nabla\cdot u = 0$, we have

$$\int J_{13}|\Delta_j G|^{q-2}\Delta_j G\,dx = 0.$$

By Hölder's inequality,

$$\left|\int J_{11}|\Delta_j G|^{q-2}\Delta_j G\right| \le \|J_{11}\|_{L^q}\|\Delta_j G\|_{L^q}^{q-1}.$$

We write the commutator in terms of the integral,

$$J_{11} = \int \Phi_j(x-y)\,(S_{k-1}u(y) - S_{k-1}u(x))\cdot\nabla\Delta_k G(y)\,dy,$$

where Φ_j is the kernel of the operator Δ_j. As in the proof of Lemma 3.3, we have, for any $0<\epsilon<1$,

$$\|J_{11}\|_{L^q} \le \||x|^{1-\epsilon}\Psi_j(x)\|_{L^1}\,\|S_{j-1}u\|_{\mathring{B}^{1-\epsilon}_{q,\infty}}\,\|\nabla\Delta_j G\|_{L^\infty}.$$

By the definition of Φ_j and Bernstein's inequality, we have

$$
\begin{aligned}
\|J_{11}\|_{L^q} &\le C\,2^{j(\epsilon+\frac{2}{q})}\,\||x|^{1-\epsilon}\Psi_0(x)\|_{L^1}\,\|S_{j-1}\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\|\Delta_j G\|_{L^\infty}\\
&\le C\,2^{j(\epsilon+\frac{2}{q})}\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\,\|\Delta_j G\|_{L^q}.
\end{aligned}
$$

Again, by Bernstein's inequality,

$$
\begin{aligned}
\|J_{12}\|_{L^q} &\le C\|\Delta_j u\|_{L^q}\|\nabla\Delta_j G\|_{L^\infty}\\
&\le C2^{j(\epsilon+\frac{2}{q})}\,\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\,\|\Delta_j G\|_{L^q};
\end{aligned}
$$

$$\begin{aligned}\|J_{14}\|_{L^q} &\le C\|\Delta_j u\|_{L^q}\|\nabla S_{j-1}G\|_{L^\infty}\\ &\le C2^{j(\epsilon+\frac{2}{q})}\,\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\sum_{m\le j-2}2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q};\end{aligned}$$

$$\begin{aligned}\|J_{15}\|_{L^q} &\le C2^{j(\epsilon+\frac{2}{q})}\sum_{k\ge j-1}2^{(j-k)(1-\frac{2}{q})}\,\|\Lambda^{1-\epsilon}\Delta_k u\|_{L^q}\|\Delta_k G\|_{L^q}\\ &\le C2^{j(\epsilon+\frac{2}{q})}\,\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\sum_{k\ge j-1}2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}.\end{aligned}$$

Combining the estimates above yields

$$\begin{aligned}\|J_1\|_{L^q} &\le C\,2^{j(\epsilon+\frac{2}{q})}\,\|\omega\|_{\mathring{B}^{-\epsilon}_{q,\infty}}\Big[\|\Delta_j G\|_{L^q}+\sum_{m\le j-2}2^{(m-j)\frac{2}{q}}\|\Delta_m G\|_{L^q}\\ &\quad+\sum_{k\ge j-1}2^{(j-k)(1-\frac{2}{q})}\,\|\Delta_k G\|_{L^q}\Big]\,\|\Delta_j G\|_{L^q}^{q-1}.\end{aligned}$$

This completes the proof of Lemma 6.9. ■

6.5 Global Bounds for $\|\omega\|_{L^1_t B^{0,a}_{\infty,1}}$ and $\|\omega\|_{L^q}$ for any $q \ge 2$

In this section we prove that ω_0 is in L^q, then the solution ω is also *a priori* in L^q at any time.

