UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

SOME TYPES AND COVERS FOR QUATERNIONIC HERMITIAN GROUPS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

Degree of

DOCTOR OF PHILOSOPHY

By

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Norman, Oklahoma
2014

SOME TYPES AND COVERS FOR
QUATERNIONIC HERMITIAN GROUPS

A DISSERTATION APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

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Acknowledgements

First, I would like to thank my advisor Dr. Alan Roche for his never ceasing support and encouragement. I am very grateful to him for not giving up on me. I would also like to thank Dr. Lucy Lifschitz, Dr. Ralf Schmidt, Dr. Jonathan Kujawa, and Dr. James Hawthorne for their willingness to serve on my doctoral committee.

A special thanks to the late Dr. Edward Cline who initially encouraged me to pursue a Ph.D. in mathematics. I would also like to thank the Department of Mathematics and its staff. Their work and support did not go unnoticed.

Lastly I thank my wife Kelsey and son James for putting up with me.

Contents

Abstract

In this dissertation we determine the reducibility of certain induced representations. We do this using Bushnell and Kutzko's method of types and covers. We consider certain quaternionic hermitian groups over a p-adic field. While the types and covers can be found in the literature, the computation of the associated Hecke algebras has not been done before.

Chapter 1

Introduction

The area of interest in this dissertation is the representation theory of reductive p-adic groups. Specifically we are interested in how Bushnell-Kutzko's theory of types and covers can be used to obtain explicit results on the reducibility of parabolically induced representations.

Representation theory is a central area of modern mathematics. In particular, the representation theory of reductive p-adic groups is central to the web of conjectures known as the Langlands program whose influence pervades current research in automorphic forms and number theory. The origins of representation theory go back to Frobenius and others in the 1890s in the study of finite groups. A key tool introduced by Frobenius is the method of induction, a way of building representations of a group from representations of subgroups. To make this an effective means of constructing or classifying representations, one needs to be wise in the choice of both the subgroups one induces from and the representations one induces.

In the case of reductive real or p-adic groups, it is natural to induce from what are called parabolic subgroups. For general linear groups, these are (up to conjugacy) the subgroups of invertible block upper-triangular matrices. A proper parabolic subgroup is not reductive but admits a canonical reductive quotient. For example, in the case of block upper-triangular matrices, this reductive quotient is isomorphic to the corresponding group of block diagonal matrices. One takes a representation of this reductive quotient, views it as a representation of the parabolic subgroup and then induces. The whole process is called parabolic induction.

In contrast to reductive real groups, a central feature of reductive p-adic groups is that there are irreducible representations that never occur as subrepresentations of parabolically induced representations. These are the *supercuspidal* representations. They serve as fundamental building blocks. Indeed, suppose Π is an irreducible representation of a reductive p-adic group G. By insights of Harish-Chandra and others it is known that there is a parabolic subgroup of G and an irreducible supercuspidal representation π of its reductive quotient such that Π occurs in the representation obtained from π via parabolic induction. Moreover, only finitely many Π are related to the supercuspidal representation π in this way.

Thus, a core problem in p-adic representation theory is to understand when and how parabolically induced representations decompose, especially when the inducing representation is supercuspidal. This is the problem I study in a very special situation.

In the case of finite groups, Mackey theory provides an efficient way of decomposing induced representations through the action of certain intertwining operators. The same operators can be used to study parabolically induced representations for p-adic groups. Their construction, however, is considerably more subtle and involves a process of analytic continuation. By work of Langlands and Shahidi [20], one knows that properties of these intertwining operators give rise to local L-functions and that these L-functions are the key to many reducibility questions. In [21] Shahidi studies certain reducibility questions in this way for the split classical groups.

Bushnell-Kutzko's method of types and covers [4] provides another way of studying reducibility questions. It relies on detailed knowledge of the internal structure of the inducing representation and certain related constructions. In circumstances where this is available, the method can lead to strikingly explicit results. Indeed, in [9] Kutzko and Morris use the method to reconsider a special case of the situation studied by Shahidi in [21] and obtain a sharper form of his results. In this dissertation, I study a situation that is analogous to the one considered by Kutzko and Morris where the underlying group is non-split. Pursuing the same basic strategy, I obtain an explicit reducibility result as described in more detail below.

With the first chapter being this introduction, in the second chapter we give some background that mainly aims to fix some notation and terminology. In the third chapter we pose the question and walk through the answer.

Our approach is local. We note that Muić and Savin in [17] have answered the reducibility question that we consider using global methods. Their work is more general in that it is not restricted to the depth zero case. However, their results are less explicit. Further, their approach only works when the underlying field has characteristic zero. Our only restriction is that the residual characteristic of the underlying field is odd.

We adopt the following conventions and notations. All rings are assumed to have multiplicative identities. All representations are complex representations. Given a category $\mathcal{C}$, by $x \in \mathcal{C}$ we mean that x is an object in the category. We often identify representations if they are equivalent as representations. In this same way, we often say that one representation is a subrepresentation of another representation if it is equivalent to a subrepresentation of the other representation.

Chapter 2

Background

In this chapter we lightly touch on some of the background needed. We will not go into great detail, but only highlight tools and results that we shall use later. The main sources for background are [5], [2], and [11].

2.1 Algebraic groups

We assume some familiarity with algebraic groups. There are several ways to present the basic notions. We adopt here the view found in [22]. For this section only, let K be an algebraically closed field. We recall that an algebraic group is an algebraic variety over K that is a group, such that the multiplication and taking inverse are morphisms of varieties. When the variety is affine we call the group an affine (or linear) algebraic group. It is a fact that every affine algebraic groups is isomorphic to some closed (in the Zariski topology) subgroup of $\mathrm{GL}_n(K)$ for some n. We follow standard abuses of notation in often not distinguishing between algebraic groups and their groups of F-points.

2.1.1 About finite groups

We now describe finite groups of Lie type as in [5]. Assume that K has characteristic p and let $q = p^e$ for some positive integer e. Let $F_q : \mathrm{GL}_n(K) \to \mathrm{GL}_n(K)$ be the homomorphism given by

$$F_q(a_{ij}) = (a_{ij}^q).$$

A map $F : G \to G$ is called a standard Frobenius map if for some embedding $i : G \to \mathrm{GL}_n(K)$ we have

$$i(F(g)) = F_q(i(g))$$

for all $g \in G$. A map $F : G \to G$ is called a Frobenius map if F^m is a standard Frobenius map for some m. For a Frobenius map $F : G \to G$ we can talk about the F-points of G:

$$G^F = \{g \in G : F(g) = g\}.$$

Then G^F is a finite subgroup of G. The finite groups we obtain in this way are called the finite groups of Lie-type. Consider the following examples.

1. Let $G = \mathrm{GL}_n(K)$. Let $i : G \to G$ be the identity map. Let $F : G \to G$ be the map: $(a_{ij}) \mapsto (a_{ij}^q)$. Then $G^F = \mathrm{GL}_n(\mathbb{F}_q)$.

2. Let $G = \mathrm{GL}_n(K)$. Let again $i : G \to G$ be the identity. Let $F : G \to G$ be the map $F : (a_{ij}) \mapsto {}^\top(a_{ij}^q)^{-1}$. Then $G^F = \{(a_{ij}) \in \mathrm{GL}_n(\mathbb{F}_{q^2}) : (a_{ij})(a_{ij}^q)^\top = 1\}$. This is the unitary group $U_n(\mathbb{F}_{q^2})$.

3. Let $G = \mathrm{Sp}_n(K)$. Let $i : G \to \mathrm{GL}_n(K)$ be inclusion. Let $F : G \to G$ be the map $F : (a_{ij}) \mapsto (a_{ij}^q)$. In this case, $G^F = \mathrm{Sp}_n(\mathbb{F}_q)$.

2.1.2 Locally profinite groups

The following is mainly based on the treatment in [2]. We say that a topological group G is profinite if it is compact, Hausdorff, and totally disconnected. We can also define a profinite group as the inverse limit of finite groups. Let $(I, \leq)$ be a partially ordered set. Let $\{H_i\}_{i\in I}$ be a collection of finite groups all equipped with the discrete topology. Let $f_{ij} : H_j \to H_i$ for $i \leq j$ be group homomorphisms. We require that f_{ii} is the identity of H_i and that $f_{ki} \circ f_{ij} = f_{kj}$ for $k \leq i \leq j$. Then we can form the inverse limit

$$\varprojlim H_n = \left\{(h_i) \in \prod H_n : f_{ij}(h_j) = h_i\right\}.$$

This is a group under $(a_i)(b_i) = (a_i b_i)$.

For p a prime, $H_n = \mathbb{Z}/p^n\mathbb{Z}$, and the maps $f_{ij} : \mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$ being given by $f_{ij} : x + p^j\mathbb{Z} \mapsto x + p^i\mathbb{Z}$, the inverse limit is the p-adic integers $\mathbb{Z}_p$.

We also have the definition of a locally profinite group. We say that a topological group G is locally profinite if it is Hausdorff and if each neighborhood of the identity in G contains a compact open subgroup.

Lemma 2.1. *A topological group is locally profinite if and only if it is locally compact and totally disconnected.*

A compact locally profinite group is a profinite group. Finite groups are locally profinite.

2.2 Facts from representation theory

We recall some basic facts from representation theory. The following is not in any way a systematic treatment, but merely serves to fix notation and highlight tools that we shall need later.

2.2.1 Smooth representations

Let G be a locally profinite group. A representation (π, V) of G is a complex vector space V together with a group homomorphism $\pi : G \to \mathrm{GL}(V)$.

Let K be a compact open subgroup of G. Then we let V^K be the set of

$\pi(K)$-fixed vectors: $V^K = \{v \in V \mid \pi(k)v = v \text{ for all } k \in K\}$. We call (π, V) a smooth representation if

$$V = \bigcup_K V^K.$$

A smooth representation (π, V) is called admissible if the spaces V^K have finite dimension.

Let

$$V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$$

be the dual of V. Define a representation (π^*, V^*) of G by

$$(\pi^*(g)v')(v) = v'(\pi(g^{-1})v)$$

for $v \in V$, $v' \in V^*$ and $g \in G$. This defines a representation of G, but it is not necessarily smooth. Therefore we consider the space of smooth vectors

$$V^\vee = (V^*)^\infty = \bigcup_K (V^*)^K$$

where the union is taken over all compact open subgroups of G. We then define the representation $(\pi^\vee, V^\vee)$ by $\pi^\vee(g)(v) = \pi^*(g)v$ for $v \in V^\vee$ and $g \in G$. This representation is smooth and we call it the *smooth dual* or the *contragredient* of (π, V).

Given a representation π of $H \leq G$ and $g \in G$ we let π^g denote the representation π^g of $H^g = g^{-1}Hg$ given by $\pi^g(h') = \pi(gh'g^{-1})$.

2.2.2 Induced representations

Let G be a locally profinite group and let H be a closed subgroup of G. Let (σ, W) be a smooth representation of H. Let

$$\begin{aligned}\mathrm{Ind}_H^G(\sigma, W) = \{f : G \to W \mid f(hg) = \sigma(h)f(g) &\text{ for all } h \in H, g \in G,\\ &\text{there is a compact open subgroup } K \leq G\\ &\text{such that } f(gk) = f(g) \text{ for all } g \in G, k \in K\}.\end{aligned}$$

Let G act on $\mathrm{Ind}_H^G(\sigma, W)$ by

$$(g.f)(x) = f(xg).$$

This defines a smooth representation of G. We also consider the space

$$\mathrm{ind}_H^G(\sigma, W) = \{f \in \mathrm{Ind}_H^G(\sigma, W) \mid f \text{ has compact support modulo } H\}.$$

The same action as above defines a smooth representation of G. We call this method of induction *compact induction*. That f has compact support modulo H means that $\mathrm{supp}(f)$ is compact in G/H. That is, if $p : G \to G/H$ is the projection map, then $p(\mathrm{supp}(f))$ is compact.

We also define *normalized induction*. Let P be a parabolic subgroup of the F-points of a reductive algebraic group defined over a p-adic field F. Let

$$\iota_P^G(\sigma, W) = \mathrm{ind}_P^G(\delta_P^{1/2} \otimes \sigma).$$

Here δ_P is a certain function: $P \to (0, \infty)$. See section 3 in [2] for more on this function. As a matter of notation we will write $\iota_P^G(\sigma)$ for $\iota_P^G(\sigma, V)$.

One important property of normalized induction is that

$$\iota_P^G(\sigma)^\vee \simeq \iota_P^G(\sigma^\vee).$$

In particular parabolic induction takes self-dual representations to self-dual representations.

2.2.3 Cuspidal representations

Definition 2.2. Let G be a finite group of Lie type. Let π be an irreducible representation of G. We call π *cuspidal* if

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G \tau) = \{0\}$$

for all proper parabolic subgroups P of G and τ representation of a Levi factor of P inflated to P.

In the following chapter we will, as mentioned earlier, study reducibily of induced representations. A key requirement is that the representation being induced has *depth zero*. The general definition of depth can be found in [16]. In chapter 3 we will state the definition in the setup that we use there.

We recall *Frobenius Reciprocity*. Let G be a locally profinite group. Let H be a open subgroup of G. Let (π, V) be a representation of G and (σ, W) a representation of H. Then

$$\mathrm{Hom}_G(\mathrm{ind}_H^G \sigma, \pi) \simeq \mathrm{Hom}_H(\sigma, \pi|_H).$$

For H a closed subgroup of G, we have

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \simeq \mathrm{Hom}_H(\pi|_H, \sigma).$$

We also recall *Mackey's Irreducibility Criterion* which is often used in conjunction with Frobenius Reciprocity. The original reference for this is [13]. Let G be a locally profinite group and H an open and compact mod center subgroup of G. Let (π, V) be a smooth irreducible representation of H. Then $\mathrm{ind}_H^G \pi$ is irreducible exactly when

$$\mathrm{Hom}(\pi^g|_{H \cap H^g}, \pi|_{H \cap H^g}) = \{0\}$$

for $g \notin H$.

2.3 Finite groups of Lie type

The following highlights certain aspects that we shall make reference to later. A fuller treatment of the theory of finite groups of Lie Type can be found in [5] where precise statements and definitions can be found. Another good resource is [6].

In the following we abuse notation and let $G = G^F$ be a finite group of Lie type with a Frobenius map $F : G \to G$. As described well in [5] we have the so-called Deligne-Lusztig (generalized) characters $R_{T,\theta}$ where T is a maximal torus in G and θ an irreducible character of G. We recall that θ is said to be in general position if the only element of $W(T) = N_G(T)/T$ that fixes θ is the identity. In the case where θ is in general position $\pm R_{T,\theta}$ is an irreducible character of G.

For $\chi \in \mathrm{Irr}(G)$ a character there is a maximal torus T and $\theta \in \mathrm{Irr}(T)$ such that $\langle \chi, R_{T,\theta}\rangle \neq 0$. Here $\langle \cdot, \cdot \rangle$ is the usual scalar product. This means that all irreducible characters of G occur as constituents of some $R_{T,\theta}$. We say that two pairs (T,θ) and (T',θ') are geometrically conjugate if

- there is a $g \in G$ such that $T = {}^gT'$ and
- if $g \in G^{F^n}$ for some n, then $\theta \circ N_{F^n/F} = \theta' \circ N_{F^n/F} \circ \mathrm{Ad}(g)$.

Here $\mathrm{Ad}(g)$ is conjugation by g and $N_{F^n/F}$ is the reduced norm.

If (T,θ) and (T',θ') are not geometrically conjugate, then $R_{T,\theta}$ and $R_{T',\theta'}$ have no irreducible components in common.

Each irreducible character of G determines a unique geometric conjugacy class $[T,\theta]$. We say that two characters are (geometrically) conjugate if in this way they give rise to the same class. An irreducible character χ is called unipotent if $\langle R_{T,1}, \chi\rangle \neq 0$. Note that all unipotent characters form a single conjugacy class. An irreducible character χ is called regular if $\langle \Gamma, \chi\rangle \neq 0$ where Γ is the Gelfand-Graev character (See chapter 14 in [6] for more on this). An irreducible character will be called semi-simple if its dual is (up to sign) a regular irreducible character. Note that the only semisimple unipotent character is the trivial character.

If (T^*, F^*) is dual to (T, F), then $\mathrm{Irr}(T^F) \simeq (T^*)^{F^*}$ (see chapter 13 in [6] for more on this). Note that the θ in the $R_{T,\theta}$'s are exactly characters of T, so using this isomorhism (of groups) we get elements of the dual torus T^*.

Suppose that (G, F) and (G^*, F^*) are in duality with corresponding dual tori T and T^*. Then we have have a bijection between the pairs $\{[T,\theta]\}$ and F^*-stable semisimple conjugacy classes in G^*.

2.4 Types and covers

The theory of types and covers is described in detail in [4]. We adopt the approach given there. We will, though, also rely on the presentations given in [11] and [3]. Let F be a non-Archimedean local field. Let G be the F-rational points of a connected, reductive, algebraic group defined over F. Then G is (isomorphic to) a closed subgroup of $GL_n(F)$ for some n. Since $GL_n(F)$ is locally profinite, G is locally profinite. We recall that this means that G is locally compact and totally disconnected, or equivalently, that each neighborhood of the identity in G contains a compact open subgroup.

We note that G is unimodular, so a left Haar measure is a right Haar measure and vice versa. We fix a Haar measure μ on G.

Let $\mathfrak{R}(G)$ be the category of smooth representations of G. That is, the objects in $\mathfrak{R}(G)$ are the smooth representations and the morphisms are G-maps between representations. Given a smooth representation π of G, there is a parabolic subgroup P of G with Levi decomposition $P = LU$ (with L the Levi factor) and an irreducible supercuspidal representation σ of L such that π is a composition factor of $\iota_P^G(\sigma)$. We note that if π is supercuspidal then $L = G$. The pair (L, σ) is not unique. We say that two pairs (L_1, σ_1) and (L_2, σ_2) are inertially equivalent if there is a $g \in G$ such that $L_2 = L_1^g$ and $\sigma_2 \otimes \chi \simeq \sigma_1^g$ for some unramified character χ of L_2. We let $[L, \sigma] = [L, \sigma]_G$ be the equivalence class and we let $\mathfrak{B}(G)$ be the set of equivalence classes. Given $\pi \in \mathfrak{R}(G)$ we then obtain a unique equivalence class $\mathfrak{I}(\pi) \in \mathfrak{B}(G)$. We call $\mathfrak{I}(\pi)$ the inertial support (or just support) of π.

For $\mathfrak{s} \in \mathfrak{B}(G)$ we let $\mathfrak{R}^{\mathfrak{s}}(G)$ be the (full) subcategory of $\mathfrak{R}(G)$ consisting of all smooth representations π' with inertial support $\mathfrak{I}(\pi') = \mathfrak{s}$. We can now state the Bernstein decomposition:

$$\mathfrak{R}(G) = \prod_{s \in \mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G).$$

Let $\mathfrak{s} \in \mathfrak{B}(G)$. Then fix L and σ such that $\mathfrak{s} = [L, \sigma]$. Then let $\mathfrak{s}_L = [\sigma, L]_L \in \mathfrak{B}(L)$. Given this setup with P a parabolic subgroup of G with Levi-factor L, we have the normalized induction functor

$$\iota_P^G : \mathfrak{R}^{\mathfrak{s}_L}(L) \longrightarrow \mathfrak{R}^{\mathfrak{s}}(G).$$

2.4.1 Types

Let K be a compact open subgroup of G. Let (ρ, W) be an irreducible smooth representation of K. For a smooth representation (π, V) of G we let V^ρ be the ρ-isotypic subspace of V. That is, V^ρ is the sum of all irreducible K-subspaces of V which are equivalent to ρ:

$$V^\rho = \sum_{W'} W'$$

where the sum is over all W' such that $(\pi \mid_K, W') \simeq (\rho, W)$.

Let $\mathcal{H}(G)$ be the space of all locally constant compactly supported functions $f : G \to \mathbb{C}$. This is a $\mathbb{C}$-algebra under convolution $\star$. That is, for $f, g \in \mathcal{H}(G)$ we have

$$(f \star g)(x) = \int_G f(y) g(y^{-1}x)\, d\mu(y).$$

We have here fixed a Haar measure μ on G. We note that $\mathcal{H}(G)$ only has an identity when G is a discrete group. For (π, V) a representation of G, $\mathcal{H}(G)$ acts on V via

$$hv = \int_G h(x)\pi(x)v d\mu(x)$$

for $h \in \mathcal{H}(G)$ and $v \in V$. We identify $\mathfrak{R}(G)$ and $\mathcal{H}(G)-$Mod.

Let e_ρ be the element in $\mathcal{H}(G)$ defined with support K such that for each $x \in K$,

$$e_\rho(x) = \frac{\dim \rho}{\mu(K)} \operatorname{tr}_W(\rho(x^{-1})).$$

Then e_ρ is an idempotent in $\mathcal{H}(G)$, that is, $e_\rho \star e_\rho = e_\rho$.

Let $\mathfrak{R}_\rho(G)$ be the (full) subcategory of $\mathfrak{R}(G)$ consisting of all (π, V) where V is generated by $V^\rho = e_\rho V$ (the ρ-isotypic vectors). That is, $(\pi, V) \in \mathfrak{R}_\rho(G)$ if and only if

$$V = \mathcal{H}(G) \star e_\rho V.$$

We can now state the definition of a type.

Definition 2.3. Let $\mathfrak{s} \in \mathfrak{B}(G)$. We say that (K, ρ) is an $\mathfrak{s}$-type in G if $\mathfrak{R}_\rho(G) = \mathfrak{R}^{\mathfrak{s}}(G)$.

2.4.2 Hecke algebras

We define the Hecke algebra:

$$\begin{aligned}\mathcal{H}(G, \rho) = \{f : G \to \operatorname{End}_{\mathbb{C}}(\rho) : &\operatorname{supp}(f) \text{ is compact and} \\ &f(k_1 g k_2) = \rho(k_1) f(g) \rho(k_2)\ \forall k_i \in K, g \in G\}.\end{aligned}$$

There is a canonical isomorphism

$$\mathcal{H}(G,\rho)\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}(W)\xrightarrow{\simeq} e_\rho\star\mathcal{H}(G)\star e_\rho,$$

and a corresponding functor

$$V\mapsto V^\rho:\quad \mathfrak{R}_\rho(G)\longrightarrow e_\rho\star\mathcal{H}(G)\star e_\rho-\mathrm{Mod}.$$

So we get a functor

$$M_\rho:\mathfrak{R}_\rho(G)\longrightarrow\mathcal{H}(G,\rho)-\mathrm{Mod}.$$

The importance of types is seen in the following theorem

Theorem 2.4. *The following are equivalent:*

1. *(K,ρ) is an $\mathfrak{s}$-type in G.*

2. *For $\pi\in\mathrm{Irr}(\mathfrak{R}^{\mathfrak{s}}(G))$ we have $\mathfrak{I}(\pi)=\mathfrak{s}$.*

3. *The functor M_ρ is an equivalence of categories.*

2.4.3 Covers

Definition 2.5. Let $P=LU$ be a parabolic subgroup with Levi factor L and let $\overline{P}=L\overline{U}$ be its opposite. We will in general denote the L-opposite of a parabolic subgroup P by $\overline{P}$. Let K be a compact open subgroup of G and let (σ,W) be an irreducible representation of K. Then we say that (K,σ) is decomposed with respect to (L,P) if the following hold:

1. $K=(K\cap\overline{U})(K\cap L)(K\cap U)$.

2. $K\cap\overline{U}$, $K\cap U\leq\ker(\sigma)$.

For (K, ρ) decomposed with respect to (L, P) we write $K_L = K \cap L$ and we write ρ_L for $\rho|_{K_L}$. Let $I_G(\rho) = \{g \in G : \text{there exists } f \in \mathcal{H}(G, \rho) \text{ such that } g \in \operatorname{supp}(f)\}$. Let $\mathcal{H}(G, \rho)_L = \{f \in \mathcal{H}(G, \rho) : \operatorname{supp}(f) \subseteq KLK\}$.

Proposition 2.6. *Let (K, ρ) decompose with respect to (L, P). Then*

1. *ρ_L is irreducible.*

2. *$I_L(\rho_L) = I_G(\rho) \cap L$.*

3. *There is an embedding*

$$T : \mathcal{H}(L, \rho_L) \longrightarrow \mathcal{H}(G, \rho)$$

   such that if $f \in \mathcal{H}(L, \rho_L)$ has support $K_L z K_L$ for some $z \in L$, then $T(f)$ has support contained in KzK.

4. *The map T induces an isomorphism of vector spaces*

$$\mathcal{H}(L, \rho_L) \xrightarrow{\simeq} \mathcal{H}(G, \rho).$$

Definition 2.7. An element $z \in L$ is called (K, P)-positive if

1. $z(K \cap U)z^{-1} \subseteq K \cap U$,

2. $z^{-1}(K \cap \overline{U})z \subseteq K \cap \overline{U}$.

Definition 2.8. An element $z \in L$ is called strongly (K, P)-positive if

1. z is (K, P)-positive,

2. z is in the center of M: $z \in Z(M)$,

3. for any compact open subgroups K_1 and K_2 of U, there is an integer $m \geq 0$ such that $z^m K_1 z^{-m} \subseteq K_2$,

4. for any compact open subgroups K_1' and K_2' of $\overline{U}$, there is an integer $m \geq 0$ such that $z^{-m} K_1' z^m \subseteq K_2$.

Let

$$\mathcal{H}(L, \rho_L)^+ = \{f \in \mathcal{H}(L, \rho_L) : \\ \text{supp}(f) \text{ consists of } (K, P)\text{-positive elements}\}.$$

The isomorphism of vector spaces T restricts to an embedding of algebras

$$T : \mathcal{H}(L, \rho_L)^+ \longrightarrow \mathcal{H}(G, \rho).$$

This embedding extends to $\mathcal{H}(L, \rho_L)$ under the following condition.

Theorem 2.9. *The embedding T extends to an embedding of algebras*

$$t : \mathcal{H}(L, \rho_L) \hookrightarrow \mathcal{H}(G, \rho)$$

if and only if $T(\phi_z)$ is invertible for some strongly (K, P)-positive element z. If this embedding exists, it is unique.

Proposition 2.10. *We have the following.*

1. *Strongly (K, P)-positive elements exist.*

2. *Given a strongly (K, P)-positive element $z \in L$, there is a unique function $\phi_z \in \mathcal{H}(L, \rho_L)$ with support $K_L z K_L$ such that $\phi_z(z)$ is the identity function on W.*

Let $\mathcal{H}(G,\rho)_L = \{f \in \mathcal{H}(G,\rho) : \mathrm{supp}(f) = KLK\}$. This is a subspace of $\mathcal{H}(G,\rho)$.

Definition 2.11. Let L be a proper Levi subgroup of G. Let K_L be a compact open subgroup of L, and let ρ_L be an irreducible smooth representation of K_L. Let K be a compact open subgroup of G and let ρ be an irreducible smooth representation of K. Then we say that (K,ρ) is a G-cover of (K_L,ρ_L) if

1. the pair (K,ρ) is decomposed with respect to (L,P) for every parabolic subgroup P of G with Levi factor L;

2. $L \cap L = K_L$, and $\rho|_L \simeq \rho_L$;

3. The embedding $T : \mathcal{H}(L,\rho_L)^+ \hookrightarrow \mathcal{H}(G,\rho)$ extends to an embedding of algebras $t' : \mathcal{H}(L,\rho_L) \hookrightarrow \mathcal{H}(G,\rho)$.

Chapter 3

Quaternionic Hermitian Groups

We now turn to the study of quaternionic hermitian groups.