Proposition 6.5.1 *Assume that the initial data (u_0, θ_0) satisfies the conditions as stated in Theorem 6.1. Then we have the following global a priori bounds. For any $T > 0$ and $t \le T$,*

$$\|\omega(t)\|_{L^1_t B^{0,a}_{\infty,1}} \le C(T), \quad \|\theta(t)\|_{B^{0,a^2}_{\infty,1}} \le C(T), \quad \|\omega(t)\|_{L^q} \le C(T),$$

where $C(T)$ are constants depending on T and the initial norms only.

In order to prove this proposition, we need the following fact.

Lemma 6.10 *Let $T > 0$ and let u be a divergence-free smooth vector field satisfying*

$$\int_0^T \|\nabla u\|_{L^\infty}\,dt < \infty.$$

Assume that θ solves

$$\partial_t\theta + u\cdot\nabla\theta = f.$$

Let $a:(0,\infty)\to(0,\infty)$ be an nondecreasing and radially symmetric function satisfying (6.6). Let $\rho\in[1,\infty]$. For any $t>0$,

$$\|\theta\|_{B^{0,a}_{\rho,1}} \le (\|\theta_0\|_{B^{0,a}_{\rho,1}} + \|f\|_{L^1_t B^{0,a}_{\rho,1}})\left(1+\int_0^t \|\nabla u\|_{L^\infty}dt\right).$$

This lemma can be proven in a similar fashion as that of Lemma 4.5 in [21]. A crucial assumption is that a satisfies (6.6).

Proof. (Proof of Proposition 6.5.1) We first explains that (6.23) in Proposition 6.4.1 implies that, for $t\le T$,

$$\|G\|_{L^1_t B^{0,a}_{\infty,1}} \le C(T).$$

In fact, if we choose $s\in[0,1)$ satisfying $s>\frac{2}{q}$ for $q\in(2,4)$ and set $\epsilon>0$ satisfying $\epsilon+\frac{2}{q}-s<0$, then

$$\begin{aligned}
\|G\|_{B^{0,a}_{\infty,1}} &\equiv \sum_{j\ge -1} a(2^j)\|\Delta_j G\|_{L^\infty} \le \sum_{j\ge -1} a(2^j)2^{\frac{2}{q}j}\,\|\Delta_j G\|_{L^q}\\
&\le \sum_{j\ge -1} a(2^j)2^{-\epsilon j}\,2^{j(\epsilon+\frac{2}{q}-s)}2^{js}\|\Delta_j G\|_{L^q} \le C\,\|G\|_{B^s_{q,1}},
\end{aligned}$$

where we have used the fact that $a(2^j)2^{-\epsilon j}\le C$ for C independent of j. Furthermore,

$$\|\omega\|_{L^1_t B^{0,a}_{\infty,1}} \le \|G\|_{L^1_t B^{0,a}_{\infty,1}} + \|\mathcal{R}_a\theta\|_{L^1_t B^{0,a}_{\infty,1}}.$$

By the definition of the norm in $B^{0,a}_{\infty,1}$ and recalling that $\mathcal{R}_a\theta$ is defined by the multiplier $a(|\xi|)\frac{i\xi_1}{|\xi|}$, we have

$$\begin{aligned}
\|\mathcal{R}_a\theta\|_{B^{0,a}_{\infty,1}} &= a(2^{-1})\,\|\Delta_{-1}\mathcal{R}_a\theta\|_{L^\infty} + \sum_{j\ge 0} a(2^j)\,\|\Delta_j\mathcal{R}_a\theta\|_{L^\infty}\\
&\le C\,\|\theta_0\|_{L^2} + \sum_{j\ge 0} a^2(2^j)\,\|\Delta_j\theta\|_{L^\infty}\\
&\le C\,\|\theta_0\|_{L^2} + \|\theta\|_{B^{0,a^2}_{\infty,1}}.
\end{aligned}$$