3.1 Setup

Let F be a non-Archimedean local field of residual characteristic not equal to 2 with uniformizer ϖ_F. Write $\nu_F : F \to \mathbb{Z} \cup \{\infty\}$ for the normalized valuation on F so that $\nu_F(\varpi_F) = 1$. Thus the ring of integers $\mathcal{O}_F = \{x \in F : \nu(x) \geq 0\}$ and the unique maximal ideal $\mathfrak{p}_F = \{x \in F : \nu(x) > 0\}$. Write q for the order of the residue field $k_F = \mathcal{O}_F/\mathfrak{p}_F$. Then $q = p^f$ for some odd prime p and some positive f. Let $D = F \oplus iF \oplus jF \oplus kF$ be the unique quaternionic division algebra over F. Here i^2, j^2, $k^2 \in F$ are non-squares in F. For $x = x_0 + ix_1 + jx_2 + kx_3$ we denote by $\bar{x}$ the involution

$$\overline{x_0 + ix_1 + jx_2 + kx_3} = x_0 - ix_1 - jx_2 - kx_3.$$

For a matrix $(g_{ij}) \in M_n(D)$, let $\bar{g} = (\bar{g}_{ij})$. We have the reduced norm $\mathrm{Nrd}_D : D \to F$. The map

$$\nu_D : D \to \mathbb{Z} \cup \{\infty\}$$

given by

$$\nu_D(x) = \nu_F(\mathrm{Nrd}_D(x))$$

is a valuation on D. As with F, we have the ring of integers $\mathcal{O}_D$, the maximal ideal $\mathfrak{p}_D$, and the finite residue field k_D. Note $[k_D : k_F] = 2$, so $|k_D| = q^2$.

Lemma 3.1. *One can choose a uniformizer ϖ_D in D such that $\varpi_D^2 = \varpi_F$.*

See [18] for a proof of this. We fix ϖ_D as in the lemma.

Definition 3.2. Let π be an irreducible supercuspidal representation of $\mathrm{GL}_n(D)$. Let $K = \mathrm{GL}_n(D)$ and $K_1 = 1 + \varpi_D M_n(\mathcal{O}_D)$. We say that π has depth zero if $\pi^{K_1} \neq \{0\}$. That is, π has depth zero exactly when it has non-trivial fixed vectors under K_1.

3.1.1 The groups

Let $\varepsilon \in \{\pm 1\}$ and let $n > 1$ be an integer. Let I_n be the $n \times n$ identity matrix and let J_ε be the $2n \times 2n$ matrix

$$J_\varepsilon = \begin{pmatrix} 0 & I_n \\ \varepsilon I_n & 0 \end{pmatrix}.$$

Define the group

$$G_\varepsilon = \{g \in \mathrm{GL}_{2n}(D) : g^* J_\varepsilon g = J_\varepsilon\}$$

where for $g = (g_{ij})$, $g^* = (\overline{g_{ij}})^\top = (\overline{g_{ji}})$. Then G_1 is an inner form of $\mathrm{Sp}(4n, F)$ and G_{-1} is an inner form of $\mathrm{SO}(4n, F)$. We will sometimes drop the subscript ε and just write G.

Let P be the standard Siegel parabolic subgroup of G_ε with standard Levi decomposition

$$P = LU_\varepsilon.$$

Thus

$$L = \left\{ \begin{pmatrix} a & 0 \\ 0 & {}^\top\bar{a}^{-1} \end{pmatrix} : a \in \mathrm{GL}_n(D) \right\} \simeq \mathrm{GL}_n(D)$$

and

$$U_\varepsilon = \left\{ \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} : \varepsilon X + \overline{X}^\top = 0 \right\}.$$

3.1.2 The question

Let $z \in \mathbb{C}$ be a complex number. Let Nrd denote the reduced norm from $M_n(D)$ to F (see [18]). The characters χ_z given by $\chi_z(g) = z^{-\nu_F(\mathrm{Nrd}(g))}$ for $g \in L$ are the unramified characters of L. We write $X_{nr}(L)$ for the group of unramified characters of L. Thus $\mathbb{C}^\times \simeq X_{nr}(L)$ via the map $z \mapsto \chi_z$.

Let π_0 be an irreducible unitary supercuspidal representation of L of depth zero and let $\chi \in X_{nr}(L)$. The question we wish to answer is

When is $\iota_P^{G_\varepsilon}(\pi_0 \otimes \chi)$ reducible?

That is, we want to find the characters χ such that this induced representation is reducible.

First, by Corollary 2 in [19], if $\iota_P^{G_\varepsilon}(\pi_0 \otimes \chi)$ is reducible for some χ, then

$${}^w\pi_0 \simeq \pi_0 \otimes \nu$$

for some $\nu \in X_{nr}(L)$ where w denotes the non-trivial element of $N_G(L)/L$, represented by $\left(\begin{smallmatrix} 0 & \varepsilon \\ 1 & 0 \end{smallmatrix}\right)$. We note the effect on an element from L:

$$^{w}\begin{pmatrix} a & 0 \\ 0 & {}^{\top}\bar{a}^{-1} \end{pmatrix} = \begin{pmatrix} {}^{\top}\bar{a}^{-1} & 0 \\ 0 & a \end{pmatrix}.$$

Thus ${}^{w}\pi_0(g) = \pi_0({}^{\top}\bar{g}^{-1})$. By Lemma 1.1 of [17], ${}^{w}\pi_0 \simeq \pi_0^{\vee}$. Suppose ${}^{w}\pi_0 \simeq \pi_0\nu$. We write $\nu = \eta/{}^{w}\eta$ for some $\eta \in X_{nr}(L)$. Note this is possible since ${}^{w}\eta = \eta^{-1}$, so we are simply choosing a square root of ν. Then ${}^{w}\pi_0{}^{w}\eta \simeq \pi_0\eta$, that is,

$$^{w}(\pi_0\eta) \simeq \pi_0\eta.$$

Hence, replacing π_0 by a suitable unramified twist, we can arrange for π_0 to be self-dual (and still unitary): $\pi_0^{\vee} \simeq \pi_0$. From now on, we assume that π_0 is self-dual.

3.2 Construction of π_0

We want to understand how π_0 is constructed. Consider the maximal compact open subgroup $K = \mathrm{GL}_n(\mathcal{O}_D)$ of L. Let $K_1 = 1 + \varpi_D M_n(\mathcal{O}_D)$. Then $K/K_1 \simeq \mathrm{GL}_n(k_D)$. Since π_0 has depth zero,

$$\pi_0|_K \supseteq \rho.$$

where ρ is the inflation to K of an irreducible cuspidal representation (σ, U) of $K/K_1 \simeq \mathrm{GL}_n(k_D) = \mathrm{GL}_n(\mathbb{F}_{q^2})$. To understand π_0 we therefore ask the question: What are the irreducible representations of $L \simeq \mathrm{GL}_n(D)$ that contain ρ upon restriction to K? First we focus on σ.

We know that the character of σ is a Deligne-Lusztig character:

$$\mathrm{char}(\sigma) = \pm R_{T,\theta}$$

for some minisotropic torus $T \simeq \mathbb{F}_{q^{2n}}^{\times}$ of $\mathrm{GL}_n(\mathbb{F}_{q^2})$ and some regular character $\theta : \mathbb{F}_{q^{2n}}^{\times} \to \mathbb{C}^{\times}$. Regularity of θ means that

$$\theta^{\gamma} \neq \theta$$

for all non-trivial γ in the Galois group $\mathrm{Gal}(\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^2})$ where ${}^{\gamma}\theta(x) = \theta(\gamma(x))$. We now fix this pair (T, θ). Since ρ is inflated from σ we write

$$\rho = \rho_{\theta}.$$

3.2.1 Structure of π_0

Let

$$\widetilde{K} = N_G(\rho) = \{x \in G \mid {}^{x}K = K \text{ and } {}^{x}\rho \simeq \rho\}.$$

Let $\widetilde{\rho}$ be any extension of ρ to $\widetilde{K}$ that contains ρ. This extension exists because $\widetilde{K}/K$ is cyclic. Then we claim that

$$\mathrm{ind}_{\widetilde{K}}^{L}\, \widetilde{\rho}$$

is irreducible. To show this we have to show that the elements intertwining $\widetilde{\rho}$ are exactly the elements from $\widetilde{K}$. Say that $g \in L$ intertwines $\widetilde{\rho}$. So

$$\mathrm{Hom}_{\widetilde{K} \cap {}^{g}\widetilde{K}}(\widetilde{\rho}|_{\widetilde{K} \cap {}^{g}\widetilde{K}}, {}^{g}\widetilde{\rho}|_{\widetilde{K} \cap {}^{g}\widetilde{K}}) \neq \{0\}.$$

Restricting to K,

$$\mathrm{Hom}_{K\cap {}^gK}(\rho|_{K\cap {}^gK}, \rho|_{K\cap {}^gK}) \neq \{0\}.$$

That is, g intertwines ρ. By "intertwining implies conjugacy" (see [7]), $g \in \widetilde{K}$. Hence by Mackey's criterion, $\mathrm{ind}_{\widetilde{K}}^{L}\, \widetilde{\rho}$ is irreducible.

Now say that $\widetilde{\rho}_1$ and $\widetilde{\rho}_2$ irreducible representations of $\widetilde{K}$ that contain ρ (upon restriction to K). Then the induced representations $\mathrm{ind}_{\widetilde{K}}^{L}\, \widetilde{\rho}_1$ and $\mathrm{ind}_{\widetilde{K}}^{L}\, \widetilde{\rho}_2$ are irreducible. Suppose that

$$\mathrm{ind}_{\widetilde{K}}^{L}\, \widetilde{\rho}_1 \simeq \mathrm{ind}_{\widetilde{K}}^{L}\, \widetilde{\rho}_2.$$

Then

$$\begin{aligned}
\{0\} &\neq \mathrm{Hom}_L(\mathrm{ind}_{\widetilde{K}}^{L}\, \widetilde{\rho}_1, \mathrm{ind}_{\widetilde{K}}^{L}\, \widetilde{\rho}_2) \\
&\simeq \mathrm{Hom}_{\widetilde{K}}(\widetilde{\rho}_1, \mathrm{ind}_{\widetilde{K}}^{L}\, \widetilde{\rho}_1|_{\widetilde{K}}) \\
&\simeq \mathrm{Hom}_{\widetilde{K}}(\widetilde{\rho}_1, \oplus_{g\in \widetilde{K}\backslash L/\widetilde{K}}\, \mathrm{ind}_{\widetilde{K}\cap {}^g\widetilde{K}}^{\widetilde{K}}\, {}^g\widetilde{\rho}_2|_{\widetilde{K}\cap {}^g\widetilde{K}}).
\end{aligned}$$

Thus there is a $g \in \widetilde{K}\backslash L/\widetilde{K}$ such that

$$\mathrm{Hom}_{\widetilde{K}\cap {}^g\widetilde{K}}(\widetilde{\rho}_1|_{\widetilde{K}\cap {}^g\widetilde{K}}, {}^g\widetilde{\rho}_2|_{\widetilde{K}\cap {}^g\widetilde{K}}) \simeq \mathrm{Hom}(\widetilde{\rho}_1, \mathrm{ind}_{\widetilde{K}\cap {}^g\widetilde{K}}^{\widetilde{K}}\, {}^g\widetilde{\rho}_2|_{\widetilde{K}\cap {}^g\widetilde{K}}) \neq \{0\}.$$

That is, g intertwines $\widetilde{\rho}_1$ with $\widetilde{\rho}_2$. As before this means that g intertwines ρ. So $g \in \widetilde{K}$ and $\widetilde{\rho}_1 \simeq \widetilde{\rho}_2$.

In sum, we have a bijection from equivalence classes of irreducible representations of $\widetilde{K}$ containing ρ to equivalence classes of irreducible representations of

L containing ρ given by

$$\widetilde{\rho} \mapsto \mathrm{ind}_{\widetilde{K}}^{L} \widetilde{\rho}.$$

We will show in the next section that $\widetilde{K} = K\langle \varpi_F \rangle$. This implies

$$\pi_0 = \mathrm{ind}_{\widetilde{K}}^{L} \widetilde{\rho}.$$

Note π_0 self-dual implies $\omega_{\pi_0}(\varpi_F) = \pm 1$, where ω_{π_0} denotes the central character of π_0. As $\widetilde{\rho}$ is completely determined by its value at ϖ_F, it follows that there are exactly two possibilities for π_0.

3.3 Computing $\widetilde{K}$

We have

$$\langle \varpi_F \rangle K = Z(G)K \leq \widetilde{K} \leq N_G(K) = \langle \varpi_D \rangle K.$$

Note the index of $\widetilde{K}$ in $\langle \varpi_D \rangle K$ is 1 or 2 since

$$|\langle \varpi_D \rangle K / \langle \varpi_F \rangle K| = 2.$$

We want to prove the following:

$$\widetilde{K} = \langle \varpi_F \rangle K.$$

That is, we want to prove that $\varpi_D \notin \widetilde{K}$. This will take some time. The proof divides naturally into two cases.

3.3.1 Two cases

We have

$$\pi_0 = \operatorname{ind}_{\widetilde{K}}^{L} \widetilde{\rho},$$
$$\pi_0^\vee = \left(\operatorname{ind}_{\widetilde{K}}^{L} \widetilde{\rho}\right)^\vee \simeq \operatorname{ind}_{\widetilde{K}}^{L} \widetilde{\rho}^\vee.$$

Thus $\widetilde{\rho}$ and $\widetilde{\rho}^\vee$ intertwine in L which implies that ρ and $\rho^\vee$ intertwine in L and so ρ and $\rho^\vee$ are conjugate in L (see [7]).

Hence we are in one of two cases:

$$\text{(A):}\ \ \rho \simeq \rho^\vee \qquad \text{or} \qquad \text{(B):}\ \ {}^{\varpi_D}\rho \simeq \rho^\vee.$$

We consider these seperately. We first need some preliminary observations. Let $k = \mathbb{F}_{q^2}$ and $l = \mathbb{F}_{q^{2n}}$, so the extension l/k has degree n.

Lemma 3.3. *Let ψ and ϕ be regular characters of $l^\times$. If*

$$\operatorname{tr}(\rho_\psi) = \operatorname{tr}(\rho_\phi)$$

on $l^\times$, then

$$\rho_\psi \simeq \rho_\phi.$$

Proof. In chapter 7 of [5] one finds the character formula for the $R_{T,\theta}$'s. Using a version from [12], we have for $\alpha \in l^\times$

$$\operatorname{tr}_{\rho_\psi}(\alpha) = c \sum_{\nu \in \operatorname{Gal}(l/k)} \psi^\nu(\alpha) = c \sum_{\nu \in \operatorname{Gal}(l/k)} \psi(\nu(\alpha))$$

where the constant c depends only on the field extension $k(\alpha)/k$. Hence, if $\operatorname{tr} \rho_\psi = \operatorname{tr} \rho_\phi$ on $l^\times$, then

$$\sum_{\nu \in \mathrm{Gal}(l/k)} \psi^\nu(\alpha) = \sum_{\nu \in \mathrm{Gal}(l/k)} \phi^\nu(\alpha),$$

for all $\alpha \in l^\times$. Now let $\Gamma = l^\times \rtimes \mathrm{Gal}(l/k)$ where $\mathrm{Gal}(l/k)$ acts on $l^\times$ in the obvious way. We form the induced representations

$$\tau_\psi = \operatorname{ind}_{l^\times}^\Gamma \psi \qquad \text{and} \qquad \tau_\phi = \operatorname{ind}_{l^\times}^\Gamma \phi.$$

Note τ_ψ and τ_ϕ are irreducible by Mackey's criterion. The support of $\operatorname{tr} \tau_\psi$ is contained in $l^\times$. Further, for $\alpha \in l^\times$,

$$\operatorname{tr} \tau_\psi(x) = \sum_{\nu \in \mathrm{Gal}(l/k)} \psi^\nu(x).$$

We have the same formula for τ_ϕ and so

$$\operatorname{tr} \tau_\psi = \operatorname{tr} \tau_\phi.$$

Hence $\tau_\psi \simeq \tau_\phi$. That is, $\operatorname{ind}_{l^\times}^\Gamma \rho_\psi$ and $\operatorname{ind}_{l^\times}^\Gamma \rho_\phi$ are isomorphic. Thus, by Frobenius Reciprocity,

$$\begin{aligned} 1 &= \dim \operatorname{Hom}_\Gamma(\operatorname{ind}_{l^\times}^\Gamma \rho_\psi, \operatorname{ind}_{l^\times}^\Gamma \rho_\phi) \\ &= \dim \operatorname{Hom}_{l^\times}((\operatorname{ind}_{l^\times}^\Gamma \rho_\psi)|_{l^\times}, \rho_\phi) \\ &= \dim \operatorname{Hom}_{l^\times}\left(\bigoplus_{\nu \in \mathrm{Gal}(l/k)} \rho_\psi^\nu, \rho_\phi \right). \end{aligned}$$

So there is a $\nu \in \Gamma$ such that $\psi^\nu = \phi$. Hence $\rho_\psi \simeq \rho_\phi$. $\square$

Proposition 3.4. *With notation as above,*

$$\rho_\theta^\vee \simeq \rho_{\theta^{-1}}.$$

Proof. For $x \in l^\times$, there is a constant c (which depends only on $k(x) = k(x^{-1})$) such that

$$\begin{aligned}
\operatorname{tr} \rho_\theta^\vee(x) &= \operatorname{tr} \rho_\theta(x^{-1}) \\
&= \sum_{\nu \in \mathrm{Gal}(l/k)} \theta^\nu(x^{-1}) \\
&= \sum_{\nu \in \mathrm{Gal}(l/k)} (\theta^{-1})^\nu(x) \\
&= \operatorname{tr} \rho_{\theta^{-1}}(x).
\end{aligned}$$

So from lemma 3.3, $\rho_\theta^\vee \simeq \rho_{\theta^{-1}}$. □

Write $\overline{\rho_\theta}$ for the representation given by $\overline{\rho_\theta}(g) = \rho_\theta(\bar{g})$. For $g = (a_{ij}) \in \mathrm{GL}_n(\mathbb{F}_{q^2})$, $\bar{g} = (a_{ij}^q)$.

Proposition 3.5. *Let $\theta : l^\times \to \mathbb{C}^\times$ be a regular character. Then*

$$\overline{\rho_\theta} \simeq \rho_{\theta^q}.$$

Proof. We recall that $l = \mathbb{F}_{q^{2n}}$ and $k = \mathbb{F}_{q^2}$. Fix the k-vector space $V = k^n$. Then $M_n(k) \simeq \mathrm{End}_k(V)$. Let $i : l \hookrightarrow M_n(k)$ be an embedding of k-algebras. Write $\phi : l \to l$ for the map $\lambda \mapsto \lambda^q$ and $\Phi : M_n(k) \to M_n(k)$ for the map

$(a_{ij}) \mapsto (a_{ij}^q)$. Consider the diagram

$$\begin{tikzcd} l \arrow[r, "i"] \arrow[d, "\phi"'] & M_n(k) \arrow[d, "\Phi"] \\ l \arrow[r, "i"'] & M_n(k). \end{tikzcd}$$

Let $f = \Phi \circ i$ and $g = i \circ \phi$. We can make V into an l-module in two ways:

- $\lambda.v = \Phi(i(\lambda))v$,
- $\lambda_* v = i \circ \phi(\lambda)v = i(\lambda^q)v$.

Note that for $\lambda \in k$ we have $\lambda.v = \Phi(i(\lambda))v = \Phi(\mathrm{diag}(\lambda))v = \mathrm{diag}(\lambda^q)v = i(\lambda^q)v = \lambda_* v$. Hence the two different actions agree on k. These l-module structures must be isomorphic as there is only one l-module (up to isomorphism) of a given dimension. Thus there is an isomorphism (of l-modules) $h : V \to V$ such that

$$h(\lambda.v) = \lambda_* h(v).$$

Note that h is k-semi-linear since the module structures agree on k. Thus, for $v \in V$

$$h(\lambda v) = \lambda^q h(v).$$

Likewise, $h^{-1}(\mu v) = \mu^q h^{-1}(v)$ for $v \in V$ and $\mu \in k$. Now, for $a \in \mathrm{End}_k(V)$ we have $hah^{-1} \in \mathrm{End}_k(V)$. We see this from

$$\begin{aligned}(hah^{-1})(\lambda v) &= ha(\lambda^q h^{-1}(v)) \\ &= h(\lambda^q a h^{-1}(v)) \\ &= \lambda^{q^2} hah^{-1}(v) \\ &= \lambda(hah^{-1})(v)\end{aligned}$$

for $\lambda \in k$. Thus

$$h(\Phi(i(\lambda))v) = i(\lambda^q)h(v).$$

That is

$$h \circ \Phi(i(\lambda)) \circ h^{-1} = i(\lambda^q).$$

Hence for $\alpha \in i(l^\times)$,

$$\begin{aligned}\mathrm{tr}\,\overline{\rho_\theta}(\alpha) &= \mathrm{tr}\,\rho_\theta(\bar{\alpha}) \\ &= \mathrm{tr}\,\rho_\theta(\Phi(\alpha)) \\ &= \mathrm{tr}\,\rho_\theta(h^{-1})\alpha^q h) \\ &= \mathrm{tr}\,\rho_\theta(\alpha^q).\end{aligned}$$

By Lemma 3.3, $\overline{\rho_\theta} \simeq \rho_{\theta^q}$. $\square$

3.3.2 Case (A): $\rho \simeq \rho^{\vee}$

We begin the treatment of Case (A). Recall $\mathrm{Gal}(l/k) = \langle \mathrm{Fr} \rangle$ where $\mathrm{Fr}(\lambda) = \lambda^{q^2}$. In this case we have $\rho_{\theta^{-1}} \simeq \rho_\theta = \rho$. Thus there exists a $\gamma \in \mathrm{Gal}(l/k)$ such that

$$\theta^\gamma = \theta^{-1}.$$

Applying γ twice we get

$$\theta^{\gamma^2} = \theta.$$

Since θ is regular, $\gamma^2 = 1$. Assume $\gamma = 1$. Then $\theta = \theta^{-1}$, so $\theta^2 = 1$ and the image of θ is contained in $\{\pm 1\}$. Then

$$\theta^{\mathrm{Fr}}(\lambda) = \theta(\lambda^{q^2}) = \theta(\lambda)^{q^2} = \theta(\lambda)$$

for all $\lambda \in \mathbb{F}_{q^2}$. That is, $\theta^{\mathrm{Fr}} = \theta$, a contradiction. Hence γ must have order 2. Thus $n = 2m$ for some m and $\gamma = \mathrm{Fr}^m$ is the unique element of order 2 in $\mathrm{Gal}(l/k)$. In all,

$$\theta^{-1} = \theta^\gamma = \theta^{\mathrm{Fr}^m} = \theta^{(q^2)^m} = \theta^{q^{2m}} = \theta^{q^n}.$$

We keep this in mind as we study case (B).

3.3.3 Case (B): ${}^{\varpi_D}\rho \simeq \rho^{\vee}$

From [2] we have that for $x \in \mathcal{O}_D$,

$$\varpi_D x \varpi_D^{-1} \equiv x^q \mod \mathfrak{p}_D.$$

That is, conjugating by ϖ_D induces the non-trivial $\mathbb{F}_q$-automorphism $\lambda \mapsto \lambda^q : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$. This extends to a map $\mathrm{GL}_n(\mathbb{F}_{q^2}) \to \mathrm{GL}_n(\mathbb{F}_{q^2})$ via

$$(a_{ij}) \mapsto (a_{ij}^q).$$

which we also denote by $\overline{(a_{ij})} = (\overline{a_{ij}})$. Recall that $\overline{\rho}$ denotes the representation

$$\overline{\rho}(g) = \rho(\overline{g}).$$

Proposition 3.6. *We have*

$${}^{\varpi_D}\rho_\theta \simeq \rho_{\theta^q}.$$

Proof. By Proposition 3.5, $\overline{\rho_\theta} \simeq \rho_{\theta^q}$. So we need to prove that $\overline{\rho_\theta} \simeq {}^{\varpi_D}\rho_\theta$. Recall that conjugating by ϖ_D is the same as raising to the power q mod $\mathfrak{p}_D$. Therefore, for $x \in \mathbb{F}_{q^{2n}}^\times$, $\mathrm{tr}\,\rho_{\theta^q}(x) = \mathrm{tr}\,\overline{\rho_\theta}(x)$ on $\mathbb{F}_{q^{2n}}^\times$. So by Lemma 3.3, $\overline{\rho_\theta} \simeq \rho_{\theta^q}$. □

Thus

$$\rho_{\theta^{-1}} \simeq \rho_\theta^\vee \simeq {}^{\varpi_D}\rho \simeq \overline{\rho_\theta} \simeq \rho_{\theta^q}.$$

As before this gives the existence of an element $\gamma \in \mathrm{Gal}(\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^2})$ such that

$$(\theta^q)^\gamma = \theta^{-1}.$$

Since the operators $\phi \mapsto \phi^q$ and $\phi \mapsto \phi^\gamma$ commute,

$$(((\theta^q)^\gamma)^q)^\gamma = \theta^{q^2\gamma^2} = \theta.$$

Therefore $\mathrm{Fr}\,\gamma^2 = 1$, or $\gamma^2 = \mathrm{Fr}^{-1}$. Recall that Fr is a generator of

$$\Gamma = \mathrm{Gal}(l/k) = \{x \mapsto x^{q^{2k}} : 0 \le k \le n-1\}.$$

Then Fr^{-1} is a generator, so γ is also a generator. The order of γ^2 is equal to $n/\gcd(2,n)$. Hence $\gcd(2,n) = 1$ and n must be odd. Say that $n = 2m+1$. Now γ is the unique element in $\gamma \in \mathrm{Gal}(\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^2})$ satisfying $\gamma^2 = \mathrm{Fr}^{-1}$. Clearly,

$$\mathrm{Fr}^n = \mathrm{Fr}^{2m+1} = 1.$$

So

$$(\mathrm{Fr}^m)^2 = \mathrm{Fr}^{-1}.$$

Hence

$$\gamma^2 = \mathrm{Fr}^{-1} = \mathrm{Fr}^{2m},$$

so

$$\gamma = \mathrm{Fr}^m.$$

Thus we obtain the same relation as in the other case:

$$\theta^{q^n} = \theta^{q^{2m+1}} = (\theta^q)^{q^{2m}} = (\theta^q)^\gamma = \theta^{-1}.$$

We can distinguish case (A) and case (B) by noting that case (A) occurs when n is even and case (B) when n is odd.

3.3.4 Showing $\widetilde{K} = \langle \varpi_F \rangle K$

Lemma 3.7. *We have*

$$\widetilde{K} = \langle \varpi_F \rangle K.$$

Proof. In both cases above, we concluded that

$$\theta^{q^n} = \theta^{-1}.$$

Assume that $\varpi_D \in \widetilde{K}$. That is ${}^{\varpi_D}\rho_\theta \simeq \rho_\theta$. From Proposition 3.6 we recall that

$${}^{\varpi_D}\rho_\theta \simeq \rho_{\theta^q}.$$

Thus there is a $\gamma \in \mathrm{Gal}(l/k)$ such that $\theta^\gamma = \theta^q$. So $\theta^{\gamma^2} = \theta^{q^2} = \theta^{\mathrm{Fr}}$. That is $\gamma^2 = \mathrm{Fr}$. As before, Fr is a generator for the Galois group $\mathrm{Gal}(l/k)$, hence γ is a generator. The order of γ^2 is $n/\gcd(2,n)$. Hence n must be odd.

Thus we are in case (B) where $\rho_\theta^\vee \simeq {}^{\varpi_D}\rho_\theta$. By assumption ${}^{\varpi_D}\rho_\theta \simeq \rho_\theta$, so $\rho_\theta \simeq \rho_\theta^\vee$. This is case (A) where n is even. Hence we have a contradiction. So $\varpi_D \notin \widetilde{K}$. $\square$

3.4 Affine roots

We now want to view ρ as a representation of certain other groups. First we relate these groups to a set of affine roots.

Let

$$S = \{ \begin{pmatrix} \mathrm{diag}(s_1, s_2, \ldots, s_n) & 0 \\ 0 & \mathrm{diag}(s_1^{-1}, s_2^{-1}, \ldots, s_n^{-1}) \end{pmatrix} : s_i \in F^\times \}$$

be a maximal split F-torus in G_ε.