By Lemma 6.10,

$$\begin{aligned}
\|\theta\|_{B^{0,a^2}_{\infty,1}} &\leq C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}} \left(1+\int_0^t \|\nabla u\|_{L^\infty} dt\right) \\
&\leq C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}} \left(1+\|u\|_{L^1_t L^2}+\|\omega\|_{L^1_t B^0_{\infty,1}}\right) \\
&\leq C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}} \left(1+\|u\|_{L^1_t L^2}+\|\omega\|_{L^1_t B^{0,a}_{\infty,1}}\right). \qquad (6.30)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\omega\|_{L^1_t B^{0,a}_{\infty,1}} &\leq \|G\|_{L^1_t B^{0,a}_{\infty,1}} + C\left(\|\theta_0\|_{L^2}+\|\theta_0\|_{B^{0,a^2}_{\infty,1}}\right) t \\
&\quad + C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}} \int_0^t \|u\|_{L^1_\tau L^2}\, d\tau + C\,\|\theta_0\|_{B^{0,a^2}_{\infty,1}} \int_0^t \|\omega\|_{L^1_\tau B^{0,a}_{\infty,1}}\, d\tau.
\end{aligned}$$

By Gronwall's inequality, $\|\omega\|_{L^1_t B^{0,a}_{\infty,1}} \leq C(T)$, which, in turn, implies that, by (6.30),

$$\|\theta(t)\|_{B^{0,a^2}_{\infty,1}} \leq C(T).$$

Now we prove the bound for $\|\omega\|_{L^q}$. From the equations of G and $\mathcal{R}_a\theta$,

$$\begin{aligned}
\|\omega\|_{L^q} &\leq \|G\|_{L^q} + \|\mathcal{R}_a\theta\|_{L^q} \\
&\leq \|G_0\|_{L^q} + \|\mathcal{R}_a\theta_0\|_{L^q} + 2\int_0^t \|[\mathcal{R}_a, u\cdot\nabla]\theta\|_{L^q}\, d\tau \\
&\leq \|G_0\|_{L^q} + \|\mathcal{R}_a\theta_0\|_{L^q} + 2\int_0^t \|[\mathcal{R}_a, u\cdot\nabla]\theta\|_{B^0_{q,1}}\, d\tau.
\end{aligned}$$

Following the steps as in the proof of Proposition 6.6, we can show that

$$\|[\mathcal{R}_a, u\cdot\nabla]\theta\|_{B^0_{q,1}} \leq C\|\omega\|_{L^q}\,\|\theta\|_{B^{0,a}_{\infty,1}} + C\,\|\theta_0\|_{L^2}\,\|u\|_{L^2}.$$

Gronwall's inequality and the bound $\|\theta\|_{L^1_t B^{0,a}_{\infty,1}} \leq C(T)$ then imply the bound for $\|\omega\|_{L^q}$. This completes the proof of Proposition 6.5.1. ■

6.6 Uniqueness and Proof of Theorem 6.1

This section proves the existence, uniqueness of the global solution. First we prove uniqueness and then we prove local existence. The local existence, uniqueness together with global bounds allow us to extend solution for all time.

Theorem 6.2 (Uniqueness) *Assume that the initial data (u_0, θ_0) satisfies the conditions stated in Theorem 6.1. Then, the solutions (u, θ) in the class*

$$u \in L^\infty([0,T]; H^1), \quad \omega \in L^\infty([0,T]; L^q) \cap L^1_T B^{0,a}_{\infty,1}, \quad \theta \in L^\infty([0,T], L^2 \cap B^{0,a}_{\infty,1}) \tag{6.31}$$

must be unique.

Proof. Assume that $(u^{(1)}, \theta^{(1)})$ and $(u^{(2)}, \theta^{(2)})$ are two solutions in the class (6.31). Let $p^{(1)}$ and $p^{(2)}$ be the associated pressure. The differences

$$u = u^{(2)} - u^{(1)}, \qquad p = p^{(2)} - p^{(1)}, \qquad \theta = \theta^{(2)} - \theta^{(1)}$$

satisfy

$$\begin{cases} \partial_t u + u^{(1)} \cdot \nabla u + u \cdot \nabla u^{(2)} + \mathcal{L} u = -\nabla p + \theta \mathbf{e}_2, \\ \partial_t \theta + u^{(1)} \cdot \nabla \theta + u \cdot \nabla \theta^{(2)} = 0. \end{cases}$$