Write $\Phi = \Phi(S, G)$ for the roots of S in G. This root system is of type C_n. Concretely, if

$$e_i : \begin{pmatrix} \mathrm{diag}(s_1, s_2, \ldots, s_n) & 0 \\ 0 & \mathrm{diag}(s_1^{-1}, s_2^{-1}, \ldots, s_n^{-1}) \end{pmatrix} \mapsto s_i,$$

then

$$\Phi = \{\pm e_i \pm e_j, \pm 2e_i \ : 0 \leq i < j \leq n\}.$$

The positive roots are

$$\Phi^+ = \{e_i + e_j, e_i - e_j, 2e_i \ : 0 \leq i < j \leq n\}$$

and we have the basis

$$\Delta = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n\}.$$

The longest root is $\tilde{\alpha} = 2e_1$. Now we consider the affine roots $a_{\alpha,k} = \alpha + k$ for α a root and k an integer. (Here by k we mean the translation by k.) Specifically, let $a_0 = 1 - \tilde{\alpha}$, and for $1 \leq i \leq n$ let $a_n = a_{\alpha_i,0}$. Then

$$\{a_0, a_1, \ldots, a_n\}$$

is a basis for the affine roots. And we have the affine Weyl group $\widetilde{W} = \langle s_{\alpha_i} : 0 \leq i \leq n \rangle$ where s_{α_i} is a fundamental reflection in a hyperplane.

3.5 The types and covers

Morris in [14] gives us the types and covers in our situation. We now describe them in some detail. We have ρ, a representation of $K = \mathrm{GL}_n(\mathcal{O}_D)$. Let $\Theta = \{a_1, \dots, a_{n-1}\}$. Let I be the Iwahori subgroup of G, that is, the set of $(2n) \times (2n)$ matrices in G of the form:

$$I = \begin{pmatrix} \mathcal{O}_D & \mathcal{O}_D & \dots & \mathcal{O}_D \\ \mathfrak{p}_D & \mathcal{O}_D & \dots & \mathcal{O}_D \\ \vdots & & \ddots & \\ \mathfrak{p}_D & \mathfrak{p}_D & \dots & \mathcal{O}_D \end{pmatrix} \cap G_\varepsilon.$$

This is the set of matrices with elements from $\mathcal{O}_D$ on the diagonal and above and elements from $\mathfrak{p}_D$ below the diagonal. Then we get the Siegel parahoric subgroup (compact open subgroup, $2n \times 2n$ matrices)

$$\mathcal{P} = \mathcal{P}_\Theta = IW_\Theta I = \begin{pmatrix} \mathcal{O}_D & \mathcal{O}_D \\ \mathfrak{p}_D & \mathcal{O}_D \end{pmatrix} \cap G.$$

Note that $\mathcal{P}$ is not maximal. We recall that ρ is a representation of $K = \mathrm{GL}_n(\mathcal{O}_D)$, and now we want to view ρ as a representation of $\mathcal{P}$. Then let $\mathcal{M} = \mathcal{P} \cap L \simeq K$. Concretely,

$$\mathcal{M} = \left\{ \begin{pmatrix} a & 0 \\ 0 & {}^\top\bar{a}^{-1} \end{pmatrix} : a \in \mathrm{GL}_n(\mathcal{O}_D) \right\}.$$

Using the isomorphism $\mathcal{M} \simeq K$ we view ρ as a representation of $\mathcal{M}$. We often abuse notation and write ρ for this representation. Later we will need to distinguish this representation, and so we use the notation $\rho_{\mathcal{M}}$. Let

$$\mathcal{N} = \mathcal{P} \cap U_\varepsilon = \{\begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix} : X \in M_n(\mathcal{O}_D), \varepsilon X + \overline{X}^\top = 0\},$$

$$\overline{\mathcal{N}} = \mathcal{P} \cap \overline{U}_\varepsilon = \{\begin{pmatrix} I_n & 0 \\ X & I_n \end{pmatrix} : X \in M_n(\mathcal{O}_D), \varepsilon X + \overline{X}^\top = 0\}.$$

Lemma 3.8. *We have*

$$\mathcal{P} = \overline{\mathcal{N}} \cdot \mathcal{M} \cdot \mathcal{N} = (\mathcal{P} \cap \overline{U_\varepsilon})(\mathcal{P} \cap L)(\mathcal{P} \cap U_\varepsilon)$$

with every element of $\mathcal{P}$ being written uniquely as a product $n' \cdot m \cdot n$ where $n' \in \overline{N}, m \in M, n \in N$.

We now extend ρ to a representation of $\mathcal{P}$ by $\rho(\bar{n}mn) = \rho(m)$. We must show that this is well-defined. Let $\mathcal{M}_1$ be the image of $K_1 = 1 + \varpi_D M_n(\mathcal{O}_D)$ under the isomorphism $K \simeq \mathcal{M}$. Thus ρ is trivial on $\mathcal{M}_1$. Note that

$$\mathcal{M} \cap \langle \overline{\mathcal{N}}, \mathcal{N} \rangle \subseteq \mathcal{M}_1.$$

Suppose $\bar{n}mn = \bar{n}'m'n' \in \mathcal{P}$. Then $m\mathcal{M}_1 = m'\mathcal{M}_1$. Hence $\rho(\bar{n}mn) = \rho(m) = \rho(m') = \rho(\bar{n}'m'n')$.

Now let $\mathfrak{s} = [L, \pi_0]_G$ and $\mathfrak{s}_L = [L, \pi_0]_L$. From Morris [15], we have:

Theorem 3.9. *(K, ρ) is an $\mathfrak{s}_L$-type in L and $(\mathcal{P}, \rho)$ is a G-cover of (K, ρ). In particular, $(\mathcal{P}, \rho)$ is an $\mathfrak{s}$-type in G.*

By the theory of types and covers, we have the following commutative diagram

$$
\begin{tikzcd}
\mathfrak{R}^{\mathfrak{s}}(G) \arrow[r, "m_G"] & \mathcal{H}(G,\rho)-\mathrm{Mod} \\
\mathfrak{R}^{\mathfrak{s}_L}(L) \arrow[r, "m_L"'] \arrow[u, "\iota_P^G"] & \mathcal{H}(L,\rho)-\mathrm{Mod}. \arrow[u, "t_{P*}"']
\end{tikzcd}
$$

We note the importance that π_0 has depth zero in the use of [15]. We recall that t_{P*} is the map $U \mapsto \mathrm{Hom}_{t_P(\mathcal{H}(L,\rho_L))}(\mathcal{H}(G,\rho), U)$ induced by the embedding of $\mathbb{C}$-algebras $t_P : \mathcal{H}(L,\rho) \hookrightarrow \mathcal{H}(G,\rho)$.

3.6 The Hecke algebra

We have now reduced the problem to finding the Hecke algebras $\mathcal{H}(L,\rho)$ and $\mathcal{H}(G,\rho)$. Since the intertwining of ρ in L is

$$
\begin{aligned}
\mathfrak{I}_L(\rho) &= \widetilde{K} = F^\times K = \langle \varpi_F \rangle K \\
&= \coprod_{i\in\mathbb{Z}} K\varpi_F^i K,
\end{aligned}
$$

we have

$$\mathcal{H}(L,\rho) \simeq \mathbb{C}[d, d^{-1}].$$

Here d is an indeterminate. It corresponds to the element in $\mathcal{H}(L,\rho)$ that has support $\varpi_F K$ and whose value at ϖ_F is 1. Let $\mathcal{D} = \mathbb{C}[d, d^{-1}]$.

Our main result is:

Theorem 3.10.

$$\mathcal{H}(G,\rho) = \langle f_0, f_1 : f_i^2 = q^n + (q^n - 1)f_i \rangle.$$

Note that the relations $\mathcal{H}(G,\rho)$ can also be presented by

$$\mathcal{H}(G,\rho) = \langle h_0, h_1 : h_i^2 = 1 + (q^{n/2} - q^{-n/2})h_i \rangle,$$

where $f_i = q^{n/2}h_i$, $i = 0, 1$.

The proof will occupy some space and we divide it up.

3.6.1 General preliminaries

We now fix $\varepsilon = 1$ and let $G = G_1$. The case of $\varepsilon = -1$ is very similar. We indicate in section 3.7 the minor changes needed to treat this case.

Let

$$N_G(\rho_{\mathcal{M}}) = \{g \in L : {}^g\mathcal{M} = \mathcal{M}, {}^g\rho_{\mathcal{M}} \simeq \rho_{\mathcal{M}}\}.$$

From 4.15 in [14],

$$\mathfrak{I}_G(\rho) = \mathcal{P} N_G(\rho_{\mathcal{M}}) \mathcal{P}.$$

We have

$$N_G(\rho_{\mathcal{M}}) \subseteq N_G(\mathcal{M}) = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \varpi_D & 0 \\ 0 & -\varpi_D^{-1} \end{pmatrix} \rangle \mathcal{M}.$$

As in the proof of Lemma 3.7 we now split the proof into two cases depending on whether n is even or n is odd.

3.6.2 Case (A)

Assume first that n is even. That means that $\rho \simeq \rho^\vee$. Also recall that ${}^{\varpi_D}\rho \simeq \overline{\rho}$.

Let

$$
\begin{aligned}
w_0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
w_D &= \begin{pmatrix} \varpi_D & 0 \\ 0 & -\varpi_D^{-1} \end{pmatrix}, \\
s_0 &= \begin{pmatrix} 0 & \varpi_D \\ -\varpi_D^{-1} & 0 \end{pmatrix} = w_D w_0, \text{ and} \\
s_1 &= \begin{pmatrix} 0 & -\varpi_D^{-1} \\ \varpi_D & 0 \end{pmatrix} = w_0 w_D.
\end{aligned}
$$

Then

$$
\begin{aligned}
{}^{w_0}\rho_{\mathcal{M}} &\simeq \overline{\rho}_{\mathcal{M}}^{\vee}, \\
{}^{w_D}\rho_{\mathcal{M}} &\simeq \overline{\rho}_{\mathcal{M}}.
\end{aligned}
$$

Since $\rho_{\mathcal{M}} \simeq \rho_M^{\vee}$, we have

$$
{}^{s_i}\rho_M \simeq \rho_M
$$

for $i = 0, 1$. Now

$$
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \notin N_G(\rho_M),
$$

so

$$
N_G(\rho_{\mathcal{M}}) \subsetneq N_G(\mathcal{M}).
$$

Thus

$$\langle s_0, s_1\rangle\mathcal{M} \subseteq N_G(\rho_{\mathcal{M}}) \subsetneq N_G(\mathcal{M}).$$

Now

$$s_0 s_1 = \begin{pmatrix} \varpi_F & 0 \\ 0 & \varpi_F^{-1} \end{pmatrix},$$

so

$$\langle s_0, s_1\rangle = \langle s_0\rangle \ltimes \langle s_0 s_1\rangle.$$

As $s_0 = w_D w_0$,

$$N_G(\mathcal{M}) = \langle w_0, w_D\rangle\mathcal{M} = \langle s_0, w_D\rangle\mathcal{M} = (\langle s_0\rangle \ltimes \langle w_D\rangle)\mathcal{M}.$$

Thus

$$|N_G(\mathcal{M})/\langle s_0, s_1\rangle\mathcal{M}| = |\langle w_D\rangle/\langle s_0 s_1\rangle| = 2.$$

We conclude that

$$N_G(\rho_{\mathcal{M}}) = \langle s_0, s_1\rangle\mathcal{M}.$$

Hence

$$\mathfrak{I}_G(\rho) = \mathcal{P} N_G(\rho_M)\mathcal{P} = \mathcal{P}\langle s_0, s_1\rangle\mathcal{P}.$$

Let $W(\mathcal{M}) = \langle w_0, s_1\rangle/\{\pm 1\} \simeq N_G(\mathcal{M})/\mathcal{M}$. That is, $W(\mathcal{M})$ is an infinite dihedral group generated as a Coxeter group by w_0 and s_1 where we abuse notation and write w_0 and s_1 for the corresponding elements in the quotient $W(\mathcal{M})$.

We now want to consider $\mathcal{P}\backslash G/\mathcal{P}$ double cosets. First, by general structure theory or direct matrix multiplications as in section 6.9 in [1], the map

$$w \mapsto \mathcal{P}w\mathcal{P} : W(\mathcal{M}) \to \mathcal{P}\backslash G/\mathcal{P}$$

is injective.

Second, let l be the length function on $W(\mathcal{M})$. We have, for example, $l(s_1) = 1$ and $l(s_1 w_0) = 2$. By [1],

- $\mathcal{P}w\mathcal{P}w'\mathcal{P} = \mathcal{P}ww'\mathcal{P}$ if $l(ww') = l(w) + l(w')$.
- $\mathcal{P}w_0\mathcal{P}w_0\mathcal{P} = \mathcal{P} \cup \mathcal{P}w_0\mathcal{P}$.

One checks that

$$s_0 = w_0 s_1 w_0^{-1}.$$

With this in mind we see that

$$\mathfrak{I}_{G_\varepsilon}(\rho) = \mathcal{P}\langle s_0, s_1\rangle\mathcal{P}.$$

Write $W(\rho_\mathcal{M})$ for $\langle s_0, s_1\rangle/\{\pm 1\}$. Then $W(\rho_\mathcal{M})$ is a subgroup of $W(\mathcal{M})$.

Under the isomorphism $W(\mathcal{M}) \simeq N_G(\mathcal{M})/\mathcal{M}$ this subgroup corresponds to $N_G(\rho_\mathcal{M})/\mathcal{M}$:

$$W(\rho_\mathcal{M}) \simeq N_G(\rho_\mathcal{M})/\mathcal{M}.$$

The group $W(\rho_{\mathcal{M}})$ is again an infinite dihedral group is generated as a Coxeter group by $\{s_0, s_1\}$ (again abusing notation). This Coxeter group has the length function l_ρ. Furthermore, if $l_\rho(ww') = l_\rho(w) + l_\rho(w')$ then $l(ww') = l(w) + l(w')$. Thus if $l_\rho(ww') = l_\rho(w) + l_\rho(w')$, then $\mathcal{P}w\mathcal{P}w'\mathcal{P} = \mathcal{P}ww'\mathcal{P}$.

For $i = 0, 1$, let f_i be any element in $\mathcal{H}(G_\varepsilon, \rho)$ with support $\mathcal{P}s_i\mathcal{P}$. By Schur's Lemma, f_i is unique up to multiplication by non-zero scalars. Each element of $W(\rho_{\mathcal{M}})$ is a word in $\{s_0, s_1\}$. Hence $\mathcal{H}(G_\varepsilon, \rho)$ is generated as a $\mathbb{C}$-algebra by f_0 and f_1. We now want to find the relations that the two generators satisfy.

Let

$$K_1 = \langle \mathcal{P}, s_1 \rangle.$$

Note that K_1 is a compact open subgroup of G.

We will show that

$$\mathcal{H}(K_1, \rho) \simeq \mathrm{End}_{\mathrm{Sp}_{2n}(k)}(\mathrm{ind}_{\overline{P}(k)}^{\mathrm{Sp}_{2n}(k)}\, \overline{\rho}).$$

To motivate the following sections, we assume for the moment that we have established this isomorphism. By Proposition 3.2 in [9], there exists a unique f_1 in $\mathcal{H}(G, \rho)$ with supp $f_1 = \mathcal{P}s_1\mathcal{P}$ such that

$$f_1^2 = \lambda 1 + (\lambda - 1) f_1$$

where λ is given as follows. The representation $\mathrm{ind}_{\overline{P}(k)}^{\mathrm{Sp}_{2n}(k)}\, \overline{\rho} = \pi_1 \oplus \pi_2$ where π_1 and π_2 are distinct irreducible representations of $\mathrm{Sp}_{2n}(k)$. Assume $\dim(\pi_1) \leq \dim(\pi_2)$. We need to find the parameter λ in the relation above to determine the Hecke algebra $\mathcal{H}(K_1, \rho)$. By [9],

$$\lambda = \frac{\dim(\pi_2)}{\dim(\pi_1)}.$$

We will compute this quotient in our situation. First we recall that θ corresponds to a semi-simple element $s^* \in T^*$ in the dual group G^*. Then by theorems 8.4.8 and 8.4.9 in [5],

$$\lambda = |C_{G^*}(s^*)|_p.$$

In section 3.9 below we explain how the finite group $\mathrm{Sp}_{2n}(k)$ appears.

3.6.2.1 Quadratic relations I

It is easier to find $\mathcal{H}(K_1^g, \rho^g) \simeq \mathcal{H}(K_1, \rho)$ for a suitable g. Let Ψ be the composition of $\mathrm{Ad}\left(\begin{smallmatrix} \varpi_D & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with reduction mod $\mathfrak{p}_D$. By "reduction mod $\mathfrak{p}_D$" we mean that we take each entry in the matrix mod $\mathfrak{p}_D$.

We need to find $\Psi(K_1)$. Recall first that

$$\mathcal{P} = \mathcal{N}\mathcal{M}\overline{\mathcal{N}}.$$

We find the image of each of these factors. Clearly,

$$\Psi(\mathcal{N}) = \begin{pmatrix} 1 & \mathfrak{p}_D \\ 0 & 1 \end{pmatrix} \mod \mathfrak{p}_D = \{1\}.$$

Writing Ad for $\mathrm{Ad}\left(\begin{smallmatrix}\varpi_D & 0\\ 0 & 1\end{smallmatrix}\right)$,

$$\mathrm{Ad}(\mathcal{M}) = \left\{ \begin{pmatrix} \varpi_D g \varpi_D^{-1} & 0 \\ 0 & {}^{\top}\overline{g}^{-1} \end{pmatrix} : g \in \mathrm{GL}_n(\mathcal{O}_D) \right\}.$$

Hence

$$\Psi(\mathcal{M}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & {}^{\top}a^{-1} \end{pmatrix} : a \in \mathrm{GL}_n(\mathbb{F}_{q^2}) \right\}.$$

Now recall that

$$\overline{\mathcal{N}} = \left\{ \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} : y + \overline{y}^{-1}, y \in M_n(\mathcal{O}_D) \right\}.$$

Then

$$\mathrm{Ad}\begin{pmatrix} \varpi_D & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y\varpi_D^{-1} & 1 \end{pmatrix}.$$

We write $x = y\varpi_D^{-1} \in M_n(\mathcal{O}_D)$ and find that

$$\begin{aligned} \overline{x}^{\top} &= \overline{y\varpi_D^{-1}}^{\top} \\ &= \overline{\varpi_D^{-1}}\,\overline{y}^{\top} \\ &= -\varpi_D^{-1}(-y) \\ &= \varpi_D^{-1} v \\ &= \varpi_D^{-1} y \varpi_D^{-1} \varpi_D \\ &= \varpi_D^{-1} x \varpi_D. \end{aligned}$$

We now reduce mod $\mathfrak{p}_D$ and use that $\varpi_D \alpha \varpi_D^{-1} = \bar{\alpha} = \alpha^q \bmod \mathfrak{p}_D$ to get (now in

$M_n(\mathbb{F}_{q^2}))$

$$x^\top = x.$$

Hence

$$\Psi(\overline{\mathcal{N}}) = \{\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} : x^\top = x, x \in M_n(\mathbb{F}_{q^2})\}.$$

We find that

$$\Psi(\mathcal{P}) = \overline{P}(\mathbb{F}_{q^2}).$$

This is the opposite standard Siegel parabolic subgroup of $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})$ and so

$$\Psi(K_1) = \langle \overline{P}(\mathbb{F}_{q^2}), \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle$$
$$\leq \mathrm{Sp}_{2n}(\mathbb{F}_{q^2}).$$

In fact, we have equality since $\overline{P}(\mathbb{F}_{q^2})$ is a maximal proper subgroup of $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})$ and $\overline{P}(\mathbb{F}_{q^2})$ does not contain $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. So the image is $\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})$:

$$\Psi(K_1) = \mathrm{Sp}_{2n}(\mathbb{F}_{q^2}).$$

Hence

$$\mathcal{H}(K_1, \rho) \simeq \mathrm{End}_{K_1}(\mathrm{ind}_{\mathcal{P}}^{K_1} \rho) \simeq \mathrm{End}_{\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})}(\mathrm{ind}_{\overline{P}(\mathbb{F}_{q^2})}^{\mathrm{Sp}_{2n}(\mathbb{F}_{q^2})} \tilde{\rho}).$$

It now follows that there is an $f_1 \in \mathcal{H}(G,\rho)$ with support $\mathcal{P}s_1\mathcal{P}$ such that

$$f_1^2 = q^n + (q^n - 1)f_1,$$

provided the parameter $\lambda = q^n$. We show that λ takes this value in section 3.9.

3.6.2.2 Quadratic relations II

We need a different approach to find the quadratic relations for f_0. Let

$$w_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$s_1 = \begin{pmatrix} 0 & -\varpi_D^{-1} \\ \varpi_D & 0 \end{pmatrix}.$$

Note that

$$w_0 s_1 = \begin{pmatrix} \varpi_D & 0 \\ 0 & -\varpi_D^{-1} \end{pmatrix}.$$

Thus

$$N_{G_\varepsilon}(\mathcal{M}) = \langle w_0, s_1 \rangle \mathcal{M},$$
$$\langle w_0, s_1 \rangle \cap \mathcal{M} = \{\pm 1\}.$$

Now we want to show that f_0 can be chosen so that it satisfies the same relation as f_1. First note that since $s_0 = w_0 s_1 w_0^{-1}$ $(w_0 = w_0^{-1})$,

$$\mathcal{P}s_0\mathcal{P} = \mathcal{P}w_0\mathcal{P}s_1\mathcal{P}w_0\mathcal{P}.$$

Now w_0 intertwines ρ with $\overline{\rho}$.

Lemma 3.11. *There exist elements $\Lambda \in \mathcal{H}(G_\varepsilon, \rho)$ and $\Upsilon \in \mathcal{H}(G_\varepsilon, \overline{\rho})$ such that*

$$\Lambda(p_1 w_0 p_2) = \rho(p_1)\Lambda(w_0)\rho(p_2),$$
$$\Upsilon(p_1 w_0 p_2) = \overline{\rho}(p_1)\Upsilon(w_0)\overline{\rho}(p_2)$$

for all $p_1, p_2 \in \mathcal{P}$ and

$$\Lambda\Upsilon = 1_\rho \in \mathcal{H}(G_\varepsilon, \rho),$$
$$\Upsilon\Lambda = 1_{\overline{\rho}} \in \mathcal{H}(G_\varepsilon, \overline{\rho}).$$

Here 1_ρ and $1_{\overline{\rho}}$ are the identities in their respective Hecke algebras.

Proof. First we write ${}^\phi g = {}^\top g^{-1}$ for any $g \in \mathrm{GL}_n(\mathbb{F}_{q^2})$. Let ρ^ϕ be the representation defined by

$$\rho^\phi(g) = \rho({}^\phi g).$$

Then $\rho^\phi \simeq \rho^\vee$. We see this by considering the characters of the representations. In particular we note that $\operatorname{tr} \rho({}^\top g^{-1}) = \operatorname{tr} \rho(g^{-1})$. Let V be the space of ρ. Thus there is an invertible map $t : V \to V$ satisfying that for all $g \in \mathrm{GL}_n(\mathbb{F}_{q^2})$

$$t \circ \rho({}^\phi g) = \rho(g) \circ t.$$

Hence

$$\begin{aligned} t^2\rho(g) &= t^2 \circ {}^{\phi}({}^{\phi}\rho)(g) \\ &= t \circ {}^{\phi}(\rho(g)) \circ t \\ &= \rho(g) \circ t^2. \end{aligned}$$

By Schur's Lemma, $t^2 = c1_V$ for some $c \in \mathbb{C}^\times$. Adjusting t by a suitable scalar multiple, we may assume $t^2 = 1_V$. Observe that for $k \in K$,

$$\begin{aligned} t\rho({}^{\top}k^{-1}) &= t^{\phi}\rho(k) \\ &= \rho(k)t. \end{aligned}$$

Hence, for $m \in \mathcal{M}$,

$$t\rho_{\mathcal{M}}({}^{\top}m^{-1}) = \rho_{\mathcal{M}}(m)t.$$

Now let Λ_1 be the map with support $\mathcal{P}w_0\mathcal{P}$ given by

$$\Lambda_1(p_1w_0p_2) = \rho(p_1)t\overline{\rho}(p_2)$$

for $p_i \in \mathcal{P}$. We need to show that Λ_1 is well-defined. To see this, suppose

$$p_1w_0p_2 = r_1w_0r_2$$

where $p_i, r_i \in \mathcal{P}$. Then

$$r_1^{-1}p_1 = w_0r_2p_2^{-1}w_0^{-1} \in \mathcal{P} \cap w_0\mathcal{P}w_0^{-1}.$$

Recall the decomposition

$$\mathcal{P} = \overline{\mathcal{N}}\mathcal{M}\mathcal{N}.$$

Write

$$r_1^{-1}p_1 = k^- k^0 k^+,$$
$$r_2 p_2^{-1} = l^+ l^0 l^-$$

for $k^-, l^- \in \overline{\mathcal{N}}$, $k^0, l^0 \in \mathcal{M}$, $k^+, l^+ \in \mathcal{N}$. We have

$$k^- k^0 k^+ = (w_0 l^+ w_0^{-1})(w_0 l^0 w_0^{-1})(w_0 l^- w_0^{-1}) \in \overline{N}MN.$$

By uniqueness of expression in $\overline{N}MN$,

$$k^0 = w_0 l^0 w_0^{-1}.$$

Further,

$$\rho(r_1^{-1}p_1) = \rho_{\mathcal{M}}(k^0),$$
$$\overline{\rho}(r_2 p_2^{-1}) = \overline{\rho}_{\mathcal{M}}(l^0).$$

We want to prove that

$$\Lambda_1(p_1 w_0 p_2) = \Lambda_1(r_1 w_0 r_2).$$

That is, we want

$$\rho(p_1)t\overline{\rho}(p_2) = \rho(r_1)t\overline{\rho}(r_2).$$

We can state this as

$$\rho(r_1^{-1}p_1)t = t\overline{\rho}(r_2 p_2^{-1}).$$

That is,

$$\rho_{\mathcal{M}}(k^0)t = t\overline{\rho_{\mathcal{M}}}(l^0).$$

But $l^0 = w_0 k^0 w_0^{-1}$ ($w_0^{-1} = w_0$), so we need

$$\rho_{\mathcal{M}}(k^0)t = t\overline{\rho_{\mathcal{M}}}(w_0 k^0 w_0^{-1}).$$

This comes from the definition of t. Hence Λ_1 is well-defined.

Likewise we obtain a well-defined function Υ_1 with support $\mathcal{P}w_0\mathcal{P}$ satisfying

$$\Upsilon_1(p_1 w_0 p_2) = \overline{\rho}(p_1)t\rho(p_2)$$

for $p_i \in \mathcal{P}$.