By Lemmas 6.11 and 6.12 below, we have the following estimates

$$\begin{aligned} \|u(t)\|_{B^0_{2,\infty}} &\le \|u(0)\|_{B^0_{2,\infty}} + C\,\|\theta\|_{L^\infty_t B^{-1,a}_{2,\infty}} \\ &\quad + C \int_0^t \|u(\tau)\|_{L^2} \, (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}) \, d\tau \end{aligned}$$

and

$$\begin{aligned} \|\theta(t)\|_{B^{-1,a}_{2,\infty}} &\le \|\theta(0)\|_{B^{-1,a}_{2,\infty}} + C \int_0^t \|\theta(\tau)\|_{B^{-1,a}_{2,\infty}} (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \, d\tau \\ &\quad + C \int_0^t \|u(\tau)\|_{L^2} \|\theta^{(2)}\|_{B^{0,a}_{\infty,1}} \, d\tau. \end{aligned}$$

In addition, we bound $\|u\|_{L^2}$ by the following interpolation inequality

$$\|u\|_{L^2} \le C\,\|u\|_{B^0_{2,\infty}} \, \log\left(1 + \frac{\|u\|_{H^1}}{\|u\|_{B^0_{2,\infty}}}\right)$$

together with $\|u\|_{H^1} \le \|u^{(1)}\|_{H^1} + \|u^{(2)}\|_{H^1}$. These inequalities allow us to conclude that

$$Y(t) \equiv \|u(t)\|_{B^0_{2,\infty}} + \|\theta(t)\|_{B^{-1,a}_{2,\infty}}$$

obeys

$$Y(t) \le 2\,Y(0) + C \int_0^t D_1(\tau) Y(\tau)\, \log\left(1 + D_2(\tau)/Y(\tau)\right)\, d\tau, \tag{6.32}$$

where

$$\begin{aligned} D_1 &= \|\theta^{(2)}\|_{B^{0,a}_{\infty,1}} + \|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}, \\ D_2 &= \|u^{(1)}\|_{H^1} + \|u^{(2)}\|_{H^1}. \end{aligned}$$

Applying Osgood's inequality to (6.32) and noticing that $Y(0) = 0$, we conclude that $Y(t) = 0$. This completes the proof of Theorem 6.2. ■

We now state and prove two estimates used in the proof of Theorem 6.2.

Lemma 6.11 *Assume that $u^{(1)}$, $u^{(2)}$, u, p and θ are defined as in the proof of Theorem 6.2 and satisfy*

$$\partial_t u + u^{(1)} \cdot \nabla u + u \cdot \nabla u^{(2)} + \mathcal{L} u = -\nabla p + \theta \mathbf{e}_2. \tag{6.33}$$

Then we have the a priori bound

$$\begin{aligned} \|u(t)\|_{B^0_{2,\infty}} &\le \|u(0)\|_{B^0_{2,\infty}} + C\, \|\theta\|_{L^\infty_t B^{-1,a}_{2,\infty}} \\ &\quad + C \int_0^t \|u(\tau)\|_{L^2} \left(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}\right) d\tau. \end{aligned} \tag{6.34}$$

Proof. (Proof of Lemma 6.11) Let $j \ge -1$ be an integer. Applying Δ_j to (6.33) and taking the inner product with $\Delta_j u$, we obtain, after integration by parts,

$$\frac{1}{2} \frac{d}{dt} \|\Delta_j u\|_{L^2}^2 + \|\mathcal{L}^{\frac{1}{2}} \Delta_j u\|_{L^2}^2 = J_1 + J_2 + J_3, \tag{6.35}$$

where

$$\begin{aligned} J_1 &= -\int \Delta_j u\, \Delta_j (u^{(1)} \cdot \nabla u)\, dx, \\ J_2 &= -\int \Delta_j u\, \Delta_j (u \cdot \nabla u^{(2)})\, dx, \\ J_3 &= \int \Delta_j u\, \Delta_j (\theta \mathbf{e}_2)\, dx. \end{aligned}$$