Now the convolution $\Lambda_1\Upsilon_1 \in \mathcal{H}(G,\rho)$ has support of contained in

$$\mathcal{P}w_0\mathcal{P}w_0\mathcal{P} = \mathcal{P} \sqcup \mathcal{P}w_0\mathcal{P}.$$

If $\Lambda_1\Upsilon_1(w_0) \neq 0$, then there is a function in $\mathcal{H}(G,\rho)$ which is supported on $\mathcal{P}w_0\mathcal{P}$. This implies ${}^{\varpi_D}\rho_{\mathcal{M}} \simeq \rho_{\mathcal{M}}$, so ${}^{\varpi_D}\rho \simeq \rho$. As we are in case (A), this is a contradiction. Hence the support of $\Lambda_1\Upsilon_1$ is contained in $\mathcal{P}$ and $\Lambda_1\Upsilon_1 = c1_\rho \in$

$\mathcal{H}(G,\rho)$ for some constant $c \in \mathbb{C}^\times$. The identity element 1_ρ is supported on $\mathcal{P}$ and satisfies

$$1_\rho(p) = \frac{1}{\mathrm{vol}(\mathcal{P})}\rho(p)$$

for $p \in \mathcal{P}$. Hence

$$\Lambda_1\Upsilon_1(1) = \frac{c}{\mathrm{vol}(\mathcal{P})}1_V.$$

We now determine the value of c.

$$\begin{aligned}
\frac{c}{\mathrm{vol}(\mathcal{P})}1_V &= \Lambda_1\Upsilon_1(1) \\
&= \int_{G_\varepsilon} \Lambda_1(g)\Upsilon_1(g^{-1})\,dg \\
&= \int_{\mathcal{P}w_0\mathcal{P}} \Lambda_1(g)\Upsilon_1(g^{-1})\,dg.
\end{aligned}$$

We use here that the support of Λ_1 and Υ_1 is $\mathcal{P}w_0\mathcal{P}$. For $g = p_1w_0p_2 \in \mathcal{P}w_0\mathcal{P}$, we have $g^{-1} = p_2^{-1}w_0p_1^{-1}$, so

$$\begin{aligned}
\Lambda_1(g)\Upsilon_1(g^{-1}) &= \rho(p_1)t\overline{\rho}(p_2)\overline{\rho}(p_2^{-1})t\rho(p_1^{-1}) \\
&= \rho(p_1)tt\rho(p_1^{-1}) \\
&= \rho(p_1)\rho(p_1^{-1}) \\
&= 1_V.
\end{aligned}$$

Thus

$$\begin{aligned}\frac{c}{\mathrm{vol}(\mathcal{P})}1_V &= \int_{\mathcal{P}w_0\mathcal{P}} 1_V \, dg \\ &= \mathrm{vol}(\mathcal{P}w_0\mathcal{P})1_V.\end{aligned}$$

Hence

$$\begin{aligned}c &= \mathrm{vol}(\mathcal{P})\,\mathrm{vol}(\mathcal{P}w_0\mathcal{P}) \\ &= \mathrm{vol}(\mathcal{P})\,\mathrm{vol}(\mathcal{P})[\mathcal{P}w_0\mathcal{P} : \mathcal{P}].\end{aligned}$$

Let $\Lambda = c^{-1/2}\Lambda_1$ and $\Upsilon = c^{-1/2}\Upsilon_1$. In all

$$\begin{aligned}\Lambda\Upsilon &= c^{-1}\Lambda_1\Upsilon_1 \\ &= c^{-1}c1_\rho \\ &= 1_\rho.\end{aligned}$$

In the same way

$$\Upsilon\Lambda = 1_{\overline{\rho}}.$$

□

We can say a bit more about the constant c in the proof above. In fact, we can compute the index $[\mathcal{P}w_0\mathcal{P} : \mathcal{P}] = [\mathcal{P} : \mathcal{P} \cap w_0\mathcal{P}w_0^{-1}]$. Here

$$\mathcal{P} \cap w_0\mathcal{P}w_0^{-1} = \begin{pmatrix} \mathcal{O}_D & \mathcal{O}_D \\ \mathfrak{p}_D & \mathcal{O}_D \end{pmatrix} \cap \begin{pmatrix} \mathcal{O}_D & \mathfrak{p}_D \\ \mathcal{O}_D & \mathcal{O}_D \end{pmatrix} = \begin{pmatrix} \mathcal{O}_D & \mathfrak{p}_D \\ \mathfrak{p}_D & \mathcal{O}_D \end{pmatrix}.$$

(This is, of course, all intersected with G_1.) Hence we see that the index of

$\mathcal{P} \cap w_0 \mathcal{P} w_0^{-1}$ is exactly the size of the set

$$N(\mathbb{F}_{q^2}) = \left\{ \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} : X \in M_n(\mathcal{O}_D/\mathfrak{p}_D), X + \overline{X}^\top = 0 \right\}.$$

The size of this set is

$$|N(\mathbb{F}_{q^2})| = q^{n^2}.$$

Note that $X = (x_{ij}) \in N(\mathbb{F}_{q^2})$ means that $x_{ji} = -x_{ij}^q = 0$ for $j > i$. For $i = j$ we get $x_{ii} + x_{ii}^q = 0$. This equation has exactly q solutions. In all, the number of elements in $N(\mathbb{F}_{q^2})$ is

$$\begin{aligned}(q^2)^{\sum_{i=1}^{n-1} i} q^n &= (q^2)^{\frac{1}{2}(n-1)n} q^n \\ &= q^{(n-1)n} q^n \\ &= q^{n^2}.\end{aligned}$$

Thus

$$c = \mathrm{vol}(\mathcal{P})^2 q^{n^2}.$$

Since $n = 2m$ is even,

$$\sqrt{c} = \mathrm{vol}(\mathcal{P}) q^{n^2/2} = \mathrm{vol}(\mathcal{P}) q^{2m^2}.$$

We can (and do) choose an element $f_1' \in \mathcal{H}(G_\varepsilon, \overline{\rho})$ such that

$$(f_1')^2 = q^n + (q^n - 1) f_1'.$$

Let $f_0 = \Lambda f_1' \Upsilon$. Then

$$\begin{aligned} f_0^2 &= (\Lambda f_1' \Upsilon)^2 \\ &= \Lambda f_1' \Upsilon \Lambda f_1' \upsilon \\ &= \Lambda (f_1')^2 \upsilon \\ &= \Lambda (q^n + (q^n - 1) f_1') \Upsilon \\ &= q^n \Lambda \Upsilon + (q^n - 1) \Lambda f_1' \Upsilon \\ &= q^n + (q^n - 1) f_0. \end{aligned}$$

3.6.3 Case (B)

We now turn to the case where $n = 2m+1$ is odd and ${}^{\varpi_D}\rho \simeq \rho^\vee$. The arguments here are broadly similar to those in case (A), so we do not provide all the details, but just point out the differences. The groups $\mathcal{N}, \mathcal{M}$, and $\overline{\mathcal{N}}$ are as before. Again we recall that $k = \mathbb{F}_{q^2}$ and $l = \mathbb{F}_{q^{2n}}$. Let

$$w_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$s_1 = \begin{pmatrix} 0 & -\varpi_D^{-1} \\ \varpi_D & 0 \end{pmatrix}.$$

Now

$$s_0 = s_1 w_0 s_1^{-1} = \begin{pmatrix} 0 & \varpi_F^{-1} \\ \varpi_F & 0 \end{pmatrix}.$$

Then

$$N_G(\mathcal{M}) = \langle w_0, s_1 \rangle \mathcal{M}$$

and

$$\mathfrak{I}_G(\rho) = \mathcal{P}\langle w_0, s_0 \rangle \mathcal{P}.$$

Let

$$K_0 = \langle \mathcal{P}, w_0 \rangle.$$

3.6.3.1 Quadratic relations I

Note $K_0 \subseteq \mathrm{GL}_{2n}(\mathcal{O}_D)$, so we can reduce K_0 mod $\mathfrak{p}_D$. We write Ψ for this operation.

As in 3.6.2.1, $\Psi(\mathcal{N})$ is trivial.

$$\Psi(\mathcal{M}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & {}^\top\bar{a}^{-1} \end{pmatrix} : a \in \mathrm{GL}_n(k) \right\}.$$

Now $\Psi(\mathcal{P})$ is a maximal parabolic subgroup of $U_{2n}(\mathbb{F}_{q^2})$, where we realize the unitary group using the form given by

$$J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

That is

$$U_{2n}(\mathbb{F}_{q^2}) = \{g \in \mathrm{GL}_{2n}(\mathbb{F}_{q^2}) : {}^{\top}\bar{g}Jg = J\}.$$

As $\Psi(w_0) \notin \Psi(\mathcal{P})$, it follows that $\Psi(K_0) = U_{2n}(\mathbb{F}_{q^2})$. Hence we need to computer the parameter λ from section 3.6.2 (page 44) and we do this in section 3.8.

3.6.3.2 Quadratic relations II

Everything works out as in section 3.6.2.2.

3.7 The $\varepsilon = -1$ case

Until now we have silently assumed that $\varepsilon = 1$. We now touch on the case when $\varepsilon = -1$. Much is the same and independent of the value of ε. Now, the group G_{-1} is realized with respect to the form corresponding to

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

We again get two cases and two sets of quadratic relations for each case.

We have $\mathcal{P} = \overline{\mathcal{N}}\mathcal{M}\mathcal{N}$ with $\mathcal{M}$ as above and

$$\mathcal{N} = \left\{ \begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix} : \overline{X}^{\top} = X \right\}, \quad \overline{\mathcal{N}} = \left\{ \begin{pmatrix} I_n & 0 \\ X & I_n \end{pmatrix} : \overline{X}^{\top} = X \right\}.$$

As in section 3.6.2.1 we get a situation where we want to reduce mod $\mathfrak{p}_D$. The key thing now is that

$$\Psi(\overline{\mathcal{N}}) = \left\{ \begin{pmatrix} I_n & 0 \\ x & I_n \end{pmatrix} : x^\top = -x \right\}.$$

For case (A) with the corresponding parahoric K, $\Psi(K)$ is the special orthogonal group $\mathrm{SO}_{2n}(\mathbb{F}_{q^2})$. As SO_{2n} does not have connected center, we cannot directly appeal to Lusztig's work to determine the parameter λ. Instead, we appeal to [9] to see that $\lambda = q^n$. For case (B) we again get the unitary group.

3.8 Unitary case

It remains to compute the parameter λ in the unitary and the symplectic cases. We start with the unitary case.

Let $G = U_{2n}(F_{q^2})$ (not to be confused with G_ε). Since the dual group $G^* \simeq G$, we will leave out the $*$ in the notation.

We realize G using the form

$$J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

That is,

$$G = \{g \in \mathrm{GL}_{2n}(\mathbb{F}_{q^2}) : {}^\top\bar{g}Jg = J\}.$$

We have

$$P = M \ltimes N$$

where

$$M = \{ \begin{pmatrix} a & 0 \\ 0 & {}^{\top}\bar{a}^{-1} \end{pmatrix} : a \in \mathrm{GL}_n(\mathbb{F}_{q^2}) \},$$
$$N = \{ \begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix} : X + \overline{X}^{\top} = 0 \}.$$

We also recall that

$$\theta^{q^n} = \theta^{-1}.$$

As mentioned in the background material, θ corresponds to an element s in $M^* \simeq M$. We write

$$s = \begin{pmatrix} \alpha & 0 \\ 0 & {}^{\top}\bar{\alpha}^{-1} \end{pmatrix} \in M.$$

Here $\alpha \in \mathbb{F}_{q^{2n}}^{\times}$, or more precisely, α is in the image of $\mathbb{F}_{q^{2n}}^{\times}$ under a fixed embedding $\mathbb{F}_{q^{2n}}^{\times} \hookrightarrow \mathrm{GL}_n(\mathbb{F}_{q^2})$. That θ is regular implies that α generates $\mathbb{F}_{q^{2n}}/\mathbb{F}_{q^2}$. That is, $\mathbb{F}_{q^{2n}} = \mathbb{F}_{q^2}[\alpha]$. Our goal is to compute

$$|C_G(s)|_p.$$

That is, we want the order of a Sylow p-subgroup of $C_G(s)$. By 3.19 in [6] the Sylow p-subgroups of $C_G(s)$ are the sets of $\mathbb{F}_{q^2}$-points of the unipotent radicals of the Borel subgroups of $C_{\mathbf{G}}(s)$. By 2.2 in [6], the Borel subgroups of $C_{\mathbf{G}}(s)$ have the form $\mathbf{B} \cap C_{\mathbf{G}}(s)$ where $\mathbf{B}$ is a Borel subgroup of $\mathbf{G}$. Hence $C_P(s) = P \cap C_G(s)$ contains a Sylow p-subgroup of $C_G(s)$. Now suppose that $x \in P$ commutes with s and that $x = mn$ with $m \in M$ and $n \in N$. Then $sx = xs$ and so $sxs^{-1} = x$.

Hence

$$sxs^{-1} = x = smns^{-1}sms^{-1}sns^{-1}.$$

Since $P = M \ltimes N$ is a direct product, s commutes with both m and n. Therefore

$$C_P(s) = C_M(s) \ltimes C_N(s).$$

The centralizer of s in M is T and thus has order prime to p. Furthermore $C_N(s)$ is a p-subgroup. In all,

$$C_N(s) = \{n \in N : sn = ns\}$$

is a Sylow p-subgroup of $C_G(s)$. We now compute $|C_N(s)|$. That is, we must compute the number of elements $n \in N$ that commute with s.

Lemma 3.12. *We have*

$$|C_N(s)| = q^n.$$

Proof. The elements of N are have the form

$$\begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix},$$

where $X + \overline{X}^{\top} = 0$. If

$$n = \begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix},$$

then $ns = sn$ gives

$$\begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & {}^{\top}\bar{\alpha}^{-1} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & {}^{\top}\bar{\alpha}^{-1} \end{pmatrix} \begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix},$$

which is equivalent to $\alpha X^{\top}\bar{\alpha} = X$. That is, we need to count the number of solutions to the following equations:

1. $\alpha X^{\top}\bar{\alpha} = X$.
2. $X + {}^{\top}\overline{X} = 0$.

Now let $\Xi = M_n(\mathbb{F}_{q^2})$. Let

$$\Xi_\varepsilon = \{X \in \Xi : {}^{\top}\overline{X} = \varepsilon X\}.$$

Then

$$\Xi = \Xi_1 \oplus \Xi_{-1},$$

since, for any $X \in \Xi$,

$$X = \frac{1}{2}(X + \overline{X}^{\top}) + \frac{1}{2}(X - \overline{X}^{\top}).$$

Let

$$\Xi(\alpha) = \{X \in \Xi : \alpha X \alpha^{\top} = X\},$$
$$\Xi_\varepsilon(\alpha) = \{X \in \Xi_\varepsilon : \alpha X \bar{\alpha}^{\top} = X\}.$$

Then

$$\Xi(\alpha) = \Xi_1(\alpha) \oplus \Xi_{-1}(\alpha).$$

Choose $\gamma \in \mathbb{F}_{q^2}$ such that $\gamma \neq 0$ and $\overline{\gamma} = -\gamma$. Then the map $\Xi_\varepsilon(\alpha) \to \Xi_{-\varepsilon}(\alpha)$ given by $X \mapsto \gamma X$ is a bijection.