By Plancherel's theorem,

$$\|\mathcal{L}^{\frac{1}{2}}\Delta_j u\|_{L^2}^2 \ge C\, 2^j a^{-1}(2^j)\, \|\Delta_j u\|_{L^2}^2,$$

where $C = 0$ in the case of $j = -1$ and $C > 0$ for $j \ge 0$. The estimate for J_3 is easy and we have, by Hölder's inequality,

$$|J_3| \le \|\Delta_j u\|_{L^2}\, \|\Delta_j \theta\|_{L^2} \le 2^j a^{-1}(2^j)\, \|\Delta_j u\|_{L^2}\, \|\theta\|_{B^{-1,a}_{2,\infty}}.$$

To estimate J_1, we need to use a commutator structure to shift one derivative to $u^{(1)}$. For this purpose, we write

$$\Delta_j(u^{(1)} \cdot \nabla u) = J_{11} + J_{12} + J_{13} + J_{14} + J_{15}, \tag{6.36}$$

where

$$\begin{aligned}
J_{11} &= \sum_{|j-k|\le 2} [\Delta_j, S_{k-1}u^{(1)} \cdot \nabla]\Delta_k u,\\
J_{12} &= \sum_{|j-k|\le 2} (S_{k-1}u^{(1)} - S_j u^{(1)}) \cdot \nabla \Delta_j \Delta_k u,\\
J_{13} &= S_j u^{(1)} \cdot \nabla \Delta_j u,\\
J_{14} &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u^{(1)} \cdot \nabla S_{k-1} u),\\
J_{15} &= \sum_{k\ge j-1} \Delta_j(\Delta_k u^{(1)} \cdot \nabla \widetilde{\Delta}_k u).
\end{aligned}$$

Since $\nabla \cdot u^{(1)} = 0$, we have

$$\int J_{13}\, \Delta_j u\, dx = 0.$$

J_{11}, J_{12}, J_{14} and J_{15} can be bounded in a similar fashion as in the proof of Lemma 6.9 and we have

$$\begin{aligned}
&\|J_{11}\|_{L^2},\, \|J_{12}\|_{L^2} \le C\, (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \|\Delta_j u\|_{L^2},\\
&\|J_{14}\|_{L^2} \le C\, (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \sum_{m\le j-1} 2^{m-j} \|\Delta_m u\|_{L^2},\\
&\|J_{15}\|_{L^2} \le C\, (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \sum_{k\ge j-1} 2^{j-k} \|\Delta_k u\|_{L^2}.
\end{aligned}$$

To estimate J_2, we write

$$\Delta_j(u \cdot \nabla u^{(2)}) = J_{21} + J_{22} + J_{23}, \tag{6.37}$$

where

$$\begin{aligned}
J_{21} &= \sum_{|j-k|\le 2} \Delta_j(S_{k-1}u \cdot \nabla \Delta_k u^{(2)}),\\
J_{22} &= \sum_{|j-k|\le 2} \Delta_j(\Delta_k u \cdot \nabla S_{k-1} u^{(2)}),\\
J_{23} &= \sum_{k\ge j-1} \Delta_j(\Delta_k u \cdot \nabla \widetilde{\Delta}_k u^{(2)}).
\end{aligned}$$

Therefore, by Hölder's inequality,

$$\begin{aligned}
&\|J_{21}\|_{L^2} \le C\,\|u\|_{L^2}\,\|\nabla\Delta_j u^{(2)}\|_{L^\infty},\\
&\|J_{22}\|_{L^2} \le C\,\|\Delta_j u\|_{L^2}(\|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}),\\
&\|J_{23}\|_{L^2} \le C\,(\|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}) \sum_{k\ge j-1} 2^{j-k}\|\Delta_k u\|_{L^2}.
\end{aligned}$$

Inserting the estimates above in (6.35), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u\|_{L^2} + C\,2^j a^{-1}(2^j)\,\|\Delta_j u\|_{L^2} \le C\,2^j a^{-1}(2^j)\,\|\theta\|_{B^{-1,a}_{2,\infty}} + K(t), \tag{6.38}$$

where

$$\begin{aligned}
K(t) \;=\;& C\,(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}})\|\Delta_j u\|_{L^2}\\
&+ C\,\|u\|_{L^2}\,\|\nabla\Delta_j u^{(2)}\|_{L^\infty} + (\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \sum_{m\le j-1} 2^{m-j}\|\Delta_m u\|_{L^2}\\
&+ C\,(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}} + \|u^{(2)}\|_{L^2} + \|\omega^{(2)}\|_{B^0_{\infty,1}}) \sum_{k\ge j-1} 2^{j-k}\|\Delta_k u\|_{L^2}.
\end{aligned}$$

Integrating (6.38) in time and taking $\sup_{j\ge -1}$, we obtain (6.34). This completes the proof of Lemma 6.11. ■

Lemma 6.12 *Assume that θ, $u^{(1)}$, u and $\theta^{(2)}$ are defined as in the proof of Theorem 6.2 and satisfy*

$$\partial_t\theta + u^{(1)}\cdot\nabla\theta + u\cdot\nabla\theta^{(2)} = 0. \tag{6.39}$$

Then we have the a priori bound

$$\begin{aligned}\|\theta(t)\|_{B^{-1,a}_{2,\infty}} &\leq \|\theta(0)\|_{B^{-1,a}_{2,\infty}} + C\int_0^t \|\theta(\tau)\|_{B^{-1,a}_{2,\infty}}(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}})\,d\tau \\ &\quad + C\int_0^t \|u(\tau)\|_{L^2}\|\theta^{(2)}\|_{B^{0,a}_{\infty,1}}\,d\tau. \qquad (6.40)\end{aligned}$$

Proof. (Proof of Lemma 6.12) Let $j \geq -1$ be an integer. Applying Δ_j to (6.39) and taking the inner product with $\Delta_j\theta$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j\theta\|^2_{L^2} = K_1 + K_2, \qquad (6.41)$$

where

$$\begin{aligned} K_1 &= -\int \Delta_j\theta\,\Delta_j(u^{(1)}\cdot\nabla\theta)\,dx, \\ K_2 &= -\int \Delta_j\theta\,\Delta_j(u\cdot\nabla\theta^{(2)})\,dx.\end{aligned}$$

To estimate K_1, we decompose $\Delta_j(u^{(1)}\cdot\nabla\theta)$ as in (6.36) and estimate each component in a similar fashion to obtain

$$\begin{aligned}|K_1| &\leq C\,\|\Delta_j\theta\|^2_{L^2}\,(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \\ &\quad + C\,\|\Delta_j\theta\|_{L^2}\,2^j a^{-1}(2^j)\,\|\theta\|_{B^{-1,a}_{2,\infty}}(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}).\end{aligned}$$

To estimate K_2, we decompose $\Delta_j(u\cdot\nabla\theta^{(2)})$ as in (6.37) and bound the components in a similar fashion to have

$$|K_2| \leq C\,\|\Delta_j\theta\|_{L^2}\,\|u\|_{L^2}2^j a^{-1}(2^j)\,\|\theta^{(2)}\|_{B^{0,a}_{\infty,1}}.$$

Combining these estimates, we find

$$\begin{aligned}\frac{d}{dt}\|\Delta_j\theta\|_{L^2} &\leq C\,2^j a^{-1}(2^j)\,\|\theta\|_{B^{-1,a}_{2,\infty}}(\|u^{(1)}\|_{L^2} + \|\omega^{(1)}\|_{B^0_{\infty,1}}) \\ &\quad + C\,\|u\|_{L^2}2^j a^{-1}(2^j)\,\|\theta^{(2)}\|_{B^{0,a}_{\infty,1}}.\end{aligned}$$