We also have the bijection from Ξ_{-1} to $C_N(s)$ given by

$$X \mapsto \begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix}.$$

Hence

$$|C_N(s)| = |\Xi_{-1}(\alpha)| = |\Xi_1(\alpha)|.$$

If we can show that $|\Xi(\alpha)| = q^{2n}$, then we are done. That is, we want to count the number of elements $X \in \Xi$ satisfying that $\alpha X = X^\top \bar{\alpha}^{-1}$ for fixed α. Consider an embedding $\phi_1 : \mathbb{F}_{q^{2n}} \hookrightarrow M_n(\mathbb{F}_{q^2})$: $\beta \mapsto m_\beta$. Let $f(x)$ be the minimal polynomial of α over $\mathbb{F}_{q^2}$. Then

$$\mathbb{F}_{q^{2n}} \simeq \mathbb{F}_{q^2}[x]/\langle f(x) \rangle.$$

Under ϕ_1, a polynomial $p(\alpha)$ in α is mapped to $p(m_\alpha)$. We get another embedding $\phi_2 : \mathbb{F}_{q^{2n}} \simeq \mathbb{F}_{q^2}[x]/\langle f(x) \rangle \hookrightarrow M_n(\mathbb{F}_{q^2})$ satisfying

$$p(\alpha) \mapsto p({}^\top\bar{\alpha}^{-1}).$$

We first must show that this map is well-defined.

Let α^* denote ${}^\top\bar{\alpha}$. Also, for

$$p(x) = \sum_j b_j x^j \in \mathbb{F}_{q^2}[x],$$

we let

$$\bar{p}(x) = \sum_j \overline{b_j} x^j.$$

We need to show that the minimal polynomial evaluated at ${}^\top\overline{m}_\alpha^{-1}$ is zero. We note that

$$f({}^\top\overline{m}_\alpha^{-1}) = f(m_\alpha^{-1})^* = (\bar{f}(m_\alpha^{-1}))^*.$$

We want to show that $\bar{f}(m_\alpha^{-1}) = 0$, equivalently $\bar{f}(\alpha^{-1}) = 0$. We have $\theta^{-1} = \theta^{q^n}$, so the same relation applies to s and also to α. Thus $\alpha^{-1} = \alpha^{q^n}$. Let $c \in \mathrm{Gal}(\mathbb{F}_{q2n}/\mathbb{F}_q)$ be the map $c(\lambda) = \lambda^{q^n}$. We have $f(\alpha) = 0$, so

$$0 = f(\alpha)^{q^n} = \bar{f}(\alpha^{q^n}) = \bar{f}(\alpha^{-1}).$$

The second equality uses the fact that n is odd. Hence the embedding is well-defined.

We now have two different embeddings ϕ_1 and ϕ_2 of l in $M_n(\mathbb{F}_{q^2})$. We want to find the number of X satisfying $\alpha X = X {}^\top\bar{\alpha}^{-1}$. That is, we want to find the number of X satisfying

$$X\phi_2(\lambda) = \phi_1(\lambda)X$$

for all $\lambda \in \mathbb{F}_{q^{2n}}$. Let $k = \mathbb{F}_{q^2}$ and $l = \mathbb{F}_{q^{2n}}$. As in the proof of Proposition 3.5, we make k^n into an l-module in two ways:

$$\lambda.v = \phi_1(\lambda)v,$$
$$\lambda_* v = \phi_2(\lambda)v.$$

We then obtain two l-modules that we denote ${}_1k^n$ and ${}_2k^n$. That means that $X\phi_2(\lambda) = \phi_1(\lambda)X$ is equivalent to

$$X \in \mathrm{Hom}_l({}_1k^n, {}_2k^n) \simeq \mathrm{Hom}_l(l, l) \simeq l,$$

so $|\mathrm{Hom}_l({}_1k^n, {}_2k^n)| = q^{2n}$. □

3.9 Symplectic case

We again let $k = \mathbb{F}_{q^2}$ and $l = \mathbb{F}_{q^{2n}}$. Let $G = \mathrm{Sp}_{2n}(\mathbb{F}_{q^2})$. The dual group $G^* = \mathrm{SO}_{2n+1}(\mathbb{F}_{q^2})$. We realize G^* using

$$J = \begin{pmatrix} 0 & 0 & I_n \\ 0 & 1 & 0 \\ I_n & 0 & 0 \end{pmatrix}.$$

That is,

$$G^* = \{g \in \mathrm{SL}_{2n+1}(\mathbb{F}_{q^2}) : {}^{\top}gJg = J\}.$$

Let

$$M^* = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & {}^\top a^{-1} \end{pmatrix} : a \in \mathrm{GL}_n(\mathbb{F}_{q^2}) \right\}$$

and

$$N^* = \left\{ \begin{pmatrix} I_n & u & X \\ 0 & 1 & -u^\top \\ 0 & 0 & I_n \end{pmatrix} : X^\top + uu^\top + X = 0 \right\}.$$

Then

$$P^* = M^* \ltimes N^*$$

is the standard Siegel parabolic subgroup. As mentioned in section 2.3, the character θ corresponds to an element

$$s^* = \begin{pmatrix} \alpha & & \\ & 1 & \\ & & {}^\top \alpha^{-1} \end{pmatrix}$$

in M^*. Here $\alpha \in \mathbb{F}_{q^{2n}}^\times$. That θ is regular means that $k[\alpha] = \mathbb{F}_{q^{2n}}$. We want to compute

$$|C_{G^*}(s^*)|_p.$$

First we observe (as in the unitary case) that

$$|C_{G^*}(s^*)| = |C_{N^*}(s^*)|.$$

Lemma 3.13. *We have*

$$|C_{N^*}(s^*)| = q^n.$$

Proof. Let

$$\begin{pmatrix} I_n & u & X \\ 0 & 1 & -u^\top \\ 0 & 0 & I_n \end{pmatrix} \in N^*.$$

Then $s^*n = ns^*$ means

1. $\alpha u = u$,
2. ${}^\top u^\top a\alpha^{-1} = {}^\top u$, and
3. $\alpha X = X^\top \alpha^{-1}$.

The first condition gives $u = 0$ since $\alpha \neq 1$ which holds because $k(\alpha) = l$. We are left with the third condition. As before, let $\Xi = M_n(k)$ and $\Xi_\varepsilon = \{X \in \Xi : {}^\top X = \varepsilon X\}$. Again,

$$\Xi = \Xi_1 \oplus \Xi_{-1}.$$

We define $\Xi(\alpha)$, $\Xi_\varepsilon(\alpha)$ as before, so that

$$\Xi(\alpha) = \Xi_1(\alpha) \oplus \Xi_{-1}(\alpha).$$

We again have a bijection from $\Xi_{-1}(\alpha) \to C_{N^*}(s^*)$ now given by

$$X \mapsto \begin{pmatrix} I_n & 0 & X \\ 0 & 1 & 0 \\ 0 & 0 & I_n \end{pmatrix}.$$

As in the unitary case, we define two l-module structures on k^n. Again let $f(x)$ be the minimal polynomial of α over k. As before, $l \simeq k[x]/\langle f(x) \rangle$. For $p(x) \in k[x]$ we define an l-action on k^n by

$$p(\alpha).v = p(\alpha)v.$$

Note this makes sense since $p(\alpha) \in M_n(k)$. We define the second l-module structure via

$$p(\alpha)_* v = p({}^\top\alpha^{-1})v = {}^\top p(\alpha^{-1})v.$$

Again this makes sense since $p(\alpha^{-1}) \in M_n(k)$. Again we need to check that this second structure is well-defined. That is, we need to check that $f(\alpha^{-1}) = 0$. Again we use that $\alpha^{q^n} = \alpha^{-1}$, so

$$\begin{aligned} 0 &= f(\alpha) \\ &= f(\alpha)^{q^n} \\ &= f(\alpha^{q^n}) \\ &= f(\alpha^{-1}). \end{aligned}$$

As before, we now have two l-module structures ${}_1k^n$ and ${}_2k^n$ on k^n. Once again

$\Xi(\alpha) \simeq l$, and so

$$|\Xi(\alpha)| = |l| = q^{2n}.$$

Since $\Xi(\alpha) = \Xi_1(\alpha) \oplus \Xi_{-1}(\alpha)$,

$$|\Xi(\alpha)| = |\Xi_1(\alpha)||\Xi_{-1}(\alpha)|.$$

In particular, each $|\Xi_\varepsilon(\alpha)|$ is non-zero. Let $X \in \Xi_{-1}(\alpha)$, $X \neq 0$. Then

$$\Xi(\alpha) = \{p(\alpha)X : p(x) \in k[x]\}.$$

We have $p(\alpha)X \in \Xi_{-1}(\alpha)$ exactly when

$$(p(\alpha)X)^\top = -p(\alpha)X.$$

Here the left hand side is

$$X^\top p(\alpha)^\top = -Xp(\alpha)^\top$$

because $X \in \Xi_{-1}(\alpha)$. The right hand side is

$$-Xp(\alpha^{-1})^\top.$$

So $p(\alpha)X$ is in $\Xi_{-1}(\alpha)$ exactly when

$$Xp(\alpha)^\top = Xp(\alpha^{-1})^\top,$$

that is, when

$$p(\alpha) = p(\alpha^{-1}).$$

Now let $\sigma \in \mathrm{Gal}(l/k)$ denote the unique element of order 2 given by $\sigma(x) = x^{q^n}$. Then $\sigma(\alpha) = \alpha^{-1}$. So the condition above is equivalent to

$$p(\alpha) = \sigma(p(\alpha)).$$

Let l_0 be the fixed field of σ. Then $\langle\sigma\rangle = \mathrm{Gal}(l/l_0)$ and so $[l : l_0] = 2$. In conclusion we get

$$\Xi_{-1}(\alpha) = \{\lambda X : \lambda \in l_0\}$$

and this set has order $|l_0| = q^n$. Hence,

$$|\Xi_1(\alpha)| = |\Xi_{-1}(\alpha)| = q^n.$$

$\square$

3.9.1 Conclusion

We have now proved Theorem 3.10 on page 38. That is, we have proved that in all cases the Hecke algebra $\mathcal{H}(G_\varepsilon, \rho)$ is generated by two elements f_0 and f_1 subject to the same relation $f_i^2 = q^n + (q^n - 1)f_i$. This will now allow us to give the answer to the question that we posed in the beginning:

When is $\iota_P^{G_\varepsilon}(\pi_0 \otimes \chi)$ reducible?

That is, for what χ is the induced representation reducible?

3.10 Answering the question

We recall the diagram

$$
\begin{tikzcd}
\mathfrak{R}^{\mathfrak{s}}(G_\varepsilon) \arrow[r, "m_{G_\varepsilon}"] & \mathcal{H}(G_\varepsilon, \rho) - \mathrm{Mod} \\
\mathfrak{R}^{\mathfrak{s}_L}(L) \arrow[u, "\iota_P^G"] \arrow[r, "m_L"'] & \mathcal{H}(L, \rho) - \mathrm{Mod}. \arrow[u, "t_{P*}"']
\end{tikzcd}
$$

We are interested in the induced representation, that is, in the functor

$$\iota_P^{G_\varepsilon} : \mathfrak{R}^{\mathfrak{s}_L}(L) \longrightarrow \mathfrak{R}^{\mathfrak{s}}(G_\varepsilon).$$

We have now translated the question into a question about the map

$$t_{P*} : \mathcal{H}(L, \rho) - \mathrm{Mod} \longrightarrow \mathcal{H}(G_\varepsilon, \rho) - \mathrm{Mod}.$$

We have found generators and relations for $\mathcal{H}(G_\varepsilon, \rho)$. Let

$$\mathcal{D} = \mathbb{C}[d, d^{-1}]$$
$$\mathcal{H} = \langle h_0, h_1 : h_i^2 = 1 + (q^{n/2} - q^{-n/2})h_i \rangle$$

with d an indeterminate. We recall from page 39 the note that with $h_i = q^{-n/2} f_i$, $\mathcal{H}$ is generated by h_0 and h_1 subject to the relation $h_i^2 = 1 + (q^{n/2} - q^{-n/2})h_i$. We recall also that $\mathcal{H}(L, \rho) \simeq \mathbb{C}[d, d^{-1}]$.

Then

$$\mathcal{H}(L, \rho) - \mathrm{Mod} \simeq \mathcal{D} - \mathrm{Mod}$$
$$\mathcal{H}(G_\varepsilon, \rho) - \mathrm{Mod} \simeq \mathcal{H} - \mathrm{Mod}.$$

Therefore, we have the map

$$\iota_* : \mathcal{D} - \mathrm{Mod} \longrightarrow \mathcal{H} - \mathrm{Mod}.$$

This map is given by $\iota_*(T) = \mathrm{Hom}_{\mathcal{H}}(\mathcal{H}, T)$ where $\mathcal{H}$ acts by right translation.

Proposition 3.14. *We have $\iota_*(d) = h_0 h_1$.*

The proof of this Proposition is like the proof of Proposition 6.4 in [9]. From Proposition 1.6 in [10], $\iota_*(\psi)$ is reducible exactly when $\psi(d) \in \{q^{-n}, -1, q^n\}$. Recall that π_0 is self-dual, and so the square of its central character ω_{π_0} is 1: $\omega_{\pi_0}^2 = 1$. That is, the image of ω_{π_0} is contained in $\{\pm 1\}$.

For χ an unramified character of L,

$$m_L(\pi_0 \otimes \chi)(d) = \omega_{\pi_0}(\varpi_F)\chi(\varpi_F).$$

If χ is trivial, then we have reducibility if and only if $\omega_{\pi_0}(\varpi_F) = -1$. Assume that χ is not trivial. We know that $\chi(x) = |\mathrm{Nrd}(x)|_F^s$. So we need

$$\begin{aligned}
\{q^{-n}, -1, q^n\} &\ni \chi(\varpi_F) \\
&= |\mathrm{Nrd}(\varpi_F)|_F^s \\
&= |\varpi_F^{2n}|_F^s \\
&= q^{-\nu_F(\varpi_F^{2n})s} \\
&= q^{-2ns}.
\end{aligned}$$

Hence we need $s \in \left\{\pm\frac{1}{2}\right\}$

We state this a bit differently. One can choose π_0 such that $\omega_{\pi_0}(\varpi_F) = 1$, and so we end with the following theorem.

Theorem 3.15. *Let π_0 be a depth zero irreducible unitary supercuspidal representation of G_ε. Choose π_0 such that it has trivial central character. Let χ be an unramified character of L. Then*

$$\iota_P^G(\pi_0 \otimes \chi)$$

is reducible if and only if $\chi(\varpi_F) = -1$ or $\chi(\cdot) = |\mathrm{Nrd}(\cdot)|_F^{\pm 1/2}$.

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