Integrating in time, multiplying by $2^{-j}a(2^j)$ and taking $\sup_{j\geq-1}$, we obtain (6.40). This completes the proof of Lemma 6.12. ∎

Proof of the main theorem

Proof. (Proof of Theorem 6.1) Uniqueness is already shown in 6.2, it suffices to establish the existence of solutions. The first step is to obtain a local (in time) solution and then extend it into a global solution through the global *a priori* bounds obtained in the previous section. The local solution can be constructed through the method of successive approximation. That is, we consider a successive approximation sequence $\{(\omega^{(n)}, \theta^{(n)})\}$ solving

$$\begin{cases} \omega^{(1)} = S_2\omega_0, \quad \theta^{(1)} = S_2\theta_0, \\ \partial_t \omega^{(n+1)} + u^{(n)} \cdot \nabla \omega^{(n+1)} + \mathcal{L}\omega^{(n+1)} = \partial_{x_1}\theta^{(n+1)}, \\ \partial_t \theta^{(n+1)} + u^{(n)} \cdot \nabla \theta^{(n+1)} = 0, \\ \omega^{(n+1)}(x,0) = S_{n+2}\omega_0(x), \quad \theta^{(n+1)}(x,0) = S_{n+2}\theta_0(x). \end{cases} \tag{6.42}$$

To show that $\{(\omega^{(n)}, \theta^{(n)})\}$ converges to a solution of (6.4), it suffices to prove that $\{(\omega^{(n)}, \theta^{(n)})\}$ obeys the following properties:

(1) There exists a time interval $[0, T_1]$ over which $\{(\omega^{(n)}, \theta^{(n)})\}$ are bounded uniformly in terms of n. More precisely, we show that

$$\|\omega^{(n)}\|_{L_t^\infty(L^2 \cap L^q) \cap L_t^1 B^{0,a}_{\infty,1}} \le C(T_1), \quad \|\theta^{(n)}\|_{L_t^\infty(L^2 \cap B^{0,a^2}_{\infty,1}) \cap L_t^1 B^{0,a}_{\infty,1}} \le C(T_1),$$

where $C(T_1)$ is a constant independent of n.

(2) There exists $T_2 > 0$ such that $\omega^{(n+1)} - \omega^{(n)}$ is a Cauchy sequence in $L_t^\infty B^{-1}_{\infty,1}$ and $\theta^{(n+1)} - \theta^{(n)}$ is Cauchy in $L_t^1 B^{-1,a}_{\infty,1}$, namely

$$\|\omega^{(n+1)} - \omega^{(n)}\|_{L_t^\infty B^{-1}_{\infty,1}} \le C(T_2)\, 2^{-n}, \quad \|\theta^{(n+1)} - \theta^{(n)}\|_{L_t^1 B^{-1,a}_{\infty,1}} \le C(T_2)\, 2^{-n}$$

for any $t \in [0, T_2]$, where $C(T_2)$ is independent of n.

If the properties stated in (1) and (2) hold, then there exists (ω, θ) satisfying

$$\omega \in L_t^\infty(L^2 \cap L^q) \cap L_t^1 B^{0,a}_{\infty,1}, \quad \theta \in L_t^\infty(L^2 \cap B^{0,a^2}_{\infty,1}) \cap L_t^1 B^{0,a}_{\infty,1},$$

$$\omega^{(n)} \to \omega \quad \text{in} \quad L_t^\infty B_{\infty,1}^{-1}, \qquad \theta^{(n)} \to \theta \quad \text{in} \quad L_t^1 B_{\infty,1}^{-1,a}$$

for any $t \leq \min\{T_1, T_2\}$. It is then easy to show that (ω, θ) solves (6.4) and we thus obtain a local solution and the global bounds in the previous sections allow us to extend it into a global solution. It then remains to verify the properties stated in (1) and (2). Property (1) can be shown as in the previous sections (Section 6.2 through Section 6.5) while Property (2) can be checked as in the proof of Theorem 6.2. We thus omit further details. This completes the proof of the main Theorem. ■

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